MINOR PRESERVING DELETABLE EDGES IN GRAPHS

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Abstract. A 3-connected graph is minimally 3-connected if removal of any edge destroys 3-connectivity. Halin proved that if $G$ is a minimally 3-connected graph with $n \geq 8$ vertices, then $|E(G)| \leq 3n - 10$, with the exception of $K_{3,n-3}$ whose size is $3n - 9$. In this paper we will extend this result to a larger class of graphs. Suppose $G$ and $H$ are simple 3-connected graphs and $G$ has a proper $H$-minor. We say $G$ is $H$-critical if removal of any edge either destroys 3-connectivity or the $H$-minor. Such graphs are useful because they are just barely outside the class of graphs with no minor isomorphic to $H$. We will prove that if $G$ is $H$-critical and $H \neq W_3$, then one of three possibilities must occur: $G/f$ is $H$-critical for some edge $f$; $G/f\setminus e$ is $H$-critical for some pair of edges $e, f$ incident to a vertex of degree 3; or $G - w$ is $H$-critical for some degree 3 vertex $w$. Moreover, if $G$ is $H$-critical, then $|E(G)| \leq |E(H)| + 3|V(G)| - |V(H)|$. Halin’s theorem will follow from this result.

1. Introduction

Graphs in this paper may have loops and parallel edges. If not, they are called simple graphs. A graph is 3-connected if at least 3 vertices must be removed to disconnect the graph. The degree of a vertex $v$, denoted by $\text{deg}(v)$, is the number of edges incident to $v$.

Suppose $G$ is a simple 3-connected graph with a simple 3-connected proper minor $H$. The motivating question that began the investigation in this paper is as follows:

When can we guarantee the existence of a set of edges $X$ in $G$ such that removal of $X$ preserves 3-connectivity and the $H$-minor?

To delete an edge $e$ remove it from the graph. The resulting graph, denoted by $G \setminus e$, is called an edge-deletion of $G$. An edge $e$ in a 3-connected graph $G$ is called deletable if $G \setminus e$ is 3-connected. A 3-connected graph is minimally 3-connected if no edge is deletable.

To contract an edge $f$ with end vertices $u$ and $v$, collapse the edge by identifying $u$ and $v$ as one vertex, and delete the loop formed. The resulting graph, denoted by $G/f$, is called an edge-contraction. A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by deleting edges (and any isolated vertices formed as a result) or contracting edges. We write $H = G \setminus X/Y$, where $X$ is the set of edges deleted and $Y$ is the set of edges contracted.

Suppose $G$ and $H$ are simple 3-connected graphs and $G$ has a proper minor isomorphic to $H$. An edge $e$ is called an $H$-deletable edge if $G \setminus e$ is 3-connected and has a minor isomorphic to $H$. A 3-connected graph
with a proper $H$-minor, but no $H$-deletable edge, is called an $H$-critical graph. Thus in an $H$-critical graph removal of an edge either destroys 3-connectedness or the $H$-minor. The concept of an $H$-critical graph is a generalization of the concept of a minimally 3-connected graph. We will show how some results from minimally 3-connected graphs can be extended to $H$-critical graphs.

The main result in this paper is a connectivity-based structural result for simple 3-connected graphs. The graph $W_{n-1}$ is the wheel graph with $n-1$ spokes. (It is customary to use the number of spokes as the subscript.)

**Theorem 1.1.** Suppose $G$ and $H$ are simple 3-connected graphs, where $|E(G)| \geq |E(H)| + 3$, and $|V(G)| \geq |V(H)| + 1$. If $G$ has an $H$-minor, then there exists a set of $H$-deletable edges $D$ such that

$$|D| \geq |E(G)| - |E(H)| - 3|V(G)| - |V(H)|$$

and a sequence of $H$-critical graphs

$$G_{|V(H)|}, \ldots, G_{|V(G)|},$$

where $G_{|V(H)|} \cong H$, $G_{|V(G)|} = G \setminus D$, and for all $i$ such that $|V(H)| + 1 \leq i \leq |V(G)|$:

(i) $G_i/f = G_{i-1}$, where $f$ is an edge;
(ii) $G_i/f \setminus e = G_{i-1}$, where $e$ and $f$ are edges incident to a vertex of degree 3; or
(iii) $G_i - w = G_{i-1}$, where $w$ is a vertex of degree 3.

The proof of Theorem 1.1 is broken down into two lemmas that are useful by themselves. We will first prove that if $G$ is $H$-critical, then one of three possibilities must occur: $G/f$ is $H$-critical for some edge $f$; $G/f \setminus e$ is $H$-critical for some pair of edges $e, f$ incident to a vertex of degree 3; or $G - w$ is $H$-critical for some degree 3 vertex $w$. Second we will prove that if $G$ is $H$-critical, then

$$|E(G)| \leq |E(H)| + 3(|V(G)| - |V(H)|).$$

This gives a bound on the size of $G$ that depends only on the number of vertices and edges of $H$ and the number of vertices of $G$. The proof of Theorem 1.1 will follow from these results.

If $H$ is the prism graph, then we can get a slightly better bound. In Section 3 we will prove that if $G$ is a prism-critical graph on $n \geq 7$ vertices. Then $|E(G)| \leq 3n - 10$. As a consequence we get the following result by Halin (see [4] and [1, p. 26]): If $G$ is a minimally 3-connected graph with $n \geq 8$ vertices, then $|E(G)| \leq 3n - 10$, with the exception of $K_{3,n-3}$ whose size is $3n - 9$. Thus the upper bound on the size of $H$-critical graphs may be viewed as a generalization of Halin’s Theorem from minimally 3-connected graphs to $H$-critical graphs.

2. Proof of Theorem 1.1

The operations opposite edge-deletion and edge-contraction are edge addition and vertex split, respectively. A graph $G$ with an edge added between non-adjacent vertices is denoted by $G + e$ and called an edge addition of $G$. To split a vertex $v$ of degree at least 4, replace $v$ with two distinct vertices $v_1$ and $v_2$, join them by a new edge $f$, and join each neighbor of $v$ to exactly one of $v_1$ or $v_2$ in such a way that both $v_1$ and $v_2$ have degree at least 3. The resulting graph, denoted by $G \circ f$, is called a vertex split of $G$. The notation comes from matroid perspective, where the focus is on edges. Thinking about graphs from the matroid perspective led to the results in this paper. Thus edge addition/vertex split are denoted in terms of edges. Note that if $G$ is simple and 3-connected, then so are its edge additions and vertex splits.

Tutte’s Wheels Theorem states that a simple graph $G$ is 3-connected if and only if $G$ is a wheel or $G$ is obtained from a wheel by a finite sequence of edge additions or vertex splits [10]. Negami extended Tutte’s Wheels Theorem by maintaining a specified 3-connected minor other than the wheel. Let $G$ and $H$ be simple 3-connected graphs, where $H$ is not a wheel. Then $G$ has an $H$-minor if and only if $G$ can be obtained from $H$ by a finite sequence of edge additions and vertex splits [7]. This result is known as
the Splitter Theorem. The Splitter Theorem for matroids (in a form that looks quite different) appeared two years earlier in a paper by Seymour [9]. The version below is by Oxley and Coullard [2]. See also [8, Corollary 12.3.1]. The hypothesis in Negami’s result that “H is not a wheel” can be replaced by “G is not a wheel and H \( \neq W_3 \).”

**Theorem 2.1. (Splitter Theorem)** Suppose \( G \) and \( H \) are simple 3-connected graphs such that \( G \) is not a wheel and \( H \neq W_3 \). Then \( G \) has an \( H \)-minor if and only if there exists a sequence of simple 3-connected graphs \( G_0, \ldots, G_t \) such that \( G_0 = H \), \( G_t \sim G \), and for \( 1 \leq i \leq t \) either

\[
G_i = G_{i-1} + e \quad \text{or} \quad G_i = G_{i-1} \circ f,
\]

where \( e \) is an edge added to a pair of non-adjacent vertices in \( G_{i-1} \) and \( f \) is the edge added when splitting a vertex in \( G_{i-1} \).

Suppose \( G \) and \( H \) are simple 3-connected graphs, where \( G \) is not a wheel and \( H \) is not a 3-wheel. We will prove that if \( G \) is \( H \)-critical, then \( G \) can be constructed from an \( H \)-critical graph \( G' \) on \( |V(G)| - 1 \) vertices using one of three operations:

- Splitting a vertex;
- Adding an edge and splitting a vertex incident to the newly added edge; or
- Adding two edges incident to a common end vertex and then splitting the common vertex.

Additional conditions are placed on the second and third operations. When an edge \( e \) is added between non-adjacent vertices only a vertex incident to \( e \) may be split, and split in such a way that \( e \) and the new edge \( f \) formed between the split vertices are incident to a vertex of degree 3 (see Figure 1). Edges \( e_1 \) and \( e_2 \) are added between non-adjacent vertices in such a way that \( e_1 \) and \( e_2 \) are incident to a common vertex, and when that common vertex is split, \( e_1 \) and \( e_2 \), and the new edge \( f \) are incident to a degree 3 vertex (see Figure 1).

**Figure 1. Diagram for Lemma 2.2**

**Lemma 2.2.** Suppose \( G \) and \( H \) are simple 3-connected graphs, where \( G \) is not a wheel and \( H \neq W_3 \). If \( G \) is \( H \)-critical, then there is an \( H \)-critical graph \( G' \) on \( |V(G)| - 1 \) vertices such that:

(i) \( G = G' \circ f \);

(ii) \( G = G' + e \circ f \), where \( e \) and \( f \) are incident to a degree 3 vertex in \( G \); or

(iii) \( G = G' + \{e_1, e_2\} \circ f \), where \( \{e_1, e_2\} \) are incident to a degree 3 vertex in \( G \).

**Proof.** Theorem 2.1 implies that we can construct a graph isomorphic to \( G \) from \( H \) by a sequence of edge additions and vertex splits. Since \( G \) has no \( H \)-deletable edges, the last operation in forming \( G \) is splitting a vertex. So

\[
G = G^+ \circ f
\]

for some graph \( G^+ \) with \( |V(G)| - 1 \) vertices. Now \( G^+ \) may have \( H \)-deletable edges. Let \( \{e_1, \ldots, e_k\} \), where \( k \geq 0 \), be the largest set of \( H \)-deletable edges in \( G^+ \) such that

\[
G' = G^+ \setminus \{e_1, \ldots, e_k\}
\]

and \( G' \) has no \( H \)-contractible edges. Then

\[
G = G' + \{e_1, \ldots, e_k\} \circ f,
\]

where \( e_1, \ldots, e_k \) are edges added between non-adjacent vertices so that \( \{e_1, \ldots, e_k\} \) are incident to a degree 3 vertex in \( G' \), and when that common vertex is split, \( e_1, \ldots, e_k \) and the new edge \( f \) are incident to a degree 3 vertex in \( G' \) (see Figure 1).
where $G'$ has $|V(G)| - 1$ vertices. Moreover, $G'$ is $H$-critical since $\{e_1, \ldots, e_k\}$ is a largest possible set of $H$-deletable in $G^+$.

Let $v$ be the vertex in $G'$ that is split into $v$ and $v'$ such that $f$ is the edge between $v$ and $v'$. If any one of the edges $e_1, \ldots, e_k$ is incident to a vertex other than $v$, then it would remain an $H$-deletable edge in $G$. Therefore each of $e_1, \ldots, e_k$ is incident to $v$. We will prove that $k \leq 2$, and in each case the specified restrictions hold.

Case (i) Suppose $k = 0$. Then $G = G' \circ f$ and there is nothing to show.

Case (ii) Suppose $k = 1$. Then $G = G' + e_1 \circ f$. Now $e_1$ is incident to $v$ and the other end vertex of $e_1$ has degree at least 4 in $G + e_1$, since $G'$ is 3-connected. Without loss of generality, suppose $e_1$ is incident with the new vertex $v'$ in $G$. If the new vertex $v'$ is incident with $e_1$ and $f$, and one other edge, then $e_1$ and $f$ are incident to a degree 3 vertex, and we are done. Otherwise $v'$ is incident with $e_1$ and $f$, as well as two more edges. Thus both end vertices of $e_1$ in $G$ have degree 4 and $e_1$ is an $H$-deletable edge in $G$; a contradiction.

Case (iii) Suppose $k = 2$. Then $G = G' + \{e_1, e_2\} \circ f$. Now $e_1$ and $e_2$ are both incident with $v$, and the other end vertices of $e_1$ and $e_2$ have degree at least 4 in $G' + \{e_1, e_2\}$, since $G'$ is 3-connected. Suppose when $v$ is split into $v$ and $v'$, one of $e_1$ and $e_2$ is incident to $v$, say $e_1$, and the other is incident to $v'$. Since degree $v$ is at least 5, when $v$ is split one of $v$ or $v'$ has at least 4 edges incident to it, say $v$. In this case $e_1$ is an $H$-deletable edge in $G$, which is a contradiction.

Thus we may assume that when $v$ is split into $v$ and $v'$, both $e_1$ and $e_2$ are incident to the same vertex, say $v'$. The new edge $f$ is also incident to $v'$. If at least one other edge is also incident to $v$, then one of $e_1$ or $e_2$ would be an $H$-deletable edge in $G$, and again we have a contradiction. The only possibility left is that $v'$ has degree 3 and is incident with $e_1$, $e_2$, and $f$ giving the situation in (ii).

Case (iv) Suppose $k \geq 3$. Then $G = G' + \{e_1, \ldots, e_k\} \circ f$. Each edge in $\{e_1, \ldots, e_k\}$ is incident to $v$, and its other end vertex has degree at least 4 in $G' + \{e_1, \ldots, e_k\}$, since $G'$ is 3-connected. When $v$ is split into $v$ and $v'$ in $G$, if the new vertex $v'$ has all the edges $e_1, \ldots, e_k$ incident to it (in addition to the new edge $f$), then again one of the edges $e_1, \ldots, e_k$ is an $H$-deletable edge in $G$. Otherwise at most two of the three edges $e_1, \ldots, e_k$ are incident to $v'$, say $e_1$ are $e_2$ are incident to $v'$, and the rest remain incident to $v$, and are therefore $H$-deletable edges in $G$; a contradiction.

Observe that in all instances, it is the newly added edges that are deletable, so the $H$-minor is never disturbed.

We can express Lemma 2.2 in terms of edge-deletions and edge-contractions. In this case we do not have to exclude the wheels in the hypothesis. A rim vertex in a wheel with at least 4 spokes has degree 3. A rim vertex in a wheel $W_{n-1}$ consists of a spoke and two adjacent rim edges, and deleting a spoke and contracting a rim edge gives $W_{n-2}$.

**Corollary 2.3.** Suppose $G$ and $H$ are simple 3-connected graphs. If $G$ is $H$-critical, then there is an $H$-critical graph $G'$ on $|V(G)| - 1$ vertices such that: $G/f = G'$, where $f$ is an edge; $G/f \setminus e = G'$, where edges $e$ and $f$ are incident to a vertex of degree 3; or $G - w = G'$, where $w$ is a vertex of degree 3.

**Proof.** If $G$ is not a wheel, then the result holds by Lemma 2.2, since in the third operation deleting three edges incident to a vertex of degree 3 amounts to deleting the vertex of degree 3. Therefore suppose $G \cong W_{n-1}$, where $n \geq 6$. Observe that the only 3-connected minor of a wheel on $n$ vertices is a wheel on fewer vertices and $W_{n-1}/r \setminus s \cong W_{n-2}$, where $r$ is a rim edge and $s$ is a spoke edge. Since a wheel is minimally 3-connected, $G$ satisfies (ii) in the statement of the corollary. Lastly, the restriction $H \neq W_3$ may be removed since $W_4/r \setminus s = W_3$. □
All $H$-critical graphs are obtained in the construction process outlined in Lemma 2.2. However, some graphs that are not $H$-critical are also obtained. To understand this consider the graph $G = \text{prism} + \{e_1, e_2\} \circ f$, shown in Figure 2 (first graph in the fourth column). Neither $G \setminus e_1$ nor $G \setminus e_2$ is 3-connected, but $G$ has a deletable edge (the edge in bold). When the bold edge, which is part of the original prism minor, is deleted, the resulting graph no longer has the original prism minor, but still has a minor isomorphic to the prism.

![Figure 2. Some 3-connected graphs with a prism minor](image)

From the above example in Figure 2 it may look like the third operation in Lemma 2.2 is not needed since the graph $G = \text{prism} + \{e_1, e_2\} \circ f$ is not $H$-critical. However, this is not true in general. Let $H$ be the prism graph and consider the graph $G = \text{prism} + e_1 \circ f_1 + \{e_2, e_3\} \circ f_2$ constructed as shown in Figure 3. It is minimally 3-connected since removing any edge destroys 3-connectivity. An exhaustive search and construction of all graphs with 8 vertices and 14 edges shows that the only way to obtain it is following the order of operations shown in Figure 3. Thus Operation (iii) in Lemma 2.2 is required.

![Figure 3. An example to show that Operation (iii) in Lemma 2.2 is essential](image)

The next lemma establishes a sharp bound on the size of $H$-critical graphs that depends only on the order of $G$ and the size and order of $H$.

**Lemma 2.4.** Suppose $G$ and $H$ are simple 3-connected graphs such that $G$ has an $H$-minor, $|V(H)| \geq 5$, and $|V(G)| \geq |V(H)| + 1$. If $G$ is $H$-critical, then $|E(G)| \leq |E(H)| + 3|V(G)| - |V(H)|$. 
Case (iii) If Case (ii) If Case (i) If

d induction hypothesis, Thus if \(d\) vertices via one of the following operations:

\(W\) wheel on \(|\ V = k\) is sharp because the third operation in Lemma 2.2 is essential, as indicated by the example of the graph edges, thereby answering the question raised in the introduction. Additionally, the bound in Lemma 2.4

Proof of Theorem 1.1. Suppose \(G \cong W_{n-1}\), where \(n \geq 5\). The only 3-connected minor of a wheel on \(n\) vertices is a wheel on fewer vertices. Thus if \(G \cong W_{n-1}\), then \(H = W_{n-k}\), where \(k \geq 2\). Observe that

\[|E(G)| = 2(n-1) = 2n - 2\]

and

\[|E(H)| + 3||V(G)| - |V(H)|| = 2(n - k) + 3[n - (n - k + 1)] = 2(n - k) + 3(k - 1) = 2n + k - 3.\]

Since \(k \geq 2\), \(2n - 2 < 2n - k + 3\), and the result holds.

Thus we may assume \(G \not\cong W_{n-1}\). The proof is by induction on \(|V(G)|\). If \(|V(G)| = |V(H)|\), then since \(G\) is \(H\)-critical, \(G \cong H\) and the result holds trivially. Assume that the result holds for graphs with \(|V(G)| - 1\) vertices. Lemma 2.2 implies that \(G\) is obtained from an \(H\)-critical graph \(G'\) on \(|V(G)| - 1\) vertices via one of the following operations:

(i) \(G = G' \circ f\);
(ii) \(G = G' + e \circ f\), where \(e\) and \(f\) are contained in a vertex triad of \(G\); or
(iii) \(G = G' + \{e_1, e_2\} \circ f\), where \(\{e_1, e_2, f\}\) is a vertex triad of \(G\).

Thus if \(d = |E(G)| - |E(G')|\), then observe that \(1 \leq d \leq 3\). Moreover, since \(G'\) is \(H\)-critical, by the induction hypothesis,

\[|E(G')| \leq |E(H)| + 3||V(G')| - |V(H)||\]
\[= |E(H)| + 3||V(G)| - |V(H)|| - 1\]
\[= |E(H)| + 3||V(G)| - |V(H)|| - 3.\]

We will consider the three cases separately.

Case (i) If \(d = 1\), then \(G = G' \circ f\), and

\[|E(G)| = |E(G')| + 1\]
\[\leq |E(H)| + 3||V(G)| - |V(H)|| - 3 + 1\]
\[= |E(H)| + 3||V(G)| - |V(H)|| - 2.\]

Case (ii) If \(d = 2\), then \(G = G' + e \circ f\), and

\[|E(G)| = |E(G')| + 2\]
\[\leq |E(H)| + 3||V(G)| - |V(H)|| - 3 + 2\]
\[= |E(H)| + 3||V(G)| - |V(H)|| - 1.\]

Case (iii) If \(d = 3\), then \(G = G' + \{e_1, e_2\} \circ f\), and

\[|E(G)| = |E(G')| + 3\]
\[\leq |E(H)| + 3||V(G)| - |V(H)|| - 3 + 3\]
\[= |E(H)| + 3||V(G)| - |V(H)||.\]

Lemma 2.4 implies that if \(G\) has size greater than \(|E(H)| + 3||V(G)| - |V(H)||\), then \(G\) has \(H\)-deletable edges, thereby answering the question raised in the introduction. Additionally, the bound in Lemma 2.4 is sharp because the third operation in Lemma 2.2 is essential, as indicated by the example of the graph \(G = prism + e_1 \circ f_1 + \{e_2, e_3\} \circ f_2\) in Figure 3.

We are now ready for the proof of Theorem 1.1. Like the Splitter Theorem, Theorem 1.1 is a useful induction tool.

Proof of Theorem 1.1. Suppose \(G\) and \(H\) are simple 3-connected graphs such that \(G\) has an \(H\)-minor, \(|E(G)| \geq |E(H)| + 3\), and \(|V(G)| \geq |V(H)| + 1\). First suppose \(G \cong W_{n-1}\), where \(n \geq 5\). Observe that \(W_{n-1}/r \backslash s \cong W_{n-2}\), where \(r\) is a rim edge and \(s\) is a spoke edge. Since the only 3-connected minor of a wheel on \(n\) vertices is a wheel on fewer vertices, and a wheel is minimally 3-connected, \(G\) satisfies (ii) in
the statement of the theorem. Therefore suppose \( G \not\cong W_{n-1} \). Let \( D \) be the set of \( H \)-deletable edges in \( G \). Lemma 2.4 implies that

\[
|D| \geq |E(G)| - |E(H)| - 3|V(G)| - |V(H)|.
\]

Let \( t = |V(G)| \) and \( G_t = G[D] \). Observe that \( G_t \) is \( H \)-critical and has \( |V(G)| \) vertices. Repeated application of Corollary 2.3 implies that there is a sequence of 3-connected \( H \)-critical graphs \( G_0, \ldots, G_t \), such that \( G_0 \cong H \), \( G_1 = G[D] \), and for \( 1 \leq i \leq t \): \( G/f = G' \) for an edge \( f \); \( G/f\setminus e = G'' \) for a pair of edges \( e \) and \( f \) incident to a vertex of degree 3; or \( G - w = G'' \) for a vertex \( w \) of degree 3. \( \square \)

3. Halin’s bound on the size of minimally 3-connected graphs

In this section we consider the situation when \( H \) is the prism graph. The first result in this section is Dirac’s well-known characterization of 3-connected graphs with no prism minor [3]. The graphs \( K'_3, n-3 \), \( K''_3, n-3 \), and \( K'''_3, n-3 \) are obtained from the complete bipartite graph \( K_3, n-3 \) with one, two, and three edges, respectively, joining the three vertices in one class. As it turns out \( K_3, n-3 \) is the only minimally 3-connected graph with no prism minor and also the only graph with \( 3n - 9 \) edges. Using this fact makes Halin’s result very straightforward.

**Theorem 3.1.** (Dirac, 1963) A simple 3-connected graph \( G \) has no prism minor if and only if \( G \) is isomorphic to \( K_5 \setminus e \), \( K_5 \), \( W_r + 1 \), for \( r \geq 3 \), \( K_3, n-3 \), \( K'_3, n-3 \), \( K''_3, n-3 \), or \( K'''_3, n-3 \), for \( n \geq 6 \).

Since the 3-connected graphs without a prism minor are known, it suffices to focus only on the 3-connected graphs with a prism minor. Therefore suppose \( G \) has a prism minor, then using Lemma 2.4 with \( H \) as the prism graph, where \( |E(H)| = 9 \) and \( |V(H)| = 5 \), we get

\[
|E(G)| \leq 9 + 3(n - 6) = 3n - 9
\]

However we can do better. The next result is proved exactly like Lemma 2.4, except that the initial conditions are slightly different.

**Theorem 3.2.** Suppose \( G \) is a simple 3-connected graph on \( n \geq 7 \) vertices with a prism minor. If \( G \) is prism-critical, then \( |E(G)| \leq 3n - 10 \).

**Proof.** The proof is by induction on \( n \geq 7 \). Suppose \( n = 7 \). Lemma 2.4 implies that the prism-critical graphs with 7 vertices are vertex splits of \( \text{prism} + e \) and \( \text{prism} + \{e_1, e_2\} \), since the prism has no vertex splits. Observe that \( \text{prism} + e \) has three vertex splits without prism-deletable edges, but \( \text{prism} + \{e_1, e_2\} \) does not have any vertex splits without prism-deletable edges. See Figure 2 which shows that the graph with 7 vertices and 12 edges obtained from the graph with 6 vertices and 11 edges using Operation (iii) of Lemma 2.2 is not prism-critical. Thus for \( n = 7 \), \( |E(G)| = 11 = 3n - 10 \).

Assume that the result holds for graphs of rank \( n - 1 \). As in the proof of Lemma 2.4, \( G \) can be obtained from a prism-critical graph on \( n - 1 \) vertices via one of the three operations in Lemma 2.2. Since \( G' \) is prism-critical, by the induction hypothesis,

\[
|E(G')| \leq 3(n - 1) - 10.
\]

Thus if \( G = G' \circ f \), then

\[
|E(G)| = |E(G')| + 1 \leq 3(n - 1) - 10 + 1 = 3n - 12.
\]

If \( G = G' + e \circ f \), then

\[
|E(G)| = |E(G')| + 2 \leq 3(n - 1) - 10 + 2 = 3n - 11.
\]

If \( G = G' + \{e_1, e_2\} \circ f \), then

\[
|E(G)| = |E(G')| + 3 \leq 3(n - 1) - 10 + 3 = 3n - 10.
\]
Halin proved that a minimally 3-connected rank $n \geq 8$ graph has size at most $3n - 9$, and the only family of size $3n - 9$ is $K_{3,p}$ [4]. This result follows from Theorem 3.2

**Theorem 3.3. (Halin 1969)** Let $G$ be a minimally 3-connected graph on $n \geq 8$ vertices. Then $|E(G)| \leq 3n - 9$. Moreover, $|E(G)| = 3n - 9$ if and only if $G \cong K_{3,n-3}$.

**Proof.** If $G$ has a prism minor, then since $G$ is minimally 3-connected by hypothesis $G$ must be a prism-critical graph. Theorem 3.2 states that $|E(G)| \leq 3n - 10$. If $G$ has no prism minor, then Theorem 3.1 (Dirac’s Theorem) implies that $G$ is $W_{n-1}$ or $K_{3,n-3}$. Observe that $|E(W_{n-1})| = 2(n-1)$ and $|E(K_{3,n-3})| = 3(n-3) = 3n - 9$. Thus $|E(G)| = 3n - 9$ if and only if $G \cong K_{3,n-3}$. □

Thus Lemma 2.4 may be viewed as extending Halin’s bound from minimally 3-connected matroids to $H$-critical graphs. In the process we obtained Theorem 1.1 which is a structural result for $H$-critical graphs.

Let $EX[H]$ be the class of graphs with no $H$-minor. Theorem 1.1 links the notion of minimally 3-connected graphs to excluded minor classes via the notion of $H$-critical graphs. Once the $H$-critical graphs are identified, then for each $n$ the smallest 3-connected members of $EX[H]$ are also identified. Adding edges to these graphs result in the extremal members of $EX[H]$. The next step in this analysis is to study the situation where $G$ has multiple minors $H_1 \ldots H_k$ and $G$ is $H_i$-critical for $i \in \{1, \ldots k\}$.

**References**


**Response to Referee’s Report**

The authors are grateful to the referees for their careful and timely report with many helpful suggestions. The second author suggested a collaboration between the authors. This was a very good suggestion because together we were able to give the paper focus and created a new structural result for $H$-critical graphs. We took the paper in a different direction than the first version.

When we presented this at a conference, a couple of topologists who were working on characterizing the graphs of graphs with no Kuratowski minors told us they may be able to use our work and also suggested the direction in the last paragraph of the paper.
We made all the changes the referees suggested. In case we didn’t, it is most likely because we changed focus and many have missed a point made by the referees. Only thing is instead of saying all graphs in the paper are simple we stated that loops and parallel edges may be allowed and specified simple everywhere. The definition of minors (mentioning that isolated vertices are deleted) comes from the matroid approach to graphs. When deleting edges in a 3-connected graph as long as the resulting graph is 3-connected no isolated vertices are formed, so it doesn’t matter in a sense. In any case we are open to making this and any other suggestions by the referees.

We considered the computation of minimally 3-connected graphs primarily as a tool to ensure our theorems were correct and were planning to publish code on github mainly and add some numbers to the Online Database of Integer Sequences. After the second referee asked if we are saying we can construct graphs faster, we realized we may have stumbled on something we didn’t expect. There are several more results that must be proved, but tentatively yes we think we can computer graphs faster than current methods. We also have to prove results on computational complexity that is a little further from our expertise and will require some sort of review of our work or suitable collaboration before submitting anything for publication; hence the tentative yes. Thank you very much for asking this question and pointing us in this direction.

Finally, if the referees wish to reject, we would appreciate a quick response. No matter what the decision we appreciate the referees insights and suggestions.