An Overview of Volatility Derivatives and Recent Developments

September 17th, 2013

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Outline

- Introduction
- The Volatility Market
- History and Development of Volatility Derivatives
- Recent Developments and Current Research Projects
- Conclusion and Future Research Directions


[?] Add a Comparison
Add
UKX:IND
Open 6619.58
High 6630.14
Low 6562.32
Close 6604.21
Print
7000
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APR
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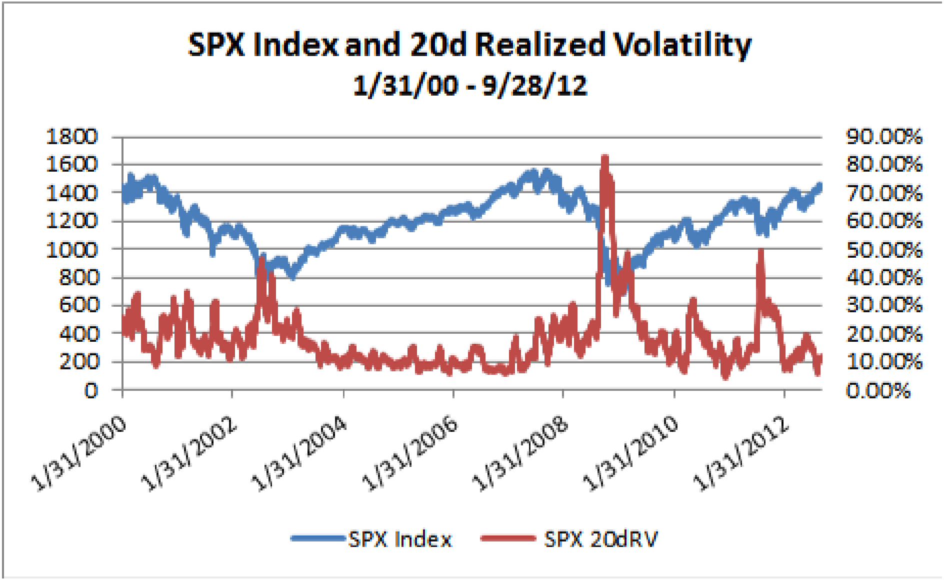
- Stock index has significant variations across time.
- Study of finance: Tradeoff between Return and Risk.
- Practical way to measure "volatility/variation": accumulated squared log stock returns:

$$RV = 100 \times \sqrt{\frac{252}{n} \sum_{i=1}^{n} \left(\ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2} \qquad (1)$$

where $0 = t_0 < t_1 < ... < t_n = T$ for a time period $[0, T]$.
- This is named Realized Volatility(Historical Volatility).


SPX Index and 20d Realized Volatility
1/31/00 - 9/28/12
1800
1600
1400
1200
1000
800
600
400
200
0
90.00%
80.00%
70.00%
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
1/31/2000
1/31/2002
1/31/2004
1/31/2006
1/31/2008
1/31/2010
1/31/2012
SPX Index
SPX 20dRV

Black-Scholes Paradigm

- Model the stock price as Geometric Brownian Motions:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

where W_t is a standard Brownian motion.
- Call option payoff: $(S_T - K)^+$. Gives you upside potential.
- The famous Black-Scholes formula:

$$\begin{aligned} C &= E[e^{-rT}(S_T - K)^+] \\ &= S_0 \mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2), \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

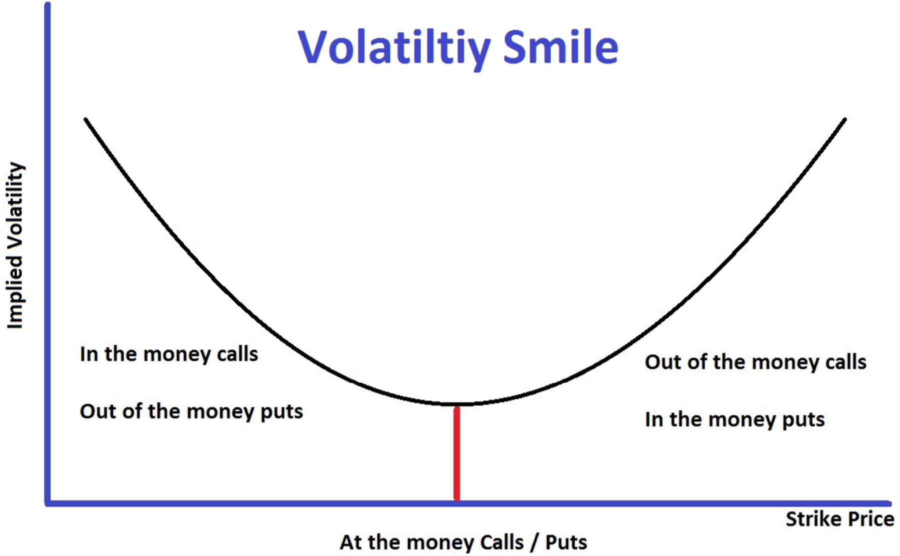
$$d_2 = d_1 - \sigma\sqrt{T}.$$

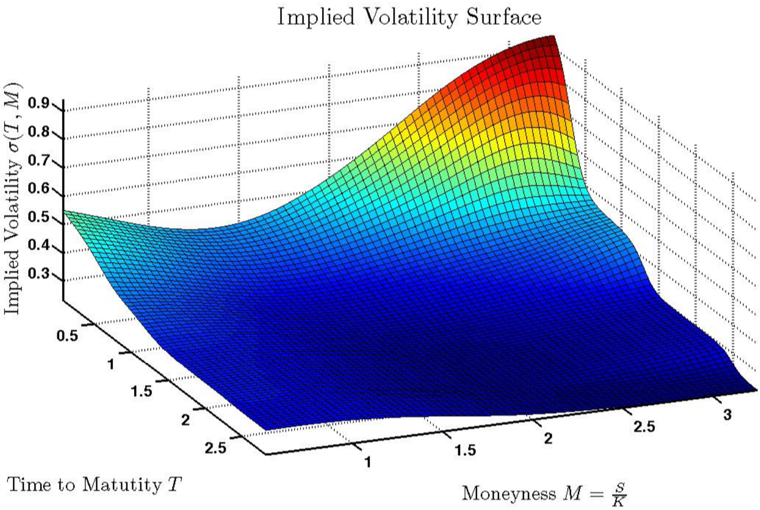
Implied Volatility: An Inverse Problem

- Denote $C = BS(\sigma)$.
- We observe the call option surface C_0, how to concisely summarize the information?
- Black-Scholes formula is monotone in σ.
- Equate $C_0 = BS(\sigma)$. Solve the **unique** "implied volatility" $\sigma_{imp} = BS^{-1}(C_0)$.
- σ_{imp} depends on strike K and time to maturity T.
- Denote $\sigma_{imp} = \sigma_{imp}(K, T)$.

Volatility Smiles :-)

- Plot σ_{imp} against the strike K. The graph is not flat.
- The graph depicts a "smile" shape.
- Draw the implied volatility against the strike and maturity: non-flat surface.
- Black-Scholes assumption is not practical.
- "All model is wrong, but some are useful!"


Volatiltiy Smile
Implied Volatility
In the money calls
Out of the money puts
Out of the money calls
In the money puts
Strike Price
At the money Calls / Puts


Implied Volatility Surface
Implied Volatility $\sigma(T, M)$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.5
1
1.5
2
2.5
Time to Matutity T
1
1.5
2
2.5
3
Moneyness $M = \frac{S}{K}$

Volatility Smiles: research topics

- Fitting the implied volatility surface instead of the option price surface.
- Use the fitted implied volatility surface to predict future option price movement or price options.
- For close-to-maturity fitting, important to know asymptotics of implied volatility as $T \to 0$.
- Can we express the implied volatility explicitly using model parameters?

Volatility Smiles: my research

- Express C_0 as an infinite analytical series of σ_{imp}:

$$C_0 = \sum_{i=0}^{\infty} a_i \sigma_{imp}^i \tag{2}$$

  and determine the coefficients $a_i, i = 0, 1, ...$
- Use Lagrange inversion theorem to represent σ_{imp} as an infinite series of C_0:

$$\sigma_{imp} = \sum_{j=0}^{\infty} b_j C_0^j, \tag{3}$$

  and determine the coefficients $b_j, j = 0, 1, ...$
- For a particular model used, $C_0 = f(\alpha_1, \alpha_2, ...)$, where α_i are model parameters.
- Plug in the above expression in (3).

A Volatility Index: VIX

- In 1993, the Chicago Board Options Exchange (CBOE) introduced VIX to measure the market's expectation of 30 day volatility implied by at-the-money $S\&P100$ Index (OEX) option prices.
- Formula[1] for VIX:

$$VIX = 100 \times \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2$$

- F: forward index level derived from index option prices;
- K_0: first strike below the forward index level F
- K_i: strike price of the ith out-of-the-money option;
- $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$;
- $Q(K_i)$: the midpoint of the bid-ask spread for each option with strike K_i.

[1] "More than you ever wanted to know about volatility swaps" by Demeterfi et al. (1999)

VOLATILITY S&P 500 (^VIX) - Chicago Options
15.85 0.08 (0.51%) Sep 6
Add to Portfolio
Enter name(s) or symbol(s)
GET CHART
COMPARE
EVENTS
TECHNICAL INDICATORS
CHART SETTINGS
RESET
Sep 5, 2013 2:04 PM - 2:09 PM EDT: ^VIX 15.73
17.50
17.00
16.50
16.00
15.50
© 2013 Yahoo! Inc.
Fri Aug 30, 2013
Tue Sep 3
Wed Sep 4
Thu Sep 5
Fri Sep 6
1D 5D 1M 3M 6M YTD 1Y 2Y 5Y Max
FROM: Aug 30 2013
TO: Sep 6 2013
-5.64%
1990
1995
2000
2005
2010

Empirical Facts about Volatility

- Declining stock prices are more likely to give rise to massive portfolio re-balancing (and thus volatility) than increasing stock prices.
- This asymmetry arises naturally from the existence of thresholds below which positions must be cut unconditionally for regulatory reasons.
- Realized volatility of traded assets displays significant variability.
- Volatility is subject to fluctuations.

How do we model volatility?

- "Stochastic Volatility" (SV) denotes a class of models where the stock price is modeled as

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t, S_0 = 1.$$

- V_t itself is another stochastic process that satisfies

$$dV_t = \mu(V_t)dt + \sigma(V_t)dW_t^{(1)}, \qquad V_0 = v_0.$$

- We assume $E[dW_t W_t^{(1)}] = \rho dt$
- In equity markets, usually this correlation level ρ is negative.

Common Stochastic Volatility Models

Some popular stochastic volatility models are

Model	$\mu(.)$	$\sigma(.)$
Heston	$\kappa(\theta - x)$	$\xi\sqrt{x}$
3/2	$\omega x - \theta x^2$	$\xi x^{3/2}$
Hull-White	μx	σx

Features of Stochastic Volatility Models

- Heston model: volatility process follows a mean-reverting Feller diffusion.
- The dynamics of the calibrated Heston model predict that: volatility can reach zero, stay at zero for some time, or stay extremely low or very high for long periods of time.
- Hull-White model: volatility process follows another Geometric Brownian Motion.
- Market calibration very likely leads to $\mu < 0$.
- The dynamics of the Hull-White stochastic volatility model predict that: both expectation and most likely value of instantaneous volatility converge to zero.

Variance Swaps

- A derivative product purely dependent on the underlying volatility.
- The variance swap is an OTC contract:

$$\text{Notional} \times \left(\frac{1}{T} \text{ Realized Variance(RV)} - \text{Strike}(K) \right)$$

- $RV = \sum_{i=0}^{n-1} \left(\ln \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2$ with $0 = t_0 < t_1 < \ldots < t_n = T$.
- "Volatility as an asset class". A tool to trade volatility and make profits.
- Like any swap, a variance swap is an OTC contract with zero upfront premium.
- In contrast to most swaps, a variance swap has a payment only at expiration.

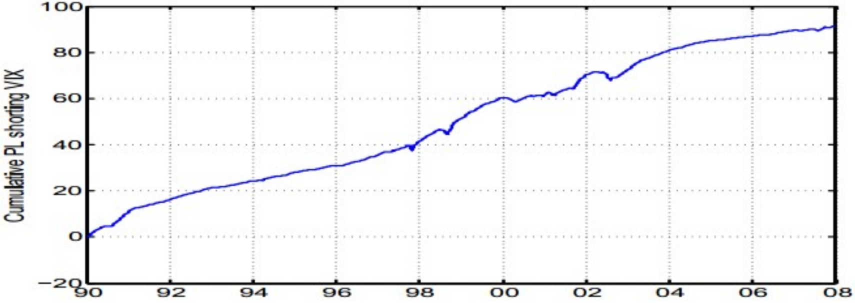
Variance Swaps (Cont'd)

- From Carr and Lee (2009): "According to Michael Weber, now with J.P. Morgan, the first volatility derivative appears to have been a variance swap dealt in 1993 by him at the Union Bank of Switzerland (UBS)."
- The emergence of variance swaps in 1998: due to the historically high implied volatilities experienced in that year.
- Hedge funds found it attractive to sell realized variance at rates that exceeded by wide margins the econometric forecasts of future realized variance based on time series analysis of returns on the underlying index.
- Variance swap rate $VS = E^Q[RV]$, where Q is the risk-neutral measure.

Variance Swaps (Cont'd)

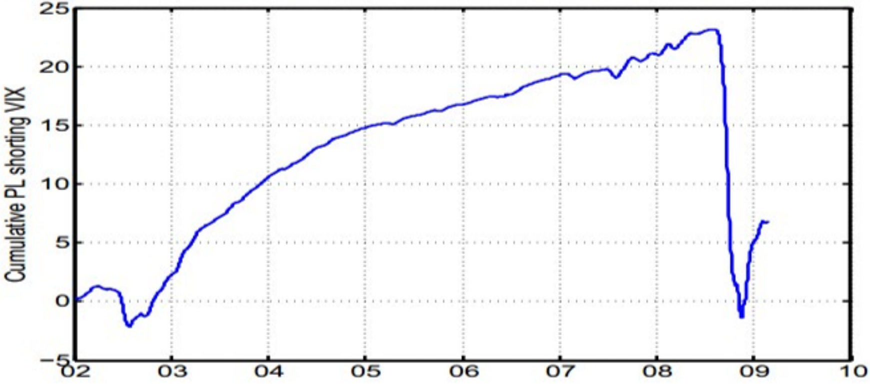
- Define the "Variance Risk premium" as $VRP = E^P[RV] - VS$, where P is the physical probability measure.
- Selling realized variance has positive alpha? Or VRP is usually negative. Generally accepted or believed by hedge funds from 1998 to 2008 (Carr and Lee (2010), Carr and Wu (2009)).
- Until the financial melt-down, when short-variance funds are wiped out due to leverage.

The PL from 1990 to 2007: Mean=$1.39k, Std=$2.17k/contract. IR= 2.2



Comparison: IR from SPX: 0.4

until Oct. 2008, when short-variance funds are wiped out due to leverage



Variance Options

- Successful rollout of variance swaps on stock indices.
- Next step: introduce variance swaps on individual stocks
- In 2005, options on realized variance was introduced. Payoff: $(RV - K)^+$.
- Bet on the realized variance level in the future.
- Other exotic payoff structure and products.

Timer Options

A (perpetual) timer option: an option with **random** maturity.

- Investors specify a variance budget B.
- The **random maturity**:

$$\tau := \inf \left\{ u > 0, \int_0^u V_s ds \geq B \right\}.$$

- The payoff at time τ:

$$\max(S_\tau - K, 0)$$

Why would timer options be attractive?

- In April 2007, Société Générale Corporate and Investment Banking (SG CIB) started to sell this timer option that allows buyers to **"specify the level of volatility used to price the instrument"**.
- Sawyer (2007) explains that "*this product is designed to give investors more flexibility and ensure they do not overpay for an option [...] But the level of implied volatility is often higher than realized volatility, reflecting the uncertainty of future market direction. [...] In fact, having analyzed all stocks in the Euro Stoxx 50 index since 2000, SG CIB calculates that 80% of three-month calls that have matured in-the-money were overpriced.*"

Price Variance Swap under Practical Models

- In practice, variance swaps are **discretely** sampled.
- Broadie and Jain (2008): a **closed-form formula** of the fair strike of the discrete variance swap for the Heston model.
- Bernard and Cui (2013): a general expression for the fair strike of a discrete variance swap in the time-homogeneous stochastic volatility model
- In several models (Heston, Hull-White, Schoebel-Zhu), we obtain explicit formula for the fair strike.
- Asymptotic expansion of the fair strike with respect to n and T.

Hong's Approach and Forward Characteristic Functions

In a presentation by Hong (2004), he looks at the forward characteristic function of the log stock price returns.

$$\phi(u) = E\left[e^{iu \ln \frac{S_{t_{i+1}}}{S_{t_i}}}\right]$$

Then after differentiation

$$E\left[\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = -\frac{d^2\phi(u)}{du^2}\ |_{u=0}$$

Then it is possible to evaluate the discrete variance swap, or in general discrete moment swaps with payoff $\sum_{i=1}^{n}\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}}\right)^m$

Research Directions along the Hong approach

- There are quite a few models where the forward characteristic function can be calculated.
- Affine processes by Duffie et al. (2000): reduced to solve systems of ODEs.
- Levy processes: from the Levy-Khinchin Formula, and stationary and independent increment property.
- An example of non-affine process: 3/2 model
- My recent research: obtain the CF for the 3/2 model, and price discrete variance swap in this model.

Bernard and Cui (2011), JCF

- Recall: $dV_t = \mu(V_t)dt + \sigma(V_t)dW_t$.
- The **joint law** of (τ, V_τ) is

$$(\tau, V_\tau) \overset{law}{\sim} \left(\int_0^B \frac{1}{X_s}ds,\ X_B\right)$$

here X_t is governed by the SDE

$$\begin{cases} dX_t = \frac{\mu(X_t)}{X_t}dt + \frac{\sigma(X_t)}{\sqrt{X_t}}dW_t, \\ X_0 = V_0 \end{cases}$$

where B is a standard Brownian motion.

Analytical Pricing of Variance Options

A variance option: $\left(\int_0^T V_t dt - K\right)^+$, $K > 0$.

- Compare with a standard call option: $(V_T - K)^+$, $K > 0$.
- Key observation: $\int_0^t V_s ds$ is **increasing** in t almost surely.
- Once "**in the money**" $\Longrightarrow$ **always** "**in the money**" afterwards.
- The **first instant** the variance option is **"in the money"**: the **first passage time** of $\int_0^T V_s ds$ to K.

Variance Option and First Hitting time of Integrated Process

- Define $\tau := \inf\{u > 0, \int_0^u V_t dt \geq K\}$.
- $\{\int_0^T V_t dt \geq K\} \iff \{\tau \leq T\}$.

$$
\begin{aligned}
C_0 &= e^{-rT} E\left[\left(\int_0^T V_t dt - K\right)^+\right] \\
&= e^{-rT} E\left[\left(\int_0^T V_t dt - K\right) \mathbb{1}\{\int_0^T \mathbf{V_t dt} \geq \mathbf{K}\}\right] \\
&= e^{-rT} E\left[\left(\left(\int_0^\tau \mathbf{V_t dt} + \int_\tau^{\mathbf{T}} \mathbf{V_t dt}\right) - K\right) \mathbb{1}\{\tau \leq \mathbf{T}\}\right] \\
&= e^{-rT} E\left[\left(\left(\mathbf{K} + \int_\tau^{\mathbf{T}} \mathbf{V_t dt}\right) - K\right) \mathbb{1}\{\tau \leq T\}\right] \\
&= e^{-rT} E\left[\left(\int_\tau^T V_t dt\right) \mathbb{1}\{\tau \leq T\}\right]
\end{aligned}
$$

Connecting Discrete to Continuous Sampling

- Define Quadratic Variation:
  $QV = \lim_{n\to\infty, \max_{i=0,1,\ldots,n-1}(t_{i+1}-t_i)\to 0} RV.$
- We know that RV converges to QV in probability. But we are interested in whether this convergence takes place in $L1$ or not.
- Jarrow et al (2013) gives preliminary answer, but for the case of $3/2$ mode it remains an open problem.
- The fair strike of the **"discrete variance swap"** is

$$\mathbf{K_d^M(n)} := \frac{1}{T} E\left[\sum_{i=0}^{n-1}\left(\ln \frac{S_{t_{i+1}}}{S_{t_i}}\right)^2\right] = \frac{1}{T} E[RV]$$

- The fair strike of the **"continuous variance swap"** is

$$\mathbf{K_c^M} := \frac{1}{T} E\left[\int_0^T V_s ds\right] = \frac{1}{T} E[QV]$$

General representation of the discrete fair strike

- I obtain a general representation of the discrete strike in terms of continuous strike.
- Define, for $n \geq 1,\ t_i = i\Delta,\ \ i = 1, 2, \ldots n = T/\Delta$ and

$$C(\Delta) = \frac{1}{T}\sum_{i=0}^{n-1} E\left[\left(\int_{t_i}^{t_i+\Delta} m^2(V_s)ds\right)^2\right].$$

- Assuming that the third moments exists,

$$\gamma(\Delta) = \frac{1}{T}\sum_{i=0}^{n-1} E\left[\left(\int_{t_i}^{t_i+\Delta} m(V_t)dW_t^{(2)}\right)^3\right].$$

- **Assumption 1**: For some $\Delta > 0,\ C(\Delta) < \infty$.

Convergence in Time-homogeneous Diffusion Case

Theorem

Assume[a] *the general time-homogeneous diffusion model, and Assumption 1. The fair strike of a discrete variance swap is given by*

$$K_d(\Delta) = K_c + r^2\Delta - rK_c\Delta + \frac{1}{4}C(\Delta) - \rho B(\Delta),$$

where

$$B(\Delta) = \frac{1}{T}\sum_{i=0}^{n-1} E\left[\left(\int_{t_i}^{t_i+\Delta} m^2(V_s)ds\right)\left(\int_{t_i}^{t_i+\Delta} m(V_t)dW_t^{(2)}\right)\right]$$
$$= \frac{1}{3}\gamma(\Delta).$$

[a]Theorem 1 of Bernard, Cui and McLeish (2013)

Theorem

We have $K_d(\Delta) \to K_c$ as $\Delta \to 0$ for all ρ if and only if Assumption 1 holds.

Proposition

In the case of time-homogeneous diffusion models, the following statements are equivalent:
(1) Assumption 1;
(2) $K_d(\Delta) \in L^1$ for some $n \geq 1$;
(3) $K_d(\Delta) \to K_c$ as $\Delta \to 0$ for all $-1 \leq \rho \leq 1$.

Conclusion

- We have provided a brief introduction to the volatility market.
- We provide an overview of stochastic volatility models.
- We present the history of volatility derivatives and motivations behind them.
- We present some current research topics in this area.

Thank You

Q & A

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