Minor preserving deletable edges in graphs

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Since today is 9/11 I'd like to start by taking a moment to think about the victims of the 9/11 attack.

Names written in the pale sky.
Names rising in the updraft amid buildings.
Names silent in stone
-Billy Collins

This is joint work with João Paulo Costalonga

The paper is available on my webpage
`http://userhome.brooklyn.cuny.edu/skingan/papers`

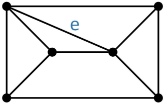
1. Introduction
   – basic terminology
2. New results
   – the two lemmas that combine to form the new theorem.
3. Previous results
   – a description of the previous results used
4. Proof idea
   – just a very rough idea
5. Conclusion by way of a picture
   – one slide

Definition 1.

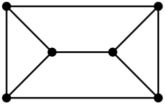
A graph G is 3-**connected** if at least 3 vertices must be removed to disconnect G.

Definition 2.

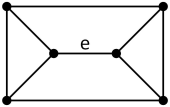
H is a **minor** of G if H can be obtained from G by deleting edges (and any isolated vertices) and contracting edges.

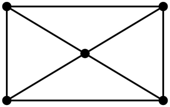


G



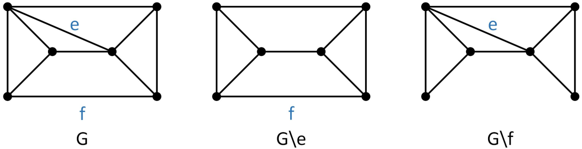
G\e





Definition 3a.

An edge in a 3-connected graph is **deletable** if $G\backslash e$ is 3-connected.

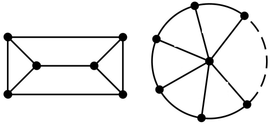


In the above figure, edge e is deletable, but edge f is not deletable.

Definition 3b.

A 3-connected graph is **minimally 3-connected** if it has no deletable edges.

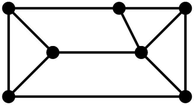
Example: Any cubic graph or wheels



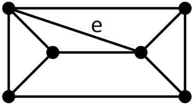
Definition 4.

Let G and H be simple 3-connected graphs such that G has a proper H-minor.

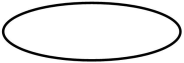
- We say e is an H-**deletable edge** if $G\backslash e$ is 3-connected and has an H-minor.
- We say G is H-**critical** if it has no H-deletable edges.



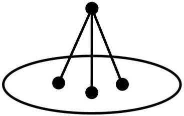
Prism-critical



Not prism-critical



Any 3-connected graph H



H-critical graph

Goals:

- Structure theorem for 3-connected graphs in terms of H-critical graphs.
- Bound on the number of elements in an H-critical graph.

2. New Results

If G is H-critical, then there is a smaller H-critical graph that can be obtained from G in a very precise manner.

Lemma 1.

Let G and H be simple 3-connected graphs such that G has a **proper** H-minor. If G is H-critical, then there exists an H-critical graph G' on $|V(G)| - 1$ vertices such that:

(i) $G/f = G'$, where f is an edge;

(ii) $G/f \backslash e = G'$, where edges e and f are incident to a degree 3 vertex; or

(iii) $G - w = G'$, where w is a vertex of degree 3.

Lemma 1 is based on:

S. R. Kingan and M. Lemos (2014), Strong Splitter Theorem , *Annals of Combinatorics*, Vol. 18 – 1, 111 – 116.

If G is H-critical, then the size of G is bounded above by the number of edges and vertices of H and the number of vertices of G.

Lemma 2.

Let G and H be simple 3-connected graphs such that G has a proper H-minor, $|V(H)| \geq 5$, and $|V(G)| \geq |V(H)| + 1$. If G is H-critical, then

$$|E(G)| \leq |E(H)| + 3[|V(G)| - |V(H)|].$$

Main Theorem (JPC, SRK 2020+)

Let G and H be simple 3-connected graphs such that G has a proper H minor, $|E(G)| \geq |E(H)| + 3$, and $|V(G)| \geq |V(H)| + 1$. Then there exists a set of H-deletable edges D such that

$$|D| \geq |E(G)| - |E(H)| - 3[|V(G|) - |V(H)|]$$

and a sequence of H-critical graphs

$$G_{|V(H)|}, \ldots, G_{|V(G)|},$$

where $G_{|V(H)|} \cong H$, $G_{|V(G)|} = G \backslash D$, and for all i such that $|V(H)| + 1 \leq i \leq |V(G)|$:

(i) $G_i / f = G_{i-1}$, where f is an edge;

(ii) $G_i / f \backslash e = G_{i-1}$, where e and f are edges incident to a vertex of degree 3; or

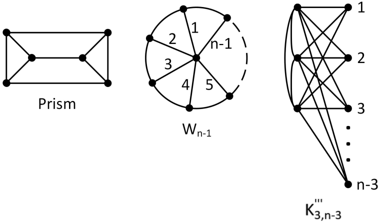
(iii) $G_i - w = G_{i-1}$, where w is a vertex of degree 3.

3. Previous results

G. A. Dirac (1963). Some results concerning the structure of graphs, *Canad. Math. Bull.* **6**, 183–210.

Theorem (Dirac 1963)

A simple 3-connected graph G has no prism minor if and only if G is isomorphic to $K_5\backslash e$, K_5, $\mathbf{W_{n-1}}$ for $n \geq 4$, $\mathbf{K_{3,n-3}}$, $K'_{3,n-3}$, $K''_{3,n-3}$, or $K'''_{3,n-3}$ for $n \geq 6$.



$\mathbf{W_{n-1}}$ and $\mathbf{K_{3,n-3}}$ are minimally 3-connected.

R. Halin (1969) Untersuchungen uber minimale n-fach zusammenhangende graphen, *Math. Ann* 182 (1969), 175–188.

Theorem (Halin, 1969)

Let G be a minimally 3-connected graph on $n \geq 8$ vertices. Then

$$|E(G)| \leq 3n - 9.$$

Moreover, $|E(G)| = 3n - 9$ if and only if $G \cong K_{3,n-3}$.

Corollary of Dirac's Theorem and Halin's Theorem

Let G be a minimally 3-connected graph with a prism minor on $n \geq 8$ vertices. Then

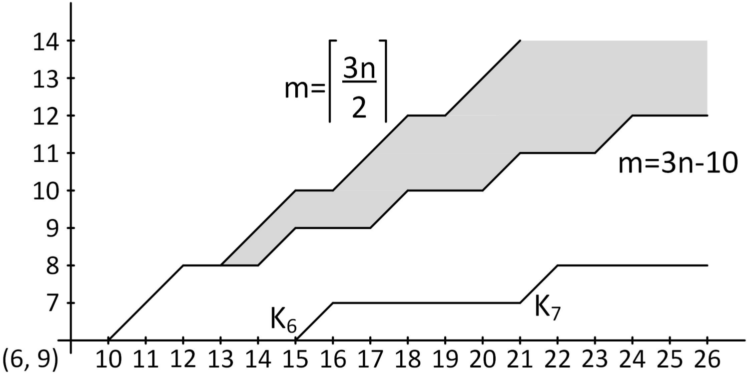
$$|E(G)| \leq 3n - 10.$$

F. Harary, The maximum connectivity of a graph. PNAS July 1, 1962 48 (7) 1142-1146.

Harary, 1962

Let G be a 3-connected graph with n vertices and m edges. Then

$$m \geq \left\lceil \frac{3n}{2} \right\rceil.$$



The class of minimally 3-connected graphs is a “sparse” class of graphs.

W. T. Tutte (1961). A theory of 3-connected graphs, *Indag. Math* **23**, 441–455.

Wheels Theorem (Tutte 1961)

Let G be a simple 3-connected graph that is not a wheel. Then there exists an element e such that either $G\backslash e$ or G/e is simple and 3-connected.

P. D. Seymour (1980). Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28**, 305–359.

S. Negami (1982). A characterization of 3-connected graphs containing a given graph. *J. Combin. Theory Ser. B* **32**, 9–22.

Splitter Theorem (Seymour 1980)

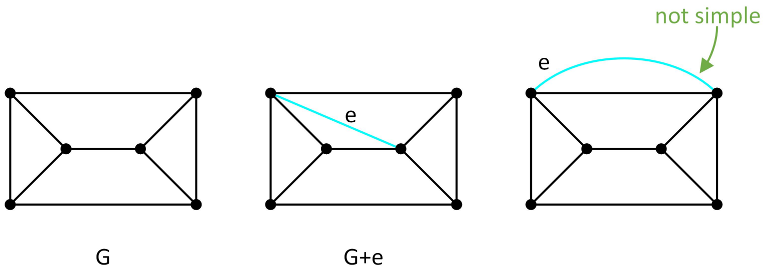
Suppose G and H are simple 3-connected graphs such that G has a proper H-minor, G is not a wheel, and $H \neq W_3$. Then there exists an element e such that $G \backslash e$ or G/e is simple, 3-connected, and has an H-minor

C. R. Coullard and J. G. Oxley, J. G. (1992). Extension of Tutte's wheels-and-whirls theorem. *J. Combin. Theory Ser. B* **56**, 130–140.

The operations that reverse deletions and contractions are edge additions and vertex splits.

Definition 5.

A graph G with an edge e added between non-adjacent vertices is denoted by $G + e$ and called a **(simple) edge addition** of G.



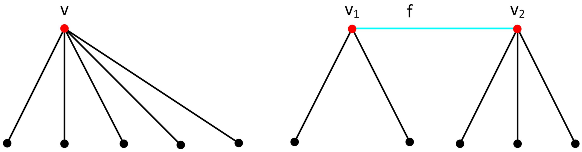
An edge addition is 3-connected.

Definition 6.

Suppose G is a 3-connected graph with a vertex v such that $deg(v) \geq 4$. To **split** vertex v,

- Divide $N_G(v)$ into two disjoint sets S and T, both of size at least 2.
- Replace v with two distinct vertices v_1 and v_2, join them by a new edge $f = v_1 v_2$; and
- Join each neighbor of v in S to v_1 and each neighbor in T to v_2.

The resulting 3-connected graph is called a **vertex split** of G and is denoted by $G \circ_{S,T} f$.



We can get a different graph depending on the assignment of neighbors of v to v_1 and v_2. By a slight abuse of notation, we can say $G \circ f$, referencing S and T only when needed. The focus is always on the edges. This is the matroid perspective.

Wheels Theorem and Splitter Theorem again, the constructive version this time. The previous renditions were the top-down version.

Wheels Theorem (again)

Let G be a simple 3-connected graph that is not a wheel. Then G can be constructed from a wheel by a finite sequence of edge additions or vertex splits

Splitter Theorem (again)

Suppose G and H are simple 3-connected graphs such that G has a proper H-minor, G is not a wheel, and $H \neq W_3$. Then G can be constructed from H by a finite sequence of edge additions and vertex splits.

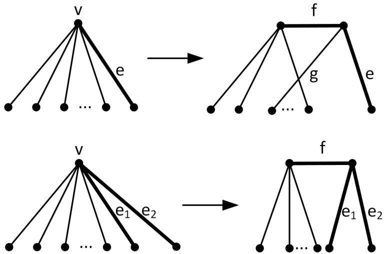
4. Proof ideas

Lemma 1 (again). Suppose G and H are simple 3-connected graphs such that G has a proper H-minor, G is not a wheel, and $H \neq W_3$. If G is H-critical, then there exists an H-critical graph G' on $|V(G)| - 1$ vertices such that:

(i) $G = G' \circ f$;

(ii) $G = G' + e \circ f$, where e and f are in a triad of G; or

(iii) $G = G' + \{e_1, e_2\} \circ f$, where $\{e_1, e_2, f\}$ is a triad of G



Proof Idea. The Splitter Theorem implies that we can construct G from H by a sequence of edge additions and vertex splits.

Since G is H-critical, the last operation in forming G is a vertex split. So

$$G = G^+ \circ f$$

for some graph G^+ with $|V(G)| - 1$ vertices.

Now G^+ may have deletable edges. Remove as many deletable edges as needed to obtain a minimally 3-connected graph

$$G' = G^+ \backslash \{e_1, \ldots, e_k\}$$

where G' has no deletable edges. Then

$$G = G' + \{e_1, \ldots e_k\} \circ f.$$

We have to prove that $k \leq 2$, and in each case the specified restrictions hold.

Lemma 2.

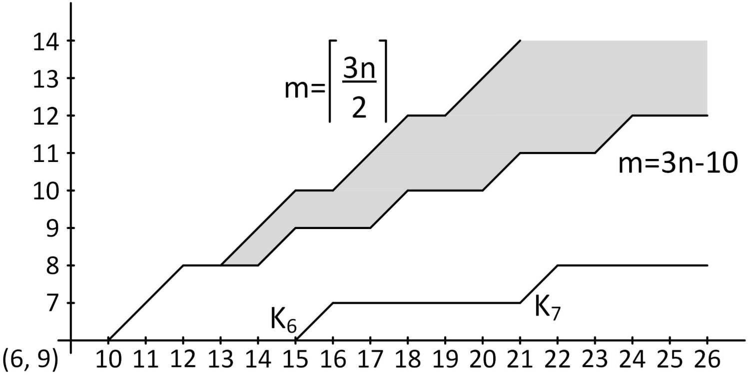
Let G and H be simple 3-connected graphs such that G has a proper H-minor, $|V(H)| \geq 5$, and $|V(G)| \geq |V(H)| + 1$. If G is H-critical, then

$$|E(G)| \leq |E(H)| + 3[|V(G)| - |V(H)|].$$

Proof Idea. The result holds for wheels. Assume G is not a wheel. The proof is by induction on $|V(G)|$. Use Lemma 1 and work through all the possibilities.

Halin's theorem for minimally 3-connected graphs follows from Dirac's theorem and Lemma 2 with $H = \textit{prism}$.

6. Conclusion by way of a picture



If H is in the grey cone of minimally 3-connected graphs, then H-critical graphs is a subset of minimally 3-connected graphs.

But H does not have to be in the grey area. H could be anywhere and we get a similar grey cone emanating from H.