A CHARACTERIZATION OF 2-NEIGHBORHOOD DEGREE LIST OF DIAMETER 2 GRAPHS

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Abstract. Let $N_2DL(v)$ denote the set of degrees of vertices at distance 2 from $v$. The 2-neighborhood degree list of a graph is a listing of $N_2DL(v)$ for every vertex $v$. A degree restricted 2-switch on edges $v_1v_2$ and $w_1w_2$, where $deg(v_1) = deg(w_1)$ and $deg(v_2) = deg(w_2)$, is the replacement of a pair of edges $v_1v_2$ and $w_1w_2$ by the edges $v_1w_2$ and $v_2w_1$ given that $v_1w_2$ and $v_2w_1$ did not appear in the graph originally. Let $G$ and $H$ be two graphs of diameter 2 on the same vertex set. We prove that $G$ and $H$ have the same 2-neighborhood degree list if and only if $G$ can be transformed into $H$ by a sequence of degree restricted 2-switches.

1. Introduction

Two graphs $G_1$ and $G_2$ are isomorphic if there is a bijection from $V(G_1)$ to $V(G_2)$ that preserves adjacencies. Two labeled graphs $G_1$ and $G_2$ are label isomorphic or identical if there is a bijection from $V(G_1)$ to $V(G_2)$ that preserves labeled adjacencies. The degree of a vertex $v$, denoted by $deg(v)$, is the number of edges incident to $v$. The distance between a pair of vertices $u$ and $v$ in a graph $G$, denoted by $d(u, v)$, is the length of the shortest path between $u$ and $v$. The diameter, denoted by $diam(G)$, is the maximum value of $d(u, v)$, where the maximum is taken over all pairs of vertices $u$ and $v$ in $G$. The eccentricity of vertex $v$, denoted by $e(v)$, is the maximum value of $d(u, v)$, where the maximum is taken over all vertices $u \neq v$.

Let $G$ be a graph with vertices $v_1, \ldots, v_n$. The degree sequence of a graph is a listing of the degrees of its vertices: $deg(v_1), deg(v_2), \ldots, deg(v_n)$. The set of all vertices adjacent to a vertex $v$ is denoted by $N(v)$ and called the neighborhood of $v$. Note that the neighborhood of $v$ does not contain $v$ itself. For each vertex $v$, the neighborhood degree list of $v$, denoted by $NDL(v)$, is the list of degrees of vertices in $N(v)$. The neighborhood degree list of $G$, denoted by $NDL(G)$, is the list of lists

$$\{NDL(v_1), NDL(v_2), \ldots, NDL(v_n)\}$$
By convention the degree sequence and the neighborhood degree list of a vertex are written in descending order. The concept of neighborhood degree list was introduced independently by Barrus and Donavan [1] and Bassler et al [2].

In this paper we generalize the notion of neighborhood degree list. Let \( N_k(v) \) be the set of vertices of distance \( k \geq 1 \) from \( v \). In this notation \( N(v) = N_1(v) \). Observe that \( k \leq \text{diam}(G) \). The \( k \)-neighborhood degree list of \( v \), denoted by \( N_kDL(v) \), is the list of degrees of vertices in \( N_k(v) \). The \( k \)-neighborhood degree list of \( G \), denoted by \( N_kDL(G) \), is the list of lists

\[
\{N_kDL(v_1), N_kDL(v_2), \ldots, N_kDL(v_n)\}.
\]

Essentially we are considering concentric balls of vertices of increasing distance centered around a vertex.

The concept of \( N_kDL \) is a strengthening of the well known distance degree sequence. For a vertex \( v \), let \( \text{deg}_0(v) = 1 \) and for \( k \geq 1 \), let \( \text{deg}_k(v) = |N_k(v)| \). The distance degree sequence of \( v \) is the sequence

\[
(\text{deg}_0(v), \text{deg}_1(v), \text{deg}_2(v), \ldots, \text{deg}_{\text{diam}(G)}(v)).
\]

It was introduced by Randic in [8] to distinguish chemical isomers by their graph structure [3]. A graph in which all the vertices have the same distance degree sequence is called distance degree regular. Distance regular graphs are regular graphs. However, not all regular graphs are distance regular. Distance regular graphs have applications in numerous areas including algebraic combinatorics and coding theory. See [4] and [10].

The motivation for our definition of \( k \)-neighborhood degree list comes from the problem of identifying fake followers on Instagram and Twitter. A New York Times article titled “The follower factory”\(^1\) explains how celebrities purchase followers from companies that create millions of such accounts and sell them as followers. Such companies are called follower factories. An Instagram influencer’s follower count (i.e. degree in the social network) may be high, but the followers may be mostly vertices of low degree. Such fake influencers would be flagged by computing their \( NDL \). In case the fake accounts have a degree greater than 1 in an effort to hide that they are fake accounts, then computing \( N_kDL \), for \( k \geq 1 \), would reveal an anomaly in the pattern of \( N_kDL \) lists. Measures of centrality like betweenness centrality, eigenvalue centrality, PageRank, etc. can also be used to compare vertices, but they are global measures designed for specialized applications. On the other hand \( N_kDL \) is a local measure. In many cases the entire graph is unknown and a local measure of influence is needed.

Although the terminology is quite different, the notion of \( N_kDL(v) \) appears in Roberio et al [9]. Approaches for inferences on graphs rely on finding ways to embed vertices into the \( n \)-dimensional real vector space (\( \mathbb{R}^n \)) so that “similar” vertices are embedded near each other. The approach described in [9] uses the number of links of a vertex to its neighbors, number of links of the neighbors to their neighbors, and so on. In other words they use \( N_kDL(v) \) as a measure of similarity between vertices and they establish experimentally that \( N_kDL(v) \) is better than state-of-the-art techniques in capturing similarity of vertices.

In this paper we give a short proof of Barrus and Donovan’s theorem characterizing graphs with the same \( NDL \) and a characterization of diameter 2 graphs with the same \( N_{2}DL \).

A 2-switch in a graph is the replacement of a pair of edges \( v_1v_2 \) and \( w_1w_2 \) by the edges \( v_1w_2 \) and \( v_2w_1 \) given that \( v_1w_2 \) and \( v_2w_1 \) did not appear in the graph originally. There may or may not be edges between pairs of vertices \( v_1, w_1 \) and \( v_2, w_2 \). The 2-switch operation is illustrated in Figure 1. It was the key idea in a result by Havel [6] and independently by Hakimi [7] that characterizes precisely the sequence of numbers that correspond to the degree sequence of a graph. See also [5]. Observe that a 2-switch on a pair of edges does not alter the degrees of the four vertices involved. Therefore a 2-switch does not alter the degree sequence of the resulting graph. If some 2-switch turns \( G \) into \( G' \), then a 2-switch on the same four vertices turns \( G' \) into \( G \).

A degree restricted 2-switch on edges \( v_1v_2 \) and \( w_1w_2 \) is a 2-switch performed when \( \text{deg}(v_1) = \text{deg}(w_1) \) and \( \text{deg}(v_2) = \text{deg}(w_2) \). (Barrus and Donavan call this an \( n \)-switch.) The next theorem is the main result in this paper.

**Theorem 1.1.** Let \( G \) and \( H \) be two graphs on \( n \) vertices with diameter 2. Then \( G \) and \( H \) have the same 2-neighborhood degree list if and only if \( G \) can be transformed into \( H \) by a sequence of degree restricted 2-switches.

2. The proof of Theorem 1.1

When generalizing \( NDL \) to \( N_{2}DL \) one problem that comes up is that the 2-switch operation can alter the diameter. Consider for example a 2-switch performed on the cube graph (circular 4-ladder) that converts it to the Mobius 4-ladder as shown in Figure 2. Observe that \( NDL \) is preserved in both graphs. However, \( N_{2}DL \) is not preserved. The diameter of the cube is 3, but when a 2-switch operation is done to obtain the Mobius 4-ladder, the diameter is reduced to 2. Thus the 2 graphs have different \( N_{2}DL \).

Let \( G \) and \( H \) be two graphs on \( n \) vertices. Berge proved that \( G \) and \( H \) have the same degree sequence if and only if \( G \) can be transformed into \( H \) by a sequence of 2-switches. See [11, p. 47]. We do not use this result, rather we use the technique that Berge uses. The next result appears in [1, Theorem 3.3]. We give a different proof based on the lexicographic ordering of the neighborhood degree list.

Let \( a \) and \( b \) be two vertices in a graph such that \( \text{deg}(a) = \text{deg}(b) = t \). Let \( N(a) = \{a_1, \ldots, a_t\} \) and \( N(b) = \{b_1, \ldots, b_t\} \), where the vertices are listed in descending order based
on their degrees. We say \( NDL(a) = NDL(b) \) if the ordered lists of degrees are the same. In other words
\[
(deg(a_1), \ldots, deg(a_t)) = (deg(b_1), \ldots, deg(b_t))
\]
We say \( NDL(a) < NDL(b) \) if using the lexicographic ordering
\[
(deg(a_1), \ldots, deg(a_t)) < (deg(b_1), \ldots, deg(b_t)).
\]
Lexicographic ordering is defined recursively. If \( deg(a_1) < deg(b_1) \), then \( NDL(a) < NDL(b) \). If \( deg(a_1) = deg(b_1) \), then the order is determined by the lexicographic order of \( (deg(a_2), \ldots, deg(a_t)) \) and \( (deg(b_2), \ldots, deg(b_t)) \). If \( deg(a_2) < deg(b_2) \), then \( NDL(a) < NDL(b) \). If \( deg(a_2) = deg(b_2) \), then check the sequences \( (deg(a_3), \ldots, deg(a_t)) \) and \( (deg(b_3), \ldots, deg(b_t)) \), and so on.

**Lemma 2.1. (Barrus and Donavan 2018)** Let \( G \) and \( H \) be two graphs on \( n \) vertices. Then \( G \) and \( H \) have the same neighborhood degree list if and only if \( G \) can be transformed into \( H \) by a sequence of degree restricted 2-switches.

**Proof.** One direction is straightforward. If \( G \) can be transformed into \( H \) by a sequence of degree restricted 2-switches, then clearly \( G \) and \( H \) have the same neighborhood degree list.

Conversely, suppose \( G \) and \( H \) have the same neighborhood degree list. The proof is by induction on \( n \geq 4 \). The result holds for graphs on 4 vertices trivially. Assume that the result holds for all graphs with \( n - 1 \) vertices.

Let \( w \) be a vertex of maximum degree \( \Delta \) in \( G \). Let \( z \) be a neighbor of \( w \) and let \( S \) be the set of all vertices that are not neighbors of \( w \), but have the same degree as \( z \). If \( S = \phi \), then proceed to the next neighbor. Otherwise suppose \( S \neq \phi \). Choose \( x \in S \) so that \( NDL(x) \) is highest among vertices of \( S \). If \( NDL(x) \leq NDL(z) \), then again proceed to the next neighbor of \( S \).

Suppose \( NDL(x) > NDL(z) \). Note that \( w \) is a vertex of maximum degree. Since \( w \) is incident to \( z \), but not to \( x \), there exists \( y \) incident to \( x \), but not to \( z \), such that \( deg(y) = deg(w) \). Thus we can perform a degree restricted 2-switch operation on \( wz \) and \( yx \). Delete \( wz \) and \( yx \) and add \( wx \) and \( yz \). (See Figure 3.) Observe that the resulting graph has the same \( NDL \) as \( G \) (and consequently the same degree sequence).

Repeat the above process for every neighbor of \( w \) to obtain a graph \( G^* \) where \( NDL(G^*) = NDL(G) \) and neighbors of \( w \) in \( G^* \) are adjacent to vertices with the same degrees as neighbors of \( w \) in \( G \), but with highest \( NDL \).
Similarly, choose a vertex \( w_H \) in \( H \) of highest degree such that \( NDL_H(w_H) = NDL_G(w) \). Such a vertex exists since \( G \) and \( H \) have the same \( NDL \). There exists a sequence of degree restricted 2-switches that transforms \( H \) into \( H^* \), where \( NDL(H^*) = NDL(H) \) and neighbors of \( w_H \) in \( H^* \) are adjacent to vertices with the same degrees as neighbors of \( w_H \) in \( H \), but with highest \( NDL \).

Observe that the degrees of the neighbors of \( w \) and \( w_H \) in \( G^* \) and \( H^* \), respectively, are the same. In addition,

\[
NDL_{H^*}(w) = NDL_H(w_H) = NDL_G(w) = NDL_{G^*}(w)
\]

Consider \( G' = G^* - w \) and \( H' = H^* - w_H \). Then \( NDL(G') = NDL(H') \). By the induction hypothesis applied to \( G' \) and \( H' \), there exists a sequence of degree restricted 2-switches that transforms \( G' \) to \( H' \). These degree restricted 2-switches do not involve \( w \) and \( w_H \), which have the same \( NDL \) in \( G^* \) and \( H^* \), respectively. So applying this sequence of degree restricted 2-switches transforms \( G^* \) to \( H^* \). Finally, we can transform \( G \) to \( H \) by transforming \( G \) to \( G^* \), then \( G^* \) to \( H^* \), and then (in reverse order) \( H \) to \( H^* \). \( \square \)

**Lemma 2.2.** Let \( G \) be a graph on \( n \) vertices with diameter 2. Then

\[
deg(v) = n - 1 - |N_2(v)|.
\]

**Proof.** Since \( G \) has diameter 2, every vertex is of distance 1 or 2 from every other vertex. So for each \( v \in V(G) \),

\[
V(G) = \{v\} \cup N(v) \cup N_2(v),
\]

where \( |N(v)| \cap |N_2(v)| = \phi \). Since \( \deg(v) = |N(v)| \),

\[
n = 1 + \deg(v) + |N_2(v)|.
\]

Therefore

\[
\deg(v) = n - 1 - |N_2(v)|.
\]

\( \square \)

The main idea in the proof of Theorem 1.1 is that if the graph has diameter 2, then we can recover \( NDL \) from \( N_2DL \) and vice versa. Moreover, we can recover the degree sequence from \( NDL \) in any graph. Let us look at an example to illustrate this point. Consider the graph \( G \) with diameter 2 shown in Figure 4. It has degree sequence \( 5, 5, 4, 4, 4, 4, 3, 3 \) and \( NDL \) and \( N_2DL \).
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Observe that $N_1(v_1) = \{v_2, v_4, v_6, v_8\}$. So the members of $N_2(v_1)$ are the rest of the vertices (except $v_1$ itself). Thus $N_2(v_1) = \{v_3, v_5, v_7\}$

![Figure 4. A diameter 2 graph](image)

**Proof of Theorem 1.1.** Suppose $G$ and $H$ have the same $N_2DL$. Then for every vertex $v_G$ in $G$, there is a vertex $v_H$ in $H$ with the same $N_2DL$ and vice-versa. Thus there is a one-to-one correspondence between the vertices of $G$ and $H$ such that for every pair of corresponding vertices $v_G$ and $v_H$,

$$N_2DL(v_G) = \{\deg_G(u) \mid u \in N_2(v_G)\},$$

$$N_2DL(v_H) = \{\deg_H(u) \mid u \in N_2(v_H)\},$$

and $N_2DL(v_G) = N_2DL(v_H)$. Observe that $|N_2(v_G)|$ is the number of entries in $N_2DL(v_G)$ and $|N_2(v_H)|$ is the number of entries in $N_2DL(v_H)$. Therefore $|N_2(v_G)| = |N_2(v_H)|$. By Lemma 2.2

$$\deg_G(v_G) = n - 1 - |N_2(v_G)| = n - 1 - |N_2(v_H)| = \deg_H(v_H).$$

Thus the degree sequence of $G$ and $H$ can be obtained from $N_2DL$. Moreover the degree sequence of $G$ and $H$ are the same.

Next observe that if $N_2DL(v) = \{\deg(u) \mid u \in N_2(v)\}$, then since the degree sequence is known, $NDL(v) = \{\deg(u) \mid u \notin \{v\} \cup N_2(v)\}$. Since $G$ and $H$ have the same $N_2DL$ and the same degree sequence, $G$ and $H$ must have the same $NDL$. Lemma 2.1 implies that $G$ can be transformed into $H$ by a sequence of degree restricted 2-switches.

Conversely, suppose $G$ can be transformed into $H$ by a sequence of degree restricted 2-switches. Lemma 2.1 implies that $NDL$ is maintained at each stage (even if the diameter changes) so at the end of the sequence of degree restricted 2-switches $G$ and $H$ have the
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Thus there is a one-to-one correspondence between the vertices of $G$ and $H$ such that for every pair of corresponding vertices $v_G$ and $v_H$,

$$NDL(v_G) = \{\deg_G(u) \mid u \in N(v_G)\},$$

$$NDL(v_H) = \{\deg_H(u) \mid u \in N(v_H)\}$$

and $NDL(v_G) = NDL(v_H)$. Observe that $|N(v_G)|$ is the number of entries in $NDL(v_G)$ and $|N(v_H)|$ is the number of entries in $NDL(v_H)$. Therefore $|N(v_G)| = |N(v_H)|$ and $\deg(v_G) = \deg(v_H)$. Thus the degree sequence of $G$ and $H$ can be obtained from $N_2DL$ and they are the same.

Next, observe that if $NDL(v) = \{\deg(u) \mid u \in N(v)\}$, then since the degree sequence is known, $N_2DL(v) = \{\deg(u) \mid u \not\in \{v\} \cup N(v)\}$. In conclusion, if $G$ and $H$ have the same $NDL$ and the same degree sequence, then since $G$ and $H$ are diameter 2 graphs they must have the same $N_2DL$. □

References

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