Abstract
Matroids are a generalization of several combinatorial objects, such as graphs, matrices, designs, linear spaces, and projective geometries. In this survey article we give a brief introduction to matroids and describe some results from a graphic perspective.

1. Introduction
Hassler Whitney introduced matroids in his 1935 paper: “On the abstract properties of linear dependence”. In defining a matroid Whitney tried to capture the fundamental properties of dependence that are common to graphs and matrices. He defined a matroid $M$ as a set $E$ of $n$ elements and a family $I$ of subsets of $E$, called independent sets, that satisfy the following postulates:

1. the empty set is independent;
2. any subset of an independent set is independent; and
3. if $I_1$ and $I_2$ are independent sets such that $|I_1| < |I_2|$, then there exists $e \in I_2 - I_1$ such that $I_1 \cup e$ is independent.

This definition is very abstract, so let us consider examples of concrete objects (relatively speaking) that satisfy these postulates. Consider an $r$ by $n$ matrix $A$ over a field $F$ with columns labeled $\{1, 2, \ldots, n\}$. Define $E$ as the set of column labels and $I$ as subsets of column labels that correspond to linearly independent sets of columns in the vector space $V(r, F)$. Then $I$ satisfies the three postulates and the resulting matroid, denoted by $M[A]$, is called the vector matroid of $A$. The first two postulates are obvious. A small explanation shows that the third postulate is also straightforward [1] (see §1.1.4). Let $W$ be the subspace of $V(r, F)$ spanned by $I_1 \cup I_2$. Then $\dim W \leq |I_1|$. Suppose, if possible, $I_1 \cup e$ is dependent for every element, $e \in I_2 - I_1$. Then $W$ is contained in the span of $I_1$. Hence, $\dim W \leq |I_1|$. Thus, we obtain $|I_2| \leq \dim W \leq |I_1| < |I_2|$: a contradiction.

Consider a specific matrix $A$ with entries from the finite field $GF(2)$.

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

In this example $E = \{1, 2, 3, 4, 5, 6, 7\}$ and $I$ consist of all seven singleton sets, all 21 pairs except $\{1, 7\}$, and the following 24 triples: $\{1, 2, 3\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 6\}, \{2, 3, 7\}, \{2, 4, 5\}, \{2, 4, 6\}, \{2, 5, 6\}, \{2, 5, 7\}, \{2, 6, 7\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 4, 7\}, \{3, 5, 6\}, \{3, 5, 7\}, \{4, 5, 7\}, \{4, 6, 7\}$, and $\{5, 6, 7\}$.

A dependent set is a subset of $E$ not in $I$. A circuit is a minimal dependent set. A basis is a maximal independent set. For the above matrix, the family of circuits, denoted by $C$, consists of $\{1, 7\}, \{1, 2, 4\}, \{1, 3, 6\}, \{2, 3, 5\}, \{2, 4, 7\}, \{3, 6, 7\}, \{4, 5, 6\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 5, 6, 7\}$, and $\{3, 4, 5, 7\}$. The family of bases, denoted by $B$, consists of the 24 triples listed previously. The circuits satisfy the following postulates:

1. No proper subset of a circuit is a circuit; and
2. If $C_1$ and $C_2$ are circuits and $e \in C_1 \cap C_2$, then there exists a circuit $C_3$ such that $C_3 \subset (C_1 \cup C_2) - e$.
The bases satisfy the following postulates:

1. At least one basis exists; and
2. If \( B_1 \) and \( B_2 \) are bases and \( e \in C_1 \cap C_2 \), then there exists an element \( f \in B_2 - B_1 \) such that \( (B_1 - e) \cup f \) is a basis.

In our matrix example, the first circuit and basis postulates are obviously satisfied by minimal linearly dependent sets. The second postulates are explained in [1] (see §§1.1.3 and 1.2.2).

A simple matroid is one in which there are no 1- or 2-element circuits. For example, the matroid above is not a simple matroid since \( \{1, 7\} \) is a circuit. A matrix will give a simple matroid if it has no zero column and no repeated columns up to scalar multiples. Simple matroids are also called combinatorial geometries. Some mathematicians prefer to dispense with the term matroid in favor of combinatorial geometry. We see in the next section that this is a quite appropriate term for simple matroids.

For a subset \( X \) of \( E \), the rank of \( X \), denoted by \( r(X) \), is the size of a maximal independent subset in \( X \). The rank of \( M \), denoted by \( r(M) \), is the size of a basis. Thus, if \( X \) is independent, then \( r(X) = |X| \) and if \( X \) is a basis, then \( r(X) = |X| = r(M) \). In the above example, \( r(M) = 3 \).

Whitney’s first definition of a matroid is in terms of rank. This definition is worth stating because it makes clear that, in its most abstract form, a matroid is simply a function from the set of subsets of an entry to the subject, conspicuously reveals the unique peculiarity of this field, namely the exceptionally large variety of cryptomorphic definitions of a matroid, each one harking back to a different mathematical Weltanschauung. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that anyone would deem impossible, were it not for the fact that matroids do exist.”

We end this section with some of Rota’s thoughts on matroids taken from his introduction to Kung’s useful collection of fundamental matroid papers [6]. “Like many of the great ideas of this century, matroid theory was invented by one of the foremost American pioneers, Hassler Whitney. His paper, which remains the best entry to the subject, conspicuously reveals the unique peculiarity of this field, namely the exceptionally large variety of cryptomorphic definitions of a matroid, each one harking back to a different mathematical Weltanschauung. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist.”
Rota; Brylawski’s Tutte-Grothendieck decomposition theory; Zaslavsky’s arrangement of hyperplanes; the varietal theory of matroids of Kahn and Kung; and the newer decomposition theory initiated by Seymour (which we discuss in Sections 3 and 4).

2. Popular Classes of Matroids

In this section we highlight popular classes of matroids. We begin with representable matroids. A rank-$r$, $n$-element matroid $M$ is called representable over a field $F$ if it is the vector matroid of a matrix $A$ with entries from $F$. Using Gaussian reduction, we can write $A$ in standard form, $[I | D]$, where $D$ is an $r$ by $(n-r)$ matrix. Suppose that $M$ is a combinatorial geometry. Then each column in $A$ corresponds to a nonzero representative vector from a 1-dimensional subspace of $V(r,F)$. So the columns of $A$ can be viewed as a subset of the projective geometry $PG(r-1,F)$. In the rest of this article we focus on the finite fields $GF(q)$, where $q$ is a power of a prime. The corresponding projective geometries are called finite Desarguesian projective geometries. Shown below are $PG(2,2)$, also known as the Fano plane, $PG(2,3)$, and $PG(3,2)$.

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

Observe that representable matroids are contained in the projective geometries in much the same way that graphs are contained in the complete graphs. Thus, for a given $q$, obtaining the $GF(q)$-representable geometries of rank $r$ is a matter of enumerating the non-isomorphic subsets of $PG(r-1,q)$. It is of course, as difficult, if not more difficult, than graph isomorphism. Formally, we say two matroids $E_1$ on $X_1$ and $E_2$ on $X_2$ are isomorphic if there is a bijection $f$ from $X_1$ to $X_2$ such that for all $X \subseteq E_1$, $X$ is independent in $E_1$ if and only if $f(X)$ is independent in $E_2$.

Next we study the class of rank three geometries. This may seem like the simplest class of matroids, but we see in a moment that there is a surprise. Shortly after the orginal paper of Whitney, MacLane [7] described how a matroid can be represented as a schematic geometric figure composed of points, lines, planes, 3-surfaces, 4-surfaces, and so forth, with certain combinatorially defined incidences. He noted that a schematic geometric figure may or may not correspond to an actual figure. Until then mathematicians tended to focus more on the realizability of the geometric figures in a particular surface and less on the combinatorial nature of the figures.

A rank-1 geometry is just a point; a rank-2, $n$-element geometry is a line with $n$ points; and a rank-3, $n$-element geometry can be drawn as a schematic two-dimensional figure consisting of points and lines that satisfy the following rules:

1. Any two distinct points belong to exactly one line;
2. Any line contains at least two distinct points;
3. No line contains all the points; and
4. There are at least two points.

Similarly, rank-4 geometries can be drawn as three-dimensional figures of points, lines, and planes, and so on. Combinatorial geometries of rank at most three are also called linear spaces. Figure 1 shows the geometric representations of well-known rank-3 geometries, such as the Fano matroid, $PG(2,3)$, the non-Fano matroid,
the Pappus matroid, and the $8_3$ configuration, which is a minimal excluded minor for $GF(5)$-representation. Note that when drawing these schematic figures only the lines with three or more points are drawn.

Every projective geometry of dimension greater than two is isomorphic with a finite Desarguesian projective geometry $[8]$. However, the existence of projective geometries of dimension two (projective planes) is a major unsolved problem. Clearly, $PG(2, q)$ exists for any prime power $q$. Moreover, projective planes of orders 2, 3, 4, 5, 7, and 8 are unique. However, starting with order 9 there exist projective planes that are not isomorphic to $PG(2, q)$. In particular, there are four non-isomorphic projective planes of order 9 $[9]$. There is no projective plane of order 6; this follows from a theorem of Bruck and Ryser $[10]$ that states: For $m = 4k + 1$ or $4k + 2$, if there exists a projective plane of order $m$, then $m$ is the sum of two squares. The converse of this theorem is not true. A computer search established that there are no projective planes of order

Figure 1: Rank three geometries with geometric and matrix representations

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Another unsolved problem is motivated by M. Hall’s 1943 result [12]: a linear space can be embedded in a projective plane. Hall’s proof makes no mention of whether the plane is finite or infinite. Welsh asked whether a finite linear space can be embedded in a finite projective plane [1] (see §14.8.1).

Figure 1 also shows the non-Pappus matroid [7], which is different from the other matroids because it is non-representable. In other words, the non-Pappus matroid cannot be embedded in $PG(2, q)$. Observe that the non-Pappus matroid is obtained from the Pappus matroid by removing one three-point line. This suggests that non-representable matroids can be obtained by taking a suitable theorem of projective geometry, drawing the configuration that it implies, and modifying the configuration so that it remains a matroid, but no longer satisfies the theorem [6] (see p.133). For more on the geometric perspective see the Introduction to Matroids on Bonin’s web page.

Our next class of matroids is the class of uniform matroids. Let $E = \{1, 2, \ldots, n\}$ and $r \leq n$. Define $C$ as the family of all $(r + 1)$-element subsets of $E$. Then $C$ clearly satisfies the circuit axioms and the resulting matroid, denoted by $U_{r,n}$, is called a uniform matroid. Observe that the bases of $U_{r,n}$ are the $r$-element sets, the independent sets are all the sets of size at most $r$, the spanning sets are all the sets of size at least $r$, and $r(X) = |X|$ if $|X| \leq r$ and $r(X) = r$ if $|X| \geq r$. For example, $U_{2,4}$ (see Figure 1) is the line with four points.

The class of paving matroids are somewhat similar to uniform matroids. A rank-$r$ matroid with no circuits of size less than $r$ is called a paving matroid. Blackburn, Crapo, and Higgs generated exhaustively the isomorphism-free matroids for $n \leq 8$ [13] and noticed that paving matroids dominated. Welsh asked whether this observation holds in general [1] (see §14.5.5). This question remains unanswered.

Our final class of matroids is the class of graphic matroids. However, before we define the graphic matroids consider the notion of duality, because it will lead us nicely into graphic matroids. The notion of duality is central in matroids. Let $M$ be a matroid on $E$ with bases family $B$. The family $B^* = \{E - B : B \in B\}$ also satisfies the basis axioms [1] (see §2.1.1). The matroid with ground set $E$ and bases family $B^*$ is called the dual of $M$ and is denoted by $M^*$. The bases, circuits, and so on of $M^*$ have the prefix “co” in front of them, as in, cocycles and cocircuits. A matroid $M$ is called self-dual if $M^* = M$. For representable matroids the dual matrix is easy to obtain. Let $M$ be a rank-$r$, $n$-element matroid $M$ represented by a matrix $A = [I_r \setminus D]$ over $F$. Then $M^*$ is represented by $[-D^T | I_{n-r}]$ [1] (see §2.2.8). The dual of a uniform matroid $U_{r,n}$ is $U_{r-n,n}$.

Thus, every matroid has a dual. This is in marked contrast to graphs since only planar graphs have duals. However, non-planar graphs, such as $K_5$ and $K_{3,3}$, when viewed as matroids, have duals. ($K_5$ is the complete graph on five vertices and $K_{3,3}$ is the complete bipartite graph with three vertices in each part of the vertex partition.) With this much information we are ready to view a graph as a matroid.

Consider the graph $G$ shown in Figure 2. Let $E$ be the set of edges $\{1, 2, 3, 4, 5, 6, 7\}$. The cycles in $G$ are $\{1, 7\}$, $\{1, 2, 4\}$, $\{1, 3, 6\}$, $\{2, 3, 5\}$, $\{2, 4, 7\}$, $\{3, 6, 7\}$, $\{4, 5, 6\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 6\}$, $\{2, 5, 6, 7\}$, and $\{3, 4, 5, 7\}$. The cycles satisfy the circuit postulate. Once again the first postulate is obvious. The second postulate requires a little explanation. Suppose $C_1$ and $C_2$ are cycles in $G$ and $e$ is a common edge. Let $e$ have endpoints $u$ and $v$. We construct a cycle in $(C_1 \cup C_2) - e$. Commencing with $u$ traverse along the common edges of $(C_1 \cup C_2) - e$ until a last common vertex (call it $w$) is reached. Commencing with $w$ traverse along $C_1$ until the next common vertex (call it $x$) is reached. The path from $w$ to $x$ along $C_1$ joined with the path $x$ to $w$ along $C_2$ forms a cycle contained in $(C_1 \cup C_2) - e$ [1] (see §1.1.7). Thus, we obtain a matroid related to the graph. We denote this matroid by $M(G)$ and call it the cycle matroid of $G$. The bases of $M(G)$ are the spanning trees of $C$. The rank of $M(G)$ is one less than the number of vertices.

Next, observe that the matrices for $M(G)$ are the same as the circuits for $M[A]$. Thus, the binary matrix $A$ contains all the information contained in the graph $G$. We can construct $A$ from $G$ as follows: select a spanning tree and label it $\{1, 2, 3\}$. These edges correspond to the three unit vectors in $A$. Since the edges $1, 2, 4$ form a cycle in $G$ the corresponding columns form a minimally dependent set in $A$. So column 4 is $(1, 1, 0)$. 

\[ \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \]
similarly, since edges 2, 3, 6 form a cycle, column 5 is (0, 1, 1) and so on. Several examples of graphs with their binary matrix representation are illustrated in Figure 2.

A matroid is called **graphic** if it can be represented as the cycle matroid of a graph. A matroid is called **cographic** if it is the dual of a graphic matroid. For example, $M^*(K_3^e)$ and $M^*(K_{3,3})$ are cographic matroids. Graphic and cographic matroids are representable over every field $[1]$ (see §5.1.2). The reader may wonder if there is a way of determining whether or not a matroid is graphic. Tutte $[14]$ gave a polynomial-time algorithm for determining whether a binary matroid is graphic. If the binary matrix is graphic, then the algorithm returns the incidence matrix of a graph otherwise it concludes that the matrix is not graphic. Later Seymour gave a polynomial-time algorithm for determining whether any matroid, not necessarily a binary matroid, is graphic $[15]$.

Another graphic approach to matroids stems from electrical engineering. As Tutte notes in $[14]$, matroid results are of practical interest to electrical engineers. For example, Bruno and Weinberg defined and studied electrical networks based on matroids rather than graphs in $[16][17]$. The reader is also referred to the book *Network Analysis and Synthesis*, by Weinberg $[18]$ and the Minty survey paper $[19]$.

Next we compare graph isomorphism with graphic matroid isomorphism. If two graphs $G_1$ and $G_2$ are isomorphic then their cycle matroids $M(G_1)$ and $M(G_2)$ are also isomorphic. Whitney $[20]$ proved that the converse is true if we replace graph isomorphism with 2-isomorphism.

Observe that there is no concept of a vertex in a matroid. The emphasis is on the edges. However, Kelmans $[21]$ noted that a non-separating cocircuit in a matroid is a reasonable analog of a vertex. The cocircuits of a graphic matroid (i.e., circuits of the dual) are the minimal cutsets in the graph. The edges incident with a vertex form a minimal cutset and their removal separates the graph into two components one of which is trivial with just one vertex. Such a cutset is different from one that disconnects the graph into non-trivial components. For example, in the graph $H$ shown in Figure 2, the edges $\{1, 2, 7\}$ and $\{1, 3, 7\}$ are both minimal cutsets but the latter separates the graph into two non-trivial components. Using the notion of non-separating cutsets, Kelmans gave a short proof of Whitney’s 2-isomorphism result. He showed that in the 3-connected case, $M(G_1) = M(G_2)$ if and only if $G_1 = G_2$. We discuss 3-connectivity in detail in the next section.

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**Figure 2: Graphs with their binary matrix representations**

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Tutte left a bounty of techniques many of which are being used fruitfully by a new generation of matroid researchers. In his *Toast to Matroids*, he wrote: “I am asked sometimes what a matroid is. I often revert to our sacred writings and recall the encounter of Alice with the Cheshire cat. At one stage the cat vanishes away, beginning with the tip of its tail and ending with the grin, which persists long after the remainder of the cat.”

For more on Tutte’s perspective the reader is referred to the survey paper Oxley’s [22], which begins with Tutte’s insightful statement: “If a theorem about graphs can be expressed in terms of edges and circuits only it probably exemplifies a more general theorem about matroids.” This paper contains many matroid results from the graphic perspective obtained by Ding, Dharmatilake, Goddyn, Lemos, Oporowski, Oxley, Reid, Robertson, Seymour, Vertigan, P.-L. Wu, and others. A MathSci search on the authors mentioned in that paper will turn up a wide variety of matroid results related to graph theory.

3. Connectivity and Minors

We commence this section with the notions of deletions and contractions in a matroid. Let $M$ be a matroid on $E$ and $X \subseteq E$. The *deletion* of $X$ from $M$, denoted by $M \setminus X$, is the matroid on $E - X$, in which $C \subseteq E - X$ is a circuit if and only if $C$ is a circuit in $M$. The *contraction* of $X$ from $M$, denoted by $M / X$, is defined to be the matroid $(M / X)^*$. A matroid $N$ is called a *minor* of $M$ if $N = M \setminus X / Y$. The classes of graphic matroids and representable matroids are closed under minors and duality.

Deletion and contraction in graphic and representable matroids is easy. In a graphic matroid $M(G)$ deleting an element corresponds to deleting an edge in $G$ (and any resulting isolated vertices). To contract an element $e$, identify the vertices of the edge $e$ in $G$ and delete the resulting loop. Thus a minor of a graphic matroid is simply a minor of the underlying graph and is obtained by deleting edges and isolated vertices and contracting edges. In a representable matroid $M(A)$, deleting an element corresponds to deleting the corresponding column in $A$ and any resulting zero rows. Contracting an element involves doing a pivot on a non-zero element of the column to turn it into a unit vector (if it is not already a unit vector) and then deleting the column, as well as the row with the non-zero element. For example, the graphs that are obtained from $K_5$ by deleting and contracting the edge labeled 1 and their corresponding matrices, are shown in Figure 3.

![Figure 3: Deleting and contracting an element in $K_5$](image)

Next we discuss matroid connectivity. For $X \subseteq E$, the *connectivity function*, $\lambda$, is defined by

$$\lambda(X) = r(X) + r(E - X) - r(M).$$

Observe that $\lambda(X) = \lambda(E - X)$. We say $M$ is $k$-connected if, for every non-empty proper subset $X$ of $E$, $\lambda(X) \geq k$. This definition of connectivity is referred to as Tutte $k$-connectivity to distinguish it from other types of $k$-connectivity. Tutte $k$-connectivity is invariant under duality [1] (see §8.1.5). Moreover, from the definition it is clear that matroid connectivity begins with 2-connectivity. So when we say connected matroid we mean 2-connected matroid. The next two results compare matroid connectivity with graph connectivity.

**Theorem 3.1:** Let $G$ be a graph with at least three vertices and no isolated vertex. Then $M(G)$ is 2-connected if and only if $G$ is 2-connected and has no loop.

**Theorem 3.2:** Let $G$ be a graph with at least three vertices, no isolated vertex, and $G \neq K_3$. Then $M(G)$ is 3-connected if and only if $G$ is 3-connected and has no loop or parallel edges.

Thus, the notions of 2- and 3-connectivity coincide for simple graphs and matroids. However, for higher values of $k$, Tutte $k$-connectivity differs from graph $k$-connectivity. A $k$-connected graph cannot have cocircuits (cutsets) of size less than $k$, but it could have circuits of small size. A $k$-connected matroid, with at least
2(k – 1)-elements, cannot have circuits and cocircuits of size less than k. This is because if C is a circuit (or a cocircuit) then \( \lambda(C) = |C| - 1 \). Hence, if \( |C| \leq k \), then \( \lambda(C) \leq k - 1 \); a contradiction to k-connectivity.

The next result by Bixby [23] describes a decomposition result for 2-connected matroids using certain operations called direct-sums and 2-sums. Let \( M_1 \) and \( M_2 \) be matroids on disjoint sets \( E_1 \) and \( E_2 \) with independent sets \( I_1 \) and \( I_2 \). The matroid \( M \) on \( E_1 \cup E_2 \) with independent family

\[
\mathcal{I} = \{ I_1 \cup I_2 : I_1 \in I_1, I_2 \in I_2 \}
\]

is called the direct-sum of \( M_1 \) and \( M_2 \). The notions of 2-sums and 3-sums require more explanation and the reader is referred to [1] (see pp.248 and 421). Informally, the direct-sum of graphs \( G_1 \) and \( G_2 \) is obtained by viewing graphs \( G_1 \) and \( G_2 \) as one graph or fusing them at a vertex. The 2-sum is obtained by fusing \( G_1 \) and \( G_2 \) across an edge and then deleting the edge. Similarly, the 3-sum is obtained by fusing \( G_1 \) and \( G_2 \) across a triangle and then deleting the triangle. The classes of graphic, cographic, and \( F \)-representable matroids are closed under direct-sums and 2-sums [1] (see §§4.2.15 and 7.1.23). Only the classes of graphic and binary matroids are closed under 3-sums [1] (see §12.4.16).

**Theorem 3.3:** Every matroid that is not 3-connected can be constructed from its 3-connected proper minors by a sequence of operations consisting of direct-sums and 2-sums.

There is no such general result for matroids that are not Tutte 4-connected. However, in 1980, Seymour proved an important decomposition result for regular matroids. A matroid is said to be regular if it can be represented by a totally unimodular matrix; that is, a matrix over the reals for which every square submatrix has determinant in \( \{0, 1, -1\} \). A regular matroid is representable over every field [1] (see §6.6.3). The class of regular matroids is closed under minors and duality [1] (see §§3.2.5 and 2.2.22). Seymour found a Tutte 4-connected regular matroid, which he called \( R_{10} \), and showed that it is the only such matroid. All other regular matroids can be decomposed as the direct sum, 2-sum, or 3-sum of \( R_{10} \) or graphic or cographic matroids.

There are several competing versions of 4-connectivity. One form that is emerging as quite useful is internal 4-connectivity. For a matroid \( M \), a partition \( (X, Y) \) of \( E \) is called a k-separation if \( |X| \geq k, |Y| \geq k \), and \( \lambda(X) \leq k - 1 \). When \( \lambda(X) = k - 1 \), we call \( X \) an exact k-separation. When \( |X| = k \) and \( \lambda(X) = k - 1 \) we call \( X \) a minimal exact k-separation. A matroid is said to be internally 4-connected if it has no non-minimal exact 3-separations. Tutte 4-connectivity does not allow any exact 3-separations. An internally 4-connected matroid may have an exact 3-separation \( (X, Y) \) provided one of \( X \) or \( Y \) has cardinality at most 3. So 3-element circuits and cocircuits are allowed in such matroids.

Returning to the decomposition of regular matroids, \( R_{10} \) is a rank-5, 10-element, self-dual matroid with a circulant pattern of three ones in each column. A binary matrix representation is shown as follows:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

**Theorem 3.4:** Every regular matroid \( M \) can be constructed by means of operations consisting of direct-sums, 2-sums, and 3-sums starting with matroids each of which is isomorphic to a minor of \( M \), and each of which is either graphic, or cographic, or isomorphic to \( R_{10} \).
Figure 1 is obtained from the Fano matroid by relaxing the circuit–hyperplane \{4, 5, 6\} and the non-Pappus matroid is obtained from the Pappus matroid by relaxing the circuit–hyperplane \{4, 5, 9\}.

**Theorem 3.5:** Let \( M \) be a 3-connected matroid with rank at least 3. For every element \( e \), neither \( M \setminus e \) nor \( M / e \) is 3-connected if and only if \( M \) is isomorphic to a wheel or a whirl.

The next result, by Seymour, called the **Splitter Theorem**, generalizes the Tutte result to matroids \cite{24}. Let \( \mathcal{M} \) be a class closed under minors and isomorphism. A matroid \( M \) is a **splitter** for \( \mathcal{M} \) if every 3-connected single element extension of \( M \) and \( M^* \) is outside \( \mathcal{M} \). The matroid \( R_{10} \) is the unique splitter for the class of regular matroids. Splitters are useful objects as we see shortly.

**Theorem 3.6:** Let \( M \) and \( N \) be 3-connected matroids such that \( N \) is a minor of \( M \), \( |E(N)| \geq 4 \); and if \( N \) is a wheel then \( M \) has no larger wheel as a minor, whereas if \( N \) is a whirl then \( M \) has no larger whirl as a minor. Then there is a sequence \( M_0, M_1, ..., M_n \) of 3-connected matroids such that \( M_0 = N \), \( M_n = N^* \), and, for all \( i \in \{0, 1, ..., n-1\} \), \( M \) is a single element deletion or a single element contraction of \( M_{i+1} \).

Thus, 3-connected matroids can be built up from a minor by a sequence of single element extensions and coextensions. We pause here for a moment and define the terms extension and coextension. If \( M \) and \( N \) are matroids on the sets \( E \cup x \) and \( E \) where \( x \not\subseteq E \), then \( M \) is an extension of \( N \) if \( M \setminus x = N \), and \( M \) is a coextension of \( N \) if \( M^* \) is an extension of \( N^* \). If \( N \) is a 3-connected matroid, then an extension \( M \) of \( N \) remains 3-connected provided \( x \) is not in a 1- or 2-element circuit of \( M \) and \( x \) is not a coloop of \( M \). Likewise, \( M \) is a 3-connected coextension of \( N \) if \( M^* \) is a 3-connected extension of \( N^* \). Hence, for 3-connected matroids represented by a matrix, obtaining 3-connected extensions is a matter of adding columns from the projective geometry and testing for isomorphism. For example, we can check that \( R_{10} \) has two non-isomorphic single element extensions both of which are non-regular. Since it is self-dual there is no need to check the coextension.

Seymour proves Theorem 3.4 by first proving that if \( M \) is a 3-connected regular matroid, then either \( M \) is graphic or cographic or \( M \) has a minor isomorphic to one of \( R_{10} \) or \( R_{12} \). Since \( R_{10} \) is a splitter for regular matroids, there are no further extensions and coextensions of \( R_{10} \) that are regular. The matroid \( R_{10} \) is the next in our list of useful binary matroids. It is a rank-6, 12-element, self-dual matroid. A matrix representation is shown as follows:

\[
\begin{bmatrix}
1 & ... & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

\( R_{12} \) has a non-minimal exact 3-separation \((X, Y)\) where \( X = \{1, 2, 5, 6, 9, 10\} \). To finish the proof Seymour showed that if a regular matroid \( M \) has a minor isomorphic to \( R_{12} \), then \( M \) has an exact 3-separation \((X, Y)\) with \( |X|, |Y| \geq 6 \). In other words the 3-separation in \( R_{12} \) remains in any regular matroid containing \( R_{12} \).

![Figure 4: Classes of matroids](image-url)
Figure 4 shows us the big picture. It may look as if the class of graphs is the smallest class of matroids, but as we see from Theorem 3.4, other classes can be built up from the graphic matroids.

4. Excluded-Minor Results

Excluded-minor results are of two types, structure driven and minor driven [25]. We begin with structure driven results. Given a minor closed class \( \mathcal{M} \) with a certain structure, the goal in such results is to characterize \( \mathcal{M} \) by finding the minimal excluded minors for \( \mathcal{M} \). We say \( \mathcal{M} \) is a minimal excluded minor for \( \mathcal{M} \) if \( \mathcal{M} \) is not in \( \mathcal{M} \), but every single-element deletion and contraction of \( \mathcal{M} \) is in \( \mathcal{M} \). The first such result is the Kuratowski characterization of planar graphs in terms of forbidden subdivisions [26]. Wagner extended Kuratowski’s characterization to minors [27], thus providing the Kuratowski-Wagner excluded-minor characterization of planar graphs.

**Theorem 4.1:** A graph is planar if and only if it has no minor isomorphic with \( K_5 \) or \( K_{3,3} \).

The next four results by Tutte can be found in [28][29].

**Theorem 4.2:** A matroid is binary if and only if it has no minor isomorphic with \( U_{2,4} \).

**Theorem 4.3:** A matroid is regular if and only if it has no minor isomorphic with \( U_{2,4}, F_7, \) or \( F_7^* \).

**Theorem 4.4:** A matroid is graphic if and only if it has no minors isomorphic with \( U_{2,4}, F_7, F_7^*, M(K_5), \) or \( M'(K_3,3) \).

**Theorem 4.5:** A matroid is cographic if and only if it has no minor isomorphic with \( U_{2,4}, F_7, F_7^*, M(K_5), \) or \( M(K_3,3) \).

In 1971, Rota made the following conjecture that remains unresolved today.

**Conjecture 4.6:** For every prime power \( q \), the class of \( GF(q) \)-representable matroids has a finite list of minimal excluded minors.

That same year, in an unpublished manuscript, Reid gave an excluded minor characterization for ternary matroids. Proofs for this result were also published by Bixby [30] and Seymour [31]. More recently, in 2000, Geelen, Gerards, and Kapoor [32] characterized \( GF(4) \)-connected matroids using the Geelen technique of blocking sequences.

**Theorem 4.7:** A matroid is ternary if and only if it has no minor isomorphic with \( U_{2,4}, U_{2,5}, F_7, \) or \( F_7^* \).

**Theorem 4.8:** A matroid is representable over \( GF(4) \) if and only if it has no minor isomorphic with \( U_{2,4}, U_{2,5}, F_7, F_7^*, P_6, P_8, \) or \( P_8^* \).

The matroid \( P_6 \) is shown in Figure 1. The matroid \( P_8 \) is a self-dual rank-4, 8-element ternary matroid. A matrix representation over \( GF(3) \) is shown as follows:

\[
\begin{bmatrix}
1 & \ldots & 4 & 5 & 6 & 7 & 8 \\
I_4 & 0 & 1 & 1 & -1 \\
& 1 & 0 & 1 & 1 \\
& 1 & 1 & 0 & 1 \\
& -1 & 1 & 1 & 0
\end{bmatrix}
\]

The matroid \( P_8^* \) is obtained from \( P_8 \) by relaxing two circuit hyperplanes.

This brings us to the matroid-minors conjecture and the matroid well-quasiordering conjecture both due to Whittle.\(^1\) Observe that the Rota conjecture would follow from the next conjecture, which is called the matroid minors conjecture.

\(^1\) The author is indebted to Geoff Whittle for presenting a superb talk on the matroid-minors conjecture at the 2003 AMS regional conference in Baton Rouge.
Conjecture 4.9: Let $\mathcal{M}$ be a minor closed class of $GF(q)$-representable matroid, where $q$ is a power of a prime. Then $\mathcal{M}$ has a finite list of minimal excluded minors.

Conjecture 4.10: $GF(q)$-representable matroids are well-quasiordered under the minor order.

It follows from these conjectures that every infinite family of $GF(q)$-representable matroids contains at least two matroids one of which is a minor of the other (that is, there is no infinite anti-chain of matroids). The matroid well-quasiordering conjecture is equivalent to showing that if $\mathcal{M}$ is a minor closed class of $GF(q)$-representable matroids, then $\mathcal{M}$ has a finite list of $GF(q)$-representable excluded minors. Thus, the matroid-minors conjecture implies the matroid well-quasiordering conjecture. In fact, the matroid-minors conjecture is equivalent to Rota’s conjecture and the matroid well-quasiordering conjecture.

The motivation for these conjectures is famous graph-minors theorem due to Robertson and Seymour: the class of graphs is well-quasiordered under the minor order. The graph-minors theorem together with the Tutte characterization of graphic matroids (Theorem 4.4) implies that the every minor closed class of graphs has a finite list of excluded minors [33]. In proving the graph-minor theorem the notion of tree-width and the accompanying notion of branch-width plays a central role. Geelen and Whittle proved that the matroid-minors conjecture holds for matroids with bounded-branch width [34].

We show how splitters fit into this. As mentioned a splitter $M$ for a 3-connected class of matroids is one that is not in the class but every single-element extension and coextension of $M$ is in the class. If a class has two splitters then clearly one cannot be contained in the other. The graph-minors theorem implies that no class of graphs can have an infinite set of splitters. Can one find an infinite set of splitters for a subclass of $GF(q)$-representable matroids? Not if the matroid-minors conjectures holds.

We now turn our attention to minor-driven results, many of which are motivated by graph theory. Two such well-known results are listed below. The first is due to Hall [35] and the second due to Dirac [36]. The family $K_{3,p}^r$ is that of the complete bipartite graphs, with three vertices in one part of the vertex partition and $p \geq 3$ in the other part. The families $K_5^{r}$, $K_5^{r},'$ or $K_5^{r},''$ are obtained from $K_3$ by adding one, two, or three edges, respectively, to the partition with three vertices.

Theorem 4.12: A simple 3-connected graph $G$ has no minor isomorphic with $K_{3,3}$ if and only if $G$ is planar or has at most five vertices.

Theorem 4.13: A simple 3-connected graph $G$ has no minor isomorphic with $(K_5 \setminus e)^r$ if and only if $G$ is isomorphic with $K_5^r$, $K_5^{r},e$, $W_r$, for $r \geq 3$, $K_3,p'$, $K_5^r,p'$, $K_5^{r},p'$, or $K_5^{r},''p'$.

Excluded minor results are often of the form “A 3-connected matroid $M$ has no minor isomorphic with (fill in your favorite matroids here) if and only if $M$ is (identify the classes here).” Some examples of matroid results are provided below. The first result is by Brylawski [37], the second by Oxley [38], and the third by the author [39]. The matroid $Z_r$ is a rank-$r$, $(2r+1)$-element, non-regular matroid. It can be represented by the binary matrix $[I_r | D]$ where $D$ has $r+1$ columns. The first $r$ columns have zeros along the diagonal and ones elsewhere. The last column is a column of ones. The matroid $H_{r}$ is also a rank-$r$, $(2r+1)$-element, non-regular matroid. It can be obtained from the Fano matroid by attaching a fan along a triangle. This useful notion of attaching fans along triangles originates with Oxley and Wu [40].

Theorem 4.14: A 3-connected binary matroid $M$ has no minor isomorphic with $M(W_3)$ if and only if $M$ is a series–parallel network.

Theorem 4.15: A 3-connected binary matroid $M$ has no minor isomorphic with $M(W_4)$ if and only if $M$ or $M^*$ is isomorphic with $M(W_3)$, $F_7$, $Z_r$, $Z_r \setminus \{v_r\}$, or $Z_r \setminus b_r$.

Theorem 4.16: A 3-connected binary matroid $M$ has no minor isomorphic with $M(K_5 \setminus e)$, $M^*(K_5 \setminus e)$, or $AG(3,2)$ if and only if $M$ or $M^*$ is isomorphic with $M(K_3,3)$, $R_{10}$, $T_{10}$, $M(W_4)$, $H_{r}$, or $H_{r} \setminus \{v_r\}$.

In the statement of Theorem 4.16, another rank 5, 10-element self-dual binary matroid $T_{10}$ shows up. A matrix representation for $T_{10}$ is:
The matroid $T_{10}$ is non-regular, internally 4-connected, and the unique splitter for the class of binary matroids with no minor isomorphic with $M(K_5\setminus e)$ or $M'(K_5\setminus e)$. This matroid was simultaneously discovered by the author and by Chula Jayavardane, a student of Neil Robertson.

It is not easy to give a complete characterization of the 3-connected members of an excluded minor class since frequently the class has too many members. One approach is to replace 3-connectivity with internal 4-connectedness.

The matroids $R_{10}$ and $T_{10}$ feature prominently in the characterization of almost-regular matroids; that is, those matroids for which either $M\setminus e$ or $M/e$ is regular for every element $e$ of $M$ [1] (see §14.8.8). Almost-graphic and almost-planar matroids are similarly defined. Graphic almost-planar matroids were characterized by Gubser [41]. An element $e$ of $M$ is called a regular element if $M\setminus e$ and $M/e$ are both regular elements. Complete characterizations of almost-regular matroids with at least one regular element, as well as the almost-graphic matroids were given by Lemos and the author [42]. The following excluded-minor characterization of almost-regular matroids with no $T_{10}$-minor is a crucial step in the proof. The matroid $T_{10}$ is almost-regular, but not almost-graphic, and has zero regular elements.

**Theorem 4.18:** A 3-connected binary almost-regular matroid $M$ has no minor isomorphic with $T_{10}$ if and only if $M$ or $M^*$ is isomorphic with $X_{12}$, or a 3-connected restriction of $S_{3n+1}$, for $n \geq 3$, $F_1(m, n, r)$ or $F_2(m, n, r)$, for $m, n, r \geq 1$.

We see that there are three distinct almost-regular families. The binary, non-regular matroid $S_{3n+1}$ is obtained by modifying the bicycle-wheel graph. The families $F_1(m, n, r)$ and $F_2(m, n, r)$ are obtained from $F_7$ by attaching fans, with appropriate basis points, along three triangles of $F_7$ with a common element. Another rank 6, 12-element matroid $X_{12}$ makes an appearance in this result. A matrix representation for $X_{12}$ is as follows:

$$
\begin{bmatrix}
1 & \ldots & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
I_6 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
& 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
& 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
& 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
& 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
$$

The matroid $X_{12}$ is a splitter for the almost-regular matroid $M$ with no minor isomorphic with $T_{10}$.

Thus, we see that there are a few special binary matroids that play a role in decomposition results. We conclude this paper with yet another such binary matroid. Hall’s 1943 result (Theorem 4.12) can be rephrased as follows: Let $G$ be a simple, 3-connected graph with a minor isomorphic with $K_5$. Then either $G = K_5$ or $G$ has a minor isomorphic with $K_{3,3}$. This result was generalized by the author [43] as follows:

**Theorem 4.19:** Let $M$ be a 3-connected binary matroid with a minor isomorphic with $M(K_5)$. Then either $M$ has a $M(K_{3,3})$ or $M(K_{3,3}^*)$ minor or $M$ is isomorphic with $M(K_{5})$, $T_{12}$, or $T_{12}/e$.

The rank 6, 12-element matroid $T_{12}$ is self-dual, non-regular, and Tutte 4-connected. A binary matrix representation is as follows:

$$
\begin{bmatrix}
1 & \ldots & 5 & 6 & 7 & 8 & 9 & 10 \\
I_5 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
& 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
& 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
& 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
& 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
$$

The matroid $T_{10}$ is non-regular, internallre 4-connected, and the unique splitter for the class of binary matroids with no minor isomorphic with $M(K_5\setminus e)$ or $M'(K_5\setminus e)$. This matroid was simultaneously discovered by the author and by Chula Jayavardane, a student of Neil Robertson.
Observe that, like $R_{10}$, $T_{12}$ has a circulant pattern of three ones. This matroid is the unique splitter for the class of matroids with no minor isomorphic with $M(K_{3,3})$ or $M'(K_{3,3})$. It has a transitive automorphism group; that is, up to isomorphism, it has a unique single-element extension and a unique single-element deletion. It also has an interesting connection to the Petersen graph.

We end this article by listing two more examples where $T_{12}$ appears. It is a minimal excluded minor for ideal binary clutters [44] and a relaxation of $T_{12}$ is a minimal excluded minor for the class of matroids that are binary and ternary [45].

References

[23] [T72] R.E. Bixby; Composition and Decomposition of Matroids and Related Topics, Ph.D. Thesis, Cornell Univer-
sity (1972).
[32] [GGKOO] J. Geelen, B. Gerards, and A. Kapoor; The excluded minors for GF(4)representable matroids. *J. Combinat-
[33] [RS] N. Robertson and P.D. Seymour; Graph minors. XX. Wagner’s conjecture. *J. Combin. Theory Ser. B* —To ap-
pear.
[44] [CGO2] G. Cornuejols and B. Guenin; Ideal binary clutters, connectivity, and a conjecture of Seymour. *SIAM J. Dis-
[45] [OW00] B. Oporowski, J.G. Oxley, and G. Whittle; On the excluded minors for the matroids that are either binary or ternary. —To appear.

References provided by author not cited in text

[e] [GOVWO] J. Geelen, J.G. Oxley, D. Vertigan, and G. Whittle; On the excluded minors for quaternary matroids, *J. Combinat-
[h] [GGW02] J. Geelen, B. Gerards, and G. Whittle; Branch width and well-quasi-ordering in matroids and graphs, *J. Combinat-

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