Insurance against Market Crashes

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Motivation

Mathematical Formalism

Insurance claims
- Drawdown insurance
- Cancellable drawdown insurance
- Drawdown insurance contingent on drawups

Reference
How to insure?
How much is the insurance?
Filtered probability space \((\Omega, \mathbb{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})\) with filtration \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\).

Price/value process: \(\{S_t\}_{t \geq 0}: \frac{dS_t}{S_t} = rdt + \sigma dW_t\).

Log price/value process: \(\{X_t\}_{t \geq 0}\) where \(X_t = \log S_t\) and \(x = X_0\).

Drawdown process: \(D_t = \overline{X}_t - X_t\), where \(\overline{X}_t = \overline{x} \lor \left(\sup_{s \in [0, t]} X_s\right)\).

Drawup process: \(U_t = X_t - \underline{X}_t\), where \(\underline{X}_t = \underline{x} \land \left(\inf_{s \in [0, t]} X_s\right)\).

Reference period’s low & high: \(\underline{x} \leq \overline{x} < \underline{x} + k\).

First hitting times of the drawdown/drawup process:

\[
\tau_D(k) = \inf\{t \geq 0 | D_t \geq k\}
\]

\[
\tau_U(k) = \inf\{t \geq 0 | U_t \geq k\}
\]

A market crash is modeled as \(\tau_D(k)\)!
Figure: July of 2011 is the reference period. $\bar{x} = 7.21$ and $x = 7.16$. Initial drawdown $y = D_0 = 0.05$. The large drawdown in August is due to the downgrade of US debt by S&P.
Insurance claims against a market crash

Let $r \geq 0$ be the risk-free interest rate, $\mathbb{Q}$ the risk-neutral measure.

- **Drawdown insurance**: time-0 value is (seen from the protection buyer)
  
  $$V_0(p) = \mathbb{E}^{\mathbb{Q}} \left\{ - \int_0^{\tau_D(k) \wedge T} pe^{-rt} dt + \alpha e^{-r \tau_D(k)} I_{\{\tau_D(k) \leq T\}} \right\}$$

- **Ways to terminate a drawdown insurance when necessary**
  - **Callable drawdown insurance**: the time-0 value is (seen from the protection buyer, $\tau$ is the cancelation time)
    
    $$V_0^c(p) = \sup_{0 \leq \tau < T} \mathbb{E}^{\mathbb{Q}} \left\{ - \int_0^{\tau_D(k) \wedge \tau} pe^{-rt} dt + \alpha e^{-r \tau_D(k)} I_{\{\tau_D(k) \leq \tau\}} - ce^{-r \tau} I_{\{\tau < \tau_D(k)\}} \right\}$$

  - **Drawdown insurance contingent on drawups**: time-0 value is (seen from the protection buyer)
    
    $$V_0^U(p) = \mathbb{E}^{\mathbb{Q}} \left\{ - \int_0^{\tau_D(k) \wedge \tau_U(k) \wedge T} pe^{-rt} dt + \alpha e^{-r \tau_D(k)} I_{\{\tau_D(k) \leq \tau_U(k) \wedge T\}} \right\}$$
Fair evaluation

- The premium is the rate $P^*$ such that the time-0 value of a insurance is zero:
  \[ v_0(P^*) = 0 \]

- Value calculation:
  \[
  v_0(p) = E^Q \left\{ - \int_0^{\tau_D(k) \wedge T} p e^{-rt} dt + \alpha e^{-r\tau_D(k)} \mathbb{I}_{\{\tau_D(k) \leq T\}} \right\} \\
  = E^Q \left\{ \left( \frac{p}{r} + \alpha \mathbb{I}_{\{\tau_D(k) \leq T\}} \right) e^{-r(\tau_D(k) \wedge T)} - \frac{p}{r} \right\} 
  
  If perpetual $T = \infty$, then
  \[
  v_0(p) = \frac{p}{r} - \left( \alpha + \frac{p}{r} \right) \xi(D_0) := -f(D_0, p) 
  
  where $\xi(y) = E^Q\{e^{-r\tau_D(k)}|D_0 = y\}$
The conditional Laplace transform $\xi(y)$

For $\mu = r - \frac{1}{2}\sigma^2$, $0 \leq y_1, y_2 < k$, $\Xi_{\mu, \sigma}^r = \sqrt{\frac{2r}{\sigma^2} + \frac{\mu^2}{\sigma^4}}$

The quantity $\xi(y) = E^Q\{e^{-r(\tau_D(k))} | D_0 = y\}$ satisfies functional equation:

$$
\xi(y_2) = e^{\frac{\mu}{\sigma^2}(y_2-k)} \frac{\sinh(\Xi_{\mu, \sigma}^r(y_2-y_1))}{\sinh(\Xi_{\mu, \sigma}^r(k-y_1))} + e^{\frac{\mu}{\sigma^2}(y_2-y_1)} \frac{\sinh(\Xi_{\mu, \sigma}^r(k-y_2))}{\sinh(\Xi_{\mu, \sigma}^r(k-y_1))} \xi(y_1)
$$

Equivalently,

$$
\Lambda(y_2) - \lambda(y_1) = e^{-\frac{\mu k}{\sigma^2}} \frac{\sinh(\Xi_{\mu, \sigma}^r(y_2-y_1))}{\sinh(\Xi_{\mu, \sigma}^r(k-y_1)) \sinh(\Xi_{\mu, \sigma}^r(k-y_2))}
$$

$\xi(0)$ is calculated by H. Taylor 1975.
More properties of $\xi(y)$

- $\xi(y)$ is increasing over $[0, k]$: continuity of path and Markov property
- Neumann condition at 0: $\xi'(0) = 0$
- ODE: Feynman-Kac
  \[
  \frac{1}{2}\sigma^2 \xi''(y) - \mu \xi' = r \xi(y)
  \]
- $\xi(y)$ is strictly convex, i.e., $\xi''(y) > 0$ for all $y \in (0, k)$
Pricing a cancellable drawdown insurance

Callable drawdown insurance: recall that the time-0 value is (seen from the protection buyer, \( \tau \) is the cancellation time, \( c \) is the cancellation fee)

\[
V_c^0(p) = \sup_{0 \leq \tau < T} \mathbb{E}^Q \left\{ -\int_0^{\tau_D(k) \wedge \tau} pe^{-rt} dt + \alpha e^{-rT_D(k)} \mathbb{I}\{\tau_D(k) \leq \tau\} - ce^{-r\tau} \mathbb{I}\{\tau < \tau_D(k)\} \right\}
\]

To find the fair premium \( p^* \), we need to first solve the above optimal stopping problem to find the value function \( V_c^0(p) \), and then solve for \( P^* \) in

\[
V_c^0(P^*) = 0
\]
Premium of cancellation

- To avoid unnecessary complications, we consider perpetual insurances, i.e., $T = \infty$

- Notice that, for any cancellation time $\tau < \tau_D(k)$,

$$
- \int_0^{\tau_D(k) \wedge \tau} pe^{-rt} dt + \alpha e^{-r\tau_D(k)} I\{\tau_D(k) \leq \tau\} - ce^{-rt} I\{\tau < \tau_D(k)\}
$$

$$
= - \int_0^{\tau_D(k)} pe^{-rt} dt + \alpha e^{-r\tau_D(k)} + \int_{\tau_D(k) \wedge \tau}^{\tau_D(k)} pe^{-rt} dt - ce^{-r\tau} I\{\tau < \tau_D(k)\} - \alpha e^{-r\tau_D(k)} I\{\tau < \tau_D(k)\}
$$

Extra premium from cancellation

- Let $V_0(p)$ be the time-0 value of a perpetual drawdown insurance, then necessarily,

$$
V_0(p) \leq V_0^c(p).
$$
The cancellable drawdown insurance as an American call type contract

- Recall that:
  \[ V_0(p) = -f(D_0, p), \text{ where } f(y) := \frac{p}{r} - \left(\alpha + \frac{p}{r}\right)\xi(y) \]

- The value function of the cancellable drawdown insurance can also be computed:
  \[ V_0^c(p) = V_0(p) + \sup_{\tau \in S} E^Q\{e^{-r\tau}(f(D_\tau) - c)\}, \quad S = \{\tau | 0 \leq \tau < \tau_D(k)\} \]

- Since \( \xi(\cdot) \) is increasing, \( f(\cdot) \) is decreasing. To avoid trivial optimal cancellation strategy \( (\tau^* \equiv \infty) \), it is necessary to have \( f(0) > 0 \). In other words,
  \[ \text{Cond} : p > \frac{r(c + \alpha \xi(0))}{1 - \xi(0)} \geq 0 \]

- Under condition \textbf{Cond}, we seek the optimal exercise time:
  \[ \sup_{\tau \in S} E^Q\{e^{-r\tau}\tilde{f}(D_\tau)\mathbb{1}_{\{\tau < \tau_D(k)\}}\}, \text{ with } \tilde{f} = f - c \]
Method of solution

- Conjecture a stopping time of the form $\tau^\theta := \tau^-_D(\theta) \land \tau_D(k) \in S$, where
  $$\tau^-_D(\theta) = \inf\{ t \geq 0 | D_t \leq \theta \}, \ 0 < \theta < k$$

- We seek a $\theta^*$ through smooth pasting
  $$\frac{\partial}{\partial y} \bigg|_{y=\theta} \mathbb{E}^Q\{ e^{-r\tau^\theta} \tilde{f}(D_{\tau^\theta}) \mathbb{I}_{\{\tau^\theta < \tau_D(k)\}} | D_0 = y \} = \tilde{f}'(\theta)$$

- Let $V(\theta^*, y) = \mathbb{E}^Q\{ e^{-r\tau^{\theta^*}} \tilde{f}(D_{\tau^{\theta^*}}) \mathbb{I}_{\{\tau^{\theta^*} < \tau_D(k)\}} | D_0 = y \}$, show that
  $$\{ e^{-r(t \land \tau_D(k))} V(\theta^*, D_{t \land \tau_D(k)}) \}_{t \geq 0}$$
  is the smallest supermartingale
  dominating $$\{ \tilde{f}(D_{t \land \tau_D(k)}) \}_{t \geq 0}$$

- Verify the cancellation strategy based on $\theta^*$ is indeed optimal
Smooth pasting

Figure: Model parameters:
\[ r = 2\%, \ p = P^* = 1.5245, \ \sigma = 30\%, \ k = 30\%, \ \alpha = 1, \ c = 0.05 \text{ and } D_0 = 10\%. \]
The “intrinsic function” \( \tilde{f}(\cdot) \) is shown in red dash line, the optimal extra premium from cancellation is shown in blue solid line. The only point determined by smooth pasting is \( \theta^* \approx 5\% \).
Theorem

Under the proposed model, there exists a unique solution $\theta^* \in (0, \theta_0)$ to equation

$$\frac{\partial}{\partial y} E^Q \{e^{-r_{\tau^\theta}} \tilde{f}(D_{\tau^\theta}) \mathbb{I}_{\{\tau^\theta < \tau_D(k)\}} | D_0 = y\} = \tilde{f}'(\theta).$$

Moreover, for any $\theta \in (\theta^*, k)$,

$$E^Q \{e^{-r_{\tau^\theta^*}} \tilde{f}(D_{\tau^\theta^*}) \mathbb{I}_{\{\tau^\theta^* < \tau_D(k)\}} | D_0 = \theta\} > \tilde{f}(\theta).$$

Here $\theta_0 \in (0, k)$ is the unique root to equation $f(\theta) = 0$.

- Mean value theorem implies existence
- Uniqueness: We use properties of $\Lambda(\cdot)$ and representation $\tilde{f}(\theta) = (\alpha + \frac{p}{r})(\xi(\theta_0) - \xi(\theta))$ to prove it.
- The last result in the theorem asserts that
  $$\{e^{-r(t \wedge \tau_D(k))} V(\theta^*, D_{t \wedge \tau_D(k)}) \}_{t \geq 0}$$
  is the smallest supermartingale dominating $\{f(D_{t \wedge \tau_D(k)}) \}_{t \geq 0}$.
Determine the fair premium implicitly

- If $\tilde{f}(0) \leq 0$, the fair premium is obtained from
  \[ V_0(P^*) = 0 \]

- If $\tilde{f}(0) > 0$, the fair premium is obtained from
  \[ V_0(P^*) + E^Q \left\{ e^{-r \tau^*} f(D_{\tau^*} \theta^*) \mathbb{I}_{\{\tau^* < \tau_D(k)\}} \right\} = 0 \]

  Notice that in this case, $\theta^* = \theta^*(P^*)$ depends on $P^*$.

- To see the dependence of $P^*$ on the size of drawdown $k$, we plot the value function $V_0^c$ on a grid of $(k, p)$, and then find the zero contour.
The value function and the fair premium $p^*$ vs. $k$ (B-S model)

**Figure:** The fair premium of the cancellable drawdown insurance decreases with respect to the drawdown strike level $k$. Model parameters: $r = 2\%$, $\sigma = 30\%$, $\alpha = 1$, $c = 0.05$ and $D_0 = 10\%$. 
Figure: Model parameters: $r = 1\%$, $\sigma = 15\%$, $\alpha = 1$, and $D_0 = 10\%$. The fair premium $p^*(c = \infty)$ for the non-callable drawdown insurance is shown in red. It is seen that the fair premium $P^*(c)$ is decreasing in the cancellation fee $c$. 
Recall that

\[ V_0^U(p) = \left( \alpha \mathbb{I}_{\{\tau_D(k) \leq \tau_U(k) \wedge T\}} + \frac{p}{r} \right) E^Q \left\{ e^{-r(\tau_D(k) \wedge \tau_U(k) \wedge T)} \right\} - \frac{p}{r} \]

For drawdown insurance contingent on drawups:

\[ P^* = \frac{r \alpha E^Q \left\{ e^{-r\tau_D(k)} \mathbb{I}_{\{\tau_D(k) \leq \tau_U(k) \wedge T\}} \right\}}{1 - E^Q \left\{ e^{-r(\tau_D(k) \wedge \tau_U(k) \wedge T)} \right\}} \]

Examine the dependence of \( P^* \) on interest rate, volatility, maturity and other model parameters.
Finite time-horizon

Using Zhang&Hadjiliadis ’10, the following probability can be obtained for drifted Brownian motion $X$

$$
\mathbb{Q}\{\tau_D(k) \leq \tau_U(k) \land T\}, \quad \mathbb{Q}\{\tau_D(k) \land \tau_U(k) \leq T\}
$$

Explicit computation of the fair premium

$$
P^* = r \alpha \int_0^T e^{-rt} \left( \frac{\partial}{\partial t} \mathbb{Q}\{\tau_D(k) \leq \tau_U(k) \land t\} \right) dt
- \int_0^T e^{-rt} \left( \frac{\partial}{\partial t} \mathbb{Q}\{\tau_D(k) \land \tau_U(k) \leq t\} \right) dt
$$
Large-time and infinite time-horizons

- For a large time-horizon $T$, it is known that

$$P^*(T) \to P^*(\infty), \text{ as } T \to \infty$$

where $P^*(\infty)$ is the fair premium for perpetual insurance

- Using Zhang & Hadjiliadis ’09, the following Laplace transform can be obtained for a general regular linear diffusion $X$

$$E^Q\{ e^{-r \tau_D(k)} \mathbb{I}_{\{ \tau_D(k) \leq \tau_U(k) \}} \}, \quad E^Q\{ e^{-r (\tau_D(k) \wedge \tau_U(k))} \}$$

- Explicit computation of the fair premium for perpetual drawdown insurance

$$P^* = P^*(\infty) = \frac{r \alpha E^Q\{ e^{-r \tau_D(k)} \mathbb{I}_{\{ \tau_D(k) < \tau_U(k) \}} \}}{1 - E^Q\{ e^{-r (\tau_D(k) \wedge \tau_U(k))} \}}$$
The fair premium $P^*$ vs. the size of drawdown $k$

**Figure:** Model parameters: $r = 1\%$, $\sigma = 15\%$, $\alpha = 1$. The fair premium $P^*(\infty)$ is shown in red. It is seen that $P^*(k, T)$ is increasing in $T$ and decreasing in $k$. 
The fair premium $p^\ast$ vs. interest rate $r$

**Figure:** Model parameters: $\sigma = 15\%$, $k = 50\%$, $\alpha = 1$. The fair premium $P^\ast(\infty)$ is shown in red. It is seen that, the fair premium $P^\ast(r)$ is eventually decreasing in $r$. 
The fair premium $P^*$ vs. volatility $\sigma$

**Figure:** Model parameters: $r = 1\%$, $k = 50\%$, $\alpha = 1$. The fair premium $P^*(\infty)$ is shown in red. Like most derivatives, the fair premium $P^*(\sigma, T)$ is increasing both in $\sigma$ and in $T$. 
Thank You!


