Zero forcing, propagation time, and throttling on a graph

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Outline

Zero forcing and its variants
- Matrices and graphs
- Standard zero forcing $Z(G)$
- PSD zero forcing $Z_+(G)$
- Skew zero forcing $Z_-(G)$
- Zero forcing numbers of families of graphs

Propagation time
- Standard propagation time $pt(G)$
- PSD propagation time $pt_+(G)$
- Skew propagation time $pt_-(G)$
- Propagation time of families of graphs

Throttling
- Throttling numbers $th(G), th_+(G), th_-(G)$
- Throttling numbers of families of graphs

Other topics

Computation

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Zero forcing and its variants

Zero forcing is a coloring game in which each vertex is initially blue or white and the goal is to color all vertices blue.

▶ The standard color change rule for zero forcing on a graph $G$ is that a blue vertex $v$ can change the color of a white vertex $w$ to blue if $w$ is the only white neighbor of $v$ in $G$.

▶ There are many variants of zero forcing, each of which uses a different color change rule.

Applications:

▶ Mathematical physics (control of quantum systems).

▶ Power domination:
  ▶ A minimum power dominating set gives the optimal placement of monitoring units in an electric network.
  ▶ Power domination is zero forcing applied to the set of initial vertices and their neighbors.

▶ Combinatorial matrix theory - illustrated in these slides.
Matrices are real. The matrix \( A = [a_{ij}] \) is \textbf{symmetric} if \( a_{ji} = a_{ij} \) and \textbf{skew symmetric} if \( a_{ji} = -a_{ij} \). Most matrices discussed are symmetric; some are skew symmetric. \( S_n(\mathbb{R}) \) is the set of \( n \times n \) real symmetric matrices.

The graph \( G(A) = (V, E) \) of \( n \times n \) symmetric or skew matrix \( A \) is

- \( V = \{1, \ldots, n\} \),
- \( E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\} \).
- Diagonal of \( A \) is ignored.

\[ A = \begin{bmatrix}
2 & -1 & 3 & 5 \\
-1 & 0 & 0 & 0 \\
3 & 0 & -3 & 0 \\
5 & 0 & 0 & 0
\end{bmatrix} \]

\[ G(A) = \begin{array}{c}
1 \quad 2 \\
4 \quad 3
\end{array} \]
The family of symmetric matrices described by a graph $G$ is

$$S(G) = \{ A \in S_n(\mathbb{R}) : G(A) = G \}.$$ 

The Inverse Eigenvalue Problem of a Graph (IEPG) is to determine all possible spectra (multisets of eigenvalues) of matrices in $S(G)$.

**Example**

A matrix in $S(P_3)$ is of the form

$$A = \begin{bmatrix} x & a & 0 \\ a & y & b \\ 0 & b & z \end{bmatrix}$$

where $a, b \neq 0$.

The possible spectra of matrices in $S(P_3)$ are all sets of 3 distinct real numbers.
Maximum multiplicity and minimum rank

Due to the difficulty of the IEPG, a simpler form called the maximum multiplicity, maximum nullity, or minimum rank problem has been studied.

The **maximum multiplicity** or maximum nullity of graph $G$ is

$$M(G) = \max\{\text{mult}_A(\lambda) : A \in S(G), \lambda \in \text{spec}(A)\}.$$  
$$= \max\{\text{null } A : A \in S(G)\}.$$  

The **minimum rank** of graph $G$ is

$$\text{mr}(G) = \min\{\text{rank } A : A \in S(G)\}.$$  

By using nullity,

$$M(G) + \text{mr}(G) = |V(G)|.$$  

The **Maximum Nullity Problem** (or **Minimum Rank Problem**) for a graph $G$ is to determine $M(G)$ (or $\text{mr}(G)$).
Zero forcing and maximum nullity

- Zero forcing starts with blue vertices (representing zeros in a null vector of a matrix) and successively colors other vertices blue.
- The zero forcing number is the minimum size of a zero forcing set.

**Theorem (BBBCCFGHHMNPSSSvdHVM 2008)**

*For every graph $G$, $M(G) \leq Z(G)$.***

- $G$ a graph with $V(G) = \{1, \ldots, n\}$ and $A \in S(G)$,
- $x \in \mathbb{R}^n$, $Ax = 0$, and $x_k = 0$ for all $k \in B \subseteq V(G)$,
- $i \in B$, $j \not\in B$, and $j$ is the only vertex $k$ such that $ik \in E(G)$ and $k \not\in B$.

Imply

$$x_j = 0$$

because equating the $i$th entries in $Ax = 0$ yields $a_{ij}x_j = 0$. 
Standard color change rule: Let $W$ be the set of (currently) white vertices. A blue vertex $v$ can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$
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N_G(v) \cap W = \{w\}.
\]

Example \((B^3; Z(T) = 4)\)
Example: Why is $Z(T) = 4$

We just showed $Z(T) \leq 4$

For trees, there is an algorithm for finding a minimum path cover and thus a minimum zero forcing set.
Variants of zero forcing

- Each type of zero forcing is a coloring game on a graph in which each vertex is initially blue or white.
- A color change rule allows white vertices to be colored blue under certain conditions.

Let $R$ be a color change rule.

- The set of initially blue vertices is $B^0 = B$.
- The set of blue vertices $B^t$ after round $t$ or time step $t$ (under $R$) is the set of blue vertices in $G$ after the color change rule is applied in $B^{t-1}$ to every white vertex independently.
- An initial set of blue vertices $B = B^0$ is an $R$ zero forcing set if there exists a $t$ such that $B^t = V(G)$ using the $R$ color change rule.
- Minimum size of an $R$ zero forcing set is the $R$ forcing number.
A real matrix is positive semidefinite matrices (PSD) if $A$ is symmetric and every eigenvalue is nonnegative.

The family of PSD described by a graph $G$ is

$$\mathcal{S}_+(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G \text{ and } A \text{ is PSD}\}.$$ 

The maximum PSD nullity of graph $G$ is

$$M_+(G) = \max\{\text{null } A : A \in \mathcal{S}_+(G)\}.$$ 

The PSD zero forcing number is $Z_+(G)$.

**Theorem (BBFHHSvdDvdH 2010)**

For every graph $G$, $M_+(G) \leq Z_+(G)$. 
PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

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Example ($B_{2+}^2$)
PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

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$$N_G(v) \cap W_i = \{w\}.$$
A matrix is **hollow** if $A$ is symmetric and every diagonal entry is 0.

A hollow matrix described by a graph $G$ is a weighted adjacency matrix of $G$.

A matrix is **skew symmetric** if $A^T = -A$.

\[
S_0(G) = \{ A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G \text{ and } A \text{ is hollow} \}.
\]

\[
S_-(G) = \{ A \in \mathbb{R}^{n \times n} : \mathcal{G}(A) = G \text{ and } A^T = -A \}.
\]

The **maximum hollow nullity** and **maximum skew nullity** of graph $G$ are

\[
M_0(G) = \max \{ \null A : A \in S_0(G) \}.
\]

\[
M_-(G) = \max \{ \null A : A \in S_-(G) \}.
\]
Skew forcing and maximum nullity

Theorem (ABDeADDDeLGGHIKNPSSW 2010 and GHHHJHJkmC 2014)

For every graph $G$, $M_-(G) \leq Z_-(G)$ and $M_0(G) \leq Z_-(G)$.

- $G$ a graph with $V(G) = \{1, \ldots, n\}$ and $A \in S_-(G)$ or $A \in S_0(G)$,
- $x \in \mathbb{R}^n$, $Ax = 0$, and $x_k = 0$ for all $k \in B \subseteq V(G)$,
- $j \not\in B$ and $j$ is the only vertex $k$ such that $ik \in E(G)$ and $k \not\in B$.

implies

$$x_j = 0$$

because equating the $i$th entries in $Ax = 0$ yields $a_{ij}x_j = 0$. 
Skew color change rule: Let $W$ be the set of (currently) white vertices. A vertex $v$ can change the color of vertex $w \in W$ to blue if

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Example ($B^{[1]}$)
Skew color change rule: Let $W$ be the set of (currently) white vertices. A vertex $v$ can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$
Theorem (four papers previously cited)

- \( n \geq 2 \), \( Z(K_n) = M(K_n) = Z_+(K_n) = M_+(K_n) = n - 1 \) and \( Z_-(K_n) = M_-(K_n) = M_0(K_n) = n - 2 \).

- \( n \geq 1 \), \( Z(K_n) = M(K_n) = Z_+(K_n) = M_+(K_n) = Z_-(K_n) = M_-(K_n) = n \).

- \( n \geq 3 \), \( Z(K_{r,n-r}) = M(K_{r,n-r}) = Z_-(K_{r,n-r}) = M_-(K_{r,n-r}) = M_0(K_{r,n-r}) = n - 2 \) and \( Z_+(K_{r,n-r}) = M_+(K_{r,n-r}) = \min(r, n - r) \).

- \( n \geq 2 \), \( Z(P_n) = M(P_n) = Z_+(P_n) = M_+(P_n) = 1 \). For even \( n \), \( Z_-(P_n) = M_-(P_n) = M_0(P_n) = 0 \) and for odd \( n \), \( Z_-(P_n) = M_-(P_n) = M_0(P_n) = 1 \).

- \( n \geq 3 \), \( Z(C_n) = M(C_n) = Z_+(C_n) = M_+(C_n) = 2 \). For even \( n \geq 4 \), \( Z_-(C_n) = M_-(C_n) = M_0(C_n) = 2 \). For odd \( n \geq 3 \), \( Z_-(C_n) = M_-(C_n) = 1 \) and \( M_0(C_n) = 0 \).
Let $R$ be a color change rule.

- The $R$-propagation time for a set $B = B^{[0]}$ of vertices, $\text{pt}_R(G, B)$, is the smallest $t$ such that $B^{[t]} = V(G)$ using the $R$ color change rule (and is infinity if this never happens).
- This is also called the number of times steps or rounds to color the graph.
- The $R$-propagation time of $G$ is

$$\text{pt}_R(G) = \min\{\text{pt}_R(G, B) : B \text{ is a minimum } R\text{-forcing set}\}$$
Standard color change rule: Let $W$ be the set of (currently) white vertices. A blue vertex $v$ can change the color of vertex $w \in W$ to blue if

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PSD propagation time \( pt_+(G) \)

PSD color change rule: Delete the currently blue vertices from the graph \( G \) and determine the components of the resulting graph; let \( W_i \) be the set of vertices of the \( i \)th component. A blue vertex \( v \) can change the color of a white vertex \( w \) to blue if

\[
N_G(v) \cap W_i = \{w\}.
\]

Example (\( B_+ = B_+^{[0]} \))
PSD propagation time \( pt_+(G) \)

PSD color change rule: Delete the currently blue vertices from the graph \( G \) and determine the components of the resulting graph; let \( W_i \) be the set of vertices of the \( i \)th component. A blue vertex \( v \) can change the color of a white vertex \( w \) to blue if

\[
N_G(v) \cap W_i = \{ w \}.
\]

Example (\( B^{[1]}_+ \))
PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

$$N_G(v) \cap W_i = \{w\}.$$
PSD propagation time $pt_+(G)$

PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

$$N_G(v) \cap W_i = \{w\}.$$
PSD propagation time $pt_+(G)$

PSD color change rule: Delete the currently blue vertices from the graph $G$ and determine the components of the resulting graph; let $W_i$ be the set of vertices of the $i$th component. A blue vertex $v$ can change the color of a white vertex $w$ to blue if

$$N_G(v) \cap W_i = \{w\}.$$ 

Example ($B_+^{[4]}$: $pt_+(T) = 4$)
Skew propagation time \( pt_-(G) \)

**Skew color change rule:** Let \( W \) be the set of (currently) white vertices. A vertex \( v \) can change the color of vertex \( w \in W \) to blue if

\[
N_G(v) \cap W = \{w\}. 
\]

**Example \((B_- = B_-^{[0]}\)**

![Graph diagram showing the skew propagation time example]
Skew color change rule: Let $W$ be the set of (currently) white vertices. A vertex $v$ can change the color of vertex $w \in W$ to blue if

$$N_G(v) \cap W = \{w\}.$$

Example ($B^{[1]}$)
Skew propagation time pt\(_{(G)}\)

Skew color change rule: Let \(W\) be the set of (currently) white vertices. A vertex \(v\) can change the color of vertex \(w \in W\) to blue if

\[
N_G(v) \cap W = \{w\}.
\]

Example \((B^{[2]}_\to:\ pt\_ (T) = 2)\)
Propagation time of complete graphs and paths

<table>
<thead>
<tr>
<th>Theorem (HHKMWY 2012, W 2015, K 2015)</th>
</tr>
</thead>
<tbody>
<tr>
<td>▶ For ( n \geq 2 ), ( \text{pt}(K_n) = \text{pt}<em>+(K_n) = \text{pt}</em>-(K_n) = 1 ).</td>
</tr>
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<td>▶ For ( n \geq 1 ), ( \text{pt}(\overline{K_n}) = \text{pt}<em>+(\overline{K_n}) = \text{pt}</em>-(\overline{K_n}) = 0 ).</td>
</tr>
<tr>
<td>▶ ( \text{pt}(K_{1,n-1}) = 2 ) and for ( 2 \leq r \leq n-2 ), ( \text{pt}(K_{r,n-r}) = 1 ). For ( 1 \leq r \leq n-1 ), ( \text{pt}<em>+(K</em>{r,n-r}) = \text{pt}<em>-(K</em>{r,n-r}) = 1 ).</td>
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<td>▶ For ( n \geq 2 ), ( \text{pt}(P_n) = n-1 ) and ( \text{pt}<em>+(P_n) = \left\lceil \frac{n-1}{2} \right\rceil ). ( \text{pt}</em>-(P_n) = \frac{n}{2} ) for even ( n ) and ( \text{pt}_-(P_n) = \left\lfloor \frac{n+1}{4} \right\rfloor ) for odd ( n ).</td>
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<tr>
<td>▶ For ( n \geq 3 ), ( \text{pt}(C_n) = \left\lceil \frac{n-2}{2} \right\rceil ) and ( \text{pt}_+(C_n) = \left\lceil \frac{n-2}{4} \right\rceil ).</td>
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<td>( \text{pt}_-(C_n) = \begin{cases} \frac{n-1}{2} &amp; \text{if } n \text{ is odd} \ \frac{n}{4} &amp; \text{if } n \equiv 0 \mod 4 \ \frac{n-2}{4} &amp; \text{if } n \equiv 2 \mod 4 \end{cases} ).</td>
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</tbody>
</table>
Throttling involves minimizing the sum of the number of resources used to accomplish a task (e.g., blue vertices) and the time needed to accomplish the task (e.g., propagation time).

Unlike propagation time of a graph, which starts with a minimum set of blue vertices, throttling often uses more blue vertices to reduce time.

Let $R$ be a color change rule.

- The $R$-propagation time for a set $B = B^0$ of vertices, $\text{pt}_R(G, B)$, is the smallest $t$ such that $B^t = V(G)$ using the $R$ color change rule (and is infinity if this never happens).

- The $R$-throttling number of a set $B$ of vertices, $\text{th}_R(G; B) = |B| + \text{pt}_R(G, B)$, is the sum of the number vertices in $B$ and the $R$-propagation of $B$.

- The $R$-throttling number of $G$ is $\text{th}_R(G) = \min_{B \subseteq V(G)} \text{th}_R(G; B) = \min_{B \subseteq V(G)} (|B| + \text{pt}_R(G, B))$. 

(Standard) throttling

- The propagation time for a set $B = B^{[0]}$ of vertices, $pt(G, B)$, is the smallest $t$ such that $B^{[t]} = V(G)$ using the (standard) zero forcing color change rule.

- The throttling number of $G$ for zero forcing is $th(G) = \min_{B \subseteq V(G)}(|B| + pt(G, B))$.

Example ($Z(T) = 4$, $pt(T) = 3$, $th(T) = 7$)
The PSD propagation time for a set $B = B^{[0]}$ of vertices, $pt_+(G, B)$, is the smallest $t$ such that $B^{[t]} = V(G)$ using the PSD color change rule.

The PSD throttling number of $G$ for zero forcing is the $th_+(G) = \min_{B \subseteq V(G)} (|B| + pt_+(G, B))$.

**Example**

$Z_+(T) = 1$ and $pt_+(T) = 4$. Using a PSD zero forcing set $B$ of 2 vertices, $pt_+(G, B) = 2$ and $th_+(T) = 2 + 2 = 4$. 
The skew propagation time for a set \( B = B^{[0]} \) of vertices, \( pt_{\_}(G, B) \), is the smallest \( t \) such that \( B^{[t]} = V(G) \) using the skew forcing color change rule.

The skew throttling number of \( G \) for zero forcing is
\[
\text{th}_{\_}(G) = \min_{B \subseteq V(G)} (|B| + pt_{\_}(G, B)).
\]

Example \((Z_{\_}(T) = 2, pt_{\_}(T) = 2, th_{\_}(T) = 4)\)
Throttling numbers of families of graphs

Theorem (BY 2013, CHKLR SVM 2019, CGH 2020)

1. For \( n \geq 1 \), \( \text{th}(K_n) = \text{th}_+(K_n) = n \). For \( n \geq 2 \), \( \text{th}_-(K_n) = n - 1 \).
2. For \( n \geq 1 \), \( \text{th}(\overline{K_n}) = \text{th}_+(\overline{K_n}) = \text{th}_-(\overline{K_n}) = n \).
3. \( \text{pt}(K_{1,n-1}) = 2 \) and for \( 2 \leq r \leq n - 2 \), \( \text{pt}(K_{r,n-r}) = 1 \). For \( 1 \leq r \leq n - 1 \), \( \text{pt}_+(K_{r,n-r}) = \text{pt}_-(K_{r,n-r}) = 1 \).
4. For \( n \geq 2 \), \( \text{th}(P_n) = \lceil 2\sqrt{n} - 1 \rceil \), \( \text{th}_+(P_n) = \lceil \sqrt{2n} - \frac{1}{2} \rceil \), and \( \text{th}_-(P_n) = \lceil \sqrt{2(n+1)} - \frac{3}{2} \rceil \).
5. For \( n \geq 3 \), \( \text{th}(C_n) = \begin{cases} \lceil 2\sqrt{n} - 1 \rceil & \text{unless } n = (2k+1)^2 \\ 2\sqrt{n} & \text{if } n = (2k+1)^2 \end{cases} \).
   \( \text{th}_+(C_n) = \lceil \sqrt{2n} - \frac{1}{2} \rceil \).
   \( \text{th}_-(C_n) = \lceil \sqrt{2n} - \frac{1}{2} \rceil \).
Relationships: standard, PSD, and skew throttling

Observation

Let $B \subseteq V(G)$ be a zero forcing set. Then,

- $B$ is a PSD forcing set and a skew forcing set.
- $Z_+(G) \leq Z(G)$ and $Z_-(G) \leq Z(G)$
- $pt_+(G, B) \leq pt(G, B)$ and $pt_-(G, B) \leq pt(G, B)$
- $th_+(G; B) \leq th(G; B)$ and $th_-(G; B) \leq th(G; B)$.
- $th_+(G) \leq th(G)$ and $th_-(G) \leq th(G)$.
- $th_+(G)$ and $th_-(G)$ are noncomparable.
- $pt_+(G)$, $pt_-(G)$, and $pt(G)$ are noncomparable (minimum values can differ).
Lower bound on $\text{th}(G)$

Theorem (Butler, Young, 2013)

Let $G$ be a graph of order $n$. Then

$$\text{th}(G) \geq \lceil 2\sqrt{n} - 1 \rceil$$

and this bound is tight.

PSD and skew are very different

- $\text{th}_+(K_{1,n-1}) = 2$.

- For any $G$ with a component of order $\geq 2$,

$$Z_-(G \circ K_1) = 0, \text{pt}_-(G \circ K_1) = 2, \text{th}_-(G \circ K_1) = 2.$$
Extreme values for th($G$)

\[
\left\lceil 2\sqrt{n} - 1 \right\rceil \leq \text{th}(G) \text{ implies the number of graphs having th}(G) = k \text{ is finite.}
\]

Remark

All the graphs having th($G$) \leq 3 are listed below.

1) th($G$) = 1 if and only if $|V(G)| = 1$.
2) th($G$) = 2 if and only if $|V(G)| = 2$.
3) th($G$) = 3 if and only if $|V(G)| = 3$ or $G = 2K_2$, $P_4$, or $C_4$.

Theorem (CK 2020+)

Let $G$ be a graph of order $n$. The following are equivalent:

1) th($G$) = $n$.
2) $G$ is a threshold graph.
3) $G$ does not have $P_4$, $C_4$, or $2K_2$ as an induced subgraph.
Theorem (CHKLRSVM 2019)

Let $G$ be a connected graph of order $n$.

1) $\text{th}_+(G) = n$ if and only if $G = K_n$.

2) $\text{th}_+(G) = n - 1$ if and only if $\alpha(G) = 2$ and $G$ does not have an induced $C_5$, house, or double diamond subgraph.

$C_5$  

house  

double diamond
Extreme values for $\text{th}_+(G)$

**Theorem (CHKLRSVM 2019)**

Let $G$ be a graph of order $n$.

1) $\text{th}_+(G) = 1$ if and only if $n = 1$.

2) $\text{th}_+(G) = 2$ if and only if $G = K_{1,n-1}$ or $G = 2K_1$.

3) For a graph $G$, $\text{th}_+(G) = 3$ if and only if at least one of the following is true:

   3.1 $G$ is disconnected and exactly of the following holds:
      
      3.1.1 $G$ is $3K_1$, or
      3.1.2 $G$ has two components, each component is a copy of $K_{1,n-1}$ or $K_1$, and at least one component has order greater than one.

   3.2 $G$ is a tree with diameter three or four, or

   3.3 $G$ is connected and there exist $v, u \in V(G)$ such that:
      
      3.3.1 $G$ has a cycle, or $G$ is a tree with diam $G = 5$,
      3.3.2 $N(u) \cup N(v) = V(G)$,
      3.3.3 $\deg(w) \leq 2$ for all $w \notin \{v, u\}$, and
      3.3.4 if $w_1, w_2 \in N(u)$ or $w_1, w_2 \in N(v)$, then $w_1$ is not adjacent to $w_2$. 

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Extreme values for $\text{th}_-(G)$

**Theorem (CGH 2020)**

Let $G$ be a graph of order $n$.

1) $\text{th}_-(G) = 1$ if and only if $G = K_1$ or $G = rK_2$ for $r \geq 1$.

2) A graph $G$ has $\text{th}_-(G) = 2$ if and only if $G$ is one of $2K_1$, $H(s, t) \sqcup rK_2$ with $r + s + t \geq 1$, or $(\tilde{G} \circ K_1) \sqcup rK_2$ where each component of $\tilde{G}$ has an edge.

3) $\text{th}_-(G) = n$ if and only if $G = nK_1$.

4) $\text{th}_-(G) = n - 1$ if and only if $G$ is a cograph, does not have an induced $2K_2$, and has at least one edge.

The graph $H(2, 3)$
Computation

There is **Sage** software that computes

- $Z(G)$, $Z_+(G)$, $Z_-(G)$,
- $pt(G)$, $pt_+(G)$, $pt_-(G)$,
- $th(G)$, $th_+(G)$, $th_-(G)$

for “small” graphs.


References


