A "CHALLENGING QUESTION" OF BJÖRNER FROM 1976: EVERY INFINITE GEOMETRIC LATTICE OF FINITE RANK HAS A MATCHING

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ABSTRACT. It is proven that every geometric lattice of finite rank greater than 1 has a matching between the points and hyperplanes. This answers a question of Pólya Prize-winner Anders Björner from the 1981 Banff Conference on Ordered Sets, which he raised as a "challenging question" in 1976.

At the famous 1981 Banff Conference on Ordered Sets—such luminaries as Erdős, Professor Garrett Birkhoff, Dilworth, Turing Award-winner D. S. Scott, Daykin, A. Garsia, R. L. Graham, C. Greene, B. Jónsson, E. C. Milner, and Oxford's H. A. Priestley attended—Björner asked (with MIT's Richard Stanley asking a question immediately afterwards, judging from the proceedings) if every geometric lattice L of finite rank [≥ 2] had a matching [13, pp. xi, xii, and 799]. Greene had proven this for finite lattices [7, Corollary 3]. Björner had proven this in special cases [4, Theorems 3 and 4]—for modular lattices and for "equicardinal lattices," i.e., lattices whose hyperplanes contained the same number of atoms. In 1976, Björner wrote, "It would be interesting to know if the result of our theorems 3 and 4 can be extended to all infinite geometric lattices, or at least to some classes of such lattices other than the modular and the equicardinal." In 1977, he proved it for lattices of rank 3 and for lattices of cardinality less than $\aleph_\omega$. The Pólya Prize-winner went on to ask at the Banff Conference if there exists a family M of pairwise disjoint maximal chains in $L \setminus \{0, 1\}$ whose union contains the set of atoms, saying, "I showed this is true for modular L, and J. Mason showed it to be true for finite L." He conjectured this in 1977 ([5, p. 18], [4, p. 10]), writing in 1976 that "[a]nother challenging question, related to the existence of matchings, is whether maximal families of pairwise disjoint maximal proper chains do exist in infinite geometric lattices (cf. [11])."

We answer Björner's 1976 question about matchings.

We selectively use some of the notation and terminology from [6] and [3, Chapter II, §8 and Chapter IV].

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Let P be a poset. Let $x, y \in P$ be such that $x \leq y$. The *closed interval* $[x, y]$ is $\{z \in P : x \leq z \leq y\}$. If $|[x, y]| = 2$, we say x is a *lower cover* of y and y is an *upper cover* of x and denote it $x \lessdot y$.

Let P be a poset with least element 0. An *atom* or *point* is a cover of 0. The set of atoms is $\mathcal{A}(P)$. If P is a poset with greatest element 1, a *co-atom*, *co-point*, or *hyperplane* is a lower cover of 1. The set of hyperplanes is $\mathcal{H}(P)$.

A poset is *semimodular* if, for all $a, b, c \in P$, $a \lessdot b, c$ and $b \neq c$ imply there exists $d \in P$ such that $b, c \lessdot d$. A *geometric lattice of finite height* is a semimodular lattice L with no infinite *chains* (totally ordered subsets)—implying L has a 0 and a 1—such that every element is a join of a subset of atoms. It is known [12, Theorem 9.4] that such an L is a complete lattice with a finite maximal chain and all maximal chains have the same size $r+1$, where r is the *height* or *rank* of L. Moreover, every element is a join of a finite set of atoms and a meet of a subset of $\mathcal{H}(L)$ (see [4, Lemma 1]). Every interval is a geometric lattice [15, §3.3, Lemma]. The rank of $\downarrow x := [0, x]$ is the *rank* $r(x)$ of $x \in L$. For $x, y \in L$, $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$ [12, Theorem 9.5]. For $x \in L$, let $\underline{x} := \mathcal{A}(L) \cap \downarrow x$ and let $\overline{x} := \mathcal{H}(L) \cap [x, 1]$.

The following is a basic fact (see [4, p. 3]).

Lemma 1. *Let L be a geometric lattice of finite height. Let $a, b \in L$ be such that $a \leq b$. Then any $x \in [a, b]$ has a* modular complement *in $[a, b]$, i.e., there exists $y \in [a, b]$ such that $x \wedge y = a$, $x \vee y = b$, and $r(x) + r(y) = r(a) + r(b)$.*

Proof. If $x = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k = b$, find $a_i \in \mathcal{A}(L) \cap \downarrow c_i \setminus \downarrow c_{i-1}$ for $i = 1, \ldots, k$. Let $y = a \vee a_1 \vee \cdots \vee a_k$. Clearly $r(y) - r(a) = k = r(b) - r(x)$, $x \vee y = b$, and $x \wedge y \geq a$. As $r(a) \leq r(x \wedge y) \leq r(x) + r(y) - r(x \vee y) = r(a) + r(b) - r(b) = r(a)$, we have $x \wedge y = a$. □

See [8, Chapters 2, 3, 5 and 8] and [9, Appendix 2, §3] for basic facts about ordinals and cardinals. If κ is a regular cardinal, a subset $\Omega \subseteq \kappa$ is *closed in* κ if for every non-empty subset $A \subseteq \Omega$, the supremum of A is κ or in Ω; it is *unbounded in* κ if the supremum of Ω is κ; it is a *club in* κ if it is both. A subset $\Omega \subseteq \kappa$ is *stationary in* κ if it intersects every club in κ; note that $|\Omega| = \kappa$.

We take our notation from [2, §§2, 4, and 6]. A *society* is a triple $\Lambda = (M_\Lambda, W_\Lambda, K_\Lambda)$ where $M_\Lambda \cap W_\Lambda = \emptyset$ and $K_\Lambda \subseteq M_\Lambda \times W_\Lambda$. If $A \subseteq M_\Lambda$ and $X \subseteq W_\Lambda$, then $K_\Lambda[A] := \{w \in W_\Lambda : (a, w) \in K_\Lambda \text{ for some } a \in A\}$, and $\Lambda[A, X] := \big(A, X, K_\Lambda \cap (A \times X)\big)$ is a *subsociety* of Λ. If $B \subseteq M_\Lambda$, then $\Lambda - B := \Lambda[M_\Lambda \setminus B, W_\Lambda]$. If Π is a subsociety, then $\Lambda/\Pi := \Lambda[M_\Lambda \setminus M_\Pi, W_\Lambda \setminus W_\Pi]$. We call a subsociety Π of Λ *saturated* if $K_\Lambda[M_\Pi] \subseteq W_\Pi$ and we denote this situation by $\Pi \lhd \Lambda$.

An *espousal* for Λ is an injective function $E : M_\Lambda \to W_\Lambda$ such that $E \subseteq K_\Lambda$. A society is *critical* if it has an espousal and every espousal is surjective.

If I is a set and $\bar{\Pi} = (\Pi_i : i \in I)$ is a family of subsocieties of Λ, then $\bigcup \bar{\Pi} := (\bigcup_{i\in I} M_{\Pi_i}, \bigcup_{i\in I} W_{\Pi_i}, \bigcup_{i\in I} K_{\Pi_i})$. Assume I is an ordinal. If $\theta \leq I$, then $\bar{\Pi}_\theta$ denotes $(\Pi_i : i < \theta)$. The sequence $\bar{\Pi}$ is *non-descending* if Π_i is a subsociety of Π_j whenever $i < j < I$; it is *continuous* if, in addition, $\bigcup \bar{\Pi}_\theta = \Pi_\theta$ for every limit ordinal $\theta < I$. If $I = J+1$, $\bar{\Pi}$ is a J-*tower in* Λ if $\bar{\Pi}$ is a continuous family of saturated subsocieties of Λ such that $\Pi_0 = (\emptyset, \emptyset, \emptyset)$.

Let Π be a subsociety of Λ. Assume $1 \leq \kappa \leq \aleph_0$. Then Π is a κ-*obstruction in* Λ if $\Pi \lhd \Lambda$ and $\Pi - A$ is critical for some $A \subseteq M_\Pi$ such that $|A| = \kappa$.

Now assume κ is a regular, uncountable cardinal. A κ-tower $\bar{\Sigma}$ in Λ is *obstructive* if, for each $\alpha < \kappa$, $\Sigma_{\alpha+1}/\Sigma_\alpha$ is either (a) a μ-obstruction in Λ/Σ_α for some $\mu < \kappa$ or (b) $(\emptyset, w, \emptyset)$ for some $w \in W_\Lambda$, and $\{\alpha < \kappa : \text{(a) holds at } \alpha\}$ is stationary in κ. We say Π is a κ-*obstruction in* Λ if $\Pi = \bigcup \bar{\Sigma}$ for an obstructive κ-tower $\bar{\Sigma}$ in Λ; by [2, Lemmas 4.2 and 4.3], $\Pi \lhd \Lambda$.

For a society Λ, $\delta(\Lambda)$ is the minimum of $\{|B| : B \subseteq M_\Lambda \text{ such that } \Lambda - B \text{ has an espousal}\}$.

We will use the following theorems of Aharoni, Nash-Williams, and Shelah:

Theorem 2. (from [2, Lemma 4.2 and Corollary 4.9a]) *If Π is a κ-obstruction, then $\delta(\Pi) = \kappa$.* □

Theorem 3. [2, Theorem 5.1] *A society Λ has an espousal if and only if it has no obstruction.* □

We will say that a geometric lattice of finite rank $r \geq 3$ has a *matching* if the society $\Big(\mathcal{A}(L), \mathcal{H}(L), \leq \cap \big(\mathcal{A}(L) \times \mathcal{H}(L)\big)\Big)$ has an espousal. (Since $\mathcal{A}(L) = \mathcal{H}(L)$ in geometric lattices of rank 2, we could say they also have a *matching*.)

Greene proved:

Theorem 4. [7, Corollary 3] *Every finite geometric lattice of rank at least 2 has a matching.* □

Björner proved:

Theorem 5. [5, Theorems 3 and 6] *Every geometric lattice of rank 3, or of finite height and cardinality less than $\aleph_\omega$, has a matching.* □

We use the following results of Björner: