On the Circuit-cocircuit Intersection Conjecture

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Abstract. Oxley has conjectured that for \(k \geq 4\), if a matroid \(M\) has a \(k\)-element set that is the intersection of a circuit and a cocircuit, then \(M\) has a \((k - 2)\)-element set that is the intersection of a circuit and a cocircuit. In this paper we prove a stronger version of this conjecture for regular matroids. We also show that the stronger result does not hold for binary matroids.

Key words. Graph, Matroid, Hamilton cycle, Regular matroid, Circuit-cocircuit intersections

1. Introduction

A circuit in a matroid, \(M\), is a minimal dependant set. A circuit in the dual matroid, usually denoted by \(M^*\), is called a cocircuit. The matroid terminology used follows Oxley [3] with the exception that three-element circuits are called triangles and three-element cocircuits are called triads. The graph notation is standard and can also be found in [3]. We denote the subgraph induced by a subset \(A\) of the vertex set of a graph as \(G[A]\). We refer to a vertex of degree one in a tree as a terminal vertex.

A set that is the intersection of a circuit and a cocircuit is called a circuit-cocircuit intersection in short. Such sets play a useful role in characterizing classes of matroids. For example, a matroid is binary if and only if every circuit-cocircuit intersection is even [3, 9.2]. Seymour strengthened this result by showing that a matroid is binary if and only if it does not have a 3-element circuit-cocircuit intersection [4]. Oxley proved that for \(k \geq 5\), if \(M\) has a \(k\)-element circuit-cocircuit intersection, then \(M\) has a 4-element circuit-cocircuit intersection [2, 2.7]. Oxley also conjectured a strengthened form of this result [3, 14.8.3]: if \(k \geq 4\) and \(M\) has a \(k\)-element set that is the intersection of a circuit and a cocircuit, then \(M\) has a \((k - 2)\)-element set that is a circuit-cocircuit intersection. A result like this would be useful toward characterizing matroids with circuit-cocircuit intersections. Such a characterization is expected to exist, but has not been found.

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A graph with a spanning circuit (a cycle that meets each vertex exactly once) is a Hamiltonian graph. Such graphs are well known and studied extensively. A matroid with a spanning circuit is likewise a Hamiltonian matroid. The circuit-cocircuit conjecture makes an assertion about a matroid with the property that both $M$ and $M^*$ are Hamiltonian.

In this paper we prove a stronger version of Oxley’s conjecture for the class of regular matroids. We prove that if a regular matroid $M$ has a spanning circuit and cospansing cocircuit, then for every element $e \in E(M)$, there exists a triangle or triad avoiding $e$. The graphic matroid version of this result was done in [1], but we include it here for completeness and because it wasn’t presented in this manner. The existence of a triangle or triad avoiding $e$ gives us the $(k - 2)$-element circuit-cocircuit intersection containing $e$. This connection may not be immediately obvious and we will explain it in detail later.

Our proof gives a constructive approach to finding a $(k - 2)$-element circuit-cocircuit intersection. It is not clear at present how to approach proving this conjecture for binary and other classes of matroids, especially since we also give examples to show that the stronger result does not hold for the class of binary matroids. We have pushed prevailing techniques as far as they can go. A probabilistic approach may be needed to resolve the conjecture, even for binary matroids, or a binary counterexample may be found of rank necessarily at least 9 and size at least 18 [1]. No one has attempted probabilistic matroid theory successfully as yet and the size of a potential counterexample is too large for existing computational techniques.

The following theorem is the main result of this paper.

**Theorem 1.1.** Suppose $M$ is a regular matroid and $X$ is a spanning circuit and cospansing cocircuit. Then for every $e \in X$, $M$ has a triangle or triad that avoids $e$ and intersects $X$ in exactly two elements.

As a corollary we obtain the following stronger version of the circuit-cocircuit conjecture for regular matroids. A proof for this corollary is given at the end of the paper.

**Corollary 1.2.** Suppose $M$ is a regular matroid with a circuit $C$ and a cocircuit $C^*$ such that $|C \cap C^*| = k \geq 4$. Then every element in $C \cap C^*$ is contained in a $(k - 2)$-element set that is the intersection of a circuit and a cocircuit.

We give two examples to show that Theorem 1.1 does not hold for non-regular matroids. However, it is not known whether or not Corollary 1.2 holds for non-regular matroids.

Consider a self-dual binary linear code having dimension $2n$ and minimum distance $n$. Such a code originates from a self-dual binary matroid $M$ with rank $2n$, $4n$ elements, circumference $2n$ (circuit of largest size), and girth $n$ (circuit of smallest size). Choose a circuit $C$ of $M$ with size $2n$. As $M$ is self-dual, it follows that $C$ is also a cocircuit of $M$. Choose different elements $e$ and $f$ belonging to $E(M) - C$. Note that $C$ is both a circuit and cocircuit of $N = M\setminus e/f$. Observe that $C$ is spanning in
both $N$ and $N^*$ because $r(N) = r(M) - 1 = 2n - 1$, $r(N^*) = r(M^*) - 1 = 2n - 1$ and $|C| = 2n$. As $M$ has girth $n$ and is self-dual, it follows that $N$ and $N^*$ both have girth at least $n - 1$. Therefore $N$ has no triangles or triads meeting $C$, provided $n \geq 5$.

Our second example is a self-dual, binary, non-regular matroid that is represented by the binary matrix $[I_9 | A]$, where $A$ is listed below:

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

In this matroid, $\{1, 2, \ldots, 10\}$ is a 10-element spanning circuit and cospanning cocircuit. Moreover, $[1, 2, 3, 4, 6, 7, 13, 18]$ is an 8-element circuit-cocircuit intersection obtained by intersecting circuit $[1, 2, 3, 4, 6, 7, 11, 13, 18]$ with cocircuit $[1, 2, 3, 4, 6, 7, 12, 13, 18]$. However, there are no three or four element circuits or cocircuits.

The proof of the main theorem relies on results proved by Seymour in his well-known paper on the decomposition of regular matroids [5]. Before stating them we need to introduce some definitions and terminology. Suppose $M_1$ and $M_2$ are simple regular matroids such that $E(M_1) \cap E(M_2) = T$ and $M_1|T = M_2|T$. If $T$ is a single element or a triangle then $T$ is a modular flat in $M_1$ and $M_2$. In this case the generalised parallel connection of $M_1$ and $M_2$, denoted by $PT(M_1, M_2)$ is defined [3, 12.4].

The 2-sum of $M_1$ and $M_2$, denoted as $M_1 \oplus_2 M_2$, is defined as $M_1 \oplus_2 M_2 = PT(M_1, M_2) \backslash p$, where $T = \{p\}$ and for $i \in \{1, 2\}$, $|E(M_i)| \geq 3$. The 3-sum of $M_1$ and $M_2$, denoted as $M_1 \oplus_3 M_2$, is defined as $M_1 \oplus_3 M_2 = PT(M_1, M_2) \backslash T$ where $T$ is a triangle which does not contain any cocircuit of $M_1$ or $M_2$ and for $i \in \{1, 2\}$, $|E(M_i)| \geq 7$.

Observe that the circuits of $M_1 \oplus_2 M_2$ are the circuits of $M_i$ that do not contain $p$, for $i \in \{1, 2\}$, and all sets of the form $C_1 \Delta C_2$, where $C_i$ is a circuit of $M_i$ and $C_1 \cap C_2 = \{p\}$. Similarly, the circuits of $M_1 \oplus_3 M_2$ are the circuits of $M_i$ that do not contain any member of $T$, and all sets of the form $C_1 \Delta C_2$, where $C_i$ is a circuit of $M_i$ and $C_1 \cap T = C_2 \cap T \neq \phi$. However, 2-sums and 3-sums differ in quite an important way. Specifically, 2-sums are invariant under duality, that is, \((M_1 \oplus_2 M_2)^* = M_1^* \oplus_2 M_2^* [3, 7.1.20(i)]\). Three-sums do not have this property.

References for the next four theorems are [3, 8.3.1], [5, 14.2], [5, 9.2], and [5, 14.3] in order.

\textbf{Theorem 1.3.} A 2-connected matroid $M$ is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids $M_1$ and $M_2$, each of which is isomorphic to a proper minor of $M$. \hfill \Box
In the next results, the matroid $R_{10}$ is the unique splitter for regular matroids [5, 7.4]. The matroid $R_{12}$ is the 3-sum of $M(K_5 \setminus x)$ and $M^*(K_{3,3})$.

**Theorem 1.4.** Let $M$ be a 3-connected regular matroid. Then either $M$ is graphic or cographic or $M$ has a minor isomorphic to one of $R_{10}$ or $R_{12}$.

**Theorem 1.5.** If $M$ is regular and $M$ has a minor isomorphic to $R_{12}$, then $M$ has an exact 3-separation $(X_1, X_2)$ with $|X_1|, |X_2| \geq 6$.

**Theorem 1.6.** Every regular matroid $M$ can be constructed by means of direct sums, 2-sums, and 3-sums, starting with matroids each of which is isomorphic to a minor of $M$, and each of which is either graphic, or cographic, or isomorphic to $R_{10}$.

We will end this section with several useful propositions on circuits and cocircuits. The first two are equivalent characterizations of binary matroids [3, 9.1.2]. The next two are in [3, 9.1.4, 9.1.3] and are required for the proof of Corollary 1.2.

**Proposition 1.7.** A matroid is binary if and only if its circuit-cocircuit intersections are even.

**Proposition 1.8.** A matroid is binary if and only if the symmetric difference of any set of circuits is a disjoint union of circuits.

**Proposition 1.9.** Let $M$ have a $k$-element set $X$ that is the intersection of a circuit and a cocircuit. Then $M$ has a minor $N$ in which $X$ is both a circuit and a cocircuit and $r(N) = r^*(N) = k - 1$.

**Proposition 1.10.** Let $N$ be a minor of a matroid $M$, and let $X$ be the intersection of a circuit and a cocircuit in $N$. Then $X$ is the intersection of a circuit and a cocircuit in $M$.

2. **Proof of the Main Theorem**

The proof of Theorem 1.1 uses a minimal counterexample approach. It is quite long so we broke it into several lemmas to make it easier to follow.

**Proof of Theorem 1.1.** Assume the result is not true and let $M$ be a minimal counterexample. Thus $M$ contains a spanning circuit, $X$, that is also a cospanning cocircuit, and an element $e \in X$, such that no triangle or triad avoids $e$ and meets $X$ in two elements.

**Lemma 2.1.** $M$ is a connected, simple, and cosimple matroid with $|X| \geq 6$, $r(M) = r^*(M) = |X| - 1$ and $|E(M)| = 2r(M) \geq 10$. 


Proof. Since \( X \) is a spanning circuit and a cospanning cocircuit, \( M \) is connected, simple and cosimple. Furthermore, \( r(M) = r^*(M) = |X| - 1 \) and \( |E(M)| = 2r(M) \). Proposition 1.7 implies that since \( X \) is both a circuit and a cocircuit and the matroid is binary the size of \( X \) is even. Suppose \( |X| = 4 \). Then \( r(M) = r^*(M) = 3 \) and \( |E(M)| = 6 \). All the binary matroids of size at most 6 are graphic [2]. So \( M \) is a graph with at least four vertices and six edges, and therefore, \( M \cong M(K_4) \). This is a contradiction because the result holds for \( M(K_4) \). Therefore, \( |X| \geq 6 \) and consequently \( |E(M)| \geq 10 \). \( \square \)

Lemma 2.2. \( M \) cannot have a 2-separation.

Proof. Suppose \( M = M_1 \oplus_2 M_2 \), where \( E(M_1) \cap E(M_2) = \{p\} \). Since \( M \) is both simple and cosimple, \( r(M_i) \geq 2 \) for all \( i \in \{1, 2\} \). Observe that \( M_i \) is connected and has fewer elements than \( M \).

Let \( X_i = X \cap E(M_i) \). Note that \( X_i \neq \phi \) since \( r(M_i) \geq 2 \). Since \( X \) is a circuit of \( M \), \( X_i \cup p \) is a circuit of \( M_i \). Observe that

\[
r(M_1) + r(M_2) - 1 = r(M) = |X| - 1 = |X_1| + |X_2| - 1.
\]

Therefore,

\[
|X_1| + |X_2| = r(M_1) + r(M_2).
\]

However, \( X_1 \) and \( X_2 \) are independent, so \( |X_1| \leq r(M_1) \) and \( |X_2| \leq r(M_2) \). So for all \( i \in \{1, 2\} \), \( |X_i| = r(M_i) \) and \( X_i \cup \{p\} \) is a spanning circuit of \( M_i \).

Since 2-sums are invariant under duality, we can show in a similar way that \( X_i \cup p \) is a cospanning cocircuit of \( M_i \). Thus \( M_i \) has a spanning circuit and cospanning cocircuit \( X_i \cup p \). Without loss of generality, we may assume \( e \in X_1 \). By the minimality of \( M_i \), the result holds for \( M_2 \). So in \( M_2 \), there is a triangle or triad that avoids \( p \) and meets \( X_2 \) in exactly two elements. However, this triangle or triad is a triangle or triad in \( M \) that meets \( X \) in exactly two elements and avoids \( e \). So the result holds for \( M \); a contradiction since \( M \) is a counterexample to the theorem. \( \square \)

Lemma 2.3. \( M \) is neither graphic nor cographic.

Proof. Suppose \( M \) is a connected graphic matroid. Then \( M \cong M(G) \), where \( G \) is a 2-connected graph without loops and parallel edges. Since \( X \) is a cocircuit, there is a partition \((A, B)\) of \( V(G) \) such that \( G[A] \) and \( G[B] \) are connected, and \( X \) is the set of edges that join vertices in \( A \) to vertices in \( B \). Since \( X \) is spanning each vertex of \( A \cup B \) meets \( X \). Since \( X \) is cospanning \( G[A] \) and \( G[B] \) are acyclic. Thus each of \( G[A] \) and \( G[B] \) is a tree with at least 3 vertices and consequently each has at least two terminal vertices (vertices of degree 1).

Let \( u_A \) and \( v_A \) be terminal vertices in \( G[A] \). Since \( G[A] \) is connected and \( |X| \geq 6 \), \( u_A \) and \( v_A \) cannot be incident to the same edge in \( G[A] \). Let \( n(u_A) \) and \( n(v_A) \) be the set of edges incident with vertices \( u_A \) and \( v_A \), respectively. Then \( |n(u_A) \cap G[A]| = 1 \) and \( |n(v_A) \cap G[A]| = 1 \). Since \( X \) is a circuit, \( |n(u_A) \cap X| = 2 \) and \( |n(v_A) \cap X| = 2 \).
Thus terminal vertices \( u_A \) and \( v_A \) give rise to disjoint triads that intersect \( X \) in exactly two elements. Additionally, observe that \( n(u_A) \cap n(v_A) = \emptyset \). So for every \( e \in X \), one of the two triads constructed above will avoid \( e \). Therefore, the result holds for \( M \); a contradiction since \( M \) is a counterexample to the theorem.

Suppose \( M \) is a connected cographic matroid. Then \( M \cong M^*(G) \) and by duality for every \( e \in X \), \( G \) has a triangle that avoids \( e \) and intersects \( X \) in exactly two elements. So a connected cographic matroid cannot be a counterexample to the theorem. \( \square \)

**Lemma 2.4.** \( M \) is not 3-connected.

**Proof.** Suppose \( M \) is 3-connected. Theorem 1.4 implies that either \( M \) is graphic or cographic or \( M \) has a minor isomorphic to one of \( R_{10} \) or \( R_{12} \). Since Lemma 2.3 implies that \( M \) can be neither graphic nor cographic, it follows that \( M \) has a minor isomorphic to \( R_{10} \) or \( R_{12} \). We can compute the circuits and cocircuits of \( R_{10} \) and check that \( R_{10} \) has no set that is both a spanning circuit and a cospanning cocircuit. Therefore \( M \not\cong R_{10} \). Moreover, \( M \) has no \( R_{10} \)-minor since \( R_{10} \) is a splitter for regular matroids \([5, 7.4]\). Therefore we may assume that \( M \) has an \( R_{12} \)-minor.

The matroid \( R_{12} \) has no circuit that is both spanning and cospanning because it is a rank 6 binary matroid and Proposition 1.8 implies that it cannot have a 7-element circuit-cocircuit intersection. Therefore, \( M \not\cong R_{12} \).

Suppose \( M \) has an \( R_{12} \)-minor. Let the ground set of this minor be \( E_1 \cup E_2 \), where \( E_1 \) and \( E_2 \) are disjoint, and \( R_{12} \) is isomorphic to the 3-sum of \( N_1 \cong M^*(K_{3,3}) \) and \( N_2 \cong M(K_5 \setminus \chi) \), where \( E(N_1) = E_1 \cup T \), \( E(N_2) = E_2 \cup T \), and \( T \) is a triangle of both \( N_1 \) and \( N_2 \) disjoint from \( E_1 \cup E_2 \). It follows from the proof of Theorem 1.4 that \( M \) contains a 3-separation \((E_1', E_2')\) so that \( M = M_1 \oplus M_2 \), where \( E(M_1) = E_1' \cup T \) and \( E(M_2) = E_2' \cup T \). Observe that \( E_1 \subseteq E_1' \) and \( E_2 \subseteq E_2' \). We can make the additional assumptions that \( e \in E(M_2) \) and \( M_1 \) cannot be further decomposed by 3-sums. Therefore, \( r(M_2) \geq 4 \) and \( M_1 \) is a 3-connected graphic or cographic matroid with \( r(M_1) \geq 3 \).

Let \( X_i = X \cap E(M_i) \) for all \( i \in \{1, 2\} \). It follows from \([5, 2.7]\) that there is a \( t \in T \) such that \( X_1 \cup t \) is a circuit of \( M_1 \) and \( X_2 \cup t \) is a circuit of \( M_2 \). The next lemma describes a property of circuits in \( M_i \).

**Lemma 2.5.** For some \( i \in \{1, 2\} \), \( X_i \cup t \) is a spanning circuit in \( M_i \) and \( X_{3-i} \cup (T - t) \) is a spanning circuit in \( M_{3-i} \).

**Proof.** Since \( M \) is the 3-sum of \( M_1 \) and \( M_2 \) and \( X \) is a spanning circuit,

\[
r(M_1) + r(M_2) - 2 = r(M) = |X| - 1 = |X_1| + |X_2| - 1
\]

and therefore

\[
|X_1| + |X_2| = r(M_1) + r(M_2) - 1.
\]

However, \( X_1 \) and \( X_2 \) are independent, so \( |X_1| \leq r(M_1) \) and \( |X_2| \leq r(M_2) \). It follows that for some \( i \in \{1, 2\} \), \( r(X_i) = r(M_i) \) and \( r(X_{3-i}) = r(M_{3-i}) - 1 \). Since \( X_i \cup t \) is a circuit and \( r(X_i \cup t) = r(M_i) \), it follows that \( X_i \cup t \) is a spanning circuit.
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It remains to show that $X_{3-i} \cup (T - t)$ is a spanning circuit in $M_{3-i}$. Let $T = \{t, s, r\}$. Suppose if possible $X_{3-i} \cup \{r, s\}$ is not a circuit of $M_{3-i}$. Then it is the disjoint union of two circuits in $M_{3-i}$. So there exists $C \subset X_{3-i}$ such that $C \cup r$ is a circuit of $M_{3-i}$. However, since $X_{i} \cup r$ is a spanning circuit of $M_{i}$, it follows that $X_{i} \cup \{r, s\}$ is also a disjoint union of two circuits of $M_{i}$. So there exists $C' \subset X_{i}$ such that $C' \cup r$ is a circuit of $M_{i}$. Therefore, $C \cup C'$ is a circuit in $M_{i}$; a contradiction since $C \cup C' \subset X$. □

Returning to the proof of Lemma 2.4, let $H = E(M) - X$ and $H_{i} = H \cap E(M_{i})$ for $i \in \{1, 2\}$. Observe that $H$ is an independent hyperplane since $X$ is a cospanning cocircuit. The next lemma describes a property of cocircuits in $M_{i}$.

**Lemma 2.6.** There exists an element $s \in T$ such that $X_{i} \cup (T - s)$ is a cocircuit of $M_{i}$ for all $i \in \{1, 2\}$.

**Proof.** Since $H$ is independent in $M$, $H_{i}$ is independent in $M_{i}$. So

$$r(H_{1}) + r(H_{2}) = |H_{1}| + |H_{2}| = |H_{1} \cup H_{2}| = |H| = r(M) - 1 = r(M_{1}) + r(M_{2}) - 3.$$  

Moreover, for $i \in \{1, 2\}$, $H_{i}$ does not span any element in $X_{i}$ since $H$ is a hyperplane and $H$ does not span any element in $X$. So $r(H_{i}) \leq r(M_{i}) - 1$. Therefore, either $r(H_{1}) = r(M_{1}) - 1$ and $r(H_{2}) = r(M_{2}) - 2$; or $r(H_{1}) = r(M_{1}) - 2$ and $r(H_{2}) = r(M_{2}) - 1$. We will assume the former; the case when the latter holds is identical.

Now $r(H_{1}) = r(M_{1}) - 1$ implies that $cl_{M_{1}}(H_{1})$ is a hyperplane. Observe that $cl_{M_{1}}(H_{1}) \neq H_{1} \cup T$ because otherwise $X_{1}$ would be a cocircuit of $M_{1}$ and therefore also of $M$; a contradiction since $X_{1} \subset X$. Observe that $cl_{M_{1}}(H_{1}) \neq H_{1}$ because otherwise $X_{1} \cup T$ would be a cocircuit of $M_{1}$; a contradiction since $M_{1}$ is binary and $T$ is a circuit with three elements. Therefore, $|cl_{M_{1}}(H_{1}) \cap T| = 1$ and we may assume that for some $s \in T$, $H_{1} \cup s$ is a hyperplane. It follows that $X_{1} \cup (T - s)$ is a cocircuit.

It remains to show that $H_{2} \cup s$ is also a hyperplane. This would mean that $X_{2} \cup (T - s)$ is a cocircuit of $M_{2}$ and the lemma will be proved. Suppose, if possible, $H_{2}$ spans $s$. Then there exists $C_{2} \subseteq H_{2}$ such that $C_{2} \cup s$ is a circuit of $M_{2}$. Moreover, since $H_{1} \cup s$ is a hyperplane, there exists $C_{1} \subseteq H_{1}$ such that $C_{1} \cup s$ is a circuit of $M_{1}$. Therefore, $C_{1} \cup C_{2} \subset H$ is a circuit of $M$; a contradiction since $H$ is independent. Therefore, $H_{2} \cup s$ is independent and since $r(H_{2}) = r(M_{2}) - 2$, it follows that $r(H_{2} \cup s) = r(M_{2}) - 1$.

Finally, we will prove that $H_{2} \cup s$ is closed in $M_{2}$. Observe that $T \not\subseteq cl_{M_{2}}(H_{2} \cup s)$ for the same reason as in the previous case. Suppose, if possible, $H_{2} \cup s$ spans an element $e \in X_{2}$. Then there exists $C_{2} \subseteq H_{2}$ such that $C_{2} \cup \{s, e\}$ is a circuit of $M_{2}$ and $C_{2} \cup \{s, s\} \subseteq H_{2} \cup \{s, e\}$. It follows that $(C_{1} \cup s) \setminus (C_{2} \cup \{s, e\}) \subset H \cup e$ is a circuit of $M$; a contradiction since $H$ is a hyperplane in $M$. Therefore, $H_{2} \cup s$ cannot span an element in $X_{2}$. It follows that $H_{2} \cup s$ is a hyperplane. □

Returning to the proof of Lemma 2.4, we see that Lemma 2.5 implies for some $t \in T$, $X_{1} \cup t$ or $X_{1} \cup (T - t)$ is a spanning circuit, $C$, of $M_{1}$. Lemma 2.6 implies for
some \( s \in T \), \( X_1 \cup (T - s) \) is a cocircuit, \( C^* \), of \( M_1 \). Observe that \( C - T = C^* - T \).

Since \( H_1 \) is independent in \( M_1 \), it follows that \( X_1 \cup T \) contains a basis of \( M_1^* \) and therefore spans \( M_1^* \). The next claim proves that under these hypotheses graphic or cographic matroids will have the required triads or triangles.

**Lemma 2.7.** Let \( M_1 \) be a 3-connected graphic or cographic matroid with a triangle \( T \) such that \( r(M_1) \geq 3 \). Let \( G \) be a spanning circuit such that \( C \cap T \neq \phi \). Let \( C^* \) be a cocircuit such that \( C^* \cap T \neq \phi \). Let \( C^* - T = C - T \), and \( C^* \cup T \) is cospanning. Then \( M_1 \) has a triad or triangle that intersects \( C - T \) in exactly two elements and avoids \( T \).

**Proof.** Suppose the result is not true and let \( M_1 \) be a counterexample. Suppose \( M_1 \) is graphic. Then \( M_1 \cong M(G) \) for some 3-connected graph \( G \) with no loops and parallel edges. Since \( C^* \) is a cocircuit of \( G \), there is a partition \((A, B)\) of \( V(G) \) such that \( C^* \) is the set of edges that join vertices in \( A \) to vertices in \( B \). Since \( |C^* \cap T| = 2 \) and \( T \) is a triangle it follows that \( |T - C^*| = 1 \). Let \( t \) be the edge in \( T - C^* \). We may assume that \( t \) joins two vertices in \( B \). Since \( C^* \cup t \) is cospanning, \( G[A] \) is a tree. If \( |A| = 1 \), say \( A = \{u_A\} \), then \( n(u_A) = C^* \). Since \( C \subseteq C^* \cup t \) and \( C^* \cup t \) contains only \( T \) as a circuit, it follows that \( C = T \). So \( r(M_1) \geq 2 \); a contradiction. Therefore \( G[A] \) has at least two terminal vertices \( u_A \) and \( v_A \). Since \( T \) is a triangle with edge \( t \) in \( G[B] \), it has just one vertex in \( G[A] \), which may be assumed to be different from \( v_A \). So \( n(v_A) \) is a triad in \( G \) that intersects \( C^* - T \) in exactly two elements and avoids \( T \). This contradicts the assumption. So \( M_1 \) cannot be graphic.

Suppose \( M_1 \) is cographic. Then there exists a 3-connected graph \( G \) with no loops and parallel edges such that \( M_1 \cong M^*(G) \). Since \( C \) is a cocircuit of \( G \), there is a partition \((A, B)\) of \( V(G) \) such that both \( G[A] \) and \( G[B] \) are connected, and \( C \) is exactly the set of edges that join vertices in \( A \) to vertices in \( B \). Since \( C \) is cospanning, it follows that \( G[A] \) and \( G[B] \) are trees. \( \square \)

**Claim 1.** Any terminal vertex of \( G[A] \) and \( G[B] \) meets an edge in \( T \).

**Proof of Claim 1.** Suppose otherwise. By relabelling if necessary, assume \( v \) is a terminal vertex of \( G[A] \) and \( n(v) \cap T = \phi \). Any edge of \( C \) that meets \( v \) is also an edge of \( C^* \). Since \( C^* \) is a cycle of \( G \), there can be at most two such edges. As \( v \) is incident with exactly one edge in \( G[A] \) it follows that \( n(v) \) is a triad of \( G \) that meets \( C \) in exactly two elements, and does not intersect \( T \). This contradicts the assumption that \( M_1 \) is a counterexample. Observe that since \( r(M_1) \geq 3 \), neither \( G[A] \) nor \( G[B] \) can consist of a single vertex. So each has at least two terminal vertices. \( \square \)

**Claim 2.** Suppose that \( v_1 \) and \( v_2 \) are distinct terminal vertices of \( G[A] \) and that they are incident with edges \( e_1, e_2 \in C \cap T \), respectively. Then \( e_1 \) and \( e_2 \) are incident with the same vertex in \( G[B] \).

**Proof of Claim 2.** Suppose otherwise. At least one of \( e_1 \) or \( e_2 \) must be incident with a terminal vertex of \( G[B] \), for otherwise we have a contradiction to Claim 1. Assume by relabelling if necessary that \( e_1 \) joins \( v_1 \) to \( u \), a terminal vertex of \( G[B] \). Both \( v_1 \)
and \( u \) have degree at least three, so there must be edges, \( f \neq e_1 \) and \( f' \neq e_1 \), such that \( f \) joins \( v_1 \) to a vertex of \( G[B] \) and \( f' \) joins \( u \) to a vertex \( G[A] \).

Since \( T \) is a minimal edge cut-set of \( G \), it follows that \( v_1 \) and \( u \) are in different components of \( G - T \). Therefore any path of \( G \) that connects \( v_1 \) to \( u \) must contain an edge of \( T \). Let \( P \) be the unique path that joins \( v_1 \) to \( u \), and that contains \( f \), but is otherwise contained in \( G[B] \). This path cannot contain \( e_1 \) or \( e_2 \), so it must contain the other edge of \( T \), which we will call \( e \). Now \( e \) cannot be in \( C \), for then \( T \subseteq C \). Thus \( e \in G[B] \). However, there is also a path \( P' \), that joins \( u \) to \( v_1 \), and that contains \( f' \), but is otherwise contained in \( G[A] \), so by the same arguments we can show \( e \in G[A] \); a contradiction because \( G[A] \cap G[B] = \Phi \).

It follows easily from Claims 1 and 2 that there cannot be five terminal vertices in \( G[A] \) and \( G[B] \). Therefore both \( G[A] \) and \( G[B] \) are paths.

Suppose that \( u_A \) and \( v_A \) are the terminal vertices of \( G[A] \) and \( u_B \) and \( v_B \) are the terminal vertices of \( G[B] \). As a consequence of Claim 1, there must be an edge \( e \in C \cap T \), that joins a terminal vertex in \( G[A] \) to a terminal vertex in \( G[B] \). By relabelling assume that \( e \) joins \( u_A \) to \( u_B \). By Claims 1 and 2, there are edges \( e_1, e_2 \in T \), such that \( e_1 \) is incident with \( v_A \), \( e_2 \) is incident with \( v_B \), and \( T = \{e, e_1, e_2\} \).

Assume that there is an internal vertex, \( w \in G[B] \). By Claim 2, the edge \( e_1 \in T \) either joins \( v_A \) to \( u_B \), or is contained in \( G[A] \). In either case \( w \) is not incident with any edge in \( C \cap T \). Therefore any edge in \( C \) that meets \( w \) is also in \( C^* \). Thus \( w \) is incident with exactly two edges in \( C \). At most one such edge can join \( w \) to \( v_A \). Let \( f \) be an edge in \( C \) that meets \( w \), but does not meet \( v_A \). Then there is a unique path that joins \( u_A \) to \( u_B \), and that contains \( f \), but is otherwise contained in \( G[A] \) and \( G[B] \). This path can contain no edge of \( T \). This contradiction shows that \( G[B] \) has no internal vertices, and is therefore a single edge. Similarly \( G[A] \) must be a single edge, and \( G \) is in fact isomorphic to \( M(K_4) \). This is a contradiction since \( M_1 \) is cographic.

\[ \square \]

Returning to the proof of Lemma 2.4, we see from Lemmas 2.5, 2.6, and 2.7 that \( M \) cannot be 3-connected.

\[ \square \]

To finish the proof of Theorem 1.1, observe that Lemmas 2.1 and 2.2 imply that the minimal counterexample is a simple, connected, regular matroid that has no 2-separation. Lemma 2.3 implies that it is neither graphic nor cographic and Lemma 2.4 implies that it is not 3-connected. This contradicts Theorem 1.6. Hence proved.

\[ \square \]

**Proof of Corollary 1.2.** Suppose \( M \) is a regular matroid with a circuit \( C \) and a cocircuit \( C^* \) such that \( |C \cap C^*| = k \geq 4 \). Proposition 1.9 implies that \( M \) has a minor \( N \) in which the set \( X = C \cap C^* \) is both a spanning circuit and a co-spanning cocircuit of size \( k \). Theorem 1.1 implies that for every element \( e \in X \), \( M \) has a triangle or triad \( P \) that avoids \( e \) and meets \( X \) in exactly two elements. Let \( P = \{p_1, p_2, p_3\} \) with \( \{p_2, p_3\} \subseteq X \). Since \( M \) is binary, Proposition 1.8 implies that \( P \cup X \) is a disjoint union of circuits and therefore must be a circuit. Observe that \( X \) contains \( e \). Thus we have a circuit \( (X - \{p_2, p_3\}) \cup p_1 \) containing \( e \) whose intersection with cocircuit \( X \) is \( k - 2 \). The argument when \( P \) is a triad is similar. Corollary 1.2 now follows from Proposition 1.10.

\[ \square \]
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References


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