Zero forcing, propagation time, and throttling on a graph

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New York Combinatorics Seminar
August 28, 2020

Outline

Zero forcing and its variants

Zero forcing is a coloring game in which each vertex is initially blue or white and the goal is to color all vertices blue.

- The standard color change rule for zero forcing on a graph G is that a blue vertex v can change the color of a white vertex w to blue if w is the only white neighbor of v in G.
- There are many variants of zero forcing, each of which uses a different color change rule.

Applications:

- Mathematical physics (control of quantum sytems).
- Power domination:
  - A minimum power dominating set gives the optimal placement of monitoring units in an electric network.
  - Power domination is zero forcing applied to the set of initial vertices and their neighbors.
- Combinatorial matrix theory - illustrated in these slides.

Matrices are real. The matrix $A = [a_{ij}]$ is symmetric if $a_{ji} = a_{ij}$ and skew symmetric if $a_{ji} = -a_{ij}$. Most matrices discussed are symmetric; some are skew symmetric.
$S_n(\mathbb{R})$ is the set of $n \times n$ real symmetric matrices.

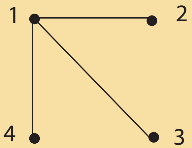
The graph $\mathcal{G}(A) = (V, E)$ of $n \times n$ symmetric or skew matrix A is

- $V = \{1, ..., n\}$,
- $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of A is ignored.

Example

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{G}(A)$



Inverse Eigenvalue Problem of a Graph (IEP-G)

The family of symmetric matrices described by a graph G is

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}.$$

The Inverse Eigenvalue Problem of a Graph (IEPG) is to determine all possible spectra (multisets of eigenvalues) of matrices in $\mathcal{S}(G)$.

Example

A matrix in $\mathcal{S}(P_3)$ is of the form

$$A = \begin{bmatrix} x & a & 0 \\ a & y & b \\ 0 & b & z \end{bmatrix} \text{ where } a, b \neq 0.$$

The possible spectra of matrices in $\mathcal{S}(P_3)$ are all sets of 3 distinct real numbers.

Maximum multiplicity and minimum rank

Due to the difficulty of the IEPG, a simpler form called the maximum multiplicity, maximum nullity, or minimum rank problem has been studied.

The maximum multiplicity or maximum nullity of graph G is

$$\begin{aligned} \mathrm{M}(G) &= \max\{\mathrm{mult}_A(\lambda) : A \in \mathcal{S}(G),\ \lambda \in \mathrm{spec}(A)\}. \\ &= \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}. \end{aligned}$$

The minimum rank of graph G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}.$$

By using nullity,

$$\mathrm{M}(G) + \mathrm{mr}(G) = |V(G)|.$$

The Maximum Nullity Problem (or Minimum Rank Problem) for a graph G is to determine $\mathrm{M}(G)$ (or $\mathrm{mr}(G)$).

Zero forcing and maximum nullity

- Zero forcing starts with blue vertices (representing zeros in a null vector of a matrix) and successively colors other vertices blue.
- The zero forcing number is the minimum size of a zero forcing set.

Theorem (BBBCCFGHHMNPSSSvdHVM 2008)

For every graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

- G a graph with $V(G) = \{1, \ldots, n\}$ and $A \in \mathcal{S}(G)$,
- $\mathrm{x} \in \mathbb{R}^n$, $A\mathrm{x} = 0$, and $x_k = 0$ for all $k \in B \subseteq V(G)$,
- $i \in B$, $j \notin B$, and j is the only vertex k such that $ik \in E(G)$ and $k \notin B$.

imply

$$x_j = 0$$

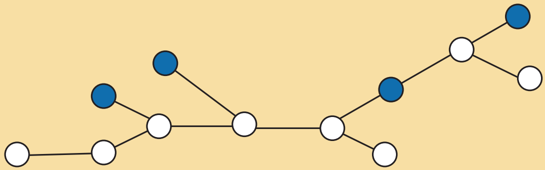
because equating the ith entries in $A\mathrm{x} = 0$ yields $a_{ij}x_j = 0$.

(Standard) zero forcing color change rule

Standard color change rule: Let W be the set of (currently) white vertices. A blue vertex v can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$

Example ($B = B^{[0]}$)



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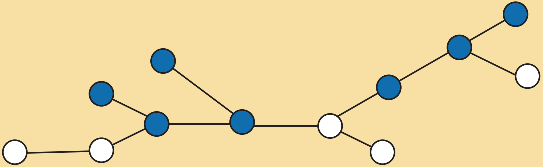
Example ($B^{[1]}$)



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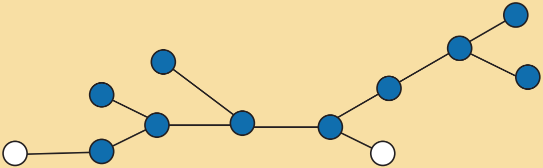
Example ($B^{[2]}$)



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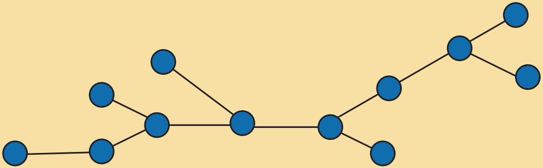
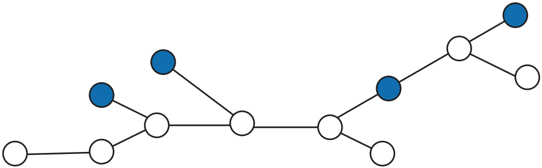
Example ($B^{[3]}$: $Z(T) = 4$)



Example: Why is $Z(T) = 4$

We just showed $Z(T) \leq 4$



For trees, there is an algorithm for finding a minimum path cover and thus a minimum zero forcing set.

Variants of zero forcing

- Each type of zero forcing is a coloring game on a graph in which each vertex is initially blue or white.
- A color change rule allows white vertices to be colored blue under certain conditions.

Let R be a color change rule.

- The set of initially blue vertices is $B^{[0]} = B$.
- The set of blue vertices $B^{[t]}$ after round t or time step t (under R) is the set of blue vertices in G after the color change rule is applied in $B^{[t-1]}$ to every white vertex independently.
- An initial set of blue vertices $B = B^{[0]}$ is an R zero forcing set if there exists a t such that $B^{[t]} = V(G)$ using the R color change rule.
- Minimum size of an R zero forcing set is the R forcing number.

Maximum PSD nullity

A real matrix is positive semidefinite matrices (PSD) if A is symmetric and every eignevalue is nonnegative.

The family of PSD described by a graph G is

$$\mathcal{S}_+(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G \text{ and } A \text{ is PSD}\}.$$

The maximum PSD nullity of graph G is

$$\mathrm{M}_+(G) = \max\{\operatorname{null} A : A \in \mathcal{S}_+(G)\}.$$

The PSD zero forcing number is $\mathrm{Z}_+(G)$.

Theorem (BBFHHSvdDvdH 2010)

For every graph G, $\mathrm{M}_+(G) \le \mathrm{Z}_+(G)$.

PSD color change rule

PSD color change rule: Delete the currently blue vertices from the graph G and determine the components of the resulting graph; let W_i be the set of vertices of the ith component. A blue vertex v can change the color of a white vertex w to blue if

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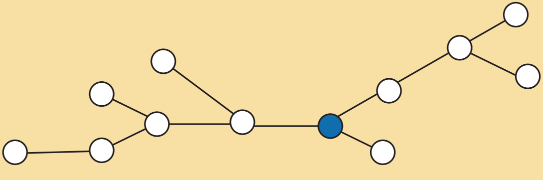
Example ($B_+ = B_+^{[0]}$)



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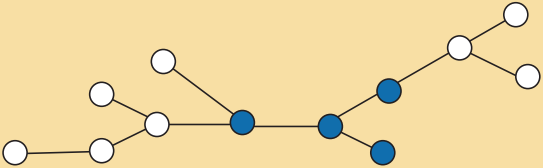

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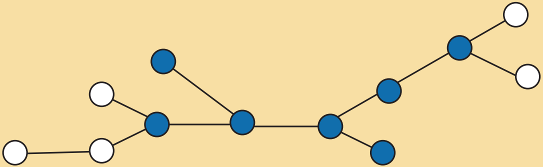

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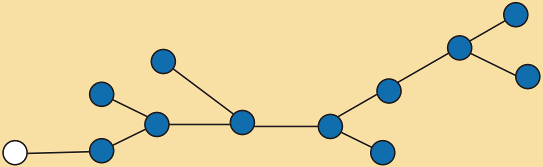

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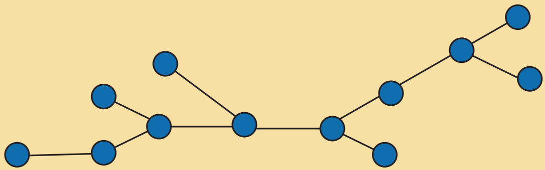
Example ($B_+^{[4]}$: $\mathrm{Z}_+(T) = 1$)



Skew and hollow symmetric maximum nullity

- A matrix is hollow if A is symmetric and every diagonal entry is 0.
- A hollow matrix described by a graph G is a weighted adjacency matrix of G.
- A matrix is skew symmetric if $A^T = -A$.

$$\begin{aligned}\mathcal{S}_0(G) &= \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G \text{ and } A \text{ is hollow}\}.\\ \mathcal{S}_-(G) &= \{A \in \mathbb{R}^{n\times n} : \mathcal{G}(A) = G \text{ and } A^T = -A\}.\end{aligned}$$

The maximum hollow nullity and maximum skew nullity of graph G are

$$\begin{aligned}\mathrm{M}_0(G) &= \max\{\operatorname{null} A : A \in \mathcal{S}_0(G)\}.\\ \mathrm{M}_-(G) &= \max\{\operatorname{null} A : A \in \mathcal{S}_-(G)\}.\end{aligned}$$

Theorem (ABDeADDeLGGHIKNPSSW 2010 and GHHHJKMcC 2014)

For every graph G, $\mathrm{M}_-(G) \leq \mathrm{Z}_-(G)$ and $\mathrm{M}_0(G) \leq \mathrm{Z}_-(G)$.

- G a graph with $V(G) = \{1, \ldots, n\}$ and $A \in \mathcal{S}_-(G)$ or $A \in \mathcal{S}_0(G)$,
- $\mathrm{x} \in \mathbb{R}^n$, $A\mathrm{x} = 0$, and $x_k = 0$ for all $k \in B \subseteq V(G)$,
- $j \notin B$ and j is the only vertex k such that $ik \in E(G)$ and $k \notin B$.

imply

$$x_j = 0$$

because equating the ith entries in $A\mathrm{x} = 0$ yields $a_{ij}x_j = 0$.

Skew zero forcing color change rule

Skew color change rule: Let W be the set of (currently) white vertices. A vertex v can change the color of vertex $w \in W$ to blue if

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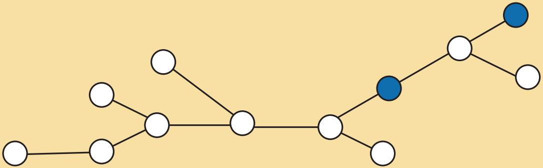

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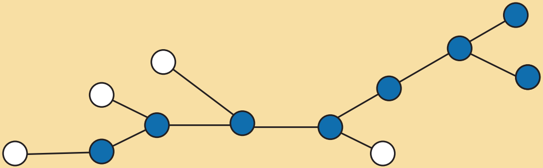

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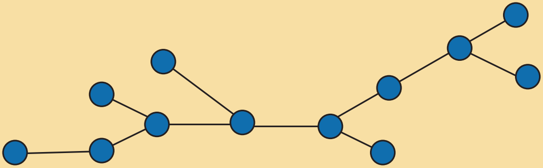
Example ($B_-^{[2]}$: $\mathrm{Z}_-(T) = 2$)



Maximum nullities and zero forcing numbers for families

Theorem (four papers previously cited)

- *For* $n \geq 2$, $\mathrm{Z}(K_n) = \mathrm{M}(K_n) = \mathrm{Z}_+(K_n) = \mathrm{M}_+(K_n) = n-1$ *and* $\mathrm{Z}_-(K_n) = \mathrm{M}_-(K_n) = \mathrm{M}_0(K_n) = n-2$.
- *For* $n \geq 1$, $\mathrm{Z}(\overline{K_n}) = \mathrm{M}(\overline{K_n}) = \mathrm{Z}_+(\overline{K_n}) = \mathrm{M}_+(\overline{K_n}) = \mathrm{Z}_-(\overline{K_n}) = \mathrm{M}_-(\overline{K_n}) = n$.
- *For* $n \geq 3$, $\mathrm{Z}(K_{r,n-r}) = \mathrm{M}(K_{r,n-r}) = \mathrm{Z}_-(K_{r,n-r}) = \mathrm{M}_-(K_{r,n-r}) = \mathrm{M}_0(K_{r,n-r}) = n-2$ *and* $\mathrm{Z}_+(K_{r,n-r}) = \mathrm{M}_+(K_{r,n-r}) = \min(r, n-r)$.
- *For* $n \geq 2$, $\mathrm{Z}(P_n) = \mathrm{M}(P_n) = \mathrm{Z}_+(P_n) = \mathrm{M}_+(P_n) = 1$. *For even* n, $\mathrm{Z}_-(P_n) = \mathrm{M}_-(P_n) = \mathrm{M}_0(P_n) = 0$ *and for odd* n, $\mathrm{Z}_-(P_n) = \mathrm{M}_-(P_n) = \mathrm{M}_0(P_n) = 1$.
- *For* $n \geq 3$, $\mathrm{Z}(C_n) = \mathrm{M}(C_n) = \mathrm{Z}_+(C_n) = \mathrm{M}_+(C_n) = 2$. *For even* $n \geq 4$, $\mathrm{Z}_-(C_n) = \mathrm{M}_-(C_n) = \mathrm{M}_0(C_n) = 2$. *For odd* $n \geq 3$, $\mathrm{Z}_-(C_n) = \mathrm{M}_-(C_n) = 1$ *and* $\mathrm{M}_0(C_n) = 0$.

Propagation time for zero forcing variants

Let R be a color change rule.

- The R-propagation time for a set $B = B^{[0]}$ of vertices, $\text{pt}_R(G, B)$, is the smallest t such that $B^{[t]} = V(G)$ using the R color change rule (and is infinity if this never happens).
- This is also called the number of times steps or rounds to color the graph.
- The R-propagation time of G is

$$\text{pt}_R(G) = \min\{\text{pt}_R(G, B) : B \text{ is a minimum } R\text{-forcing set.}\}$$

(Standard) propagation time pt(G)

Standard color change rule: Let W be the set of (currently) white vertices. A blue vertex v can change the color of vertex $w \in W$ to blue if

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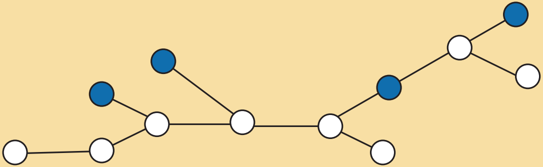
Example ($B = B^{[0]}$)



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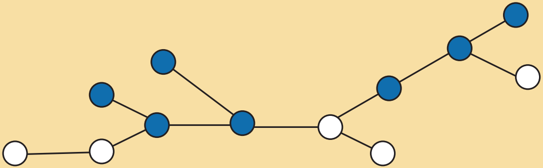

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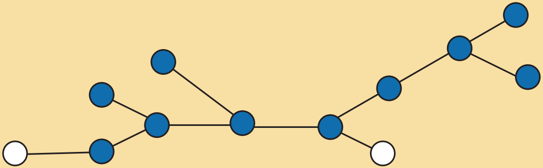

(Standard) propagation time $\mathrm{pt}(G)$

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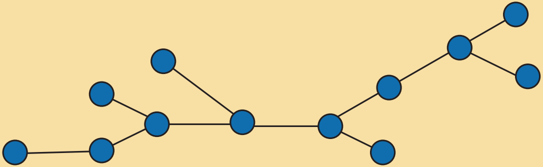
Example ($B^{[3]}$: $\mathrm{pt}(T) = 3$)



PSD propagation time $\mathrm{pt}_+(G)$

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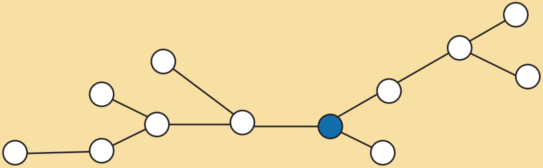

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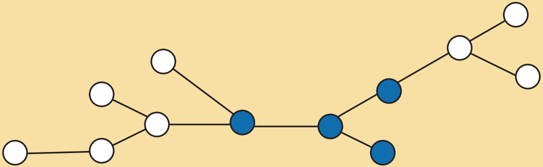

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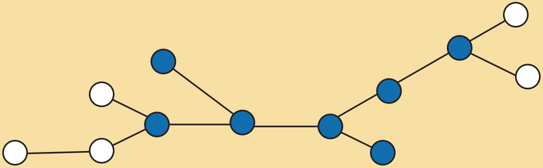

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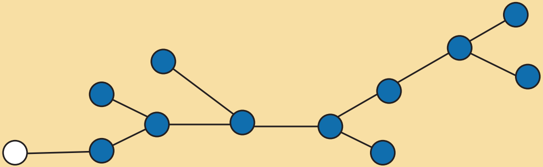

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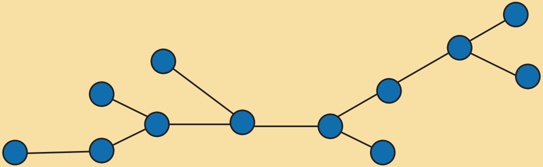
Example ($B_+^{[4]}$: $\mathrm{pt}_+(T) = 4$)



Skew propagation time $\mathrm{pt}_-(G)$

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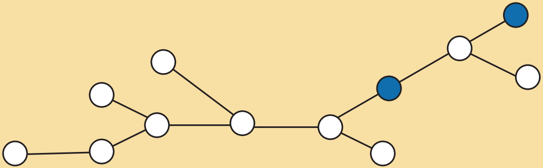
Example ($B_- = B_-^{[0]}$)



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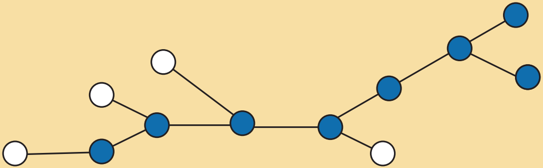

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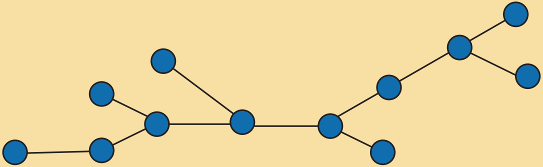


Propagation time of complete graphs and paths

Theorem (HHKMWY 2012, W 2015, K 2015)

- *For* $n \geq 2$, $\text{pt}(K_n) = \text{pt}_+(K_n) = \text{pt}_-(K_n) = 1$.
- *For* $n \geq 1$, $\text{pt}(\overline{K_n}) = \text{pt}_+(\overline{K_n}) = \text{pt}_-(\overline{K_n}) = 0$.
- $\text{pt}(K_{1,n-1}) = 2$ *and for* $2 \leq r \leq n-2$, $\text{pt}(K_{r,n-r}) = 1$.
  For $1 \leq r \leq n-1$, $\text{pt}_+(K_{r,n-r}) = \text{pt}_-(K_{r,n-r}) = 1$.
- *For* $n \geq 2$, $\text{pt}(P_n) = n-1$ *and* $\text{pt}_+(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$.
  $\text{pt}_-(P_n) = \frac{n}{2}$ *for even* n *and* $\text{pt}_-(P_n) = \left\lfloor \frac{n+1}{4} \right\rfloor$ *for odd* n.
- *For* $n \geq 3$, $\text{pt}(C_n) = \left\lceil \frac{n-2}{2} \right\rceil$ *and* $\text{pt}_+(C_n) = \left\lceil \frac{n-2}{4} \right\rceil$.

$$\text{pt}_-(C_n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{4} & \text{if } n \equiv 0 \mod 4 \\ \frac{n-2}{4} & \text{if } n \equiv 2 \mod 4 \end{cases}.$$

Throttling

Throttling involves minimizing the sum of the number of resources used to accomplish a task (e.g., blue vertices) and the time needed to accomplish the task (e.g., propagation time).

Unlike propagation time of a graph, which starts with a minimum set of blue vertices, throttling often uses more blue vertices to reduce time.

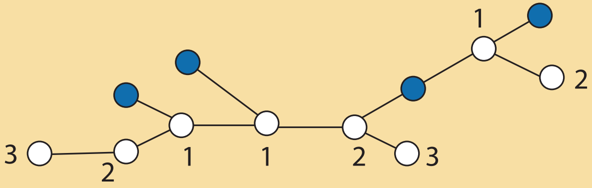
Let R be a color change rule.

- The R-propagation time for a set $B = B^{[0]}$ of vertices, $\text{pt}_R(G, B)$, is the smallest t such that $B^{[t]} = V(G)$ using the R color change rule (and is infinity if this never happens).
- The R-throttling number of a set B of vertices, $\text{th}_R(G; B) = |B| + \text{pt}_R(G, B)$, is the sum of the number vertices in B and the R-propagation of B.
- The R-throttling number of G is $\text{th}_R(G) = \min_{B \subseteq V(G)} \text{th}_R(G; B) = \min_{B \subseteq V(G)} (|B| + \text{pt}_R(G, B))$.

(Standard) throttling

- The propagation time for a set $B = B^{[0]}$ of vertices, $\text{pt}(G, B)$, is the smallest t such that $B^{[t]} = V(G)$ using the (standard) zero forcing color change rule.
- The throttling number of G for zero forcing is $\text{th}(G) = \min_{B \subseteq V(G)}(|B| + \text{pt}(G, B))$.

Example ($\text{Z}(T) = 4, \text{pt}(T) = 3, \text{th}(T) = 7$)

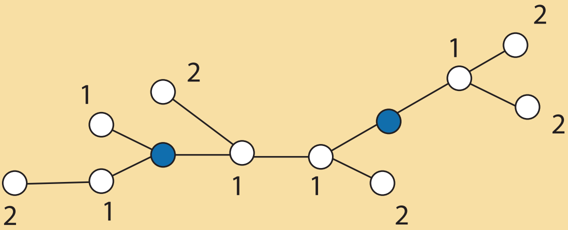


PSD throttling

- The PSD propagation time for a set $B = B^{[0]}$ of vertices, $\mathrm{pt}_+(G, B)$, is the smallest t such that $B^{[t]} = V(G)$ using the PSD color change rule.
- The PSD throttling number of G for zero forcing is the $\mathrm{th}_+(G) = \min_{B \subseteq V(G)}(|B| + \mathrm{pt}_+(G, B))$.

Example

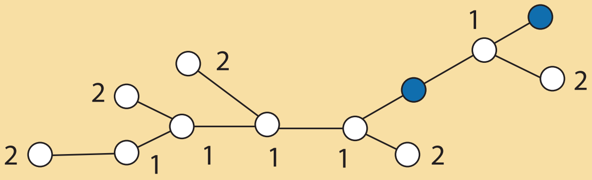
$\mathrm{Z}_+(T) = 1$ and $\mathrm{pt}_+(T) = 4$. Using a PSD zero forcing set B of 2 vertices, $\mathrm{pt}_+(G, B) = 2$ and $\mathrm{th}_+(T) = 2 + 2 = 4$.



Skew throttling

- The skew propagation time for a set $B = B^{[0]}$ of vertices, $\mathrm{pt}_-(G, B)$, is the smallest t such that $B^{[t]} = V(G)$ using the skew forcing color change rule.
- The skew throttling number of G for zero forcing is $\mathrm{th}_-(G) = \min_{B \subseteq V(G)}(|B| + \mathrm{pt}_-(G, B))$.

Example ($\mathrm{Z}_-(T) = 2, \mathrm{pt}_-(T) = 2, \mathrm{th}_-(T) = 4$)



Throttling numbers of families of graphs

Theorem (BY 2013, CHKLRSVM 2019, CGH 2020)

- *For* $n \geq 1$, $\text{th}(K_n) = \text{th}_+(K_n) = n$. *For* $n \geq 2$, $\text{th}_-(K_n) = n - 1$.
- *For* $n \geq 1$, $\text{th}(\overline{K_n}) = \text{th}_+(\overline{K_n}) = \text{th}_-(\overline{K_n}) = n$.
- $\text{pt}(K_{1,n-1}) = 2$ *and for* $2 \leq r \leq n-2$, $\text{pt}(K_{r,n-r}) = 1$. *For* $1 \leq r \leq n-1$, $\text{pt}_+(K_{r,n-r}) = \text{pt}_-(K_{r,n-r}) = 1$.
- *For* $n \geq 2$, $\text{th}(P_n) = \lceil 2\sqrt{n} - 1 \rceil$, $\text{th}_+(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$, *and* $\text{th}_-(P_n) = \left\lceil \sqrt{2(n+1)} - \frac{3}{2} \right\rceil$.
- *For* $n \geq 3$, $\text{th}(C_n) = \begin{cases} \lceil 2\sqrt{n} - 1 \rceil & \textit{unless } n = (2k+1)^2 \\ 2\sqrt{n} & \textit{if } n = (2k+1)^2 \end{cases}$.
  $\text{th}_+(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$.
  $\text{th}_-(C_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil$.

Relationships: standard, PSD, and skew throttling

Observation

Let $B \subseteq V(G)$ be a zero forcing set. Then,

- *B is a PSD forcing set and a skew forcing set.*
- $\mathrm{Z}_+(G) \le \mathrm{Z}(G)$ *and* $\mathrm{Z}_-(G) \le \mathrm{Z}(G)$
- $\mathrm{pt}_+(G, B) \le \mathrm{pt}(G, B)$ *and* $\mathrm{pt}_-(G, B) \le \mathrm{pt}(G, B)$
- $\mathrm{th}_+(G; B) \le \mathrm{th}(G; B)$ *and* $\mathrm{th}_-(G; B) \le \mathrm{th}(G; B)$.
- $\mathrm{th}_+(G) \le \mathrm{th}(G)$ *and* $\mathrm{th}_-(G) \le \mathrm{th}(G)$.
- $\mathrm{th}_+(G)$ *and* $\mathrm{th}_-(G)$ *are noncomparable.*
- $\mathrm{pt}_+(G)$, $\mathrm{pt}_-(G)$, *and* $\mathrm{pt}(G)$ *are noncomparable (minimum values can differ).*

Lower bound on $\mathrm{th}(G)$

Theorem (Butler, Young, 2013)

Let G be a graph of order n. Then

$$\mathrm{th}(G) \geq \lceil 2\sqrt{n} - 1 \rceil$$

and this bound is tight.

PSD and skew are very different

- $\mathrm{th}_+(K_{1,n-1}) = 2$.
- For any G with a component of order ≥ 2,

$$\mathrm{Z}_-(G \circ K_1) = 0, \mathrm{pt}_-(G \circ K_1) = 2, \mathrm{th}_-(G \circ K_1) = 2.$$

Extreme values for th(G)

$\lceil 2\sqrt{n} - 1 \rceil \leq \mathrm{th}(G)$ implies the number of graphs having $\mathrm{th}(G) = k$ is finite.

Remark

All the graphs having $\mathrm{th}(G) \leq 3$ *are listed below.*

1) $\mathrm{th}(G) = 1$ *if and only if* $|V(G)| = 1$.
2) $\mathrm{th}(G) = 2$ *if and only if* $|V(G)| = 2$.
3) $\mathrm{th}(G) = 3$ *if and only if* $|V(G)| = 3$ *or* $G = 2K_2$, P_4, *or* C_4.

Theorem (CK 2020+)

Let G be a graph of order n. The following are equivalent:

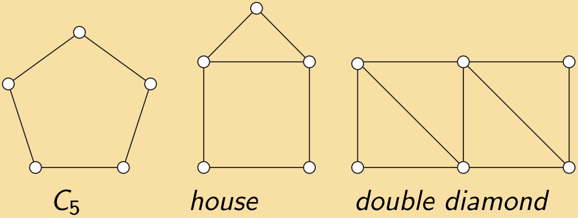
1) $\mathrm{th}(G) = n$.
2) *G is a threshold graph.*
3) *G does not have P_4, C_4, or $2K_2$ as an induced subgraph.*

Extreme values for $\mathrm{th}_+(G)$

Theorem (CHKLRSVM 2019)

Let G be a connected graph of order n.

1) $\mathrm{th}_+(G) = n$ *if and only if* $G = K_n$.
2) $\mathrm{th}_+(G) = n - 1$ *if and only if* $\alpha(G) = 2$ *and* G *does not have an induced* C_5, *house, or double diamond subgraph.*



Extreme values for $\mathrm{th}_+(G)$

Theorem (CHKLRSVM 2019)

Let G be a graph of order n.

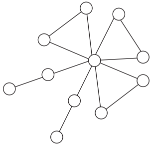
1) $\mathrm{th}_+(G) = 1$ *if and only if* $n = 1$.
2) $\mathrm{th}_+(G) = 2$ *if and only if* $G = K_{1,n-1}$ *or* $G = 2K_1$.
3) *For a graph G,* $\mathrm{th}_+(G) = 3$ *if and only if at least one of the following is true:*
   - 3.1 *G is disconnected and exactly of the following holds:*
     - 3.1.1 *G is $3K_1$, or*
     - 3.1.2 *G has two components, each component is a copy of $K_{1,n-1}$ or K_1, and at least one component has order greater than one.*
   - 3.2 *G is a tree with diameter three or four, or*
   - 3.3 *G is connected and there exist $v, u \in V(G)$ such that:*
     - 3.3.1 *G has a cycle, or G is a tree with diam $G = 5$,*
     - 3.3.2 $N(u) \cup N(v) = V(G)$,
     - 3.3.3 $\deg(w) \leq 2$ *for all* $w \notin \{v, u\}$, *and*
     - 3.3.4 *if $w_1, w_2 \in N(u)$ or $w_1, w_2 \in N(v)$, then w_1 is not adjacent to w_2.*

Extreme values for $\mathrm{th}_-(G)$

Theorem (CGH 2020)

Let G be a graph of order n.

1) $\mathrm{th}_-(G) = 1$ *if and only if* $G = K_1$ *or* $G = rK_2$ *for* $r \geq 1$.
2) *A graph G has* $\mathrm{th}_-(G) = 2$ *if and only if G is one of* $2K_1$, $H(s,t) \sqcup rK_2$ *with* $r + s + t \geq 1$, *or* $\left(\widetilde{G} \circ K_1\right) \sqcup rK_2$ *where each component of* $\widetilde{G}$ *has an edge.*
3) $\mathrm{th}_-(G) = n$ *if and only if* $G = nK_1$.
4) $\mathrm{th}_-(G) = n - 1$ *if and only if G is a cograph, does not have an induced* $2K_2$, *and has at least one edge.*



The graph $H(2,3)$

Computation

There is Sage software that computes

- $\mathrm{Z}(G), \mathrm{Z}_+(G), \mathrm{Z}_-(G)$,
- $\mathrm{pt}(G), \mathrm{pt}_+(G), \mathrm{pt}_-(G)$,
- $\mathrm{th}(G), \mathrm{th}_+(G), \mathrm{th}_-(G)$

for “small” graphs.

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