Restrictions as stabilizers

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1. Introduction

The matroid terminology in general follows Oxley [1]. Two \( r \times n \) matrices \( A \) and \( A' \) representing the same matroid \( M \) over a field \( GF(q) \) are said to be projectively equivalent representations of \( M \) if one can be obtained from the other by elementary row operations and column scaling. Otherwise \( A \) and \( A' \) are projectively inequivalent. The matrices \( A \) and \( A' \) are said to be equivalent representations of \( M \) if one can be obtained from the other by elementary row operations, column scaling, or field automorphisms. Otherwise they are called inequivalent representations. For prime fields, projective equivalence and equivalence coincide as prime fields do not have non-trivial automorphisms. For fields of order \( q \), where \( q \) is a prime power, projective equivalence is a refinement of equivalence. So a result that holds for projective equivalence automatically holds for equivalence.

A plane in a \( GF(q) \)-representable matroid is called a large plane if it has at least \( 2q \) points. A line is called a long line if it has at least \( q \) points. We denote the single-element extension of a
matroid $N$ by element $e$ as $N + e$ [1, p. 254]. We say $N$ stabilizes $N + e$ over $GF(q)$ if no $GF(q)$-representation of $N$ can be extended to two projectively inequivalent $GF(q)$-representations of $N + e$. Observe that if $N$ has $k$ projectively inequivalent $GF(q)$-representations, then $N + e$ has at most $k$ projectively inequivalent representations. Further if $N$ stabilizes all its single-element extensions, then it stabilizes all its extensions. We define stability for coextensions in a similar manner. The notion of projectively inequivalent representations.

Corollary 1.2. If $M$ is a 3-connected $GF(q)$-representable matroid with a flat $F$ and let $N = M\mid F$ be 3-connected. If $N$ stabilizes its single-element extensions over $GF(q)$, then $N$ stabilizes $M$.

Corollary 1.3. Suppose $M$ is a 3-connected $GF(q)$-representable matroid with a large plane $P$ as a restriction, but no long line as a restriction. Then $P$ stabilizes $M$.

2. Connectivity preliminaries

The usual tool for 3-connectivity is Seymour’s Splitter Theorem [3]. However, it is not possible to use it in the analysis here, since a restriction of $M$ is a specific type of minor and we want to maintain the restriction minor throughout the extension/coextension process all the way up to $M$. The Splitter Theorem allows for isomorphic minors in the construction process. We may run into problems in our setting if we move to an isomorphic minor rather than keeping the specific restriction minor. Thus we develop a theorem for restrictions that is similar to the Splitter Theorem.

A matroid is vertically 3-connected if it is connected and whenever $(X, E - X)$ is a 2-separation, either $r(X) = 1$ or $r(E - X) = 1$. In other words either $X$ or $E - X$ is a parallel set [1, p. 278]. We denote by $\tilde{M}$ the simple matroid associated with $M$ obtained by deleting loops and deleting all but one element from each non-trivial parallel class. We denote by $\tilde{M}$ the cosimple matroid associated with $M$ obtained by contracting all coloops and contracting all but one element from each non-trivial series class [1, p. 296]. The next two results are Bixby’s Lemma [1, 8.4.6] and Tutte’s Triangle Lemma [1, 8.4.9].

Lemma 2.1. Suppose $M$ is a 3-connected matroid and $e$ is an element of $M$. Then either $M\setminus e$ or $M/e$ has no non-minimal 2-separations. Moreover, in the first case, $\tilde{M}\setminus e$ is 3-connected, while in the second case, $\tilde{M}/e$ is 3-connected.
Lemma 2.2. Suppose $M$ is a 3-connected matroid having at least four elements and suppose that \{e, f, g\} is a triangle of $M$ such that neither $M \setminus e$ nor $M / f$ is 3-connected. Then $M$ has a triad that contains $e$ and exactly one of $f$ and $g$.

Theorem 2.3. Suppose $M$ is a vertically 3-connected matroid and $N = M|X$ is a 3-connected proper restriction of $M$. Then there is an element $e \in E(M) - X$ such that $M \setminus e$ or $M / e$ is vertically 3-connected and has $N$ as a restriction minor.

Proof. Suppose $cl_M(X) = E(M)$. Then deleting any element in $E(M) - X$ preserves 3-connectivity and maintains $N$ as a restriction minor. Further, suppose $M$ has a non-trivial parallel class. Then since $N$ is 3-connected, we may delete an element from such a class preserving vertical 3-connectivity and keeping $N$ as a restriction minor. Thus we may assume that $cl_M(X) \neq E(M)$ and $M$ is 3-connected.

Let $e \in E(M) - cl(X)$. Note that both $M \setminus e$ and $M / e$ have $N$ as a minor. By Bixby’s Lemma (Lemma 2.1) either $M \setminus e$ is 3-connected or $M / e$ is 3-connected. In the latter case $M / e$ is vertically 3-connected and we are done. Therefore, assume the former, that is $M / e$ is 3-connected up to series pairs. If $M \setminus e$ has no series pairs then $M \setminus e$ is itself 3-connected and again we are done. So assume \{f, g\} is a series pair. Observe that \{f, g\} $\cap$ $cl(X)$ = $\phi$, because otherwise $N$ would not be 3-connected. Since $M$ is 3-connected, \{e, f, g\} is a triad outside the span of $X$ and $M / f$ and $M / g$ both have $N$ as a restriction minor. By Tutte’s Triangle Lemma (Lemma 2.2) if neither $M / f$ nor $M / g$ is 3-connected, then \{f, g\} is part of a fan with at least four elements. It follows from properties of fans [2] that either $M / f$ or $M / g$ is vertically 3-connected as required. ∎

Corollary 2.4. Suppose $M$ is a vertically 3-connected matroid and $F$ is a flat such that $M|F$ is 3-connected. Then there is a sequence of vertically 3-connected matroids

$$M|F = N_0, N_1, N_2, \ldots, N_k = M$$

such that $N_i$ is a single-element extension or coextension of $N_{i-1}$ for $i \in \{1, \ldots, k\}$.

Finally, a major tool in the proof of Theorem 1.1 is Tutte’s Linking Lemma [4].

Theorem 2.5. Let $M$ be a matroid on ground set $E$ and let $A$ and $B$ be disjoint subsets of $E$. Suppose there is no 2-separation $(X, E - X)$ of $M$ with $A \subseteq X$ and $B \subseteq E - X$. Then $M$ has a minor $N$ with ground set $A \cup B$ such that $N|A = M|A$, $N|B = M|B$, and $r_N(A) + r_N(B) - r(N) \geq 2$.

3. Proof of the main results

Proof of Theorem 1.1. Since $M$ is 3-connected and $N = M|F$ is 3-connected, by Corollary 2.4 there is a sequence of vertically 3-connected matroids

$$M|F = N_0, N_1, N_2, \ldots, N_k = M$$

such that $N_i$ is a single-element extension or coextension of $N_{i-1}$ for $i \in \{1, \ldots, k\}$. Suppose for some $j \in \{1, \ldots, k\}$, $N_j$ is a single-element extension of $N_{j-1}$ and $N_{j-1}$ does not stabilize $N_j$. By definition there is another element $f$ such that $N_{j-1} + f = N_{j-1} + e = N_j$. In other words $e$ and $f$ are clones. Observe that there is no 2-separation $(X, Y)$ of $E(M)$ with $F \subseteq X$ and $\{e, f\} \subseteq Y$. By Tutte’s Linking Lemma there is a minor $N'$ with ground set $F \cup \{e, f\}$ such that $N'|F = M|F = N$ and $N'|\{e, f\} = M|\{e, f\}$ and $r_N(F) + r_N(\{e, f\}) - r(N') \geq 2$. So $r(N') = r(F)$ and $\{e, f\}$ are clones in $N'$. This contradicts the fact that $N$ stabilizes its single-element extensions. The argument is similar for coextensions. Thus $M|F$ stabilizes $M$. ∎

The proof of Corollary 1.2 follows easily from Theorem 1.1 because an $n$ point line is a rank-2 flat and in order for it to be $GF(q)$-representable, $n \leq q + 1$. However, while it is clear that a line with at
least \( q \) points in a \( GF(q) \)-representable matroid merits being called a long line (only one more point can be added) it is not apparent why a plane with at least \( 2q \) elements deserves to be called a “large” plane. The following result makes this clear.

**Proposition 3.1.** Let \( M \) be a rank-3 simple \( GF(q) \)-representable matroid with at least \( 2q \) elements. Then \( M \) stabilizes its simple single-element extensions.

**Proof.** Suppose \( M \) is a rank-3 simple \( GF(q) \)-representable matroid with at least \( 2q \) elements that does not stabilize its single-element extension \( M + e \). Then there is an element \( f \) such that \( M + e = M + f \) and \( e \) and \( f \) are clones in \( M + \{ e, f \} \). Since \( M + \{ e, f \} \) is \( GF(q) \)-representable, \( e \) can be on at most \( q + 1 \) lines. But since \( e \) and \( f \) are clones, none of these lines other than the line containing both \( e \) and \( f \) can contain three or more points. So the line containing \( e \) and \( f \) can have at most \( q + 1 \) points and outside this line there can be at most \( q \) points. It follows that \( M + \{ e, f \} \) has at most \( 2q + 1 \) points. Therefore, \( M \) has at most \( 2q - 1 \) elements; a contradiction to the hypothesis. \( \square \)

**Proof of Corollary 1.3.** Observe that \( M \) is a 3-connected matroid with a large plane \( P \) as a restriction. Proposition 3.1 implies that \( P \) stabilizes its single-element extensions. Further since \( P \) has no \( q \) point lines, \( M | P \) must be 3-connected since a rank-3 simple matroid is 3-connected if and only if it is not covered by two lines. Theorem 1.1 implies that \( P \) stabilizes \( M \). \( \square \)

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**References**