

# Approximation Methods in Derivatives Pricing

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- A brief overview of approximation methods
- Timer option price approximation
  - Perpetual
  - Finite-maturity
- Conclusion

# Why approximation methods?

- Speed is money
- Fastest MC/PDE is often too slow
  - Real time calibration and pricing
  - CVA calculations in which prices per path at each future point need to be computed. Nested MC is a nightmare
- Analytic study adds understanding
  - Super-hedge
  - Asymptotic behavior
  - Price properties such as convexity, continuity, monotonicity, etc

- Perturbation, usually through PDE
- Probabilistic approach, e.g., moment matching, linear projection
- Lower and upper bounds, interpolation
- Other heuristic approach

- PDE from Feynman-Kac. Or perturb an expectation
- In physics, small/large parameter could be interaction strength  $\lambda$ , number of particles in a system ( $1/N$ -approximation in QCD), the dimension of space ( $\epsilon$ -approximation in QFT), Plank constant (WKB approximation)
- In finance: volatility, volatility of volatility, interest rates, time, correlation, relative initial prices in a spread option, strike price
- In systems with no apparent small parameters, searching for one takes effort. Li, Deng and Zhou (2008, 2010) approximate the price of spread options using the curvature of the exercise boundary hyper-surface
- regular/singular. Li (2010) expands the transition density of a diffusion using small  $t$  (singular, expands around a Dirac- $\delta$ )

# Probabilistic approach

- Widely used in finance industry
- Approximate densities as normal, bivariate-normal, lognormal, or other tractable ones
  - Moment matching for Asian options
  - Gaussian copula for credit derivatives
  - Mixed lognormal for matching volatility smile/skew
- Project random variable  $X$  as a linear function of  $Y$  with some normal noise

## Lower and upper bounds

- Sometimes we cannot price a derivative. But we can get lower and upper bounds by no-arbitrage considerations. For example,  $(S_1 + S_2 - K)^+ \leq (S_1 - K_1)^+ + (S_1 - K_2)^+$
- Sometimes relies on inequalities such as GM-AM inequality, comonotonicity, Hölder's inequality, Young's inequality. For example, arithmetic-mean price Asian option is more expensive than geometric-mean price Asian
- Results often useful for super-hedge considerations
- Bounds can sometimes be very tight. American option pricing
- Li (2010) considers writing American option price as a linear combination of two simple bounds, and solves the combination coefficient approximately through the PDE it satisfies

## Other heuristic approach

- Often different approaches are mixed together.  
Approximations within approximations
- Consider different regions. Pasting approximations together  
(need to be careful with Greeks)
- In LMM, drift freezing is frequently used
- Li and Mercurio (2013) approximate finite-maturity timer option price as a linear combination of plain-vanilla price and perpetual timer option price based on the spirit of matched asymptotic expansion
- Not completely rigorous, but better than the alternative of doing nothing



- Realized variance is defined as

$$\sum_{i=1}^N \left[ \log \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right) \right]^2 \quad (1)$$

- A timer option contract has a pre-specified variance budget  $B$ . Perpetual timer options are similar to plain-vanilla options, except that they are only exercisable at random time  $\tau^B$  when  $B$  is first reached
- Finite-maturity timer options are exercisable at time  $\tau := \min(\tau^B, T)$ , where  $T$  is a maximum maturity specified in the contract

- One-factor time-homogenous stochastic volatility model:

$$dS_u = (r - \delta)S_u du + \sqrt{V_u}S_u dW_u^S \quad (2)$$

$$dV_u = a(V_u) du + \eta b(V_u) dW_u^V \quad (3)$$

Constant correlation  $\rho$  between the two Brownian motions



$$\xi_u = \xi + \int_0^u V_s ds, \quad \tau^B := \inf \{u > 0 : \xi_u = B\} \quad (4)$$

- We want to compute

$$C_{\text{perp}} = \mathbb{E} \left[ e^{-r\tau^B} (S_{\tau^B} - K)^+ \right], \quad C_{\text{fin}} = \mathbb{E} \left[ e^{-r\tau} (S_{\tau} - K)^+ \right]$$

- Monte Carlo. Can be extremely time-consuming, although Bernard and Cui (2011) made improvements
- Multi-dimensional numerical integration as in Lee (2008), Li (2013), and Liang, Lemmens and Temepere (2011). Only works for specific models. Assumed  $\delta = 0$ , or even  $r = 0$
- PDE approach. Could be slow (3+1 dimensions)
- Analytic approximation as in Saunders (2010). However, not very accurate even under extremely large  $\kappa$

# Perpetual timer options

- Pricing PDE with boundary condition  $C(S, B, V) = (S - K)^+$ :

$$\begin{aligned} VC_\xi + a(V)C_V + \frac{1}{2}\eta^2 b^2(V)C_{VV} \\ + (r - \delta)SC_S + \frac{1}{2}VS^2C_{SS} + \rho\eta\sqrt{V}b(V)SC_{SV} - rC = 0 \end{aligned} \quad (5)$$

- Existing measures  $\mathbb{Q}'$ ,  $\widehat{\mathbb{Q}}$  and  $\widehat{\mathbb{Q}}'$  such that

$$\begin{aligned} C(S, \xi, V) &= \mathbb{E}^{\mathbb{Q}} [e^{-r\tau} S_\tau \mathbf{1}_{S_\tau > K}] - K \mathbb{E}^{\mathbb{Q}} [e^{-r\tau} \mathbf{1}_{S_\tau > K}] \\ &= S \mathbb{E}^{\mathbb{Q}'} e^{-\delta\tau} \cdot \mathbb{E}^{\widehat{\mathbb{Q}}} [\mathbf{1}_{S_\tau > K}] - K \mathbb{E}^{\mathbb{Q}} e^{-r\tau} \cdot \mathbb{E}^{\widehat{\mathbb{Q}}} [\mathbf{1}_{S_\tau > K}] \\ &:= Se^{-\delta T'} N(d^+) - Ke^{-rT} N(d^-) \end{aligned} \quad (6)$$

- If  $r = \delta = 0$  or  $\eta = 0$ , we have exact forms for  $d^\pm$

- It's true that  $\mathcal{T} = \mathcal{T}(\xi, V)$  and  $\mathcal{T}' = \mathcal{T}'(\xi, V)$ . We write:

$$d^{\pm} := d^{\pm}(S, \mathcal{T}, \mathcal{T}', \Sigma) = \frac{\log(S/K) + r\mathcal{T} - \delta\mathcal{T}'}{\Sigma} \pm \frac{1}{2}\Sigma$$

where we postulate  $\Sigma = \Sigma(\xi, V)$

- Plugging the solution  $C = Se^{-\delta\mathcal{T}'} N(d^+) - Ke^{-r\mathcal{T}} N(d^-)$  into the PDE, and collecting the  $N(d^+)$ ,  $N(d^-)$  and  $n(d^+)$  terms, we get three interconnected PDEs for  $\mathcal{T}$ ,  $\mathcal{T}'$  and  $\Sigma$
- We assume small  $\eta$  and solve those PDEs using perturbation

# Perpetual timer options

- Under  $\eta = 0$ ,

$$C(S, \xi, V) = Se^{-\delta T} N(d^+) - Ke^{-rT} N(d^-) \quad (7)$$

with

$$d^\pm = \frac{\log(Se^{(r-\delta)T}/K)}{\sqrt{B-\xi}} \pm \frac{1}{2}\sqrt{B-\xi} \quad (8)$$

Here  $\mathcal{T} = \mathcal{T}(\xi, V)$  is the solution of the first-order PDE

$$V\mathcal{T}_\xi + a(V)\mathcal{T}_V + 1 = 0 \quad (9)$$

with the boundary condition  $\mathcal{T}(B, V) = 0$

- For nonzero  $\eta$ , to lowest orders in  $\eta$ , we get

$$V\mathcal{T}_\xi + a(V)\mathcal{T}_V + \frac{1}{2}\eta^2 b^2(V) [\mathcal{T}_{0,VV} - r(\mathcal{T}_{0,V})^2] + 1 = o(\eta^2)$$

$$V\mathcal{T}'_\xi + a'(V)\mathcal{T}'_V + \frac{1}{2}\eta^2 b^2(V) [\mathcal{T}'_{0,VV} - \delta(\mathcal{T}'_{0,V})^2] + 1 = o(\eta^2)$$

with  $a'(V) := a(V) + \eta\rho\sqrt{V}b(V)$ , and

$$V(\Sigma^2)_\xi + a(V)(\Sigma^2)_V + V + 2\eta\rho(r - \delta)\sqrt{V}b(V)\mathcal{T}_{0,V} = \mathcal{O}(\eta^2)$$

- All three first-order PDEs can be solved exactly using method of characteristics

- Write

$$\begin{aligned}\mathcal{T}(\xi, V) &\approx \mathcal{T}_0(\xi, V) + \eta^2(\mathcal{H}_0(\xi, V) - r\mathcal{H}_1(\xi, V)) \\ \mathcal{T}'(\xi, V) &\approx \mathcal{T}'_0(\xi, V) + \eta^2(\mathcal{H}'_0(\xi, V) - \delta\mathcal{H}'_1(\xi, V))\end{aligned}$$

and

$$\Sigma^2(\xi, V) = B - \xi + 2\eta\rho(r - \delta)G(\xi, V)$$

- The functions needed above can be worked out for many models in our general class, including Heston and 3/2-models



# Perpetual timer options

- For example, in Heston, we have ( $\mathcal{T}'_0$ ,  $\mathcal{H}'_0$  and  $\mathcal{H}'_1$  are similar)

$$\mathcal{T}_0 = \frac{1}{\kappa} \log R$$

$$\mathcal{H}_0 = \frac{(R-1) [2R^2z^2 + R(2-5z-2z^2) - 2-z]}{4\kappa^2 R^2 (1+z)^3 \theta} + \frac{3z \log R}{2\kappa^2 (1+z)^3 \theta}$$

$$\mathcal{H}_1 = \frac{(R-1)(1+2R^2z + R(2z-3))}{4\kappa^3 R^2 (1+z)^2 \theta} - \frac{(2z-1) \log R}{2\kappa^3 (1+z)^2 \theta}$$

$$G = \frac{(1-R)(Rz-1) + R(z-1) \log R}{\kappa^2 R (1+z)}$$

with

$$R := e^{z-z_0 + \kappa \frac{B}{\theta}}, \quad z_0 := \frac{V_0 - \theta}{\theta}, \quad z := W \left( z_0 e^{z_0} e^{-\kappa \frac{B}{\theta}} \right)$$

# Perpetual timer options - Numerical results

$$r = 1.5\%, \delta = 3\%, S = 100,$$
$$V_0 = B = 0.087, \theta = 0.09, \kappa = 2, \eta = 0.375.$$

$K$	$\rho$	MC	$\eta = 0$	Error	Approx	Error
90	-0.5	15.265	15.605	2.23%	15.261	-0.03%
	0	15.444	15.605	1.04%	15.435	-0.06%
	0.5	15.599	15.605	0.03%	15.601	0.01%
100	-0.5	10.466	10.763	2.84%	10.465	-0.01%
	0	10.637	10.763	1.18%	10.632	-0.06%
	0.5	10.796	10.763	-0.30%	10.792	-0.04%
110	-0.5	6.973	7.221	3.56%	6.975	0.03%
	0	7.125	7.221	1.35%	7.123	-0.03%
	0.5	7.271	7.221	-0.68%	7.267	-0.06%

# Perpetual timer options

- The form  $C = Se^{-\delta T'} N(d^+) - Ke^{-rT} N(d^-)$  has many attractive features:
  - Black-Scholes like, easy to interpret quantities
  - Easy Greek computation, such as Delta and Gamma, since

$$Se^{-\delta T'} n(d^+) - Ke^{-rT} n(d^-) = 0$$

For example,  $\Delta = e^{-\delta T'} N(d^+)$

- Reduces to exact formulas in special cases
- Put timer is consistently approximated as

$$P = Ke^{-rT} N(-d^-) - Se^{-\delta T'} N(-d^+)$$

- We also approximated the joint moment generating function of  $(S_{\tau^B}, \tau^B)$

# Finite-maturity timer options

- It can be shown that for small  $\eta$ ,  $\tau^B$  is approximately normal:

$$\mu(B) = \mathcal{T}_0 + \eta^2 \mathcal{H}_0, \quad \sigma^2(B) = 2\eta^2 \mathcal{H}_1 \quad (10)$$

- The approximation is in the following sense

$$M_{\tau^B}(\lambda) \equiv \mathbb{E}e^{\lambda\tau^B} = e^{\lambda(\mathcal{T}_0 + \eta^2 \mathcal{H}_0) + \lambda^2 \eta^2 \mathcal{H}_1} + o(\eta^2) \quad (11)$$

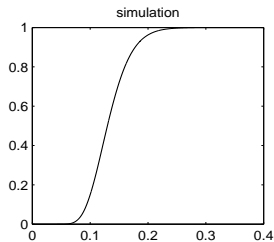
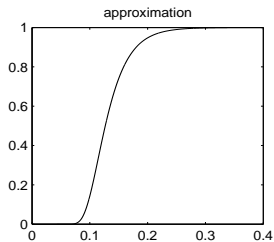
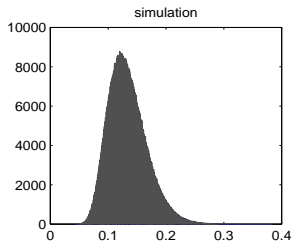
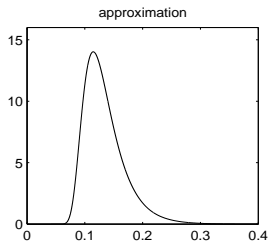
- Derivation is through a perturbation for  $\Pi(\xi, V) := M_{\tau^B}(\lambda)$

$$V\Pi_\xi + a(V)\Pi_V + \frac{1}{2}\eta^2 b^2(V)\Pi_{VV} + \lambda\Pi = 0 \quad (12)$$

- Distribution of  $\xi_T$  can be approximated through duality

$$\{\tau^x > T\} = \{\xi_T < x\} \quad (13)$$

# Finite-maturity timer options



- PDE approach is difficult. We switch to a probabilistic approach. We first work with  $\rho = 0$
- Assume  $\rho = 0$ . We can write  $C_{\text{fin}} = C_{\text{fin}}^B + C_{\text{fin}}^T$ , where

$$C_{\text{fin}}^B = \mathbb{E} \left[ C^{\text{BS}}(S, K, r, \delta, \tau^B, B) 1_{\{\tau^B < T\}} \right] \quad (14)$$

$$C_{\text{fin}}^T = \mathbb{E} \left[ C^{\text{BS}}(S, K, r, \delta, T, \xi_T) 1_{\{\xi_T < B\}} \right] \quad (15)$$

- Given the distribution of  $\tau^B$  (and hence that of  $\xi_T$  by duality), we can evaluate the above integrals numerically (one is actually in closed form)

# Finite-maturity timer options ( $\rho = 0$ )

$$C_{\text{fin}}^B \approx S \mathbf{I}(a_+, b, -\delta, \mu(B), \sigma(B)) - K \mathbf{I}(a_-, b, -r, \mu(B), \sigma(B))$$

where

$$a_{\pm} = \frac{\log(S/K)}{\sqrt{B}} \pm \frac{\sqrt{B}}{2}, \quad b = \frac{r - \delta}{\sqrt{B}}$$

and  $\mathbf{I}(a, b, s, m, \Sigma)$  is given by

$$\mathbf{I} = e^{ms + \Sigma^2 s^2 / 2} N_2 \left( \frac{T - (m + s\Sigma^2)}{\Sigma}, \frac{a + b(m + s\Sigma^2)}{\sqrt{1 + b^2\Sigma^2}}; -\frac{b\Sigma}{\sqrt{1 + b^2\Sigma^2}} \right)$$

$$C_{\text{fin}}^T \approx C^{\text{BS}}(S, K, r, \delta, T, B) \tilde{F}_{\xi_T}(B) - S e^{-\delta T} \int_0^{\sqrt{B}} n(d_1(T, y^2)) \tilde{F}_{\xi_T}(y^2) dy$$

# Finite-maturity timer options ( $\rho = 0$ )

$T$	$K$	$\eta = 0.125$			$\eta = 0.250$			$\eta = 0.375$		
		Approx	MC	Error	Approx	MC	Error	Approx	MC	Error
0.5	90	13.267	13.265	0.01	13.230	13.235	-0.04	13.117	13.180	-0.47
	100	7.885	7.884	0.02	7.832	7.838	-0.08	7.684	7.756	-0.92
	110	4.352	4.351	0.03	4.312	4.319	-0.15	4.186	4.260	-1.73
1.0	90	15.418	15.417	0.01	15.156	15.164	-0.05	14.856	14.889	-0.22
	100	10.544	10.542	0.01	10.240	10.247	-0.06	9.897	9.928	-0.31
	110	7.000	6.999	0.02	6.702	6.709	-0.10	6.370	6.402	-0.51
1.5	90	15.586	15.586	0.00	15.523	15.515	0.05	15.379	15.358	0.14
	100	10.749	10.749	0.00	10.697	10.685	0.11	10.558	10.523	0.33
	110	7.211	7.211	0.00	7.169	7.156	0.18	7.046	7.008	0.55
2.0	90	15.586	15.586	0.00	15.530	15.533	-0.02	15.435	15.445	-0.06
	100	10.749	10.749	0.00	10.706	10.708	-0.02	10.631	10.635	-0.03
	110	7.211	7.211	0.00	7.178	7.180	-0.02	7.123	7.122	0.01
10.0	90	15.586	15.586	0.00	15.530	15.530	0.00	15.437	15.444	-0.05
	100	10.749	10.749	0.00	10.706	10.706	-0.01	10.634	10.637	-0.04
	110	7.211	7.211	0.00	7.178	7.180	-0.02	7.125	7.125	0.00



# Finite-maturity timer options

- For  $\rho \neq 0$ . We use

$$C_{\text{fin}}(\rho) \approx C_{\text{perp}}(\rho) \frac{C_{\text{fin}}^B(0)}{C_{\text{perp}}(0)} + C_{\text{vanilla}}(\rho) \frac{C_{\text{fin}}^T(0)}{C_{\text{vanilla}}(0)}$$

- The approximation uses the  $\rho$  dependence for perpetual timer options when  $T$  is large, and uses the  $\rho$  dependence for plain-vanilla options when  $B$  is large
- Goes to right limits when  $T \ll \mu(B)$ , or  $T \gg \mu(B)$
- Exact when  $\eta = 0$  or  $\rho = 0$

# Finite-maturity timer options

$\rho$	$K$	$\eta = 0.125$			$\eta = 0.250$			$\eta = 0.375$		
		Approx	MC	Error	Approx	MC	Error	Approx	MC	Error
$T = 0.5$										
-0.5	90	13.334	13.333	0.01	13.357	13.363	-0.04	13.287	13.373	-0.65
	100	7.853	7.852	0.01	7.767	7.771	-0.05	7.587	7.663	-1.00
	110	4.226	4.225	0.01	4.062	4.062	-0.01	3.832	3.877	-1.18
0.5	90	13.196	13.195	0.01	13.089	13.093	-0.03	12.915	12.963	-0.37
	100	7.916	7.916	0.01	7.894	7.903	-0.12	7.775	7.850	-0.95
	110	4.475	4.474	0.01	4.548	4.560	-0.28	4.508	4.614	-2.30
$T = 1.0$										
-0.5	90	15.407	15.441	-0.22	15.148	15.223	-0.49	14.858	14.998	-0.94
	100	10.481	10.517	-0.34	10.117	10.194	-0.75	9.716	9.836	-1.22
	110	6.894	6.920	-0.38	6.482	6.534	-0.79	6.041	6.121	-1.30
0.5	90	15.426	15.448	-0.14	15.151	15.203	-0.34	14.822	14.920	-0.66
	100	10.605	10.629	-0.23	10.356	10.414	-0.56	10.057	10.162	-1.03
	110	7.104	7.123	-0.27	6.909	6.960	-0.74	6.669	6.762	-1.39
$T = 1.5$										
-0.5	90	15.523	15.525	-0.02	15.405	15.415	-0.07	15.234	15.276	-0.28
	100	10.689	10.691	-0.02	10.579	10.583	-0.04	10.379	10.407	-0.26
	110	7.158	7.159	-0.01	7.059	7.055	0.06	6.854	6.853	0.01
0.5	90	15.648	15.647	0.01	15.637	15.639	-0.01	15.509	15.527	-0.12
	100	10.808	10.806	0.02	10.811	10.811	0.01	10.725	10.734	-0.08
	110	7.263	7.262	0.02	7.276	7.274	0.04	7.227	7.227	0.01

- Explicit formulas for perpetual and finite-maturity timer options. Accurate and fast
- Approximate distributions of  $\tau^B$  and  $\xi_T$ . Can be used to price other derivatives (Li, 2013)
- Approximations for joint moment generating function of  $(S_{\tau^B}, \tau^B)$
- Possible future work:
  - Extending the class of models we consider
  - What if  $\eta$  is large?
  - Characterizing the measures  $\widehat{\mathbb{Q}}$  and  $\widehat{\mathbb{Q}}'$ ?
  - Other approaches?