Instructions. You are allowed to use a one-sided 8.5" × 11" set of notes. You must show all your work and cross out any work you do not want graded. Except on Problems 1 and 2, make sure you give a rigorous explanation for every solution. If you don’t have enough space to complete a problem, use the back of a page and indicate clearly which problem it is you are doing there. If anything you write is logically unclear or sloppily written, you will not get any credit for it. By writing your name below, you are pledging that you have not given or received unauthorized help on this exam.

Name: ________________________________

1. (14 points) True or False? Circle the correct answer, but do not give any explanation.

(a) If a time series is Gaussian, then it is stationary.
TRUE / FALSE

(b) If $\bar{X} = (X_1, X_2) \sim N \left( \bar{0}, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{pmatrix} \right)$, then $X_1 \sim N(0, 1/2)$.
TRUE / FALSE

(c) If $\{X_t\}_{t \in \mathbb{Z}}$ is an MA(3) process, then the time series $\{Y_t\}_{t \in \mathbb{Z}}$ defined for every $t \in \mathbb{Z}$ by
$$Y_t = X_t - 14X_{t-1} + 148X_{t-2} - X_{t-58}$$
is stationary.
TRUE / FALSE

(d) If $\{Y_t\}$ is white noise, then $\{(-1)^tY_t\}$ is white noise.
TRUE / FALSE

(e) If $X_t$ is defined for all $t \in \mathbb{Z}$ by the equation $X_t = \frac{1}{2}X_{t-1} + Z_t$, where $\{Z_t\}_{t \in \mathbb{Z}}$ is white noise, then
$$\rho_X(4) = \frac{1}{16}.$$TRUE / FALSE

(f) If $\{Z_t\}_{t \in \mathbb{Z}}$ is white noise, then the time series defined by
$$X_t = Z_t^2$$
is stationary.
TRUE / FALSE

(g) If $\{X_t\}_{t \in \mathbb{Z}}$ and $\{Y_t\}_{t \in \mathbb{Z}}$ are independent stationary time series, then $\{X_t + Y_t\}_{t \in \mathbb{Z}}$ is stationary.
TRUE / FALSE
2. (8 points) In each row below are the graphs of the sample a.c.f. and sample p.a.c.f of a same time series, based on 10,000 samples. Write next to each pair the number of the correct process among those defined at the bottom of the page, but do not explain. There is exactly one sensible answer. \( \{Z_t\}_{t \in \mathbb{Z}} \sim WN(0, \sigma^2) \) is normal white noise.

Answer:

1. \( X_t = Z_t \)
2. \( X_t = Z_t - \frac{Z_{t-1}}{4} + \frac{Z_{t-2}}{2} \)
3. \( X_t = \cos \left( \frac{\pi t}{4} \right) \)
4. \( X_t = X_{t-1} + Z_t \)
5. \( X_t = Z_t - Z_{t-1} \)
6. \( X_t = \frac{X_{t-1}}{8} + \frac{X_{t-2}}{4} - \frac{X_{t-3}}{6} + Z_t \)
3. (22 points) Suppose \( \{Z_t\}_{t \in \mathbb{Z}} \) is white noise and for \( t \in \mathbb{Z} \),

\[
X_t = \frac{1}{3}X_{t-2} + Z_t.
\]

(a) Is \( \{X_t\}_{t \in \mathbb{Z}} \) stationary?

(b) Explain clearly why \( \{X_t\}_{t \in \mathbb{Z}} \) invertible.
(c) Explain clearly why \( \{X_t\}_{t \in \mathbb{Z}} \) causal and write the time series in its causal form.
4. (20 points) Let \( \{Z_t\} \) be a sequence of independent normal random variables, each with mean 0 and variance \( \sigma^2 \), and let \( c \in \mathbb{R} \) be a constant. Which, if any, of the following processes are stationary? For each stationary process, specify the mean and autocovariance function. Note: The fact that \( E[Z_t^3] = 0 \) and \( E[Z_t^4] = 3\sigma^4 \) for all \( t \in \mathbb{Z} \) might be helpful.

(a) \( X_t = Z_t e^{ct} \)

(b) \( X_t = Z_t^2 Z_{t-1}^2 \)
5. (36 points) Suppose the data set below is a realization from a time series $X$.

$$1.2, 1, 0, -1, 1, 0, -1, -1.2, -1, 1.$$  

(a) Compute the sample autocovariance at lag 1 and the sample autocorrelation at lag 1. In other words, compute $\gamma_e(1)$ and $\rho_e(1)$.

(b) Use the Box-Pierce statistic to test at the 5% level of significance whether the time series could be white noise. Say clearly what your hypotheses are, what assumptions you are making about the data, and what your conclusion is. Note: One of the following values might be helpful: $\chi^2_{0.95,1} = 3.841, \chi^2_{0.95,2} = 5.991$.  

(c) Use the turning point test to determine whether the time series could be white noise. Say clearly what your hypotheses are, what assumptions you are making about the data, and what your conclusion is. Note: One of the following values might be helpful: $z_{0.05} = 1.645, z_{0.025} = 1.96$.

(d) Can you think of any reasons why either of the tests above might be inappropriate? Give a clear explanation.

Bonus Problem: (3 points) On the homework, you showed that for an AR(1) process, Var $\bar{X}$ is smaller when $\phi = -0.9$ than when $\phi = 0.9$. Explain clearly in your own words why you might have expected that this would be the case.