

## QUICKEST DETECTION IN COUPLED SYSTEMS\*

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**Abstract.** This work considers the problem of quickest detection of signals in a coupled system of  $N$  sensors, which receive continuous sequential observations from the environment. It is assumed that the signals, which are modeled by general Itô processes, are coupled across sensors, but that their onset times may differ from sensor to sensor. Two main cases are considered: in the first one signal strengths are the same across sensors while in the second one they differ by a constant. The objective is the optimal detection of the first time at which any sensor in the system receives a signal. The problem is formulated as a stochastic optimization problem in which an extended minimal Kullback–Leibler divergence criterion is used as a measure of detection delay, with a constraint on the mean time to the first false alarm. The case in which the sensors employ cumulative sum (CUSUM) strategies is considered, and it is proved that the minimum of  $N$  CUSUMs is asymptotically optimal as the mean time to the first false alarm increases without bound. In particular, in the case of equal signal strengths across sensors, it is seen that the difference in detection delay of the  $N$ -CUSUM stopping rule and the unknown optimal stopping scheme tends to a constant related to the number of sensors as the mean time to the first false alarm increases without bound. Alternatively, in the case of unequal signal strengths, it is seen that this difference tends to zero.

**Key words.** Kullback–Leibler divergence, CUSUM, quickest detection

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**1. Introduction.** We are interested in the problem of quickest detection of the onset of a signal in a system of  $N$  sensors. We consider the situation in which, although the observations in one sensor can affect the observations in another, the onset of a signal can occur at different times (i.e., change points) in each of the  $N$  sensors; that is, the change points differ from sensor to sensor. As an example in which this situation arises consider a system of sensors monitoring the health of a physical structure in which fault conditions are manifested by vibrations in the structure. Before a change affects a given sensor, we have only noise in that sensor. Then, after a change, the system is vibrating and thus the signal received in any location reflects a vibrating system. Thus, observations at any given sensor are coupled with those

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received in other locations. The change points observed at different sensors can occur at different times because the source of the vibrations (i.e., the excitation) may arrive at different structural elements at different times. Relevant literature related to such models includes, for example, [1, 2, 3, 7, 14, 15, 24].

We assume that, after rescaling by a constant factor, the probability law of the observations is the same across sensors. This assumption, although seemingly restrictive, is realistic in view of the fact that the system of sensors is coupled. The constant then reflects the fact that signal strength can differ in different locations in the system of sensors. We model the signals through continuous-time Itô processes. The advantage of such models is the fact that they can capture complex dependencies in the observations. For example, an autoregressive process is a special case of the discrete-time equivalent of an Ornstein–Uhlenbeck process which, in turn, is a special case of an Itô process. Other special cases of this model include Markovian models and linear state-space systems commonly used in vibration-based structural analysis and health monitoring problems [1, 2, 3, 7, 14, 15, 24]. It is important to stress that the fact that the system of  $N$  sensors is coupled makes the probabilistic treatment of the problem equivalent to the one in which all observations become available in one location. The reason is that one integrated information flow is sufficient for describing such a system.

Our objective is to detect the first onset of a signal in such a system. In other words, we wish to detect the first change point in the system of sensors described above. Our problem is thus concerned with online change detection. We assume that the change points are unknown constants and adopt a min-max approach to the detection of the minimal one. We set up the problem as a stochastic optimization problem in which the objective is to minimize an extended Kullback–Leibler distance which serves as a detection delay subject to a lower bound in the mean time to the first false alarm. So far, of the min-max-type of a setup only the case of an uncoupled system of observations has been considered and only in the case that the signals across sensors are modeled as independent Brownian motions with constant drifts [13].<sup>1</sup> Our problem is also similar to the one discussed in [5] which is concerned with the detection of the minimum of two Poisson disorder times, but in which a Bayesian approach to modeling the change points is taken. Another interesting filtering approach to change-point detection is developed in a Bayesian framework in [30]. Recently the case was also considered of change points that propagate in a sensor array [26]. However, in this configuration the propagation of the change points depends on the *unknown* identity of the first sensor affected and considers a restricted Markovian mechanism of propagation of the change. For a summary of the latest work on the topic of quickest detection, please consult [25].

In this paper we consider the case in which the change points can be different and do not propagate in any specific configuration. The objective is to detect the minimum (i.e., the first) of the change points. We demonstrate that, in the situation described above, at least asymptotically, the minimum of  $N$  cumulative sums (CUSUMs) [22, 29], is asymptotically optimal in detecting the minimum of the  $N$  different change points, as the mean time to the first false alarm tends to infinity, with respect to an appropriately extended Kullback–Leibler divergence criterion [21] that incorporates the possibility of  $N$  different change points. The results in this paper generalize those of [13] in that the minimum of  $N$  CUSUMs is asymptotically optimal not only in the case in which the observations across sensors are independent but also in the case in

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<sup>1</sup>The situation with correlated Brownian motions is studied in [31, 32].

which they are coupled. In fact in this paper we not only generalize the asymptotic optimality of the  $N$ -CUSUM stopping rule to a coupled system of observations but also to more general Itô dynamics rather than the restricted Brownian motion model of observations.

In the next section we formulate the problem and identify the quantities of interest. In section 3 we analyze the properties of the proposed  $N$ -CUSUM stopping rule. In section 4, we provide asymptotic expansions of the quantities of interest as the threshold parameters of the  $N$ -CUSUM stopping rule tend to infinity. In section 5, we demonstrate that the minimum of  $N$  CUSUM stopping rules is asymptotically optimal in an extended Kullback–Leibler sense both in the case of equal and in the case of unequal signal strengths. The proofs of lemmas and propositions that are omitted can be found in the appendices. Throughout the paper, we denote by  $\mathbb{R} = (-\infty, \infty)$ ,  $\mathbb{R}_+ = [0, \infty)$  and  $\overline{\mathbb{R}}_+ = [0, \infty]$ .

**2. Mathematical formulation.** We sequentially observe the processes  $\{Z_t^{(i)}\}_{t \geq 0}$  for all  $i \in \{1, \dots, N\}$ . In order to formalize this problem we consider filtered probability space  $(\Omega, \mathbb{F}, \mathbb{P})$  with filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$  and  $\mathcal{F}_t = \sigma\{(Z_s^{(1)}, \dots, Z_s^{(N)}); s \leq t\}$ .

The processes  $\{Z_t^{(i)}\}_{t \geq 0}$  for all  $i \in \{1, \dots, N\}$  are Itô processes satisfying

$$(2.1) \quad dZ_t^{(i)} = \begin{cases} dw_t^{(i)}, & t \leq \tau_i, \\ \alpha_t^{(i)} dt + dw_t^{(i)}, & t > \tau_i, \end{cases}$$

where the  $\{w_t^{(i)}\}_{t \geq 0}$ 's are independent standard Brownian motions, and  $\tau_i s \in \overline{\mathbb{R}}_+$  denote the times of change points, which are assumed to be deterministic constants but otherwise unknown. Throughout, we will denote the minimum of the  $\tau_i$ 's by<sup>2</sup>

$$\tilde{\tau} := \min_{1 \leq i \leq N} \{\tau_i\}.$$

Finally, we assume that  $\alpha_t^{(i)} = \alpha_i(t; Z^{(1)}, \dots, Z^{(N)})$  for some predictable real-valued function  $\alpha_i$  on  $\mathbb{R}_+ \times (C(\mathbb{R}_+))^N$ , satisfying the Lipschitz condition and linear growth condition (see, for example, Theorem 2.9 on page 289 of [16] or Theorem 2.1 on page 375 of [28]).

The case considered in this paper is that in which there exist constants  $\{c_i\}_{1 \leq i \leq N}$  such that the drift processes after the  $\tau_i$ 's are proportional to one another. More specifically, without loss of generality, we assume the following.

*Assumption 1.* There exist  $1 = c_1 \leq |c_2| \leq \dots \leq |c_N|$  such that the processes  $|\alpha_t^{(1)}|, |c_2 \alpha_t^{(2)}|, \dots, |c_N \alpha_t^{(N)}|$  are modifications of one another under  $\mathbb{P}$ .

In particular, we examine two cases that are treated separately in the following; the first one corresponds to signal-strength symmetry across sensors and the second one corresponds to across sensors. The former is described mathematically by assuming  $|c_i| = 1$  for all  $i \in \{1, \dots, N\}$  and the latter, can be captured mathematically by assuming  $\max_{2 \leq i \leq N} |c_i| > 1$ .

Although Assumption 1 is mathematically necessary for the main results of this paper, in most applications the nature of the  $\alpha_t^{(i)}$ 's is the same across  $i$ . In fact, in most applications the signal observed across all sensors  $i$  is the same but can be felt at different strengths (the constants  $c_i$  represent the different strengths) in different sensors. Our formulation treats the general case in which the signal can arrive at different points in time in different sensors.

<sup>2</sup>As a convention, if  $\tau_i = \infty$  for all  $i \in \{1, \dots, N\}$ , we let  $\tilde{\tau} = \infty$ .

In what follows we describe an example of two sensors each meant to capture two coupled waves that are emitted as one seismic signal after the onset of an earthquake, namely, the S wave and the T-wave (see, for instance, the discussion following page 392 in [4]). These waves are not only buried in noise but have the tendency to interact with each other making the detection of their onset particularly difficult and giving obvious rise to a coupled system. These waves in practice may actually have different onsets as discussed in [4]. Let us now represent by  $Z_t^{(1)}$  and  $Z_t^{(2)}$  the observations received in the sensors meant to detect the onset of the earthquake. Before the onset of the earthquake both observations are simply noise. After the onset of the earthquake a signal will be emitted but may be received at different points in time in the different sensors. The models used to capture signals are often autoregressive [4] whose continuous-time equivalents are Ornstein–Uhlenbeck processes. Thus we need to allow for  $\alpha_t^{(i)}$  to depend on  $Z_t^{(i)}$ . Of course more general models would be necessary in order to capture dependence on past observations as well. This is the reason that in our model (2.1), the  $\alpha_t^{(i)}$  are generalized to depend on the whole of the path of the observations  $Z_t^{(i)}$  up to time  $t$ . Moreover, a stronger intensity in one of the two waves is known to result in a stronger intensity in the other [4]. This could then be modeled by allowing both  $\alpha_t^{(1)}$  and  $\alpha_t^{(2)}$  to have a positive dependence on  $Z_t^{(2)}$  and  $Z_t^{(1)}$ , respectively (note that this dependence does not have to be linear and could in general cases depend on prior observations too—not just the ones at the current time). Even though the two waves are different, they do give rise to the same signal which may be felt with a different strength in each of the sensors and thus the restriction  $c_2\alpha_t^{(2)} = \alpha_t^{(1)}$  with  $c_2 > 0$ , although it may appear restrictive, is in fact not far from modeling reality. We notice of course that in this example we do not have to impose the absolute value condition as both  $\alpha_t^{(1)}$  and  $\alpha_t^{(2)}$  will have the same sign.

Another application is that of a vibrating mechanical system where each sensor may be monitoring a different location on the structure. The signal causing the vibration can be felt at different times in different locations and also with different strengths  $c_i$ . Let  $Z_t^{(i)}$  represent the observations at sensor  $i$ . At first there is no signal and the sensors only observe noise. After the onset of the signal, which may be felt at different times in different sensors, a signal arrives. A signal causing a vibration in one location may also affect the observations received in another. This leads to feedback [1, 2, 3], which can then be modeled by allowing  $\alpha_t^{(i)}$  to depend on the observations received at all sensors. Also the signal may be more pronounced in one sensor than in another depending on the location of the sensor. This can be captured by allowing the  $c_i$ 's to be different across sensors. In fact, by allowing the  $c_i$ 's to have different signs we are able to capture the situation in which the sensors are hit by the signal from opposite directions.

The filtration  $\mathbb{F}$  is the filtration generated by the observations received by all sensors. Thus by requiring that  $\alpha_t^{(i)}$  be  $\mathcal{F}_t$ -measurable for all  $i$ , we have managed to capture the coupled nature of the system. In particular, in the special case in which, say,  $\alpha_t^{(1)} = -r \sum_{i=1}^N Z_t^{(i)}$ , (2.1) describes a process that displays an autoregressive (or its continuous equivalent [23]) behavior in  $\{Z_t^{(1)}\}_{t \geq 0}$ , while still being coupled with the observations received by the other sensors. More specifically, the magnitude of each increment of the process  $\{Z_t^{(1)}\}_{t \geq 0}$  at each instant  $t$  is not only affected by  $Z_t^{(1)}$  but also by  $Z_t^{(i)}$ ,  $i = 2, \dots, N$ , the observations at sensors  $2, \dots, N$ . This couples the observations received in sensor 1 with those received in sensors  $2, \dots, N$  at each instant  $t$  and results in a system of interdependent sensors. We notice that the special

case described above can also be written in the form of a linear state-space model as follows:

$$(2.2) \quad dZ_t^{(i)} = -r\mathbf{1}_{\{t>\tau_i\}} \left( \sum_{j=1}^N Z_t^{(j)} \right) dt + dw_t^{(i)}, \quad i \in \{1, \dots, N\}.$$

Autoregressive models and, more generally, linear state-space models have been used to capture seismic signals, navigation systems, vibrating mechanical systems, etc. [4]. The generality of (2.1) however is much greater than the special case described above. This is seen in the fact that  $\alpha_t^{(i)}$  at each instant  $t$  can depend on the totality of the observed paths of each of the signals received up to time  $t$ . Notice that this linear model clearly satisfies both the Lipschitz condition and the linear growth condition. To incorporate the different signal strength in each of the sensor observations, we could update the present model in (2.2) by having different constants in each row of the second  $N \times N$  matrix appearing in the equation of the linear system above.

To formulate our problem mathematically, we will work with the canonical space  $(C(\mathbb{R}_+))^N$  endowed with the filtration  $\mathbb{B} = \{\mathcal{B}_t\}_{t \geq 0}$ , where  $\mathcal{B}_t = \sigma(\omega(s); s \leq t)$  is the Borel  $\sigma$ -algebra generated by the the joint coordinate mapping  $\omega = (\omega_1, \dots, \omega_N)$  for  $Z_s^{(i)}(\omega) = \omega_i(s)$ ,  $s \geq 0$ ,  $i \in \{1, \dots, N\}$ .

We introduce a family of probability measures on  $((C(\mathbb{R}_+))^N, \mathbb{B})$ :  $\{P_{\mathbf{t}}\}_{\mathbf{t} \in (\overline{\mathbb{R}_+})^N}$ , where  $P_{\mathbf{t}}$  for a fixed  $\mathbf{t} = (\tau_1, \dots, \tau_N) \in (\overline{\mathbb{R}_+})^N$  corresponds to the measure generated on  $(C(\mathbb{R}_+))^N$  by the  $N$ -dimensional process  $\{(Z_t^{(1)}, \dots, Z_t^{(N)})\}_{t \geq 0}$  when the change in  $Z^{(i)}$  occurs at time point  $\tau_i$ , for all  $i \in \{1, \dots, N\}$ . In particular, the case that  $\tau_i = \infty$  for all  $i \in \{1, \dots, N\}$  corresponds to the no change regime, where  $(Z_t^{(1)}, \dots, Z_t^{(N)})_{t \geq 0}$  is an  $N$ -dimensional Brownian motion; and the measure generated on  $(C(\mathbb{R}_+))^N$  is the  $N$ -dimensional Wiener measure, which we denote by  $P_{\mathbf{t}_\infty}$  for  $\mathbf{t}_\infty := (\infty, \dots, \infty)$ . Now we can restate Assumption 1 as the following.

*Assumption 1'.* There exist  $1 = c_1 \leq |c_2| \leq \dots \leq |c_N|$  such that the processes  $|\alpha_1(t; \omega)|, |c_2 \alpha_2(t; \omega)|, \dots, |c_N \alpha_N(t; \omega)|$  are modifications of one another under  $P_{\mathbf{t}}$  for all  $\mathbf{t} \in (\overline{\mathbb{R}_+})^N$ .

We will also assume the drift functions  $\alpha_i(\cdot; \cdot)$  satisfy the Novikov condition.

*Assumption 2.* For all  $t \in \mathbb{R}_+$ , we have

$$(2.3) \quad E_{\mathbf{t}_\infty} \left\{ \exp \left( \frac{1}{2} \int_0^t \sum_{i=1}^N (\alpha_i(s; \omega))^2 ds \right) \right\} < \infty.$$

By Theorem 1.10 on page 371 of [28], we know that, if the change points satisfy  $0 \leq \tau_1 < \tau_2 < \min_{2 < i \leq N} \{\tau_i\}$ , we have

$$\begin{aligned} \frac{dP_{(\tau_1, \tau_2, \dots, \tau_N)}}{dP_{(\infty, \infty, \dots, \infty)}} \Big|_{\mathcal{B}_t} &= \frac{dP_{(\tau_1, \infty, \dots, \infty)}}{dP_{(\infty, \infty, \dots, \infty)}} \Big|_{\mathcal{B}_t} \\ &= \exp(u_1(t) - u_1(t \wedge \tau_1)) \quad \forall t \in [0, \tau_2], \text{ and} \\ \frac{dP_{(\tau_1, \tau_2, \tau_3, \dots, \tau_N)}}{dP_{(\tau_1, \infty, \infty, \dots, \infty)}} \Big|_{\mathcal{B}_t} &= \frac{dP_{(\tau_1, \tau_2, \infty, \dots, \infty)}}{dP_{(\tau_1, \infty, \infty, \dots, \infty)}} \Big|_{\mathcal{B}_t} \\ &= \exp(u_2(t) - u_2(\tau_2)) \quad \forall t \in \left[ \tau_2, \min_{2 < i \leq N} \{\tau_i\} \right], \end{aligned}$$

and so on, where, for all  $i \in \{1, \dots, N\}$ ,

$$(2.4) \quad u_i(t) := \int_0^t \alpha_i(s; \omega) d\omega_i(s) - \int_0^t \frac{1}{2} (\alpha_i(s; \omega))^2 ds.$$

In general, for all  $\mathbf{t} = (\tau_1, \dots, \tau_N) \in (\overline{\mathbb{R}}_+)^N$  we have that

$$(2.5) \quad \frac{dP_{\mathbf{t}}}{dP_{\mathbf{t}_\infty}} \Big|_{\mathcal{B}_t} = \exp \left( \sum_{i=1}^N (u_i(t) - u_i(t \wedge \tau_i)) \right) \quad \forall t \in \mathbb{R}_+.$$

Our objective is to find a  $\mathbb{B}$ -stopping rule  $T$  that balances the trade-off between a small detection delay subject to a lower bound on the mean time to the first false alarm and will ultimately detect  $\tilde{\tau}$ .

To this effect, we propose a generalization of the  $J_{KL}$  of [21], namely,

$$(2.6) \quad J_{KL}^{(N)}(T) := \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\overline{\mathbb{R}}_+)^N \\ \tilde{\tau} < \infty}} \operatorname{esssup}_{\tilde{\tau}} E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tilde{\tau}\}} \int_{\tilde{\tau}}^T \frac{1}{2} (\alpha_1(s; \omega))^2 ds \Big| \mathcal{B}_{\tilde{\tau}} \right\},$$

where the supremum over  $\mathbf{t} = (\tau_1, \dots, \tau_N)$  is taken over the set in which  $\tilde{\tau} < \infty$ . That is, we consider the worst detection delay over all possible realizations of paths of the  $N$ -tuple of stochastic processes  $\{(Z_t^{(1)}, \dots, Z_t^{(N)})\}_{t \geq 0}$  up to  $\tilde{\tau}$  and then consider the worst detection delay over all possible  $N$ -tuples  $\mathbf{t} = (\tau_1, \dots, \tau_N)$  over a set in which at least one of them is forced to take a finite value. This is because  $T$  is a stopping rule meant to detect the minimum of the  $N$  change points and therefore if one of the  $N$  processes undergoes a regime change, any unit of time by which  $T$  delays in reacting, should be counted towards the detection delay. This gives rise to the following stochastic optimization problem:

$$(2.7) \quad \inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T) \quad \text{with } \mathbb{B}_\gamma = \left\{ \mathbb{B}\text{-stopping rule } T : E_{\mathbf{t}_\infty} \left\{ \int_0^T \frac{1}{2} (c_N^{-1} \alpha_1(s; \omega))^2 ds \right\} \geq \gamma \right\}.$$

The constraint in the above optimization problem is a measure on the first time to the first false alarm. To see this, one can recognize the functional  $\int_0^t \frac{1}{2} (\alpha_s^{(i)})^2 ds$  as a measure of accumulation of energy of the signal [18] up to time  $t$ . The above constraint involves the expected value of this functional under the regime of no signal up to the time of stopping and is thus a measure of a mean time to the first false alarm [18, 21, 25]. Similarly, the criterion (2.6) captures the accumulation of energy of the signal after the first instance at which a signal becomes available. It is possible to motivate this criterion through a Kullback–Leibler distance. To be more specific, for any bounded stopping rule  $T$  such that

$$c_i^2 \int_0^T \alpha_i(s; \omega)^2 ds = \int_0^T \alpha_1(s; \omega)^2 ds < \infty, \quad P_{\mathbf{t}\text{-a.s.}}, \quad \forall \mathbf{t} = (\tau_1, \dots, \tau_N) \in (\overline{\mathbb{R}}_+)^N,$$

we can apply (2.5) to obtain that

$$\begin{aligned} & E_{\mathbf{t}} \left\{ \log \frac{dP_{\mathbf{t}}}{dP_{\mathbf{t}_\infty}} \Big|_{\mathcal{B}_T} \Big| \mathcal{B}_{\bar{\tau}} \right\} \\ &= E_{\mathbf{t}} \left\{ \sum_{i=1}^N (u_i(T) - u_i(T \wedge \tau_i)) \Big| \mathcal{B}_{\bar{\tau}} \right\} \\ &= \sum_{i=1}^N E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_i\}} \left( \int_{\tau_i}^T \alpha_i(s; \omega) d\omega_i(s) - \int_{\tau_i}^T \frac{1}{2} (\alpha_i(s; \omega))^2 ds \right) \Big| \mathcal{B}_{\bar{\tau}} \right\} \\ &= \sum_{i=1}^N E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_i\}} \int_{\tau_i}^T \frac{1}{2} (\alpha_i(s; \omega))^2 ds \Big| \mathcal{B}_{\bar{\tau}} \right\}, \end{aligned}$$

giving rise to the criterion

$$\mathcal{J}(T) := \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\bar{\mathbf{R}}_+)^N \\ \bar{\tau} < \infty}} \text{essup} \sum_{i=1}^N E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_i\}} \int_{\tau_i}^T \frac{1}{2} (\alpha_i(s; \omega))^2 ds \Big| \mathcal{B}_{\bar{\tau}} \right\},$$

which can be upper bounded by the criterion in (2.6) as follows:

(2.8)

$$\begin{aligned} \mathcal{J}(T) &\leq \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\bar{\mathbf{R}}_+)^N \\ \bar{\tau} < \infty}} \sum_{i=1}^N \text{essup} E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_i\}} \int_{\tau_i}^T \frac{1}{2} (\alpha_i(s; \omega))^2 ds \Big| \mathcal{B}_{\bar{\tau}} \right\} \\ &= \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\bar{\mathbf{R}}_+)^N \\ \bar{\tau} < \infty}} \sum_{i=1}^N \text{essup} E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_i\}} (c_i)^{-2} \int_{\tau_i}^T \frac{1}{2} (\alpha_1(s; \omega))^2 ds \Big| \mathcal{B}_{\bar{\tau}} \right\} \\ &\leq N \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\bar{\mathbf{R}}_+)^N \\ \bar{\tau} < \infty}} \text{essup} E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_i\}} \int_{\tau_i}^T \frac{1}{2} (\alpha_1(s; \omega))^2 ds \Big| \mathcal{B}_{\bar{\tau}} \right\} \\ &\leq N J_{KL}^{(N)}(T). \end{aligned}$$

Similarly, for the false alarms we notice that

$$\begin{aligned} (2.9) \quad E_{\mathbf{t}_\infty} \left\{ \sum_{i=1}^N \int_0^T \frac{1}{2} (\alpha_i(s; \omega))^2 ds \right\} &\geq N \min_{1 \leq i \leq N} E_{\mathbf{t}_\infty} \left\{ \int_0^T \frac{1}{2} (\alpha_i(s; \omega))^2 ds \right\} \\ &= N E_{\mathbf{t}_\infty} \left\{ \int_0^T \frac{1}{2} (c_N^{-1} \alpha_1(s; \omega))^2 ds \right\}. \end{aligned}$$

Dividing the upper bound in (2.8) and the lower bound in (2.9) by the number of sensors  $N$  results in the optimization problem given in (2.7).

Inspired by the optimality of equalizer rules as seen in [13, 31], we will examine the performance under (2.6), of an (almost) equalizer rule  $T$  which reacts at exactly the same time regardless of which change takes place first. To be more specific, let us define

$$(2.10) \quad J_j^{(N)}(T) := \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\bar{\mathbf{R}}_+)^N \\ \tau_j = \bar{\tau} < \infty}} \text{essup} E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T > \tau_j\}} \int_{\tau_j}^T \frac{1}{2} (\alpha_1(s; \omega))^2 ds \Big| \mathcal{B}_{\tau_j} \right\}$$

for all  $j \in \{1, \dots, N\}$ . That is,  $J_j^{(N)}(T)$  is the detection delay of the stopping rule  $T$  when  $\tau_j = \tilde{\tau}$  is the first change point. Then for all stopping rules  $T \in \mathbb{B}_\gamma$ ,

$$(2.11) \quad J_{KL}^{(N)}(T) = \max \left\{ J_1^{(N)}(T), J_2^{(N)}(T), \dots, J_N^{(N)}(T) \right\}.$$

We will restrict our attention to stopping rules  $T \in \mathbb{B}_\gamma$  that achieve the false alarm constraint with equality [20, 21]:

$$(2.12) \quad E_{t_\infty} \left\{ \int_0^T \frac{1}{2} (c_N^{-1} \alpha_1(s; \omega))^2 ds \right\} = \gamma,$$

and the “equalizer rule” condition

$$(2.13) \quad J_j^{(N)}(T) = J_1^{(N)}(T) + o(1) \quad \forall j \in \{1, 2, \dots, N\},$$

as  $\gamma \rightarrow \infty$ . Clearly, from (2.11) we know that, for such stopping rules, we may simplify the performance index  $J_{KL}^{(N)}(T)$  as

$$J_{KL}^{(N)}(T) = J_1^{(N)}(T) + o(1),$$

as  $\gamma \rightarrow \infty$ . While, given the complexity due to the dimensionality of the problem, it seems a formidable task to determine the exactly optimal stopping rule for (2.7), asymptotic expansions like (2.13) will make it possible to find an asymptotically optimal stopping rule for our problem (2.7).

To construct such a candidate stopping for problem (2.7), recall that in the case of  $N = 1$ , the drift,  $\alpha_1(t; \omega_1)$ , is measurable with respect to the filtration generated by the coordinate mapping  $Z_t^{(1)}(\omega) = \omega_1(t)$ ,  $t \geq 0$ , which we denote by  $\mathbb{G}^{(1)} = \{\mathcal{G}_t^{(1)}\}_{t \geq 0}$ . Then it is known from [21] that the problem in (2.7), when  $N = 1$ , is solved by the so-called CUSUM stopping rule:

$$(2.14) \quad S_\nu^{(1)} = \inf\{t \geq 0 : y(t) = \nu\} \quad \text{with} \quad y(t) = \sup_{0 \leq s \leq t} \log \frac{dP_s}{dP_\infty} \Big|_{\mathcal{G}_t^{(1)}},$$

where  $P_s$  is the measures generated by  $\{Z_t^{(1)}\}_{t \geq 0}$  on  $C(\mathbb{R}_+)$  when the change point  $\tau_1 = s$ , and  $P_\infty$  is the Wiener measure on  $C(\mathbb{R}_+)$ . The threshold  $\nu > 0$  in (2.14) is chosen so that  $E_\infty \left\{ \int_0^{S_\nu^{(1)}} \frac{1}{2} (\alpha_1(s; \omega_1))^2 ds \right\} = \gamma$ . Furthermore, with

$$(2.15) \quad g(\nu) := e^\nu - \nu - 1,$$

the threshold  $\nu$  is the unique positive solution to  $g(\nu) = \gamma$ , and the optimal detection delay is given by

$$J_{KL}^{(1)}(S_\nu^{(1)}) = E_0 \left\{ \int_0^{S_\nu^{(1)}} \frac{1}{2} (\alpha_1(s; \omega_1))^2 ds \right\} = g(-\nu).$$

In the proof in [21] that proves the optimality of the CUSUM stopping rule (2.14) to the one-dimensional equivalent of (2.7), one vital assumption is necessary for the finiteness (see Theorem 1 of [21]) of the CUSUM stopping rule (2.14) and the martingale properties of stochastic integrals, i.e., for  $\tau_1 \in \{0, \infty\}$ , and any  $t \in \mathbb{R}_+$ ,

$$P_{\tau_1} \left( \int_0^\infty \frac{1}{2} (\alpha_1(s; \omega_1))^2 ds = \infty \right) = P_{\tau_1} \left( \int_0^t \frac{1}{2} (\alpha_1(s; \omega_1))^2 ds < \infty \right) = 1.$$



Intuitively, a physical interpretation of the above assumption is that the signal over the whole positive real line has infinite energy, and thus the CUSUM stopping rule  $S_\nu^{(1)}$  in (2.14) will go off in finite time with probability 1.

In this work, we will thus assume analogous conditions.

*Assumption 3.* For all  $\tau_i \in \{0, \infty\}$ ,  $i \in \{1, \dots, N\}$ , and any  $t \in \mathbb{R}_+$ ,

$$(2.16) \quad P_{(\tau_1, \dots, \tau_N)} \left( \int_0^\infty \frac{1}{2} (\alpha_1(s; \omega))^2 ds = \infty \right) = 1,$$

$$(2.17) \quad P_{(\tau_1, \dots, \tau_N)} \left( \int_0^t \frac{1}{2} (\alpha_1(s; \omega))^2 ds < \infty \right) = 1.$$

We comment that (2.17) is implied by (2.3), since the measures  $P_{\tau_1, \dots, \tau_N} |_{\mathcal{B}_t}$  are all equivalent under (2.3).

A class of functions  $\alpha_i(\cdot; \cdot)$ 's that satisfy (2.3), (2.16), and (2.17) are the uniformly bounded Lipschitz continuous functions that are also uniformly bounded away from 0. It can be easily shown that linear state-space models of the form (2.2) also satisfy (2.3), (2.16), and (2.17).

**LEMMA 2.1.** *If  $\alpha_1(t; \omega) = \sum_{i=1}^N \beta_i \omega_i(t)$ , and  $\alpha_j(t; \omega) = c_j \alpha_1(t; \omega)$  for some real constants  $\beta_i, c_j$  such that  $\sum_{i=1}^N \beta_i^2 > 0$  and  $|c_N| \geq \dots \geq |c_2| \geq c_1 = 1$ , then (2.3), (2.16), and (2.17) hold.*

The optimality of the CUSUM stopping rule in the presence of only one observation process suggests that a CUSUM type of stopping rule might display similar optimality properties in the case of multiple observation processes. In particular, an intuitively appealing rule, when the detection of  $\tilde{\tau} = \min_i \{\tau_i\}$  is of interest, is the  $N$ -dimensional variant of the CUSUM stopping rule  $T_{\tilde{h}} = \min_i \{T_{h_i}^{(i)}\}$ , where  $T_{h_i}^{(i)}$  is the CUSUM stopping rule for the process  $\{Z_t^{(i)}\}_{t \geq 0}$  for  $i \in \{1, \dots, N\}$ . That is, we use what is known as a multichart CUSUM stopping rule [22], which can be written as

$$(2.18) \quad T_{\tilde{h}} = \inf \left\{ t \geq 0 : \max \left\{ \frac{y_1(t)}{h_1}, \dots, \frac{y_N(t)}{h_N} \right\} \geq 1 \right\}.$$

Here for any  $i \in \{1, \dots, N\}$ ,  $h_i > 0$  is the threshold for process  $\{y_i(t)\}_{t \geq 0}$ , and  $\{y_i(t)\}_{t \geq 0}$  is defined as (see (2.4)–(2.5))

$$(2.19) \quad y_i(t) := \sup_{0 \leq s \leq t} \log \frac{dP_{(\infty, \dots, s, \dots, \infty)}}{dP_{\mathbf{t}_\infty}} \Big|_{\mathcal{B}_t} = \sup_{0 \leq s \leq t} \{u_i(t) - u_i(s)\} = u_i(t) - m_i(t) \\ = \int_0^t \alpha_i(s; \omega) d\omega_i(s) - \frac{1}{2} \int_0^t (\alpha_i(s; \omega))^2 ds - m_i(t),$$

where  $P_{(\infty, \dots, s, \dots, \infty)}$  is corresponding to the change points  $\tau_j = \infty$  for all  $j \neq i$  and  $\tau_i = s$ ,  $u_i(t)$  is given in (2.4), and  $m_i(t) := \inf_{0 \leq s \leq t} u_i(s)$ . For any fixed large  $\gamma > 0$ , the  $N$ -dimensional threshold vector  $\tilde{h} = (h_1, \dots, h_N) \in (\mathbb{R}_+)^N$  is to be determined so that both (2.12) and (2.13) hold.

We will give an analytical characterization of the stopping rule  $T_{\tilde{h}}$  in the next section.

**3. Multichart CUSUM stopping rule  $T_{\tilde{h}}$ .** In this section, we study several property of the multichart CUSUM stopping rule  $T_{\tilde{h}}$  defined in (2.18). First, we show its a.s. finiteness, and compute the Kulback–Leibler detection and false-alarm

divergence of  $T_{\hbar}$ . We then establish an upper bound for  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$ , which will be used in section 5 for asymptotic analysis. Throughout this section, we will denote by  $Y(t) := (y_1(t), \dots, y_N(t))$  for  $y_i(t)$ 's defined in (2.19).

Let us begin by introducing a class of functions that will be used in the main results. For any fixed vectors  $\mathcal{S} = (\mathcal{S}_1, \dots, \mathcal{S}_N) \in \{\pm 1\}^N$  and  $\hbar \in (\mathbb{R}_+)^N$ , we define  $f_{\mathcal{S}, \hbar}(\cdot)$  as a  $C^2$  function on  $D_{\hbar} := [0, h_1] \times \dots \times [0, h_N]$ , satisfying the following PDE:

$$(3.1a) \quad \begin{cases} \sum_{i=1}^N \frac{1}{c_i^2} \frac{\partial^2 f_{\mathcal{S}, \hbar}}{\partial y_i^2} + \sum_{i=1}^N \frac{\mathcal{S}_i}{c_i^2} \frac{\partial f_{\mathcal{S}, \hbar}}{\partial y_i} = -1 \quad \forall (y_1, \dots, y_N) \in D_{\hbar}, \end{cases}$$

$$(3.1b) \quad \begin{cases} \frac{\partial f_{\mathcal{S}, \hbar}}{\partial y_i} \Big|_{y_i=0} = f_{\mathcal{S}, \hbar} \Big|_{y_i=h_i} = 0. \end{cases}$$

Using the Feynman–Kac representation of  $f_{\mathcal{S}, \hbar}$ , we obtain the following useful monotonicity properties.

LEMMA 3.1. *For any  $\mathcal{S} = (\mathcal{S}_1, \dots, \mathcal{S}_N) \in \{\pm 1\}^N$ , the function  $f_{\mathcal{S}, \hbar}$  defined in (3.1a)–(3.1b) is decreasing in  $y_i$  for each  $i$ . If  $\mathcal{S}' = (\mathcal{S}'_1, \dots, \mathcal{S}'_N) \in \{\pm 1\}^N$  such that  $\mathcal{S}'_i \leq \mathcal{S}_i$  for any  $i = 1, \dots, N$ , then for all  $(y_1, \dots, y_N) \in D_{\hbar}$ ,*

$$(3.2) \quad 0 \leq f_{\mathcal{S}, \hbar}(y_1, \dots, y_N) \leq f_{\mathcal{S}', \hbar}(y_1, \dots, y_N) \leq f_{\mathcal{S}', \hbar}(0, \dots, 0) < \infty.$$

Moreover,  $f_{\mathcal{S}, \hbar}(0, \dots, 0)$  is strictly increasing in  $h_i$  for each  $i$ .

The following technical lemma will also be useful for the localization argument later.

LEMMA 3.2. *For any fixed  $\tau, \tau' \in \mathbb{R}_+$ , and a fixed  $\mathbf{t} = (\tau_1, \dots, \tau_N) \in (\overline{\mathbb{R}}_+)^N$ , we have*

$$P_{\mathbf{t}} \left( \int_{\tau}^{\infty} \frac{1}{2} (\alpha_1(s; \omega))^2 ds = \infty \mid \mathcal{B}_{\tau'} \right) = 1.$$

THEOREM 3.3. *For any fixed vector  $\hbar \in \mathbb{R}_+^N$ ,  $\mathbf{t} = (\tau_1, \dots, \tau_N) \in \overline{\mathbb{R}}_+^N$  such that  $\tilde{\tau} = \min_i \{\tau_i\} < \infty$ , the multichart CUSUM stopping rule  $T_{\hbar}$  is a.s. finite:*

$$(3.3) \quad P_{\mathbf{t}}(T_{\hbar} < \infty \mid \mathcal{B}_{\tilde{\tau}}) = 1, \quad P_{\mathbf{t}}\text{-a.s.}$$

Define  $\mathcal{S}(\mathbf{t}; s) := (\varphi(\tau_1, s), \dots, \varphi(\tau_N, s)) \in \{\pm 1\}^N$  with  $\varphi(\tau, s) := \mathbf{1}_{\{\tau < s\}} - \mathbf{1}_{\{\tau \geq s\}}$  for  $\tau \in \overline{\mathbb{R}}_+$  and  $s \in \mathbb{R}_+$ . Let us relabel (without repetition) the change points  $\tau_1, \dots, \tau_N$  in increasing order: for some  $l \in \{1, \dots, N - 1\}$ ,

$$\tilde{\tau} = \tau_{(1)} < \tau_{(2)} < \dots < \tau_{(l)} < \infty = \tau_{(l+1)}.$$

Then we have

$$(3.4) \quad E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T_{\hbar} \geq \tilde{\tau}\}} \int_{\tilde{\tau}}^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \mid \mathcal{B}_{\tilde{\tau}} \right\} \\ = \mathbf{1}_{\{T_{\hbar} \geq \tilde{\tau}\}} \sum_{j=1}^l E_{\mathbf{t}} \{ [f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}), \hbar}(Y(\tau_{(j)} \wedge T_{\hbar})) - f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}), \hbar}(Y(\tau_{(j+1)} \wedge T_{\hbar}))] \mid \mathcal{B}_{\tilde{\tau}} \}.$$

Similarly, for a fixed  $t \in \mathbb{R}_+$ , we have

$$(3.5) \quad P_{\mathbf{t}_{\infty}}(T_{\hbar} < \infty \mid \mathcal{B}_t) = 1, \quad P_{\mathbf{t}_{\infty}}\text{-a.s.},$$

$$(3.6) \quad E_{\mathbf{t}_{\infty}} \left\{ \mathbf{1}_{\{T_{\hbar} \geq t\}} \int_t^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \mid \mathcal{B}_t \right\} = f_{\mathcal{S}(\mathbf{t}_{\infty}; t), \hbar}(y_1(t), \dots, y_N(t)) \mathbf{1}_{\{T_{\hbar} \geq t\}}.$$

*Proof.* From Lemma 3.2, the stopping rules defined as

$$t_n := \inf \left\{ t \geq \tilde{\tau} : \int_{\tilde{\tau}}^t \frac{1}{2} (\alpha_1(s; \omega))^2 ds \geq n \right\}, \quad n \geq 1,$$

are  $P_t$ -a.s. finite, and so is  $T_h^n := T_h \wedge t_n$ .

Let us apply Itô's lemma to  $\{f_{S(t; \tau_{(j)}, \hbar)}(Y(t))\}_{\tau_{(j)} \wedge T_h^n \leq t < \tau_{(j+1)} \wedge T_h^n}$  for any  $j \in \{1, \dots, l\}$ . Using (3.1a) we have that

$$\begin{aligned} (3.7) \quad & E_t \{ f_{S(t; \tau_{(j)}, \hbar)}(Y(\tau_{(j+1)} \wedge T_h^n)) - f_{S(t; \tau_{(j)}, \hbar)}(Y(\tau_{(j)} \wedge T_h^n)) | \mathcal{B}_{\tilde{\tau}} \} \mathbf{1}_{\{T_h^n \geq \tilde{\tau}\}} \\ &= E_t \left\{ \int_{\tau_{(j)} \wedge T_h^n}^{\tau_{(j+1)} \wedge T_h^n} \sum_{i=1}^N \left[ \frac{\partial f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i} dy_i(s) \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \frac{\partial^2 f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i^2} (\alpha_i(s; \omega))^2 ds \right] | \mathcal{B}_{\tilde{\tau}} \right\} \mathbf{1}_{\{T_h^n \geq \tilde{\tau}\}}. \end{aligned}$$

From (2.1), (2.4), and (2.19) we know that, for all  $i \in \{1, \dots, N\}$ ,  $dy_i(t) = du_i(t) - dm_i(t)$  and

$$du_i(s) = \alpha_i(s; \omega) dB_i(s) + \frac{1}{2} \varphi(\tau_i, s) (\alpha_i(s; \omega))^2 ds,$$

where  $B_i(t) = \omega_i(t) - \mathbf{1}_{\{t > \tau_i\}} \int_{\tau_i}^t \alpha_i(s; \omega) ds$ ,  $t \geq 0$ , is a  $\mathbb{B}$ -standard Brownian motion under  $P_t$ . On the other hand, from Assumption 1', we have for all  $i \in \{1, \dots, N\}$  that

$$du_i(s) = \alpha_i(s; \omega) dB_i(s) + \frac{1}{2} \varphi(\tau_i, s) (c_i^{-1} \alpha_1(s; \omega))^2 ds.$$

By substituting the above formula into (3.7), we obtain

$$\begin{aligned} (3.8) \quad & E_t \left\{ \int_{\tau_{(j)} \wedge T_h^n}^{\tau_{(j+1)} \wedge T_h^n} \left[ \frac{1}{2} \sum_{i=1}^N \left( \frac{\varphi(\tau_i, \tau_{(j)})}{c_i^2} \frac{\partial f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i} + \frac{1}{c_i^2} \frac{\partial^2 f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i^2} \right) (\alpha_1(s; \omega))^2 ds \right. \right. \\ & \quad \left. \left. + \sum_{i=1}^N \left( \frac{\partial f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i} \alpha_i(s; \omega) dB_i(s) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\partial f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i} dm_i(s) \right) \right] | \mathcal{B}_{\tilde{\tau}} \right\} \mathbf{1}_{\{T_h^n \geq \tilde{\tau}\}} \\ &= E_t \left\{ \int_{\tau_{(j)} \wedge T_h^n}^{\tau_{(j+1)} \wedge T_h^n} \left[ \frac{1}{2} (\alpha_1(s; \omega))^2 ds \right. \right. \\ & \quad \left. \left. + \sum_{i=1}^N \left( \frac{\partial f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i} \alpha_i(s; \omega) dB_i(s) \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\partial f_{S(t; \tau_{(j)}, \hbar)}}{\partial y_i} dm_i(s) \right) \right] | \mathcal{B}_{\tilde{\tau}} \right\} \mathbf{1}_{\{T_h^n \geq \tilde{\tau}\}}, \end{aligned}$$

where we used (3.1a) in the equality. Notice that in the event  $\{T_h^n < \tau_{(j)}\}$ , the integral inside the above expectation is 0; and in the event  $\{T_h^n \geq \tilde{\tau}\} \cap \{T_h^n \geq \tau_{(j)}\} = \{T_h^n \geq$

$\tau_{(j)}\}$ , we have

$$\begin{aligned} E_{\mathbf{t}} & \left\{ \int_{\tau_{(j)} \wedge T_{\bar{h}}^n}^{\tau_{(j+1)} \wedge T_{\bar{h}}^n} (\alpha_i(s; \omega))^2 \left( \frac{\partial f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})}}{\partial y_i} \right)^2 ds | \mathcal{B}_{\bar{\tau}} \right\} \\ & = c_i^{-2} E_{\mathbf{t}} \left\{ \int_{\tau_{(j)} \wedge T_{\bar{h}}^n}^{\tau_{(j+1)} \wedge T_{\bar{h}}^n} (\alpha_1(s; \omega))^2 \left( \frac{\partial f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})}}{\partial y_i} \right)^2 ds | \mathcal{B}_{\bar{\tau}} \right\} \\ & \leq c_i^{-2} n \max_{1 \leq k \leq l} \|f_{\mathcal{S}(\mathbf{t}; \tau(k), \bar{h})}\|^2, \end{aligned}$$

which is finite. Here  $\|f\| := \max_{1 \leq i \leq N} \sup_{y \in D_{\bar{h}}} |\frac{\partial f}{\partial y_i}|$  for any  $f \in C^1(D_{\bar{h}})$ . We have that the conditional expectation of the stochastic integral in (3.8) is 0. Moreover, since  $f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(\cdot)}$  satisfies the Neumann condition (3.1b), and the measure  $dm_i(s) = 0$  off the random set  $\{s : y_i(s) = 0\}$ , we have that  $\int_{\tau_{(j)} \wedge T_{\bar{h}}^n}^{\tau_{(j+1)} \wedge T_{\bar{h}}^n} \frac{\partial f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})}}{\partial y_i} dm_i(s) = 0$ ,  $P_{\mathbf{t}}$ -a.s. for any  $i \in \{1, \dots, N\}$ . Therefore, from (3.7) and (3.8) we have

$$\begin{aligned} (3.9) \quad E_{\mathbf{t}} & \{f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(Y(\tau_{(j+1)} \wedge T_{\bar{h}}^n))} - f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(Y(\tau_{(j)} \wedge T_{\bar{h}}^n))} | \mathcal{B}_{\bar{\tau}}\} \mathbf{1}_{\{T_{\bar{h}}^n \geq \bar{\tau}\}} \\ & = -E_{\mathbf{t}} \left\{ \int_{\tau_{(j)} \wedge T_{\bar{h}}^n}^{\tau_{(j+1)} \wedge T_{\bar{h}}^n} \frac{1}{2} (\alpha_1(s; \omega))^2 ds | \mathcal{B}_{\bar{\tau}} \right\} \mathbf{1}_{\{T_{\bar{h}}^n \geq \bar{\tau}\}}. \end{aligned}$$

From Lemma 3.1 we know that the  $f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(y_1, \dots, y_N)}$ 's are nonnegative and decreasing in  $y_i$  and  $f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})|_{y_i=h_i}} = 0$  for each  $i \in \{1, \dots, N\}$ , thus we have that

$$\begin{aligned} E_{\mathbf{t}} & \left\{ \int_{\tau_{(j)} \wedge T_{\bar{h}}^n}^{\tau_{(j+1)} \wedge T_{\bar{h}}^n} \frac{1}{2} (\alpha_1(s; \omega))^2 ds | \mathcal{B}_{\bar{\tau}} \right\} \mathbf{1}_{\{T_{\bar{h}}^n \geq \bar{\tau}\}} \\ & = E_{\mathbf{t}} \{f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(Y(\tau_{(j)} \wedge T_{\bar{h}}^n))} - f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(Y(\tau_{(j+1)} \wedge T_{\bar{h}}^n))} | \mathcal{B}_{\bar{\tau}}\} \mathbf{1}_{\{T_{\bar{h}}^n \geq \bar{\tau}\}} \\ & \leq E_{\mathbf{t}} \{f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(Y(\tau_{(j)} \wedge T_{\bar{h}}^n))} | \mathcal{B}_{\bar{\tau}}\} \leq f_{\mathcal{S}(\mathbf{t}; \tau_{(j)}, \bar{h})(0, \dots, 0)} < \infty. \end{aligned}$$

As  $n \rightarrow \infty$ ,  $t_n$  tends to  $\infty$  and  $T_{\bar{h}}^n$  tends to  $T_{\bar{h}}$ ,  $P_{\mathbf{t}}$ -a.s. Using the bounded convergence theorem we have that

$$\begin{aligned} \infty & > f_{\mathcal{S}(\mathbf{t}; \tau_{(l)}, \bar{h})(0, \dots, 0)} \geq E_{\mathbf{t}} \left\{ \int_{\tau_{(l)} \wedge T_{\bar{h}}}^{T_{\bar{h}}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds | \mathcal{B}_{\bar{\tau}} \right\} \mathbf{1}_{\{T_{\bar{h}} \geq \bar{\tau}\}} \\ & \geq E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T_{\bar{h}} = \infty\}} \int_{\tau_{(l)}}^{\infty} \frac{1}{2} (\alpha_1(s; \omega))^2 ds | \mathcal{B}_{\bar{\tau}} \right\}. \end{aligned}$$

We now claim that  $P_{\mathbf{t}}(T_{\bar{h}} = \infty | \mathcal{B}_{\bar{\tau}}) = 0$ . This is because, otherwise, from the above inequality we would have

$$\int_{\tau_{(l)}}^{\infty} (\alpha_1(s; \omega))^2 ds = 0, \quad P_{\mathbf{t}}\text{-a.s. on } \{T_{\bar{h}} = \infty\},$$

which contradicts Lemma 3.2. This proves (3.3).

Finally, using the bounded convergence theorem and (3.3), we can obtain (3.4) from (3.9).

Using similar arguments we can prove (3.5) and (3.6).  $\square$

Using Lemma 3.1 and Theorem 3.3, we can explicitly compute the  $J_{KL}^{(N)}$  index for the multichart CUSUM stopping rule  $T_{\bar{h}}$ . To this end, we introduce  $\mathcal{S}^{(j)} =$

$(\mathcal{S}_1^{(j)}, \dots, \mathcal{S}_N^{(j)})$  with  $\mathcal{S}_i^{(j)} = \mathbf{1}_{\{i=j\}} - \mathbf{1}_{\{i \neq j\}}$ , for any  $i = 1, \dots, N$  and  $j = 0, 1, \dots, N$ . We then have the following proposition.

PROPOSITION 3.4. *For the multichart stopping rule  $T_{\hbar}$ , we have that*

$$(3.10) \quad J_j^{(N)}(T_{\hbar}) = f_{\mathcal{S}^{(j)}, \hbar}(0, \dots, 0), \quad j \in \{1, \dots, N\},$$

where  $J_j^{(N)}$  is defined in (2.10).

Proof. First recall from (2.10) that, for all  $j \in \{1, \dots, N\}$ ,

$$J_j^{(N)}(T_{\hbar}) = \sup_{\substack{\mathbf{t}=(\tau_1, \dots, \tau_N) \in (\mathbb{R}_+)^N \\ \tau_j = \tilde{\tau} < \infty}} \text{esssup } E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T_{\hbar} \geq \tau_j\}} \int_{\tau_j}^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \middle| \mathcal{B}_{\tau_j} \right\}.$$

Without loss of generality, we give the proof for the case  $N = 2$ . The argument can be easily generalized to treat a general  $N$ . Notice that for  $N = 2$ , we have for  $j = 1, 2$  that

$$(3.11) \quad J_j^{(2)}(T_{\hbar}) = \sup_{\substack{\mathbf{t}=(\tau_1, \tau_2) \in (\mathbb{R}_+)^2 \\ \tau_j = \tilde{\tau} < \infty}} \text{esssup } E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T_{\hbar} \geq \tau_j\}} \int_{\tau_j}^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \middle| \mathcal{B}_{\tau_j} \right\}.$$

We assume without loss of generality that  $\tau_1 = \tilde{\tau}$ . To prove the claim in the proposition, we first show that, for any fixed  $\mathbf{t} = (\tau_1, \tau_2)$  such that  $\tau_1 \in \mathbb{R}_+$  and  $\tau_1 = \tilde{\tau}$ , we have that

$$(3.12) \quad E_{\mathbf{t}} \left\{ \mathbf{1}_{\{T_{\hbar} \geq \tau_1\}} \int_{\tau_1}^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \middle| \mathcal{B}_{\tau_1} \right\} \leq f_{(1, -1), \hbar}(0, 0), \quad P_{\mathbf{t}\text{-a.s.}}$$

To show this, we use Theorem 3.3 to obtain

$$\begin{aligned} & E_{\mathbf{t}} \left\{ \int_{\tau_1}^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \middle| \mathcal{B}_{\tau_1} \right\} \mathbf{1}_{\{T_{\hbar} \geq \tau_1\}} \\ &= E_{\mathbf{t}} \left\{ \int_{\tau_1}^{T_{\hbar} \wedge \tau_2} \frac{1}{2} (\alpha_1(s; \omega))^2 ds + \int_{T_{\hbar} \wedge \tau_2}^{T_{\hbar}} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \middle| \mathcal{B}_{\tau_1} \right\} \mathbf{1}_{\{T_{\hbar} \geq \tau_1\}} \\ &= \mathbf{1}_{\{T_{\hbar} \geq \tau_1\}} \left( E_{(\tau_1, \infty)} \{ \mathbf{1}_{\{T_{\hbar} \geq \tau_2\}} [f_{(1, -1), \hbar}(Y(\tau_1)) - f_{(1, -1), \hbar}(Y(\tau_2))] \middle| \mathcal{B}_{\tau_1} \} \right. \\ &\quad \left. + E_{(\tau_1, \infty)} \{ \mathbf{1}_{\{T_{\hbar} \geq \tau_2\}} f_{(1, 1), \hbar}(Y(\tau_2)) \middle| \mathcal{B}_{\tau_1} \} \right. \\ &\quad \left. + E_{(\tau_1, \infty)} \{ \mathbf{1}_{\{T_{\hbar} < \tau_2\}} f_{(1, -1), \hbar}(Y(\tau_1)) \middle| \mathcal{B}_{\tau_1} \} \right). \end{aligned}$$

Using Lemma 3.1, we have that

$$f_{(1, 1), \hbar}(Y(\tau_2)) \mathbf{1}_{\{T_{\hbar} \geq \tau_2\}} \leq f_{(1, -1), \hbar}(Y(\tau_2)) \mathbf{1}_{\{T_{\hbar} \geq \tau_2\}} \leq f_{(1, -1), \hbar}(0, 0) \mathbf{1}_{\{T_{\hbar} \geq \tau_2\}}, \quad P_{(\tau_1, \infty)\text{-a.s.}},$$

from which we obtain (3.12). On the other hand, applying Theorem 3.3 to  $(\tau_1, \tau_2) = (0, \infty)$ , we have

$$E_{0, \infty} \left\{ \frac{1}{2} \int_0^{T_{\hbar}} (\alpha_1(s; \omega))^2 ds \right\} = f_{(1, -1), \hbar}(0, 0).$$

This suggests that (from (3.11)),

$$J_1^{(2)}(T_{\hbar}) = f_{(1,-1),\hbar}(0, 0).$$

Therefore, (3.10) holds for  $j = 1$  and  $N = 2$ .  $\square$

We now establish an upper bound for  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$  using the  $N$ -CUSUM rule. First, let us define

$$H(\gamma) = \{ \hbar \in (\mathbb{R}_+)^N : f_{S^{(0),\hbar}}(0, \dots, 0) \geq (c_N)^2 \gamma \}.$$

The asymptotic analysis in the next section will show that  $H(\gamma) \neq \emptyset$ . From (3.6) in Theorem 3.3, we know that for any  $\hbar \in H(\gamma)$ ,

$$E_{t_\infty} \left\{ \int_0^{T_{\hbar}} \frac{1}{2} (c_N^{-1} \alpha_1(s; \omega))^2 ds \right\} = \frac{1}{(c_N)^2} f_{S^{(0),\hbar}}(0, \dots, 0) \geq \gamma.$$

Hence,  $T_{\hbar} \in \mathbb{B}_\gamma$  for any  $\hbar \in H(\gamma)$ , and we trivially have

$$(3.13) \quad J_{KL}^{(N)}(T_{\hbar}) \geq \inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T) \quad \forall \hbar \in H(\gamma).$$

From (2.13) and Proposition 3.4, we will look for a vector  $\hbar \in H(\gamma)$  for the multichart CUSUM stopping rule (18) so that

$$(3.14) \quad f_{S^{(j),\hbar}}(0, \dots, 0) = f_{S^{(j),\hbar}}(0, \dots, 0) + o(1) \quad \forall j \in \{1, 2, \dots, N\},$$

as  $\gamma \rightarrow \infty$ .

While it is not clear whether or not there is an  $\hbar \in H(\gamma)$  such that (3.14) holds and  $f_{S^{(0),\hbar}}(0, \dots, 0) = (c_N)^2 \gamma$ , we will give an explicit linear constraint on all coordinates of  $\hbar$ , namely, (5.13), under which (3.14) holds asymptotically as  $\hbar \rightarrow \infty$ . The stopping rule chosen in this way gives us an easily computable upper bound for  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$ . We will also prove in later sections that this stopping rule is asymptotically optimal as  $\gamma \rightarrow \infty$ .

**4. Asymptotic analysis.** In this section we will analyze the asymptotic behavior of  $f_{S^{(j),\hbar}}(0, \dots, 0)$ , as all  $h_i$ 's tend to  $\infty$ . As a byproduct, we show that the set  $H(\gamma) \neq \emptyset$ . The key analysis is based on the following integral representation of  $f_{S^{(j),\hbar}}$ .

LEMMA 4.1. *For any  $\hbar \in (0, \infty)^N$ , and any  $y_i \in [0, h_i]$  for all  $i \in \{1, \dots, N\}$ , we have*

$$f_{S,\hbar}(y_1, \dots, y_N) = \int_0^\infty \prod_{i=1}^N K_{S_i, \epsilon_i} \left( \frac{\epsilon_i t}{c_i^2}, \frac{y_i}{h_i} \right) dt,$$

where  $\epsilon_i := \frac{1}{h_i}$  and  $K_{S_i, \epsilon_i}(t, z)$  is the  $C^2$  function on  $[0, 1]$  satisfying the following PDE: for any  $z \in [0, 1]$ ,

$$(4.1a) \quad \begin{cases} \frac{\partial K_{S_i, \epsilon_i}}{\partial t} = \epsilon_i \frac{\partial^2 K_{S_i, \epsilon_i}}{\partial z^2} + S_i \frac{\partial K_{S_i, \epsilon_i}}{\partial z}, \\ (4.1b) \quad K_{S_i, \epsilon_i}|_{t=0} = \frac{\partial K_{S_i, \epsilon_i}}{\partial z} \Big|_{z=0} = K_{S_i, \epsilon_i}|_{z=1} = 0. \end{cases}$$

Moreover, for any  $c, \epsilon > 0$ ,  $K_{\pm 1, \epsilon}(t, 0)$  is a decreasing function, bounded between 0 and 1, and satisfies

$$(4.2) \quad \int_0^\infty K_{+1, \epsilon}(c^{-2}\epsilon t, 0) dt = c^2 \left( \frac{1}{\epsilon} - 1 + e^{-1/\epsilon} \right).$$

To examine the asymptotic behavior of  $K_{S_i, \epsilon_i}(c_i^{-2}\epsilon_i t, 0)$  as  $h_i = \frac{1}{\epsilon_i} \rightarrow \infty$  or  $\epsilon_i \downarrow 0$ , we transform (4.1a) to a heat equation and then obtain an expansion of  $K_{S_i, \epsilon_i}(t, 0)$  using the eigenfunctions of the associated Sturm–Liouville problem. Details of this expansion are given in the appendix. Using the asymptotic expansion, we obtain an estimate for  $f_{S^{(0)}, \bar{h}}(0, \dots, 0)$  as the  $h_i$ 's tend to  $\infty$ .

PROPOSITION 4.2. As  $h_i = \frac{1}{\epsilon_i} \rightarrow \infty$ , we have

$$(4.3) \quad \left| f_{S^{(0)}, \bar{h}}(0, \dots, 0) - \frac{1}{\sum_{i=1}^N c_i^{-2} e^{-h_i}} \right| \leq \mathcal{O}(1/\epsilon_{\max}),$$

where  $\epsilon_{\max} = \max_i \{\epsilon_i\}$ .

Clearly, the estimate in (4.3) tends to  $\infty$  as the  $h_i$ 's tend to  $\infty$ , hence we have  $H(\gamma) \neq \emptyset$ .

Similarly, using Lemma 4.1 for  $S = S^{(j)}$ ,  $j = 1, \dots, N$ , we have

$$f_{S^{(j)}, \bar{h}}(0, \dots, 0) = \int_0^\infty K_{+1, \epsilon_j}(c_j^{-2}\epsilon_j t, 0) \cdot \left( \prod_{i \neq j} K_{-1, \epsilon_i}(c_i^{-2}\epsilon_i t, 0) \right) dt.$$

In the limit for  $\epsilon_j \rightarrow 0, \epsilon_i \rightarrow 0$ , the value of the integral is governed by the integral of the function  $K_{+1, \epsilon_j}(c_j^{-2}\epsilon_j t, 0)$  in the sense that

$$f_{S^{(j)}, \bar{h}}(0, \dots, 0) = \int_0^\infty K_{+1, \epsilon_j}(c_j^{-2}\epsilon_j t, 0) dt + o(1) = c_j^2(h_j - 1) + o(1).$$

More precisely, we have the following result.

PROPOSITION 4.3. As  $h_i = \frac{1}{\epsilon_i} \rightarrow \infty$ , we have

$$(4.4) \quad |f_{S^{(j)}, \bar{h}}(0, \dots, 0) - c_j^2(h_j - 1)| \leq \mathcal{O} \left( \frac{1}{\epsilon_j^2} \sum_{i=1}^N e^{-1/\epsilon_i} \right).$$

**5. Asymptotic optimality of the  $N$ -CUSUM rule.** In this section we prove the asymptotic optimality of the  $N$ -CUSUM rule as  $\gamma \rightarrow \infty$ . This is accomplished by establishing a lower bound for  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$ , and then showing that the difference between this lower bound and the upper bound in (3.13) is bounded as  $\gamma \rightarrow \infty$ .

PROPOSITION 5.1. For any stopping rule  $T \in \mathbb{B}_\gamma$ , we have  $J_{KL}^{(N)}(T) \geq g(-\nu^*)$ , where  $\nu^*$  is the unique positive solution to  $g(\nu^*) = (c_N)^2 \gamma$  and the function  $g$  is as defined in (2.15).

*Proof.* We begin by defining an auxiliary performance measure

$$(5.1) \quad J_{KL}^{(N,1)}(T) = \sup_{\tau < \infty} \text{essup} E_{(\tau, \infty, \dots, \infty)} \left\{ \mathbf{1}_{\{T \geq \tau\}} \int_\tau^T \frac{1}{2} (\alpha_1(s; \omega))^2 ds \mid \mathcal{B}_\tau \right\},$$

where  $T$  is an arbitrary  $\mathbb{B}$ -stopping rule. Notice that, for any  $\mathbb{F}$ -stopping rule  $T \in \mathbb{B}_\gamma$ , by construction, we have that

$$J_{KL}^{(N)}(T) \geq J_{KL}^{(N,1)}(T).$$

Now let  $T_\nu^{(1)}$  be the first passage time of  $y_1$  to some threshold  $\nu > 0$ ,

$$T_\nu^{(1)} := \inf\{t \geq 0 : y_1(t) \geq \nu\}.$$

Then trivially, we have  $T_\nu := T \wedge T_\nu^{(1)} \leq T$ , and that

$$J_{KL}^{(N,1)}(T) \geq J_{KL}^{(N,1)}(T_\nu).$$

To finish the proof, we will show that for any fixed  $\epsilon > 0$ , there exists a  $\nu > 0$  such that

$$(5.2) \quad J_{KL}^{(N,1)}(T_\nu) \geq g(-\nu^*) - \epsilon.$$

To that end, we use similar arguments to those in Theorem 3.3, for  $\tau_1 = \tau < \infty = \tau_2, \dots, \tau_N$ , by applying Itô's lemma to  $\{g(-y_1(t))\}_{\tau \wedge T_\nu \leq t < T_\nu}$  to obtain that  $T_\nu < \infty$ ,  $P_{(\tau, \infty, \dots, \infty)}$ -a.s., and on the event  $\{T_\nu > \tau\}$ ,

$$(5.3) \quad E_{(\tau, \infty, \dots, \infty)} \left\{ \int_\tau^{T_\nu} \frac{1}{2} (\alpha_1(s; \omega))^2 ds | \mathcal{B}_\tau \right\} = E_{(\tau, \infty, \dots, \infty)} \{g(-y_1(T_\nu)) - g(-y_1(\tau)) | \mathcal{B}_\tau\}.$$

Using Girsanov's theorem at the finite stopping rule  $T_\nu \wedge n$  for some  $n > \tau$ , we have that

$$(5.4) \quad \mathbf{1}_{\{T_\nu \geq \tau\}} = \mathbf{1}_{\{T_\nu \geq \tau\}} E_{\mathbf{t}_\infty} \{e^{u_1(T_\nu \wedge n) - u_1(\tau)} | \mathcal{B}_\tau\}$$

and

$$(5.5) \quad \begin{aligned} \mathbf{1}_{\{T_\nu \geq \tau\}} E_{(\tau, \infty, \dots, \infty)} \{g(-y_1(T_\nu \wedge n)) - g(-y_1(\tau)) | \mathcal{B}_\tau\} \\ = \mathbf{1}_{\{T_\nu \geq \tau\}} E_{\mathbf{t}_\infty} \{e^{u_1(T_\nu \wedge n) - u_1(\tau)} [g(-y_1(T_\nu \wedge n)) - g(-y_1(\tau))] | \mathcal{B}_\tau\}. \end{aligned}$$

Notice that  $g(-y)$  is increasing,  $y_1(t \wedge T_\nu) \in [0, \nu]$ , and  $u_1(T_\nu \wedge n) - u_1(\tau) \leq u_1(T_\nu \wedge n) - m_1(T_\nu \wedge n) = y_1(T_\nu \wedge n) \leq \nu$ ; thus we have that

$$\begin{aligned} e^{u_1(T_\nu \wedge n) - u_1(\tau)} &\leq e^\nu < \infty, \\ |g(-y_1(T_\nu \wedge n)) - g(-y_1(\tau))| &\leq 2g(-\nu) < \infty. \end{aligned}$$

Applying the bounded convergence theorem to (5.4)–(5.5) as  $n \rightarrow \infty$ , we obtain that, in the event  $\{T_\nu \geq \tau\}$ ,

$$\begin{aligned} 1 &= E_{\mathbf{t}_\infty} \{e^{u_1(T_\nu) - u_1(\tau)} | \mathcal{B}_\tau\}, \text{ and} \\ &E_{(\tau, \infty, \dots, \infty)} \{g(-y_1(T_\nu)) - g(-y_1(\tau)) | \mathcal{B}_\tau\} \\ &= E_{\mathbf{t}_\infty} \{e^{u_1(T_\nu) - u_1(\tau)} [g(-y_1(T_\nu)) - g(-y_1(\tau))] | \mathcal{B}_\tau\}. \end{aligned}$$

In view of (5.1) and (5.3), we have that (by substituting  $s = \tau$ )

$$(5.6) \quad \begin{aligned} J_{KL}^{(N,1)}(T_\nu) E_{\mathbf{t}_\infty} \{e^{u_1(T_\nu) - u_1(s)} | \mathcal{B}_s\} \mathbf{1}_{\{T_\nu \geq s\}} \\ \geq E_{\mathbf{t}_\infty} \{e^{u_1(T_\nu) - u_1(s)} [g(-y_1(T_\nu)) - g(-y_1(s))] | \mathcal{B}_s\} \mathbf{1}_{\{T_\nu \geq s\}}. \end{aligned}$$



Following the same arguments as in Theorem 2 of [21], we integrate both sides of the above inequality with respect to  $(-dm_1(s))$  for all  $s \in [0, T_\nu]$ , then take the expectation under  $P_{t_\infty}$ . We obtain that

$$J_{KL}^{(N,1)}(T_\nu)E_{t_\infty} \left\{ e^{u_1(T_\nu)} \int_0^{T_\nu} e^{-u_1(s)} (-dm_1(s)) \right\} \geq E_{t_\infty} \left\{ e^{u_1(T_\nu)} \int_0^{T_\nu} e^{-u_1(s)} [g(-y_1(T_\nu)) - g(-y_1(s))] (-dm_1(s)) \right\}.$$

Using the fact that  $dm_1(s) = 0$  off the set  $\{s : y_1(s) = 0\} = \{s : u_1(s) = m_1(s)\}$ , and  $g(0) = 0$ , we obtain that

$$J_{KL}^{(N,1)}(T_\nu)E_{t_\infty} \{e^{y_1(T_\nu)} - e^{u_1(T_\nu)}\} \geq E_{t_\infty} \{[e^{y_1(T_\nu)} - e^{u_1(T_\nu)}]g(-y_1(T_\nu))\}.$$

On the other hand, letting  $s = 0$  in (5.6) we have that

$$J_{KL}^{(N,1)}(T_\nu)E_{t_\infty} \{e^{u_1(T_\nu)}\} \geq E_{t_\infty} \{e^{u_1(T_\nu)}g(-y_1(T_\nu))\}.$$

Combining the above two inequalities, we have

$$(5.7) \quad J_{KL}^{(N,1)}(T_\nu)E_{t_\infty} \{e^{y_1(T_\nu)}\} \geq E_{t_\infty} \{e^{y_1(T_\nu)}g(-y_1(T_\nu))\}.$$

Similarly, by applying Itô's lemma to  $\{g(y_1(t))\}_{0 \leq t < T_\nu}$ , we obtain that  $P_{t_\infty}(T_\nu < \infty | \mathcal{B}_s) = 1$  for any fixed  $s \in \mathbb{R}_+$ , and

$$E_{t_\infty} \left\{ \int_0^{T_\nu} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \right\} = E_{t_\infty} \{g(y_1(T_\nu))\}.$$

On the other hand, using the monotone convergence theorem, as  $\nu \rightarrow \infty$ , we have  $T_\nu \uparrow T$ , and

$$\lim_{\nu \rightarrow \infty} E_{t_\infty} \left\{ \int_0^{T_\nu} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \right\} = E_{t_\infty} \left\{ \int_0^T \frac{1}{2} (\alpha_1(s; \omega))^2 ds \right\} \geq (c_N)^2 \gamma.$$

As a result, there exists a  $\nu$  large enough, such that

$$E_{t_\infty} \{g(y_1(T_\nu))\} = E_{t_\infty} \left\{ \int_0^{T_\nu} \frac{1}{2} (\alpha_1(s; \omega))^2 ds \right\} \geq (c_N)^2 \gamma - \epsilon.$$

Now let us consider the nonnegative function  $p(y) := e^y [g(-y) - g(-\nu^*)] - g(y) + g(\nu^*)$ . We trivially have that  $E_{t_\infty} \{p(y_1(T_\nu))\} \geq 0$ , which implies that

$$\begin{aligned} E_{t_\infty} \{e^{y_1(T_\nu)}g(-y_1(T_\nu))\} &\geq E_{t_\infty} \{e^{y_1(T_\nu)}\}g(-\nu^*) + E_{t_\infty} \{g(y_1(T_\nu))\} - g(\nu^*) \\ &= E_{t_\infty} \{e^{y_1(T_\nu)}\}g(-\nu^*) + E_{t_\infty} \{g(y_1(T_\nu))\} - (c_N)^2 \gamma \geq E_{t_\infty} \{e^{y_1(T_\nu)}\}g(-\nu^*) - \epsilon \\ &\geq E_{t_\infty} \{e^{y_1(T_\nu)}\}[g(-\nu^*) - \epsilon], \end{aligned}$$

since  $E_{t_\infty} \{e^{y_1(T_\nu)}\} \geq 1$ . The above inequality and (5.7) together imply (5.2). This completes the proof.  $\square$

In view of (2.11), (3.13), and Propositions 3.4 and 5.1, we have the following inequalities:

$$(5.8) \quad \max\{f_{S^{(1)},\bar{h}}(0, \dots, 0), \dots, f_{S^{(N)},\bar{h}}(0, \dots, 0)\} = J_{KL}^{(N)}(T_{\bar{h}}) \geq \inf_{T \in \mathbb{B}_\gamma} J_{KL}(T) \geq g(-\nu^*),$$

where  $\bar{h}$  satisfies (5.13) and

$$(5.9) \quad f_{S^{(0)},\bar{h}}(0, \dots, 0) = (c_N)^2 \gamma,$$

and  $\nu^*$  solves

$$(5.10) \quad g(\nu) = (c_N)^2 \gamma.$$

In the following we will establish asymptotic optimality of  $T_{\bar{h}}$  by examining the rate at which the lower and upper bounds in (5.8) approach each other as  $\gamma \rightarrow \infty$  [10, 13, 12]. To this end, we will need to derive the asymptotic behavior of the quantity in (5.8) and (5.9). This is done in the following section.

We will distinguish two cases: the first case is the one of signal-strength symmetry and the second one is the one of signal-strength asymmetry.

**5.1. Signal-strength symmetry.** We capture the signal-strength symmetric case by assuming that  $|c_i| = 1$  for all  $i \in \{1, \dots, N\}$ . In this case, it is natural to consider a simplification of (2.18), namely, the one in which  $h_1 = \dots = h_N$ , since such a choice satisfies (3.14). For a large fixed  $\gamma$ , let us denote by  $h(\gamma)$  the common threshold determined by (5.9). The asymptotic analysis in the last section ensures the existence of  $h(\gamma)$ . The uniqueness of this  $h(\gamma)$  follows from Lemma 3.1. Throughout this subsection, we will slightly abuse our notation to use  $h(\gamma)$  to denote the vector  $(h(\gamma), \dots, h(\gamma))$ .

First we notice that (5.9) simplifies to

$$(5.11) \quad f_{S^{(0)},h(\gamma)}(0, \dots, 0) = \gamma,$$

and (5.8) simplifies to

$$(5.12) \quad f_{S^{(1)},h(\gamma)}(0, \dots, 0) \geq \inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T) \geq g(-\nu^*).$$

We will demonstrate that the difference between the upper and lower bounds in (5.12) is bounded by a constant as  $\gamma \rightarrow \infty$ , with  $h(\gamma)$  and  $\nu^*$  satisfying (5.11) and (5.10), respectively.

**PROPOSITION 5.2.** *As  $\gamma \rightarrow \infty$ ,*

$$f_{S^{(1)},h(\gamma)}(0, \dots, 0) = \log \gamma + \log N - 1 + o(1).$$

*Proof.* Using (4.3) and (4.4), the result follows.  $\square$

Consequently, we have the following result.

**THEOREM 5.3.** *The difference in the optimal detection delay  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$  and the detection delay of  $T_{h(\gamma)}$  of (2.18) with  $h(\gamma)$  satisfying (5.11) is bounded above by  $\log N$ , as  $\gamma \rightarrow \infty$ .*

*Proof.* From  $g(\nu^*) = \gamma$  it is easily seen that

$$g(-\nu^*) = \log \gamma - 1 + o(1).$$

The result follows from (5.12) and Proposition 5.2.  $\square$

*Remark 1.* The asymptotic optimality of the  $N$ -CUSUM stopping rule in Theorem 5.3 is of the same strength as the one of Theorem 1 in [13]. In other words, the difference in the performance of the unknown optimal stopping rule and the  $N$ -CUSUM stopping rule is bounded above by a constant as  $\gamma \rightarrow \infty$ .

**5.2. Signal-strength asymmetry.** In this section we treat the general case  $\max_{i \geq 2} |c_i| > 1$ . Without loss of generality we can assume that  $1 = c_1 = |c_2| = \dots = |c_K|$  for some  $K < N$  and  $\min_{K < i \leq N} |c_i| > 1$ .

Recall that the optimal multichart CUSUM thresholds must satisfy (3.14). While it seems a formidable task to solve for  $\bar{h} = (h_1, \dots, h_N)$  explicitly from this constraint and (5.9), thanks to the asymptotic analysis in section 4, we can give a simple explicit linear constraint on the  $h_i$ 's, which will lead to (3.14) asymptotically as the  $h_i$ 's tend to  $\infty$ .

PROPOSITION 5.4. *For  $\bar{h} = (h_1, \dots, h_N)$  such that*

$$(5.13) \quad c_1^2(h_1 - 1) = c_2^2(h_2 - 1) = \dots = c_N^2(h_N - 1),$$

*the condition (3.14) holds asymptotically as  $h_1 \rightarrow \infty$ . Moreover,*

$$(5.14) \quad J_{KL}^{(N)}(T_{\bar{h}}) = h_1 - 1 + o(1),$$

*as  $h_1 \rightarrow \infty$ .*

*Proof.* This is a mere consequence of (4.4).  $\square$

*Remark 2.* Proposition 5.4 is similar to the result obtained in Lemma 2 of [13]. More specifically, the result in (5.13) is similar to the one obtained in Lemma 2 of [13] with  $c_i^2 = \frac{2}{\mu_i^2}$  for  $i = 1, 2, \dots, N$  and that of (27) in [13] is the same as that of (5.14) for  $c_1^2 = \frac{2}{\mu_1^2}$  and  $c_1 = 1$ . Please notice that the detection delay criterion  $J_{KL}^{(N)}$  of (2.6) incorporates the different signal strengths by including a maximization over all  $i \in \{1, \dots, N\}$ .

This leads to the following result.

PROPOSITION 5.5. *For  $\bar{h}$  satisfying (5.9) and (5.13), we have*

$$(5.15) \quad J_{KL}^{(N)}(T_{\bar{h}}) = \log \gamma + 2 \log |c_N| + \log K - 1 + o(1),$$

*as  $\gamma \rightarrow \infty$ .*

*Proof.* The existence and uniqueness of the  $\bar{h}$  determined by (5.9) and (5.13) can be seen from Lemma 3.1 and (4.3). We can obtain (5.15) from Proposition 5.4 and (4.3).  $\square$

We notice that in the special case in which  $K = 1$ , Proposition 5.5 implies that

$$(5.16) \quad J_{KL}^{(N)}(T_{\bar{h}}) = \log \gamma + 2 \log |c_N| - 1 + o(1).$$

As a result of Proposition 5.5, we obtain the following two theorems that assert the asymptotic optimality of  $T_{\bar{h}}$  of (2.18).

THEOREM 5.6. *Suppose that  $\min_{2 \leq i \leq N} |c_i| > 1$  and that  $\bar{h}$  satisfies (5.9) and (5.13). Then the difference in the optimal detection delay  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$  and the detection delay of  $T_{\bar{h}}$ ,  $J_{KL}^{(N)}(T_{\bar{h}})$  of (2.18) converges to 0 as  $\gamma \rightarrow \infty$ .*

*Proof.* The asymptotic lower bound in (5.8) is  $g(-\nu^*)$  with  $g(\nu^*) = (c_N)^2 \gamma$ , which implies that

$$(5.17) \quad g(-\nu^*) = \log \gamma + 2 \log |c_N| - 1 + o(1).$$

From (5.8) and (5.16) we obtain

$$J_{KL}^{(N)}(T_h) - \inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T) \leq o(1),$$

as  $\gamma \rightarrow \infty$ .  $\square$

**THEOREM 5.7.** *Suppose that  $\min_{K < i \leq N} |c_i| > |c_1| = \dots = |c_K| = 1$  and that  $h$  satisfies (5.9) and (5.13). Then the difference in the optimal detection delay  $\inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T)$  and the detection delay of  $T_h$ , the  $J_{KL}^{(N)}(T_h)$  of (2.18), are bounded above by  $\log K$ , as  $\gamma \rightarrow \infty$ .*

*Proof.* The asymptotic lower bound in (5.8) is  $g(-\nu^*)$ . From (5.8), (5.17), and Proposition 5.5 we obtain

$$J_{KL}^{(N)}(T_h) - \inf_{T \in \mathbb{B}_\gamma} J_{KL}^{(N)}(T) \leq \log K + o(1),$$

as  $\gamma \rightarrow \infty$ .  $\square$

*Remark 3.* The results of Theorems 5.6 and 5.7 show the same strength in asymptotic optimality of the  $N$ -CUSUM rule (2.18) for  $c_1^2 = \frac{2}{\mu_1^2}$  and  $c_1 = 1$  as that of the proposed rule in [13].

**6. Conclusion.** In this paper we have established asymptotic optimality of the  $N$ -CUSUM stopping rule (2.18) as the solution to the stochastic optimization problem of (2.7). This asymptotic optimality is summarized in Theorems 5.3, 5.6, and 5.7. The resemblance of these results to the results of [13] are mathematically not surprising since the Kullback–Leibler information number is included in the criterion and thus does not appear in the formulas for the detection delay of  $T_h$ . Yet, the problem in this paper is strikingly more general than the one treated in [13]. In particular, the processes that are treated in this paper are general Itô processes which can capture signals of general dependencies. More importantly however, in this setup the processes are allowed to be coupled with each other, which provides a natural way in which to capture most systems since it is the case that signals received in one sensor can affect what is seen by another. The results in this work are yet another case [8, 9, 10, 11] which captures the robustness of the  $N$ -CUSUM stopping rule in the problem of quickest detection.

**Appendix A. Proof of Lemma 2.1.** Since the  $\alpha_t^{(i)}$ 's are linear functions of the  $Z_t^{(j)}$ 's, we know that the  $\alpha_t^{(i)}$ 's satisfy the Novikov condition (2.3) from the discussion on page 234 of [17].

To show that (2.16) holds, we introduce a new process  $Y_t := \alpha_t^{(1)}$ . An important observation is that  $\{Y_t\}_{t \geq 0}$  is a Ornstein–Uhlenbeck process under any fixed  $P_{\tau_1, \dots, \tau_N}$ ,  $\tau_i \in \{0, \infty\}$ :

$$(A.1) \quad dY_t = \lambda Y_t dt + \sigma dW_t, \quad \lambda = \left( \sum_{i=1}^N \mathbf{1}_{\{\tau_i=0\}} c_i \beta_i \right) \in \mathbb{R}, \quad \sigma = \sqrt{\sum_{i=1}^N \beta_i^2} \in \mathbb{R}_+,$$

where  $\{W_t\}_{t \geq 0}$  is a standard Brownian motion under  $\mathbb{P}$ :

$$W_t = \frac{1}{\sqrt{\sum_{i=1}^N \beta_i^2}} \sum_{i=1}^N \beta_i w_t^{(i)}.$$

To facilitate later calculation, let us denote by  $Q^{(\lambda)}$  the law of  $Y_t$  in (A.1), by  $E^{Q^{(\lambda)}}$  the expectation under  $Q^{(\lambda)}$ , and  $\mathcal{F}_t^Y = \sigma(Y_s; s \leq t)$ . It is then sufficient to obtain

(2.16) from

$$(A.2) \quad Q^{(\lambda)} \left( \int_0^t Y_s^2 ds < \infty \right) = 1 = Q^{(\lambda)} \left( \int_0^\infty Y_s^2 ds = \infty \right) \quad \forall t \in \mathbb{R}_+.$$

In the following we prove that (A.2) holds.

Using the Girsanov theorem, for any  $t \in \mathbb{R}_+$  and  $q > \frac{3\lambda^2}{2\sigma^2}$ ,

$$(A.3) \quad \begin{aligned} & E^{Q^{(\lambda)}} \left\{ \exp \left( -q \int_0^t Y_s^2 ds \right) \right\} \\ &= E^{Q^{(0)}} \left\{ \exp \left( -q \int_0^t Y_s^2 ds \right) \frac{dQ^{(\lambda)}}{dQ^{(0)}} \Big|_{\mathcal{F}_t^Y} \right\} \\ &= E^{Q^{(0)}} \left\{ \exp \left( - \left( q + \frac{1}{2} \frac{\lambda^2}{\sigma^2} \right) \int_0^t Y_s^2 ds + \frac{\lambda}{\sigma^2} \int_0^t Y_s dY_s \right) \right\} \\ &= e^{\frac{\lambda}{\sigma^2} Y_0^2 - \lambda t} E^{Q^{(0)}} \left\{ \exp \left( - \left( q + \frac{1}{2} \frac{\lambda^2}{\sigma^2} \right) \int_0^t Y_s^2 ds + \frac{\lambda}{\sigma^2} Y_t^2 \right) \right\} \\ &= e^{\frac{\lambda}{\sigma^2} Y_0^2 - \lambda t} \frac{1}{\sqrt{\cosh(bt) - \frac{2\lambda}{b} \sinh(bt)}} \exp \left( - \frac{Y_0^2}{2\sigma^2} \frac{1 - \frac{2\lambda}{b} \coth(bt)}{\coth(bt) - \frac{2\lambda}{b}} \right) < \infty, \end{aligned}$$

where  $b := \sqrt{2q\sigma^2 + \lambda^2}$ . Here we used (2.1) on page 18 of [19]. Therefore, the first equality of (A.2) holds. Moreover, let  $t \rightarrow \infty$  in (A.3) and applying the bounded convergence theorem, we have that

$$E^{Q^{(\lambda)}} \left\{ \exp \left( -q \int_0^\infty Y_s^2 ds \right) \right\} = 0.$$

Hence, the second equality in (A.2) also holds.

**Appendix B. Proof of Lemma 3.1.** To prove Lemma 3.1, the following result will be useful.

LEMMA B.1. *Suppose  $X$  and  $Y$  are two independent random variables, and  $\text{Supp}(X) = \mathbb{R}_+$ . Then  $P(X < Y) < 1$ .*

*Proof.* Since  $\text{Supp}(X) = \mathbb{R}_+$ , for any  $c \in \mathbb{R}_+$ , we have  $P(X < c) < 1$ . Let us denote  $F_Y(y) = P(Y \leq y)$ , then

$$\begin{aligned} P(X < Y) &= \int_{0+}^\infty P(X < y) dF_Y(y) \\ &= \int_0^{c-} P(X < y) dF_Y(y) + \int_c^\infty P(X < y) dF_Y(y) \\ &\leq P(X < c)P(Y < c) + P(Y \geq c) < 1. \end{aligned}$$

This completes the proof.  $\square$

We are ready to give the proof of Lemma 3.1. Without loss of generality, we prove the lemma for  $N = 2$ .

The main tool we will use in this proof is the Feynman–Kac representation of the solution to an elliptic PDE. In particular, fix a filtered probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$  with two independent standard Brownian motions  $\{B_t^{(1)}\}_{t \geq 0}$  and  $\{B_t^{(2)}\}_{t \geq 0}$ . For any

$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) \in \{\pm 1\}^2$ ,  $c_2 \neq 0$ , consider two independent reflected Brownian motions defined as

$$X_t^{\mathcal{S}_1} := X_0 + \mathcal{S}_1 t + \sqrt{2}B_t^{(1)} - \inf_{0 \leq s \leq t} \{(X_0 + \mathcal{S}_1 s + \sqrt{2}B_s^{(1)}) \wedge 0\},$$

$$Y_t^{\mathcal{S}_2} = Y_0 + \mathcal{S}_2 c_2^{-2} t + \sqrt{2}|c_2|^{-1} B_t^{(2)} - \inf_{0 \leq s \leq t} \{(Y_0 + \mathcal{S}_2 c_2^{-2} s + \sqrt{2}|c_2|^{-1} B_s^{(2)}) \wedge 0\}.$$

We denote their first passage times by

$$T_a^{X^{\mathcal{S}_1}} = \inf\{t \geq 0 : X_t^{\mathcal{S}_1} \geq a\} \quad \forall a \in \mathbb{R},$$

$$T_b^{Y^{\mathcal{S}_2}} = \inf\{t \geq 0 : Y_t^{\mathcal{S}_2} \geq b\} \quad \forall b \in \mathbb{R}.$$

For convenience, we will also denote

$$P^{x,\cdot}(\cdot) = P(\cdot | X_0 = x), \quad P^{\cdot,y}(\cdot) = P(\cdot | Y_0 = y), \quad P^{x,y}(\cdot) = P(\cdot | X_0 = x, Y_0 = y).$$

Then by the Feynman–Kac theorem, for any  $(x, y) \in [0, h_1] \times [0, h_2]$ ,

$$0 \leq E^{x,y}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\} = f_{\mathcal{S},h}(x, y),$$

where  $f_{\mathcal{S},h}$  solves (3.1a)–(3.1b). So it suffices to prove that, for  $(x', y') \in [0, x] \times [0, y]$ ,  $0 < h'_1 < h_1$ ,  $0 < h'_2 < h_2$ , and  $\mathcal{S}'_i \leq \mathcal{S}_i$ ,  $i = 1, 2$ ,

$$(B.1) \quad E^{x,y}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\} \leq E^{x',y'}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\} \wedge E^{x,y}\{T_{h_1}^{X^{\mathcal{S}'_1}} \wedge T_{h_2}^{Y^{\mathcal{S}'_2}}\},$$

$$(B.2) \quad E^{0,0}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\} > E^{0,0}\{T_{h'_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\} \vee E^{0,0}\{T_{h_1}^{X^{\mathcal{S}'_1}} \wedge T_{h'_2}^{Y^{\mathcal{S}'_2}}\}.$$

In the following we will prove (B.1) and (B.2) hold.

Using continuity of the sample path and Markov shifting operator, we have

$$T_{h_1}^{X^{\mathcal{S}_1}} = T_x^{X^{\mathcal{S}_1}} + T_{h_1}^{X^{\mathcal{S}_1}} \circ \theta(T_x^{X^{\mathcal{S}_1}}) \geq T_{h_1}^{X^{\mathcal{S}_1}} \circ \theta(T_x^{X^{\mathcal{S}_1}}), \quad P^{x',\cdot}\text{-a.s.},$$

$$T_{h_2}^{Y^{\mathcal{S}_2}} = T_y^{Y^{\mathcal{S}_2}} + T_{h_2}^{Y^{\mathcal{S}_2}} \circ \theta(T_y^{Y^{\mathcal{S}_2}}) \geq T_{h_1}^{Y^{\mathcal{S}_2}} \circ \theta(T_y^{Y^{\mathcal{S}_2}}), \quad P^{\cdot,y'}\text{-a.s.}$$

Hence we have the first inequality in (B.1). Similarly, for any  $y'' \in [0, h_2]$ ,

$$T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}} > T_{h'_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}} = 0, \quad P^{h'_1,y''}\text{-a.s.}$$

Thus,

$$E^{h'_1,y''}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\} > 0.$$

Using Lemma B.1 we know that  $P^{0,0}(T_{h'_1}^{X^{\mathcal{S}_1}} < T_{h_2}^{Y^{\mathcal{S}_2}}) > 0$ . It follows that

$$E^{0,0}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\}$$

$$= E^{0,0}\{T_{h'_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\}$$

$$+ \int_0^{h_2} P^{0,0}\left(T_{h'_1}^{X^{\mathcal{S}_1}} < T_{h_2}^{Y^{\mathcal{S}_2}}, Y_{T_{h'_1}^{X^{\mathcal{S}_1}}}^{\mathcal{S}_2} \in dy''\right) E^{h'_1,y''}\{T_{h_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\}$$

$$> E^{0,0}\{T_{h'_1}^{X^{\mathcal{S}_1}} \wedge T_{h_2}^{Y^{\mathcal{S}_2}}\}.$$

This proves (B.2).

Finally, using Lemma 3 of [10], we have

$$X_t^{\mathcal{S}'_1} \leq X_t^{\mathcal{S}_1}, \quad Y_t^{\mathcal{S}'_2} \leq Y_t^{\mathcal{S}_2} \quad \forall t \geq 0, \quad P\text{-a.s.}$$

Hence,

$$T_{h_1}^{X^{S_1}} \leq T_{h_1}^{X^{S'_1}}, T_{h_2}^{Y^{S_2}} \leq T_{h_2}^{Y^{S'_2}}, P\text{-a.s.},$$

which implies the second inequality in (B.1). The finiteness of  $f_{S,\hbar}(0,0)$  follows from that of  $E^{0,\cdot}\{T_{h_1}^{X^{S_1}}\}$ . The latter can be easily shown to be equal to  $g(-S_1 h_1) < \infty$  (see, for example, [21]).

**Appendix C. Proof of Lemma 3.2.** From (2.16) we have that, for any fixed  $\tau < \infty$  and  $\mathbf{t} = (\tau_1, \dots, \tau_N) \in \{0, \infty\}^N$ ,  $\int_\tau^\infty \frac{1}{2}(\alpha_1(s;\omega))^2 ds = \infty$ ,

$$P_{\mathbf{t}}\left(\int_\tau^\infty \frac{1}{2}(\alpha_1(s;\omega))^2 ds = \infty\right) = 1.$$

Now without loss of generality, assume that  $0 \leq \tau_1, \dots, \tau_r < \infty$  and  $\tau_{r+1} = \dots = \tau_N = \infty$ , we have from (2.5) that

$$\left. \frac{dP_{\tau_1, \dots, \tau_r, \tau_{r+1}, \dots, \tau_N}}{dP_{0, \dots, 0, \infty, \dots, \infty}} \right|_{\mathcal{B}_t} = \exp\left(-\sum_{i=1}^r u_i(t \wedge \tau_i)\right) > 0 \quad \forall t \in \mathbb{R}_+.$$

Hence,  $P_{\mathbf{t}}(\int_t^\infty \frac{1}{2}(\alpha_1(s;\omega))^2 ds = \infty) = 1$  for any  $\mathbf{t} = (\tau_1, \dots, \tau_N) \in (\overline{\mathbb{R}_+})^N$ . Moreover, for any fixed  $\tau' < \infty$ ,

$$1 = P_{\mathbf{t}}\left(\int_t^\infty \frac{1}{2}(\alpha_1(s;\omega))^2 ds = \infty\right) = E_{\mathbf{t}}\left\{P_{\mathbf{t}}\left(\int_t^\infty \frac{1}{2}(\alpha_1(s;\omega))^2 ds = \infty \mid \mathcal{B}_{\tau'}\right)\right\},$$

it must be that  $P_{\mathbf{t}}(\int_t^\infty \frac{1}{2}(\alpha_1(s;\omega))^2 ds = \infty \mid \mathcal{B}_{\tau'}) = 1$ .

**Appendix D. Proof of Lemma 4.1.** Without loss of generality, we prove the lemma for  $N = 2$ .

Using notation from the proof of Lemma 3.1, we have that

$$\begin{aligned} f_{S,\hbar}(x,y) &= E^{x,y}\left\{T_{h_1}^{X^{S_1}} \wedge T_{h_2}^{Y^{S_2}}\right\} = \int_0^\infty P^{x,y}\left(T_{h_1}^{X^{S_1}} > t, T_{h_2}^{Y^{S_2}} > t\right) dt \\ &= \int_0^\infty P^{x,\cdot}\left(T_{h_1}^{X^{S_1}} > t\right) \cdot P^{\cdot,y}\left(T_{h_2}^{Y^{S_2}} > t\right) dt. \end{aligned}$$

Hence, it suffices to show that

$$\begin{aligned} P^{x,\cdot}\left(T_{h_1}^{X^{S_1}} > t\right) &= K_{S_1, \frac{1}{h_1}}\left(\frac{c_1^{-2}t}{h_1}, \frac{x}{h_1}\right), \\ P^{\cdot,y}\left(T_{h_2}^{Y^{S_2}} > t\right) &= K_{S_2, \frac{1}{h_2}}\left(\frac{c_2^{-2}t}{h_2}, \frac{y}{h_2}\right). \end{aligned}$$

However, it is easily seen that both sides of the above equations satisfy the backward Kolmogorov equation (4.1a) and boundary condition (4.1b). The assertion now follows from the uniqueness theorem for PDEs.

Finally,

$$\int_0^\infty K_{+1, \frac{1}{h_1}}\left(\frac{c_1^{-2}t}{h_1}, 0\right) dt = \int_0^\infty P^{0,\cdot}(T_{h_1}^{X^{S_1}} > t) dt = E^{0,\cdot}\{T_{h_1}^{X^{S_1}}\} = g(-h_1).$$

Hence, (4.2) holds for  $i = 1$ . The general case can be similarly proven.

**Appendix E. Proofs of Propositions 4.2 and 4.3.** We now prove the asymptotic results in Propositions 4.2 and 4.3. To this end, we transform the function  $K_{\pm, \epsilon}(t, z)$  defined in (4.1a)–(4.1b) to the solution of a heat equation. Using the Sturm–Liouville theory, we are able to express  $K_{\pm 1, \epsilon}(t, z)$  as a series. From these series, all necessary asymptotic expansions can be derived directly.

**THEOREM E.1.** *Let  $0 < \epsilon < 1/2$  and let  $K_{-1, \epsilon}$  be the solution to (4.1a)–(4.1b) for  $\mathcal{S}_i = -1$  and  $\epsilon_i = \epsilon > 0$ . Then, for  $t > 0$ ,*

$$(E.1) \quad K_{-1, \epsilon}(t, 0) = K_{-1, \epsilon}^0(t) + H_{\epsilon}(t), \quad H_{\epsilon}(t) := \sum_{n=1}^{\infty} K_{-1, \epsilon}^n(t),$$

and the functions  $K_{-1, \epsilon}^0, K_{-1, \epsilon}^1, \dots$ , are explicitly given by

$$(E.2) \quad K_{-1, \epsilon}^0(t) = A_{\epsilon}^0 e^{(\omega^2 \epsilon - \frac{1}{4\epsilon})t},$$

$$(E.3) \quad A_{\epsilon}^0 = \frac{1}{\frac{1-e^{-4\omega}}{2\omega} - 2e^{-2\omega}} (1 - e^{-2\omega})^2 \frac{1}{\omega + \frac{1}{2\epsilon}} e^{\omega - 1/(2\epsilon)},$$

$$K_{-1, \epsilon}^n(t) = A_{\epsilon}^n e^{-1/(2\epsilon)} \frac{\sin \omega_n}{\omega_n} e^{-(\omega_n^2 \epsilon + \frac{1}{4\epsilon})t}, \quad n = 1, 2, \dots,$$

$$A_{\epsilon}^n = \frac{8\epsilon^2 \omega_n^2}{4\epsilon^2 \omega_n^2 + 1 - 2\epsilon},$$

where  $\omega$  and the  $\omega_n$ 's are, respectively, the positive solutions to the transcendental equations

$$(E.4) \quad \tanh \omega = 2\epsilon\omega,$$

$$(E.5) \quad \tan \omega_n = 2\epsilon\omega_n.$$

*Proof.* We begin by transforming (4.1a) to the heat equation: direct calculation shows that the solution  $K_{-1, \epsilon}$  can be written as

$$K_{-1, \epsilon}(t, z) = e^{\frac{1}{2\epsilon}z - \frac{1}{4\epsilon}t} u(t, z),$$

where  $u$  is the solution of

$$(E.6a) \quad \begin{cases} \frac{\partial u}{\partial t} = \epsilon \frac{\partial^2 u}{\partial z^2}, \end{cases}$$

$$(E.6b) \quad \begin{cases} u(0, z) = e^{-\frac{1}{2\epsilon}z}, \quad \left( \frac{1}{2\epsilon}u + \frac{\partial u}{\partial z} \right) \Big|_{z=0} = u|_{z=1} = 0. \end{cases}$$

The problem (E.6a)–(E.6b) can be solved using the Sturm–Liouville theory [6, 27]. More specifically, consider the ordinary differential equation  $-\epsilon \frac{d^2 \hat{u}}{dz^2} = \lambda \hat{u}$ , supplemented with the separated boundary conditions  $\hat{u}(1) = 0$  and  $\hat{u}(0)/(2\epsilon) + \hat{u}'(0) = 0$ . Clearly, the problem is a regular Sturm–Liouville problem, such that the corresponding normalized eigenfunctions form an orthonormal basis of the Hilbert space  $\mathcal{L}^2([0, 1])$  (see, for example, Theorem 12 on page 204 of [27]). Hence,  $u$  defined in (E.6a)–(E.6b) can be written as

$$(E.7) \quad u(t, z) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n \epsilon t} \phi_n(z),$$

where the  $\phi_n$ 's are the orthonormal eigenfunctions of the operator  $L := d^2/dx^2$  with the specified boundary conditions on  $\mathcal{L}^2([0, 1])$  and the  $a_n$ 's are given as projections



of the initial conditions via

$$(E.8) \quad a_n = \int_0^1 \phi_n(z) e^{-z/(2\epsilon)} dz.$$

It will be shown below that  $L$  has exactly one positive eigenvalue  $\omega^2$  given by the transcendental equation (E.4) and negative eigenvalues  $-\omega_n^2$  given by the corresponding equations (E.5). After computing the eigenfunctions explicitly, the  $a_n$ 's can be obtained directly.

We now proceed to compute the positive eigenvalue. To this end, we solve  $d^2\phi_0/dz^2 = \omega^2\phi_0$  with the Dirichlet boundary condition at  $z = 1$  and the mixed boundary condition  $\phi_0/(2\epsilon) + (d\phi_0/dz) = 0$  at  $z = 0$ . Clearly, we can write the solution of  $\phi_0$  as

$$\phi_0(z) = Ae^{\omega z} + Be^{-\omega z},$$

and then use the boundary conditions to find the constants  $A, B$ , and  $\omega$ . This yields the relations

$$(E.9) \quad A = -Be^{-2\omega}, \quad \tanh \omega = 2\epsilon\omega.$$

It can be easily seen that, for  $\epsilon < 1/2$ , we have

$$\frac{d}{d\omega} (\tanh \omega - 2\epsilon\omega) = \frac{1}{\cosh^2 \omega} - 2\epsilon,$$

which monotonically decreases from  $1 - 2\epsilon > 0$  to  $-2\epsilon$  as  $\omega$  increases from 0 to  $\infty$ . Hence, the transcendental equation in (E.9) has a unique positive solution, which is also the unique positive eigenvalue of  $L$ . The corresponding eigenfunction  $\phi$  is then given by

$$(E.10) \quad \phi_0(z) = A (e^{-\omega z} - e^{-2\omega} e^{\omega z}),$$

and the constant  $A$  is fixed by the normalization requirement

$$\int_0^1 \phi_0(z)^2 dz = 1$$

yielding after straightforward algebra

$$(E.11) \quad A = \frac{1}{\sqrt{\frac{1-e^{-4\omega}}{2\omega} - 2e^{-2\omega}}}.$$

Using (E.7), (E.8), (E.10), and (E.11), the formulas for  $K_{-1,\epsilon}^0$  and  $A_\epsilon^0$  in (E.2) and (E.3) follow directly. The calculations for the eigenfunctions corresponding to the negative eigenvalues are similar, and we omit them.  $\square$

Using a similar argument to that in the proof of Theorem E.1, we can obtain the following result.

**THEOREM E.2.** *Let  $0 < \epsilon < 1/2$  and let  $K_{+1,\epsilon}$  be the solution to (4.1a)–(4.1b). Then, for  $t > 0$ ,*

$$(E.12) \quad K_{+1,\epsilon}(t, 0) = e^{-\frac{t}{4\epsilon}} \sum_{n=1}^{\infty} K_{+1,\epsilon}^n(t),$$

and the functions  $K_{+1,\epsilon}^n$  are explicitly given by

$$K_{+1,\epsilon}^n(t) = e^{1/(2\epsilon)} \frac{\sin \omega'_n}{\omega'_n} \frac{8\epsilon^2(\omega'_n)^2}{4\epsilon^2(\omega'_n)^2 + 1 + 2\epsilon} e^{-(\omega'_n)^2 \epsilon t},$$

where the  $\omega'_n$  are the positive solutions of the transcendental equation

$$\tan \omega'_n = -2\epsilon \omega'_n.$$

Using Theorem E.1, it can be shown that the leading term of the asymptotic expansion of the function  $K_{-1,\epsilon}$  for small  $\epsilon$  is given by  $K_{-1,\epsilon}^0$ . In fact, we have the following corollary.

COROLLARY E.3. *Under the assumptions of Theorem E.1 we have*

$$|H_\epsilon(t)| \leq 2e^{-1/(2\epsilon)} \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\epsilon n^2 \pi^2 t}, \quad \int_0^\infty |H_\epsilon(t)| dt \leq \frac{2}{\epsilon \pi^3} e^{-1/(2\epsilon)} \zeta(3),$$

where  $\zeta(3)$  is Apéry's constant.

*Proof.* We verify this inequality by direct calculation. Using the explicit expansion for  $H_\epsilon(t)$  in (E.1), we find that

$$|H_\epsilon(t)| \leq e^{-1/(2\epsilon)} \sum_{n=1}^{\infty} A_\epsilon^n \frac{1}{\omega_n} e^{-(\frac{1}{4\epsilon} + \epsilon \omega_n^2)t} \leq 2e^{-1/(2\epsilon)} \sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\epsilon n^2 \pi^2 t},$$

where we have made use of the fact that  $\omega_n \in [n\pi, n\pi + \pi/2]$ . It follows that

$$\begin{aligned} \int_0^\infty |H_\epsilon(t)| dt &\leq 2e^{-1/(2\epsilon)} \sum_{n=1}^{\infty} \frac{1}{n\pi} \int_0^\infty e^{-\epsilon n^2 \pi^2 t} dt \\ &= \frac{2e^{-1/(2\epsilon)}}{\epsilon \pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{2e^{-1/(2\epsilon)}}{\epsilon \pi^3} \zeta(3). \quad \square \end{aligned}$$

Similarly, the tail of the integral of  $K_{+1,\epsilon}(t, 0)$  is exponentially small as  $\epsilon \rightarrow 0+$ .

COROLLARY E.4. *Under the assumption of Theorem E.2, we have*

$$\int_6^\infty K_{+1,\epsilon}(t, 0) dt \leq \frac{14}{\epsilon \pi^3} e^{-1/\epsilon} \zeta(3).$$

*Proof.* Using the explicit expansion for  $K_{+1,\epsilon}(t, 0)$  in (E.12), we find that

$$\begin{aligned} \int_6^\infty K_{+1,\epsilon}(t, 0) dt &\leq \sum_{n=1}^{\infty} e^{1/(2\epsilon)} \left| \frac{\sin \omega'_n}{\omega'_n} \right| \frac{8\epsilon^2(\omega'_n)^2}{4\epsilon^2(\omega'_n)^2 + 1 + 2\epsilon} \int_6^\infty e^{-(\frac{1}{4\epsilon} + (\omega'_n)^2 \epsilon)t} dt \\ &\leq \sum_{n=1}^{\infty} 2e^{1/(2\epsilon)} \frac{1}{\omega'_n} \frac{4\epsilon}{1 + 4\epsilon^2(\omega'_n)^2} e^{-(\frac{3}{2\epsilon} + 6(\omega'_n)^2 \epsilon)} \\ &< \sum_{n=1}^{\infty} \frac{2}{\epsilon} e^{-1/\epsilon} \frac{1}{(\omega'_n)^3} \leq \frac{2}{\epsilon} e^{-1/\epsilon} \frac{1}{\pi^3} \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3} = \frac{14}{\epsilon} e^{-1/\epsilon} \frac{1}{\pi^3} \zeta(3), \end{aligned}$$

where we have made use of the fact that  $\omega'_n \in [n\pi - \pi/2, n\pi]$ .  $\square$

LEMMA E.5. *Let  $\omega$  be the unique positive solution to  $\tanh \omega = 2\epsilon \omega$  for  $0 < \omega < \frac{1}{2}$ . Then there exists a constant  $C > 0$  such that*

$$\left| \omega - \frac{1}{2\epsilon} + \frac{1}{\epsilon} e^{-1/\epsilon} \right| \leq \frac{C}{\epsilon^2} e^{-2/\epsilon}.$$

*Proof.* We begin by rewriting the equation that  $\omega$  satisfies as

$$f(\omega) = 0, \text{ where } f(x) := x - \frac{1}{2\epsilon} + \left(x + \frac{1}{2\epsilon}\right)e^{-2\epsilon}.$$

We then employ a Newton–Raphson iteration with an initial value  $x_0 = \frac{1}{2\epsilon}$ , and define  $\{x_n\}$  recursively using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots$$

The result now follows from the fact that  $|f''(x)/f'(x)|$  is uniformly bounded on  $\mathbb{R}_+$ .  $\square$

**Proof of Proposition 4.2.** We now use Theorem E.1 to prove Proposition 4.2. To this end, let us introduce

$$(E.13) \quad \tilde{f}_{S^{(0),h}} := \int_0^\infty \prod_{i=1}^N K_{-1,\epsilon_i}^0(c_i^{-2}\epsilon_i t) dt.$$

Using Corollary E.3, it can be shown that the leading term in the asymptotic expansion of  $f_{S^{(0),h}}(0, \dots, 0)$  is  $\tilde{f}_{S^{(0),h}}$ , which can be computed using explicit formulas in Theorem E.1.

Without lose of generality, we prove the results for the case  $N = 2$ .

Using the fact that  $0 \leq K_{-1,\epsilon}(t, 0) \leq 1$  for all  $t, \epsilon > 0$ , and that  $\int |a(t)b(t)|dt \leq \int |a(t)|dt \cdot \int |b(t)|dt$  for any integrable functions  $a(t)$  and  $b(t)$ , we have

(E.14)

$$\begin{aligned} & |f_{S^{(0),h}}(0, 0) - \tilde{f}_{S^{(0),h}}| \\ & \leq \int_0^\infty |K_{-1,\epsilon_1}(c_1^{-2}\epsilon_1 t, 0) \cdot H_{\epsilon_2}(c_2^{-2}\epsilon_2 t)|dt + \int_0^\infty |K_{-1,\epsilon_2}(c_2^{-2}\epsilon_2 t, 0) \cdot H_{\epsilon_1}(c_1^{-2}\epsilon_1 t)|dt \\ & \quad + \int_0^\infty |H_{\epsilon_1}(c_1^{-2}\epsilon_1 t) \cdot H_{\epsilon_2}(c_2^{-2}\epsilon_2 t)|dt \\ & \leq \int_0^\infty |H_{\epsilon_2}(c_2^{-2}\epsilon_2 t)|dt + \int_0^\infty |H_{\epsilon_1}(c_1^{-2}\epsilon_1 t)|dt + \int_0^\infty |H_{\epsilon_1}(c_1^{-2}\epsilon_1 t) \cdot H_{\epsilon_2}(c_2^{-2}\epsilon_2 t)|dt \\ & \leq \int_0^\infty |H_{\epsilon_2}(c_2^{-2}\epsilon_2 t)|dt + \int_0^\infty |H_{\epsilon_1}(c_1^{-2}\epsilon_1 t)|dt \\ & \quad + \int_0^\infty |H_{\epsilon_1}(c_1^{-2}\epsilon_1 t)|dt \cdot \int_0^\infty |H_{\epsilon_2}(c_2^{-2}\epsilon_2 t)|dt \\ & \leq \frac{2c_2^2}{\epsilon_2^2\pi^3}\zeta(3)e^{-\frac{1}{2\epsilon_2}} + \frac{2c_1^2}{\epsilon_1^2\pi^3}\zeta(3)e^{-\frac{1}{2\epsilon_1}} + \frac{4(c_1c_2)^2}{(\epsilon_1\epsilon_2)^2\pi^6}\zeta^2(3)e^{-\frac{1}{2\epsilon_1}-\frac{1}{2\epsilon_2}} \\ & = \mathcal{O}\left(\frac{1}{\epsilon_{\max}^4}e^{-\frac{1}{\epsilon_{\max}}}\right), \end{aligned}$$

where we used Corollary E.3 in the last inequality.

On the other hand, let us denote by  $\omega^{(i)}$  the unique positive solution to transcendental equation  $\tanh \omega^{(i)} = 2\epsilon_i\omega^{(i)}$  for a fixed  $\epsilon_i \in (0, \frac{1}{2})$ . Using the Newton–Raphson iteration and Lemma E.5, we can write

$$(E.15) \quad \omega^{(i)} = \frac{1}{2\epsilon_i} - \frac{1}{\epsilon_i}e^{-1/\epsilon_i} + r_i, \text{ where } |r_i| \leq \frac{C_i}{\epsilon_i^2}e^{-2/\epsilon_i},$$

for some constant  $C_i > 0$ . Using (E.15), straightforward algebra shows that the coefficient  $A_{\epsilon_i}^0$  defined in (E.3) is exponentially close to 1: there exists a constant  $\tilde{C}_i > 0$  such that

$$|A_{\epsilon_i}^0 - 1| \leq \frac{\tilde{C}_i}{\epsilon_i} e^{-1/\epsilon_i}.$$

Similarly, using (E.15) we obtain the decay rate of function  $K_{-1,\epsilon}^0(t)$  defined in (E.2), as  $\epsilon_i \rightarrow 0+$ ,

$$\epsilon_i \left( \omega^{(i)} \right)^2 - \frac{1}{4\epsilon_i} = -\frac{1}{\epsilon_i} e^{-1/\epsilon_i} + r_i + 2e^{-1/\epsilon_i} + \epsilon_i r_i^2 < 0.$$

Using (E.2) and the last equation, we obtain

$$K_{-1,\epsilon_i}^0(c_i^{-2}\epsilon_i t) = A_{\epsilon_i}^0 \exp\left(-c_i^{-2}e^{-1/\epsilon_i}t + (r_i + 2e^{-1/\epsilon_i} + \epsilon_i r_i^2)c_i^{-2}\epsilon_i t\right).$$

With this representation, it follows directly that

$$\begin{aligned} \text{(E.16)} \quad \tilde{f}_{S^{(0)},\hbar} &= \int_0^\infty K_{-1,\epsilon_1}^0(c_1^{-2}\epsilon_1 t) K_{-1,\epsilon_2}^0(c_2^{-2}\epsilon_2 t) dt \\ &= \frac{A_{\epsilon_1}^0 \cdot A_{\epsilon_2}^0}{\sum_{i=1}^2 [c_i^{-2}e^{-1/\epsilon_i} - (r_i + 2e^{-1/\epsilon_i} + \epsilon_i r_i^2)c_i^{-2}\epsilon_i]} \\ &= \frac{1}{\sum_{i=1}^2 c_i^{-2}e^{-1/\epsilon_i}} + \mathcal{O}(1/\epsilon_{\max}). \end{aligned}$$

This result now follows from (E.14) and (E.16).

**Proof of Proposition 4.3.** Below we use Theorems E.1 and E.2 to prove Proposition 4.3.

Using Lemma 4.1, we have

$$\begin{aligned} \text{(E.17)} \quad &|f_{S^{(j)},\hbar}(0,0) - c_j^2(h_j - 1 + e^{-h_j})| \\ &= \left| \int_0^\infty K_{+1,\epsilon_j}(c_j^{-2}\epsilon_j t, 0) \cdot \left( \prod_{i \neq 1} K_{-1,\epsilon_i}(c_i^{-2}\epsilon_i t, 0) \right) dt - \int_0^\infty K_{+1,\epsilon_j}(c_j^{-2}\epsilon_j t, 0) dt \right| \\ &\leq \int_0^{6c_j^2\epsilon_j^{-1}} K_{+1,\epsilon_j}(c_j^{-2}\epsilon_j t, 0) \left| \prod_{i \neq j} K_{-1,\epsilon_i}(c_i^{-2}\epsilon_i t, 0) - 1 \right| dt \\ &\quad + 2 \int_{6c_j^2\epsilon_j^{-1}}^\infty K_{+1,\epsilon_j}(c_j^{-2}\epsilon_j t, 0) dt \\ &\leq \int_0^{6c_j^2\epsilon_j^{-1}} K_{+1,\epsilon_j}(c_j^{-2}\epsilon_j t, 0) \left( 1 - \prod_{i \neq j} K_{-1,\epsilon_i}(c_i^{-2}\epsilon_i t, 0) \right) dt + \mathcal{O}\left(\frac{1}{\epsilon_j^2} e^{-1/\epsilon_j}\right) \\ &\leq \frac{6c_j^2}{\epsilon_j} \left( 1 - \prod_{i \neq j} K_{-1,\epsilon_i}\left(\frac{6c_i^{-2}\epsilon_i}{c_j^{-2}\epsilon_j}, 0\right) \right) + \mathcal{O}\left(\frac{1}{\epsilon_j^2} e^{-1/\epsilon_j}\right), \end{aligned}$$

where we used Corollary E.4 in the second inequality. The third inequality follows from the fact that  $K_{-1,\epsilon_i}(t, 0)$  is bounded between 0 and 1, and it is decreasing in  $t$ .

To bound the first term in (E.17), we use Theorem E.1, Corollary E.3, (E.16), and the inequality  $1 - e^{-|x|} \leq |x|$  to obtain that

$$\left| 1 - \prod_{i \neq j} K_{-1, \epsilon_i} \left( \frac{6c_i^{-2} \epsilon_i}{c_j^{-2} \epsilon_j}, 0 \right) \right| \leq \sum_{i \neq j} \frac{6c_i^{-2} e^{-1/\epsilon_i}}{c_j^{-2} \epsilon_j} (1 + o(1)) = \mathcal{O} \left( \frac{1}{\epsilon_j} \sum_{i \neq j} e^{-1/\epsilon_i} \right).$$

The result now follows.

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