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Formulas for stopped diffusion processes with stopping times based on drawdowns and drawups

Libor Pospisil, Jan Vecer*, Olympia Hadjiliadis

Department of Statistics, Columbia University, 1255 Amsterdam Avenue, NY 10027, USA
Department of Mathematics, Brooklyn College (C.U.N.Y.), Ingersoll Hall, NY 11209, USA

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Abstract

This paper studies drawdown and drawup processes in a general diffusion model. The main result is a formula for the joint distribution of the running minimum and the running maximum of the process stopped at the time of the first drop of size a. As a consequence, we obtain the probabilities that a drawdown of size a precedes a drawup of size b and vice versa. The results are applied to several examples of diffusion processes, such as drifted Brownian motion, Ornstein–Uhlenbeck process, and Cox–Ingersoll–Ross process.

MSC: primary 91B28; secondary 60G44

Keywords: Diffusion process; Drawdowns and drawups; Stopping time

1. Introduction

In this article, we study properties of a general diffusion process $\{X_t\}$ stopped at the first time when its drawdown attains a certain value a. Let us denote this time as $T_D(a)$. The drawdown of a process is defined as the current drop of the process from its running maximum. We present two main results here. First, we derive the joint distribution of the running minimum and the running maximum stopped at $T_D(a)$. Second, we calculate the probability that a drawdown of

* Corresponding author at: Department of Statistics, Columbia University, 1255 Amsterdam Avenue, NY 10027, USA. Tel.: +1 212 851 2145; fax: +1 212 851 2164.
E-mail addresses: lp2185@columbia.edu (L. Pospisil), vecer@stat.columbia.edu (J. Vecer), ohadjiliadis@brooklyn.cuny.edu (O. Hadjiliadis).

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size a precedes a drawup of size b, where the drawup is defined as the increase of $\{X_t\}$ over the running minimum. All formulas are expressed in terms of the drift function, the volatility function, and the initial value of $\{X_t\}$. In addition to the main theorems, this paper contains other results that help us to understand the behavior of diffusion processes better. For example, we relate the probability that the drawup process stopped at $T_D(a)$ is zero to the expected running minimum stopped at $T_D(a)$.

We apply the results to several examples of diffusion processes: drifted Brownian motion, Ornstein–Uhlenbeck process (OU), and Cox–Ingersoll–Ross process (CIR). These examples play important roles in change point detection and in finance. We also discuss how the results presented in this paper are related to the problem of quickest detection and identification of two-sided changes in the drift of general diffusion processes.

Our results extend several theorems stated and proved in [1–3]. These results include the distribution of a diffusion process stopped at the first time it hits either a lower or an upper barrier, and the distribution of the running maximum of a diffusion process stopped at time $T_D(a)$. The formulas for a drifted Brownian motion presented here coincide with the results in [4]. The approach used in [4] is based on a calculation of the expected first passage times of the drawdown and drawup processes to levels a and b. However, while this approach applies to a drifted Brownian motion, it cannot be extended to a general diffusion process. In this paper, we derive the joint distribution of the running maximum and minimum stopped at $T_D(a)$, which can be obtained for a general diffusion process. Subsequently, we use this result to calculate the probability that a drawdown precedes a drawup.

Properties of drawdown and drawup processes are of interest in change point detection, where the goal is to test whether an abrupt change in a parameter of a dynamical system has occurred. Drawdowns and drawups of the likelihood ratio process serve as test statistics for hypotheses about the change point. Details can be found, for example, in [5–7].

The concept of a drawdown has been also been studied in applied probability and in finance. The underlying diffusion process usually represents a stock index, an exchange rate, or an interest rate. Some characteristics of its drawdown, such as the expected maximum drawdown, can be used to measure the downside risks of the corresponding market. The distribution of the maximum drawdown of a drifted Brownian motion was determined in [8]. Cherny and Dupire [9] derived the distribution of a local martingale and its maximum at the first time when the corresponding range process attains value a. Salminen and Vallois [10] derived the joint distribution of the maximum drawdown and the maximum drawup of a Brownian motion up to an independent exponential time. Vecer [11] related the expected maximum drawdown of a market to directional trading. Several authors, such as Grossman and Zhou [12], Cvitanic and Karatzas [13], and Chekhlov et al. [14], discussed the problem of portfolio optimization with drawdown constraints. Meilijson [15] used stopping time $T_D(a)$ to solve an optimal stopping problem based on a drifted Brownian motion and its running maximum. Obloj and Yor [16] studied properties of martingales with representation $H(M_t, \bar{M}_t)$, where M_t is a continuous local martingale and $\bar{M}_t$ its supremum up to time t. Nikeghbali [17] associated the Skorokhod stopping problem with a class of submartingales which includes drawdown processes of continuous local martingales.

This paper is structured in the following way: notation and assumptions are introduced in Section 2. In Section 3, we derive the joint distribution of the running maximum and the running minimum stopped at the first time that the process drops by a certain amount (Theorem 3.1), and in Section 4, we calculate the probability that a drawdown of size a will precede a drawup of size b (Theorems 4.1 and 4.2). Special cases, such as drifted Brownian motion, Ornstein–Uhlenbeck process, Cox–Ingersoll–Ross process, are discussed in Section 5. The relevance of the result in

Section 4 to the problem of quickest detection and identification of two-sided alternatives in the drift of general diffusion processes is also presented in Section 5. Finally, Section 6 contains concluding remarks.

2. Drawdown and drawup processes

In this section, we define drawdown and drawup processes in a diffusion model and present the main assumptions.

Consider an interval $I = (l, r)$, where $-\infty \leq l < r \leq \infty$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{W_t\}$ a Brownian motion, and $\{X_t\}$ a unique strong solution of the following stochastic differential equation:

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x \in I, \tag{1}$$

where $X_t \in I$ for all $t \geq 0$. Moreover, we will assume that functions $\mu(.)$ and $\sigma(.)$ meet the following conditions:

$$\sigma(y) > 0, \quad \text{for } \forall y \in I, \tag{2}$$

$$\int_x^r \frac{\Psi(x, z)}{\int_{z-a}^{z} \Psi(x, y)\mathrm{d}y}\mathrm{d}z = \infty, \quad \text{for all } a > 0 \text{ such that } x - a \in I, \tag{3}$$

$$\int_l^x \frac{\Psi(x, z)}{\int_{z}^{z+b} \Psi(x, y)\mathrm{d}y}\mathrm{d}z = \infty, \quad \text{for all } b > 0 \text{ such that } x + b \in I, \tag{4}$$

where $\Psi(u, z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}$ and $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$. Drifted Brownian motion, Ornstein–Uhlenbeck process, and Cox–Ingersoll–Ross process are examples of diffusion processes satisfying these assumptions. Functions $\mu(.)$ and $\sigma(.)$ will be referred to as the drift and the volatility functions. Note that a process given by (1) has the strong Markov property. If we need to emphasize that x is the starting value of $\{X_t\}$, we will write $\mathbb{P}_x[\,.\,]$.

Let us define the running maximum, $\{M_t\}$, and the running minimum, $\{m_t\}$, of process $\{X_t\}$ as:

$$M_t = \sup_{s\in[0,t]} X_s, \qquad m_t = \inf_{s\in[0,t]} X_s.$$

The drawdown and the drawup of $\{X_t\}$ are defined as:

$$DD_t = M_t - X_t, \qquad DU_t = X_t - m_t.$$

We denote by $T_D(a)$ and $T_U(b)$ the first passage times of the processes $\{DD_t\}$ and $\{DU_t\}$ to the levels a and b respectively, where $a > 0$, $b > 0$, $x - a \in I$, and $x + b \in I$. We set $T_D(a) = \infty$ or $T_U(b) = \infty$ if process DD_t does not reach a or process DU_t does not reach b:

$$T_D(a) = \inf\{t \geq 0; DD_t = a\}, \qquad T_U(b) = \inf\{t \geq 0; DU_t = b\}.$$

Conditions (3) and (4) ensure that

$$\mathbb{P}_x[T_D(a) < \infty] = \lim_{v\to r-} \mathbb{P}_x[M_{T_D(a)} \leq v] = 1,$$
$$\mathbb{P}_x[T_U(b) < \infty] = \lim_{u\to l+} \mathbb{P}_x[m_{T_U(b)} > u] = 1.$$

Thus, we assume that $T_D(a) < \infty$ and $T_U(b) < \infty$ almost surely for any $a > 0$ and $b > 0$, such that $x - a \in I$ and $x + b \in I$.

In the following sections, we derive the joint distribution of $(m_{T_D(a)}, M_{T_D(a)})$ (Section 3) and a formula for the probability $\mathbb{P}_x[T_D(a) < T_U(b)]$ (Section 4).

3. Joint distribution of the running minimum and the running maximum stopped at $T_D(a)$

The distribution of random variable $M_{T_D(a)}$ was derived in [3]. In our paper, we focus on the joint distribution of $(m_{T_D(a)}, M_{T_D(a)})$.

Note that the running minimum stopped at time $T_D(a)$ is bounded by $x-a$ and $x : x-a \leq m_{T_D(a)} \leq x$. The joint distribution of the running minimum and maximum stopped at time $T_D(a)$ will be denoted as $\overline{H}$:

$$\overline{H}_x(u, v) = \mathbb{P}_x[m_{T_D(a)} > u, M_{T_D(a)} > v],$$

where $u \in [x-a, x]$ and $v \in [x, \infty)$. In the following theorem, we will express $\overline{H}$ in terms of function $\Psi(u, z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}$, where $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$.

Theorem 3.1. *Let $a > 0$ such that $x - a \in I$. The random variables $m_{T_D(a)}$ and $M_{T_D(a)}$ have the following joint distribution:*

$$\overline{H}_x(u, v) = \frac{\int_u^x \Psi(u, z)\mathrm{d}z}{\int_u^{u+a} \Psi(u, z)\mathrm{d}z}\, \mathrm{e}^{-\int_{u+a}^{v} \frac{\Psi(u+a,z)}{\int_{z-a}^{z} \Psi(u+a,y)\mathrm{d}y}\mathrm{d}z}, \tag{5}$$

where $u \in [x-a, x]$, $v \in [u+a, \infty)$, $\Psi(u, z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}$ and $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$. If $u \in [x-a, x]$ and $v \in [x, u+a)$, then:

$$\overline{H}_x(u, v) = \frac{\int_u^x \Psi(u, z)\mathrm{d}z}{\int_u^{u+a} \Psi(u, z)\mathrm{d}z}. \tag{6}$$

Proof. The process $\{X_t\}$ is given by (1) and $X_0 = x$. First, let us assume that $u \in [x-a, x]$ and $v \in [u+a, \infty)$. The event $\left\{m_{T_D(a)} > u, M_{T_D(a)} > v\right\}$ occurs if and only if the process $\{X_t\}$ attains $u + a$ without dropping below u and then exceeds v before the drawdown reaches a. Due to the Markov property of the process $\{X_t\}$, we can write the probabilities of these events as follows:

$$\begin{aligned} \overline{H}_x(u, v) &= \mathbb{P}_x[m_{T_D(a)} > u, M_{T_D(a)} > v] \\ &= \mathbb{P}_x[X_{\tau(u,u+a)} = u + a]\, \mathbb{P}_{u+a}[M_{T_D(a)} > v] \\ &= \frac{\int_u^x \Psi(u, z)\mathrm{d}z}{\int_u^{u+a} \Psi(u, z)\mathrm{d}z}\, \mathrm{e}^{-\int_{u+a}^{v} \frac{\Psi(u+a,z)}{\int_{z-a}^{z} \Psi(u+a,y)\mathrm{d}y}\mathrm{d}z}, \end{aligned} \tag{7}$$

where $\tau(u, u+a) = \inf\{t \geq 0; X_t = u \text{ or } X_t = u+a\}$. The formula for the first probability in (7) follows from [1], page 110. The second probability in (7), representing the survival function of $M_{T_D(a)}$, was derived in [3], page 602. Finally, if $v < u + a$, we have $\left\{m_{T_D(a)} > u, M_{T_D(a)} > v\right\} = \left\{m_{T_D(a)} > u\right\} = \left\{X_{\tau(u,u+a)} = u + a\right\}$ because $M_{T_D(a)} \geq m_{T_D(a)} + a$. Thus,

$$\overline{H}_x(u, v) = \frac{\int_u^x \Psi(u, z)\mathrm{d}z}{\int_u^{u+a} \Psi(u, z)\mathrm{d}z}. \quad \diamond$$

The distribution function, the survival function, and the density function of $m_{T_D(a)}$ will be denoted as F, $\overline{F}$, and f, respectively:

$$\overline{F}_x(u) = 1 - F_x(u) = \mathbb{P}_x[m_{T_D(a)} > u], \qquad f_x(u) = \frac{\mathrm{d}F_x(u)}{\mathrm{d}u},$$

where $u \in [x-a, x]$. We can derive the marginal distribution of $m_{T_D(a)}$ from the results in Theorem 3.1.

Corollary 3.2. *Let $a > 0$ such that $x - a \in I$. The distribution function, the density function, and the expected value of random variable $m_{T_D(a)}$ are:*

$$\overline{F}_x(u) = \frac{\int_u^x \Psi(u,z)\mathrm{d}z}{\int_u^{u+a} \Psi(u,z)\mathrm{d}z}, \tag{8}$$

$$f_x(u) = \frac{\Psi(u,u+a)\int_u^x \Psi(u,z)\mathrm{d}z + \int_x^{u+a} \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2}, \tag{9}$$

$$\mathbb{E}_x\left[m_{T_D(a)}\right] = x - \int_{x-a}^x \frac{\int_x^{u+a} \Psi(u,z)\mathrm{d}z}{\int_u^{u+a} \Psi(u,z)\mathrm{d}z}\mathrm{d}u, \tag{10}$$

where $u \in [x-a, x]$, $\Psi(u,z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}$, and $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$.

Let us denote the distribution function and the survival function of the running maximum stopped at $T_D(a)$ as G and $\overline{G}$:

$$\overline{G}_x(v) = 1 - G_x(v) = \mathbb{P}_x[M_{T_D(a)} > v], \quad x \le v < r.$$

Note that the joint distribution of $(m_{T_D(a)}, M_{T_D(a)})$ can be represented by the marginal distributions $\overline{F}$ and $\overline{G}$:

$$\overline{H}_x(u,v) = \overline{F}_x(u)\,\overline{G}_{u+a}(v), \tag{11}$$

where $u \in [x-a, x]$ and $v \in [u+a, \infty)$. Let us calculate the derivative of $\overline{H}_x(u,v)$ with respect to u, which will be used in the proof of the main theorem.

Lemma 3.3. *Let $a > 0$ such that $x - a \in I$. Let $u \in [x-a, x]$ and $v \ge u + a$. Then*

$$-\frac{\partial \overline{H}_x}{\partial u}(u,v) = \mathrm{e}^{-\int_{u+a}^v \frac{\Psi(u+a,z)}{\int_{z-a}^z \Psi(u+a,y)\mathrm{d}y}\mathrm{d}z}\,\frac{\int_x^{u+a} \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2}, \tag{12}$$

where $\Psi(u,z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}$ and $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$.

Proof. According to (11),

$$\frac{\partial \overline{H}_x}{\partial u}(u,v) = \frac{\partial \overline{F}_x(u)}{\partial u}\overline{G}_{u+a}(v) + \overline{F}_x(u)\frac{\partial \overline{G}_{u+a}(v)}{\partial u}.$$

Note that the function Ψ has the following property: $\Psi(a,b)\Psi(b,c) = \Psi(a,c)$. Therefore, $\frac{\int_\cdot^\cdot \Psi(u,z)\mathrm{d}z}{\int_\cdot^\cdot \Psi(u,z)\mathrm{d}z} = \frac{\int_\cdot^\cdot \Psi(C,z)\mathrm{d}z}{\int_\cdot^\cdot \Psi(C,z)\mathrm{d}z}$ and $\frac{\Psi(u,y)}{\int_\cdot^\cdot \Psi(u,z)\mathrm{d}z} = \frac{\Psi(C,y)}{\int_\cdot^\cdot \Psi(C,z)\mathrm{d}z}$ for any constant C. Thus, the first variable of Ψ is redundant in such fractions and can be omitted during the calculation of their derivative with respect to u. Using formula (9), we have

$$\frac{\partial \overline{F}_x(u)}{\partial u} = -f_x(u) = \frac{-\int_x^{u+a} \Psi(u,z)\mathrm{d}z - \Psi(u,u+a)\int_u^x \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2}.$$

The derivative of $\overline{G}_{u+a}(v)$ with respect to u is given by:

$$\begin{aligned}\frac{\partial \overline{G}_{u+a}(v)}{\partial u} &= \frac{1}{\int_u^{u+a} \Psi(u+a,z)\mathrm{d}z} \mathrm{e}^{-\int_x^{u+a} \frac{\Psi(u+a,z)}{\int_{z-a}^{z} \Psi(u+a,y)\mathrm{d}y}\mathrm{d}z} \\ &= \frac{\Psi(u,u+a)}{\int_u^{u+a} \Psi(u,z)\mathrm{d}z}\, \overline{G}_{u+a}(v).\end{aligned}$$

Combining these results yields formula (12). ◇

Formula (9) allows us to decompose the density of $m_{T_D(a)}$ into two parts:

$$\begin{aligned} f_x(u)\mathrm{d}u &= \mathbb{P}_x[DU_{T_D(a)} > 0, m_{T_D(a)} \in (u, u+\mathrm{d}u)] \\ &\quad + \mathbb{P}_x[DU_{T_D(a)} = 0, m_{T_D(a)} \in (u, u+\mathrm{d}u)]. \end{aligned} \tag{13}$$

The set $\left\{DU_{T_D(a)} = 0\right\}$ corresponds to the event that the process attained its running minimum at time $T_D(a) : X_{T_D(a)} = m_{T_D(a)}$. In the following lemma, we calculate the probabilities introduced in (13).

Lemma 3.4. *Let $a > 0$ such that $x - a \in I$. Then*

$$\mathbb{P}_x[DU_{T_D(a)} > 0, m_{T_D(a)} \in (u, u+\mathrm{d}u)] = \frac{\int_x^{u+a} \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2}\mathrm{d}u, \tag{14}$$

$$\mathbb{P}_x[DU_{T_D(a)} = 0, m_{T_D(a)} \in (u, u+\mathrm{d}u)] = \frac{\Psi(u,u+a)\int_u^x \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2}\mathrm{d}u, \tag{15}$$

$$\mathbb{P}_x[DU_{T_D(a)} = 0] = \int_{x-a}^{x} \frac{\Psi(u,u+a)\int_u^x \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2}\mathrm{d}u, \tag{16}$$

where $\Psi(u,z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}$ and $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$.

Proof. Let us use the relationship $M_{T_D(a)} = m_{T_D(a)} + DU_{T_D(a)} + a$ to rewrite probability (14) in terms of the function $\overline{H}_x(u,v)$:

$$\begin{aligned}\mathbb{P}_x[DU_{T_D(a)} > 0, m_{T_D(a)} \in (u, u+\mathrm{d}u)] &= \mathbb{P}_x[M_{T_D(a)} > u+a, m_{T_D(a)} \in (u, u+\mathrm{d}u)] \\ &= -\frac{\partial \overline{H_x}}{\partial u}(u, u+a)\mathrm{d}u\end{aligned}$$

for $u \in [x-a, x]$. Thus, result (14) follows from Lemma 3.3. Formula (15) can be obtained using the decomposition of f in (13) and Eqs. (9) and (14). The result (15) also implies (16):

$$\mathbb{P}_x[DU_{T_D(a)} = 0] = \int_{x-a}^{x} \mathbb{P}_x[DU_{T_D(a)} = 0, m_{T_D(a)} \in (u, u+\mathrm{d}u)]. \quad \diamond$$

Remark 3.5. If $a > 0$ such that $x - a \in I$,

$$\mathbb{P}_x[DU_{T_D(a)} = 0] = -\frac{\partial}{\partial a}\mathbb{E}_x\left[m_{T_D(a)}\right]. \tag{17}$$

Proof. Formula (17) can be verified by calculating the derivative of $\mathbb{E}_x[m_{T_D(a)}]$, which is given by (10), and comparing the result with (16). ◇

Let us heuristically explain Remark 3.5 using the following expression:

$$m_{T_D(a)} = (M_{T_D(a)} - a)\,\mathbb{I}_{\{DU_{T_D(a)}=0\}} + m_{T_D(a)}\mathbb{I}_{\{DU_{T_D(a)}>0\}}.$$

Shifting a by a small number h has an impact on $m_{T_D(a)}$ only if the process $\{X_t\}$ attained its running minimum m at time $T_D(a)$:

$$DU_{T_D(a)} = X_{T_D(a)} - m_{T_D(a)} = 0.$$

In this case, $m_{T_D(a)} = M_{T_D(a)} - a$ and the change in $m_{T_D(a)}$ is $-h$ because the running maximum $M_{T_D(a)}$ remains the same. On the other hand, if $\{X_t\}$ is greater than $m_{T_D(a)}$, that is, $DU_{T_D(a)} > 0$, then small changes in a do not affect $m_{T_D(a)}$. As a result,

$$m_{T_D(a+h)} - m_{T_D(a)} \approx -h\,\mathbb{I}_{\{DU_{T_D(a)}=0\}}$$

for h small. Applying the expected value to this relationship and letting $h \to 0$ lead to Eq. (17).

The knowledge of the joint distribution $\overline{H}_x(u, v)$ allows us to determine the distribution and the expected value of the range process, $R_t = M_t - m_t$, stopped at time $T_D(a)$.

Corollary 3.6. *Let $a > 0$ such that $x - a \in I$. The distribution of the range process $R_t = M_t - m_t$, stopped at time $T_D(a)$ is:*

$$\mathbb{P}_x[R_{T_D(a)} > r] = \int_{x-a}^{x} \overline{G}_{u+a}(u + r)\frac{\int_x^{u+a} \Psi(u, z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u, z)\mathrm{d}z\right)^2}\mathrm{d}u, \tag{18}$$

$$\mathbb{P}_x[R_{T_D(a)} = a] = \int_{x-a}^{x} \frac{\Psi(u, u + a)\int_u^{x} \Psi(u, z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u, z)\mathrm{d}z\right)^2}\mathrm{d}u, \tag{19}$$

where $r > a$. The expected value of $R_{T_D(a)}$:

$$\mathbb{E}_x[R_{T_D(a)}] = \int_x^{\infty} \overline{G}_x(v)\mathrm{d}v + \int_{x-a}^{x} \frac{\int_x^{u+a} \Psi(u, z)\mathrm{d}z}{\int_u^{u+a} \Psi(u, z)\mathrm{d}z}\mathrm{d}u, \tag{20}$$

where

$$\overline{G}_{u+a}(v) = \mathrm{e}^{-\int_{u+a}^{v} \frac{\Psi(u+a,z)}{\int_{z-a}^{z} \Psi(u+a,y)\mathrm{d}y}\mathrm{d}z},$$

$$\Psi(u, z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}, \quad \textit{and} \quad \gamma(y) = \frac{\mu(y)}{\sigma^2(y)}.$$

4. Probability of a drawdown preceding a drawup

In this section, we derive formulas for the probabilities that a drawdown of size a precedes a drawup of size b and vice versa. The calculation is based on the knowledge of the joint distribution of $(m_{T_D(a)}, M_{T_D(a)})$, which appears in Theorem 3.1.

Theorem 4.1. *Assume that $\{X_t\}$ is a unique strong solution of Eq. (1) and that conditions (2)–(4) are satisfied. Let $b \geq a > 0$ such that $x - a \in I$ and $x + b \in I$. Then:*

$$\mathbb{P}_x[T_D(a) < T_U(b)] = 1 - \int_{x-a}^{x} \overline{G}_{u+a}(u+b) \frac{\int_x^{u+a} \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2} \mathrm{d}u, \tag{21}$$

$$\mathbb{P}_x[T_D(a) > T_U(b)] = \int_{x-a}^{x} \overline{G}_{u+a}(u+b) \frac{\int_x^{u+a} \Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a} \Psi(u,z)\mathrm{d}z\right)^2} \mathrm{d}u, \tag{22}$$

where

$$\overline{G}_{u+a}(v) = \mathrm{e}^{-\int_{u+a}^{v} \frac{\Psi(u+a,z)}{\int_{z-a}^{z} \Psi(u+a,y)\mathrm{d}y} \mathrm{d}z},$$

$$\Psi(u,z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}, \quad \textit{and} \quad \gamma(y) = \frac{\mu(y)}{\sigma^2(y)}.$$

Proof. Let $b \geq a > 0$. First, we will show that

$$\{T_D(a) < T_U(b)\} = \left\{0 < DU_{T_D(a)} < b - a\right\} \cup \left\{DU_{T_D(a)} = 0\right\}. \tag{23}$$

The range of the process $\{X_t\}$ at t is defined as $R_t = M_t - m_t$, which implies $R_t = DU_t + DD_t$. One can also prove that

$$R_t = \max\left\{\sup_{[0,t]} DU_u,\ \sup_{[0,t]} DD_u\right\}. \tag{24}$$

The process on the right-hand side of (24) is non-decreasing and equals zero at time $t = 0$. Moreover, it increases only if $\sup_{[0,t]} DU_u$ or $\sup_{[0,t]} DD_u$ changes, which occurs when either $X_t = M_t$ or $X_t = m_t$. In this case, the right-hand side is $M_t - m_t$, which justifies (24).

According to the formula (24), $DU_{T_D(a)} + a = \max\left\{\sup_{[0,T_D(a)]} DU_u, a\right\}$. Thus, $DU_{T_D(a)} = \max\left\{\sup_{[0,T_D(a)]} DU_u - a,\ 0\right\}$ and

$$\begin{aligned}\{T_D(a) < T_U(b)\} &= \left\{\sup_{[0,T_D(a)]} DU_u < b\right\} \\ &= \left\{0 < DU_{T_D(a)} < b - a\right\} \cup \left\{DU_{T_D(a)} = 0\right\},\end{aligned}$$

which proves (23). Furthermore, using the relationship $M_{T_D(a)} = m_{T_D(a)} + DU_{T_D(a)} + a$, we have:

$$\begin{aligned}&\mathbb{P}_x[DU_{T_D(a)} > b - a, m_{T_D(a)} \in (u, u + \mathrm{d}u)] \\ &\quad = \mathbb{P}_x[M_{T_D(a)} > u + b, m_{T_D(a)} \in (u, u + \mathrm{d}u)] = -\frac{\partial \overline{H}_x}{\partial u}(u, u + b)\mathrm{d}u.\end{aligned}$$

Now let us calculate the probability of the event $\{T_D(a) < T_U(b)\}$:

$$\begin{aligned}\mathbb{P}_x[T_D(a) < T_U(b)] &= \mathbb{P}_x[DU_{T_D(a)} = 0] + \mathbb{P}_x[0 < DU_{T_D(a)} < b - a] \\ &= 1 - \mathbb{P}_x[DU_{T_D(a)} > b - a] \\ &= 1 - \int_{x-a}^{x} \mathbb{P}_x[DU_{T_D(a)} > b - a, m_{T_D(a)} \in (u, u + \mathrm{d}u)] \\ &= 1 - \int_{x-a}^{x} \mathbb{P}_x[M_{T_D(a)} > u + b, m_{T_D(a)} \in (u, u + \mathrm{d}u)] \\ &= 1 - \int_{x-a}^{x} \left\{ -\frac{\partial \overline{H_x}}{\partial u}(u, u + b) \right\} \mathrm{d}u. \end{aligned} \tag{25}$$

The derivative of $\overline{H}$ is calculated in Lemma 3.3. If we replace $\left\{ -\frac{\partial \overline{H_x}}{\partial u}(u, u + b) \right\}$ in (25) with that result, we obtain formula (21). Probability (22) is the complement of (21). ◇

If $b < a$, the formula for $\mathbb{P}_x[T_D(a) < T_U(b)]$ is a modification of (21).

Theorem 4.2. *Assume that $\{X_t\}$ is a unique strong solution of Eq. (1) and that conditions (2)–(4) are satisfied. Let $0 < b < a$ such that $x - a \in I$ and $x + b \in I$. Then:*

$$\mathbb{P}_x[T_D(a) < T_U(b)] = \int_{x}^{x+b} G^*_{v-b}(v - a) \frac{\int_{v-b}^{x} \Psi(v, z)\mathrm{d}z}{\left(\int_{v-b}^{v} \Psi(v, z)\mathrm{d}z\right)^2} \mathrm{d}v, \tag{26}$$

$$\mathbb{P}_x[T_D(a) > T_U(b)] = 1 - \int_{x}^{x+b} G^*_{v-b}(v - a) \frac{\int_{v-b}^{x} \Psi(v, z)\mathrm{d}z}{\left(\int_{v-b}^{v} \Psi(v, z)\mathrm{d}z\right)^2} \mathrm{d}v, \tag{27}$$

where

$$G^*_{v-b}(u) = \mathrm{e}^{-\int_u^{v-b} \frac{\Psi(v-b,z)}{\int_z^{z+b} \Psi(v-b,y)\mathrm{d}y} \mathrm{d}z},$$

$$\Psi(u, z) = \mathrm{e}^{-2\int_u^z \gamma(y)\mathrm{d}y}, \quad \textit{and} \quad \gamma(y) = \frac{\mu(y)}{\sigma^2(y)}.$$

Proof. The proof is analogous to the proof of Theorem 4.1. ◇

The procedure we used in the proof of Theorem 4.1 allows us to interpret Eqs. (21) and (22) as follows:

$$\begin{aligned}\mathbb{P}_x[T_D(a) < T_U(b)] &= 1 - \int_{x-a}^{x} \mathbb{P}_x[M_{T_D(a)} > u + b, m_{T_D(a)} \in (u, u + \mathrm{d}u)] \\ &= \int_{x-a}^{x} \mathbb{P}_x[M_{T_D(a)} \le u + b, m_{T_D(a)} \in (u, u + \mathrm{d}u)], \\ \mathbb{P}_x[T_D(a) > T_U(b)] &= \int_{x-a}^{x} \mathbb{P}_x[M_{T_D(a)} > u + b, m_{T_D(a)} \in (u, u + \mathrm{d}u)]. \end{aligned}$$

Let us discuss this interpretation. If $m_{T_D(a)}$ lies in a neighborhood of u, the event of $\{T_D(a) > T_U(b)\}$ coincides with $\left\{M_{T_D(a)} > u + b\right\}$. Probability $\mathbb{P}_x[T_D(a) > T_U(b)]$ is then the integral of $\mathbb{P}_x[M_{T_D(a)} > u + b, m_{T_D(a)} \in (u, u + \mathrm{d}u)]$ over all possible values of $m_{T_D(a)}$.

Table 1
Function $\Psi(u,z)$ for examples of diffusion processes.

Process	I	$\mu(y)$	$\sigma(y)$	$\Psi(u,z)$
Brownian motion	$\mathbb{R}$	μ	σ	$\mathrm{e}^{-\frac{2\mu}{\sigma^2}(z-u)}$
Ornstein–Uhlenbeck process	$\mathbb{R}$	$\kappa(\theta-y)$	σ	$\mathrm{e}^{\frac{\kappa}{\sigma^2}\left[(z-\theta)^2-(u-\theta)^2\right]}$
Cox–Ingersoll–Ross process	$(0,\infty)$	$\kappa(\theta-y)$	$\sigma\sqrt{y}$	$\left(\frac{z}{u}\right)^{-\frac{2\kappa\theta}{\sigma^2}}\mathrm{e}^{\frac{2\kappa}{\sigma^2}(z-u)}$

When $a=b$ in (21) and (22), we have $\{T_D(a)<T_U(a)\}=\left\{DU_{T_D(a)}=0\right\}$:

$$\mathbb{P}_x[T_D(a)<T_U(a)]=\int_{x-a}^{x}\frac{\Psi(u,u+a)\int_u^x\Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a}\Psi(u,z)\mathrm{d}z\right)^2}\mathrm{d}u,$$

$$\mathbb{P}_x[T_D(a)>T_U(a)]=\int_{x-a}^{x}\frac{\int_x^{u+a}\Psi(u,z)\mathrm{d}z}{\left(\int_u^{u+a}\Psi(u,z)\mathrm{d}z\right)^2}\mathrm{d}u.$$

5. Application of the results

In this section, we apply the results from Corollary 3.2 and Theorem 4.1 to the following examples of diffusion processes: Brownian motion, Ornstein–Uhlenbeck process and Cox–Ingersoll–Ross process. We also present an application of the result in Theorem 4.1 to the problem of quickest detection and identification of two-sided changes in the drift of general diffusion processes.

5.1. *Examples of diffusion processes*

The formulas for $\overline{G}_x(v)$, $\overline{F}_x(v)$, $\overline{H}_x(u,v)$, and $\mathbb{P}_x[T_D(a)<T_U(b)]$ depend on function $\Psi(u,z)$. Table 1 shows specific forms of this function for several examples of diffusion processes $\mathrm{d}X_t=\mu(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}W_t$, where $X_t\in I$ and $X_0=x$.

We assume that conditions (2)–(4) are satisfied for all these processes. In the cases of a drifted Brownian motion and an Ornstein–Uhlenbeck process, the conditions hold true for any combination of the parameters. In the case of a Cox–Ingersoll–Ross process, we need to make an additional assumption: $k\theta>\sigma^2/2$.

One can derive an analytical expression of the function $\overline{H}_x(u,v)$ and the probability $\mathbb{P}_x[T_D(a)<T_U(b)]$ for Brownian motion:

$$\overline{H}_x(u,v)=\frac{\mathrm{e}^{\frac{2\mu}{\sigma^2}a}-\mathrm{e}^{\frac{2\mu}{\sigma^2}(u-(x-a))}}{\mathrm{e}^{\frac{2\mu}{\sigma^2}a}-1}\exp\left\{-(v-x)\frac{\frac{2\mu}{\sigma^2}}{\mathrm{e}^{\frac{2\mu}{\sigma^2}a}-1}\right\},$$

where $u\in[x-a,x]$ and $v\in[u+a,\infty)$, implying

$$\overline{G}_x(v)=\mathbb{P}_x[M_{T_D(a)}>v]=\exp\left\{-(v-x)\frac{\frac{2\mu}{\sigma^2}}{\mathrm{e}^{\frac{2\mu}{\sigma^2}a}-1}\right\},\quad v\in[x,\infty),$$

$$\overline{F}_x(u) = \mathbb{P}_x[m_{T_D(a)} > u] = \frac{e^{\frac{2\mu}{\sigma^2}a} - e^{\frac{2\mu}{\sigma^2}(u-(x-a))}}{e^{\frac{2\mu}{\sigma^2}a} - 1}, \quad u \in [x-a, x],$$

$$\mathbb{E}_x\left[m_{T_D(a)}\right] = x - \frac{\sigma^2}{2\mu} + \frac{a}{e^{\frac{2\mu}{\sigma^2}a} - 1},$$

$$\mathbb{P}_x[T_D(a) < T_U(b)] = 1 - \exp\left\{-(b-a)\frac{\frac{2\mu}{\sigma^2}}{e^{\frac{2\mu}{\sigma^2}a} - 1}\right\}\frac{e^{\frac{2\mu}{\sigma^2}a} - \frac{2\mu}{\sigma^2}a - 1}{e^{\frac{2\mu}{\sigma^2}a} + e^{-\frac{2\mu}{\sigma^2}a} - 2},$$

where $b \geq a > 0$. If $a = b$,

$$\mathbb{P}_x[T_D(a) < T_U(a)] = \frac{e^{-\frac{2\mu}{\sigma^2}a} + \frac{2\mu}{\sigma^2}a - 1}{e^{\frac{2\mu}{\sigma^2}a} + e^{-\frac{2\mu}{\sigma^2}a} - 2}.$$

Random variable $M_{T_D(a)}$ has an exponential distribution on $[x, \infty)$ and $m_{T_D(a)}$ has a truncated exponential distribution on $[x-a, x]$. Note that the formula for $\mathbb{P}_x[T_D(a) < T_U(b)]$ is identical with the results presented in [4].

When the drift μ equals zero, the formulas further reduce to:

$$\overline{H}_x(u, v) = \mathbb{P}_x[m_{T_D(a)} > u, M_{T_D(a)} > v] = \frac{x-u}{a}e^{-\frac{v-(u+a)}{a}},$$

where $u \in [x-a, x]$ and $v \in [u+a, \infty)$, implying

$$\overline{G}_x(v) = \mathbb{P}_x[M_{T_D(a)} > v] = e^{-\frac{v-x}{a}}, \quad v \in [x, \infty),$$

$$\overline{F}_x(u) = \mathbb{P}_x[m_{T_D(a)} > u] = \frac{x-u}{a} \quad \text{and} \quad \mathbb{E}_x\left[m_{T_D(a)}\right] = x - \frac{a}{2},$$

$$\mathbb{P}_x[T_D(a) < T_U(b)] = 1 - \frac{1}{2}e^{-\frac{b-a}{a}},$$

where $b \geq a > 0$. If $a = b$,

$$\mathbb{P}_x[T_D(a) < T_U(a)] = \mathbb{P}_x[T_D(a) > T_U(a)] = \frac{1}{2}.$$

Hence, $M_{T_D(a)}$ has an exponential distribution on $[x, \infty)$ with parameter $\frac{1}{a}$ and $m_{T_D(a)}$ has a uniform distribution on $[x-a, x]$.

Calculation of $\mathbb{P}_x[T_D(a) < T_U(b)]$ for an Ornstein–Uhlenbeck process and a Cox–Ingersoll–Ross process involves numerical integration.

In Figs. 1, 3 and 5, we have plotted densities of $M_{T_D(a)}$ and $m_{T_D(a)}$ for various diffusion processes. Figs. 2, 4 and 6 capture dependence of the probability $\mathbb{P}_x[T_D(a) < T_U(b)]$ on the parameters of the processes.

Let us discuss the interpretation of Fig. 4, which shows the probability $\mathbb{P}_x[T_D(1) < T_U(1)]$ as a function of κ/σ^2. When $\kappa = 0$, the drift term of $\{X_t\}$ vanishes and the probability is $\frac{1}{2}$. Moreover, if the process starts at its long-term mean, $x = \theta$, it is symmetric and $\mathbb{P}_x[T_D(1) < T_U(1)] = \frac{1}{2}$ for any value of κ/σ^2. Now let us assume that $x = \theta + 1$. As κ/σ^2 increases, the drift term will prevail over the volatility term and the process will be pushed down from x to θ. As a result, a drawdown of size 1 will tend to occur before a drawup of size 1, which explains the convergence of $\mathbb{P}_x[T_D(a) < T_U(a)]$ to 1 as $\kappa/\sigma^2 \to \infty$. A similar reasoning can be used to justify the convergence of $\mathbb{P}_x[T_D(1) < T_U(1)]$ to 0 if $x = \theta - 1$.

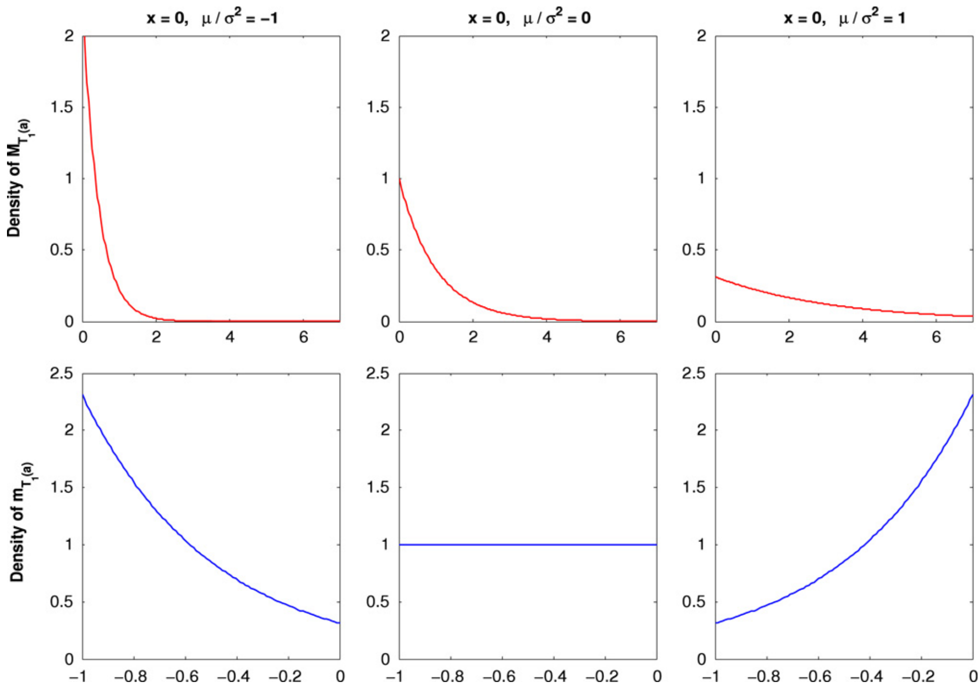

Fig. 1. Densities of $M_{T_D(a)}$ and $m_{T_D(a)}$, where $a = 1$, for a drifted Brownian motion: $X_t = \mu t + \sigma W_t$. The densities depend on the parameters through ratio μ/σ^2. $M_{T_D(a)}$ has an exponential distribution on $[0, \infty)$, while $m_{T_D(a)}$ has a uniform distribution on $[-1, 0]$ if $\mu = 0$ and a truncated exponential distribution on $[-1, 0]$ otherwise.

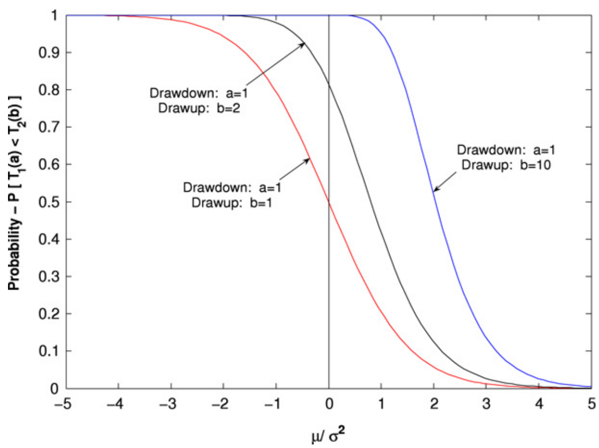

Fig. 2. Probability $\mathbb{P}_x[T_D(a) < T_U(b)]$ as a function of μ/σ^2 for different values of a and b. We assume that $\{X_t\}$ is a drifted Brownian motion, $X_t = \mu t + \sigma W_t$. Note that $\mathbb{P}_x[T_D(1) < T_U(1)] = 0.5$ for $\mu = 0$.

5.2. The problem of quickest detection and identification

In this example, we present the problem of quickest detection and identification of two-sided changes in the drift of a general diffusion process. More specifically, we give precise

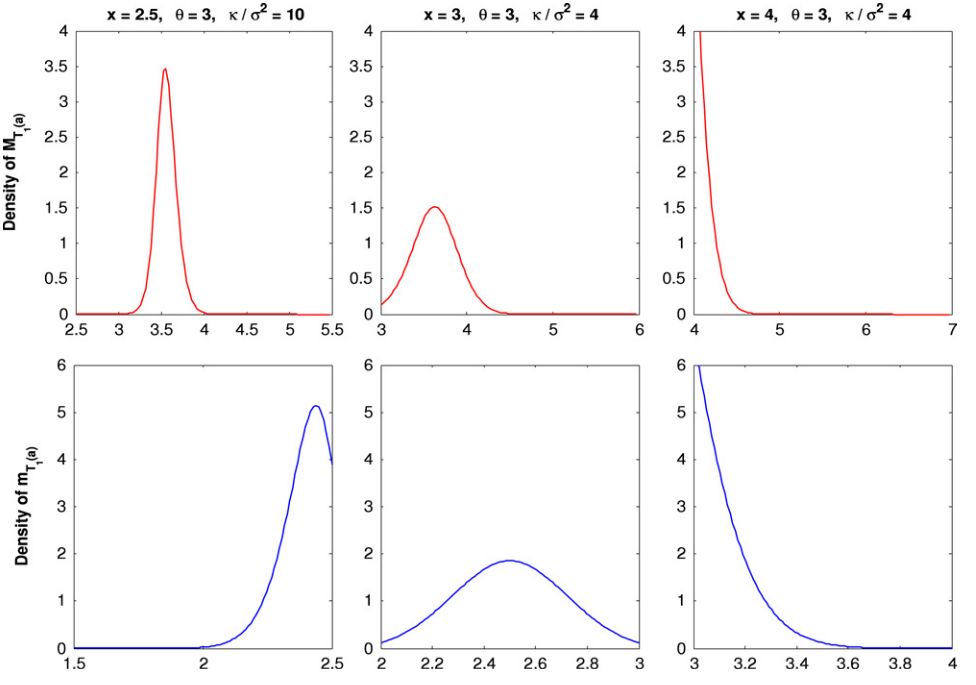

Fig. 3. Densities of $M_{T_D(a)}$ and $m_{T_D(a)}$, where $a = 1$, assuming that $\{X_t\}$ is an Ornstein–Uhlenbeck process: $\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$, $X_0 = x$. The densities depend on parameters κ and σ through ratio κ/σ^2. Note that if $x = \theta$, process $\{X_t\}$ is symmetric and consequently, $m_{T_D(a)}$ has a symmetric distribution.

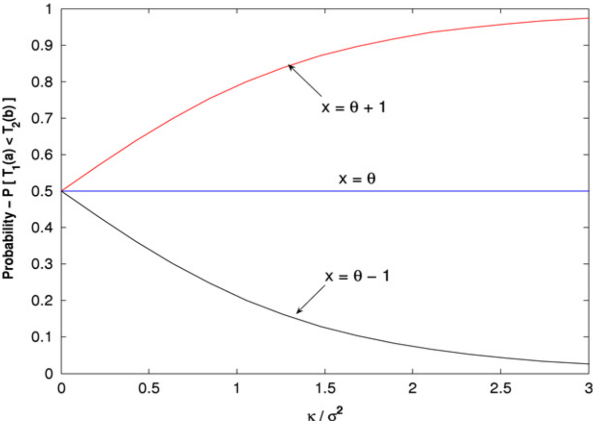

Fig. 4. Probability $\mathbb{P}_x[T_D(a) < T_U(b)]$, where $a = b = 1$, as a function of κ/σ^2. Process $\{X_t\}$ is an Ornstein–Uhlenbeck process: $\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$, $X_0 = x$. If $x = \theta$, then the process is symmetric and the probability is 0.5 for any value of κ/σ^2.

calculations of the probability of misidentification of two-sided alternatives. In particular, let $\{X_t\}$ be a diffusion process with the initial value $X_0 = x$ and the following dynamics up to a

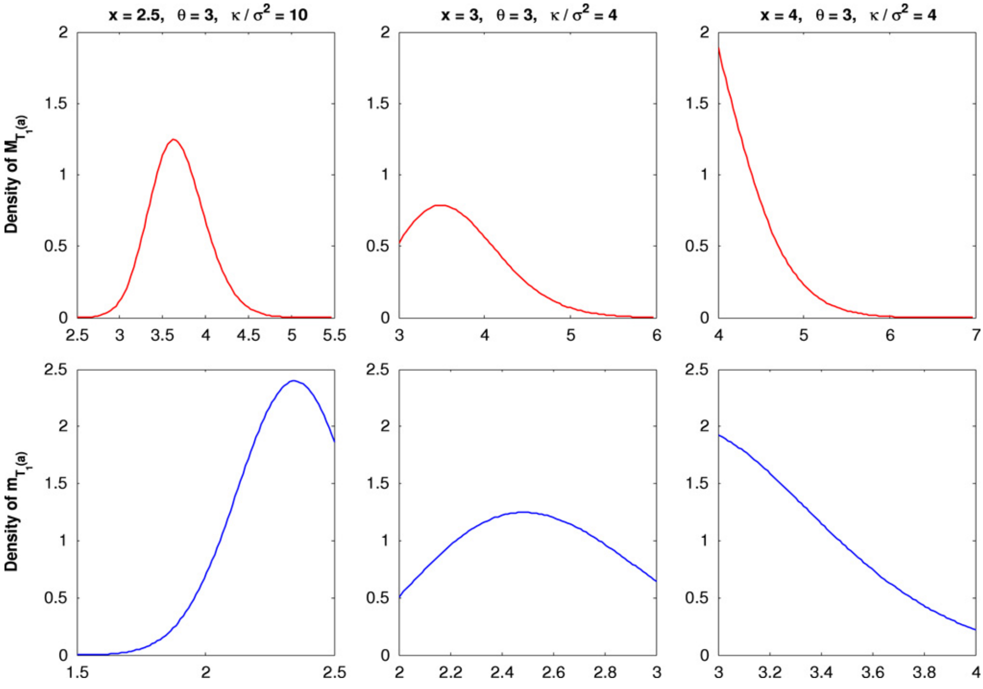

Fig. 5. Densities of $M_{T_D(a)}$ and $m_{T_D(a)}$, where $a = 1$. $\{X_t\}$ is a Cox–Ingersoll–Ross process: $\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t$, $X_0 = x$. We use the same values of parameters as in Fig. 3.

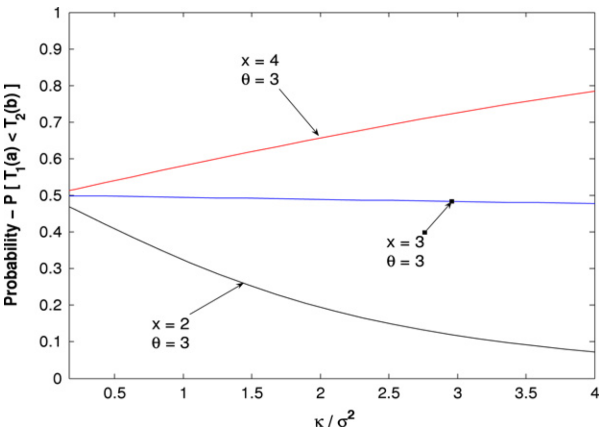

Fig. 6. Probability $\mathbb{P}_x[T_D(a) < T_U(b)]$, where $a = b = 1$, as a function of κ/σ^2, where $\kappa\theta > \sigma^2/2$. Process $\{X_t\}$ is a Cox–Ingersoll–Ross process: $\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t$, $X_0 = x$. We use the same values of parameters as in Fig. 4.

deterministic time τ:

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t, \quad t \le \tau. \tag{28}$$

For $t > \tau$, the process evolves according to one of the following stochastic differential equations:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \quad t > \tau, \tag{29}$$

$$dX_t = -\mu(X_t)dt + \sigma(X_t)dW_t \quad t > \tau. \tag{30}$$

with initial condition $y = X_\tau$. We assume that the functions $\mu(.)$ and $\sigma(.)$ are known and the stochastic differential equations (28)–(30) satisfy conditions (2)–(4) stated in Section 1.

The time of the regime change, τ, is deterministic but unknown. We observe the process $\{X_t\}$ sequentially and our goal is to identify which regime is in effect after τ.

In this context suppose that the first passage time of the drawup process to a threshold a, $T_U(a)$, can be used as a means of detecting the change of dynamics of $\{X_t\}$ from (28) to (29). Similarly, suppose that the first passage time of the drawdown process to a threshold b, $T_D(b)$ may be used as a means of detecting the change of dynamics of $\{X_t\}$ from (28) to (30) (see [7, 5]). The simplest example is when $\mu(X_t) = \mu$.

The probability measures $\mathbb{P}_x^{\tau,(1)}$ and $\mathbb{P}_x^{\tau,(2)}$ are the measures generated on the space of continuous functions $C[0, \infty)$ by the process $\{X_t\}$, if the regime changes at time τ from (28) to (29) and from (28) to (30), respectively. The stopping rule proposed and used widely in the literature for detecting such a change is known as the two-sided CUSUM (Cumulative Sum) test, $T(a) = \min\{T_D(a), T_U(a)\}$. This rule was proposed in 1959 by Barnard [22]. Its properties have been widely studied by many authors [18–21,7] and a version of this rule was also proven asymptotically optimal in [6]. It is thus the rule that has been established in the literature for detecting two-sided changes in the set-up described above.

Theorem 4.1 can be used to compute the probability of a false identification of the change. More specifically,

$$\begin{aligned}\mathbb{P}_x^{0,(1)}[T(a) = T_D(a)] &= \mathbb{P}_x^{0,(1)}[T_D(a) \leq T_U(a)] \\ &= \int_{x-a}^{x} \frac{\Psi(u, u+a) \int_u^x \Psi(u, z)dz}{\left(\int_u^{u+a} \Psi(u, z)dz\right)^2} du,\end{aligned} \tag{31}$$

with $\Psi(u, z) = e^{-2\int_u^z \gamma(y)dy}$ and $\gamma(y) = \frac{\mu(y)}{\sigma^2(y)}$, expresses the probability that an alarm indicating that the regime switched to (30) will occur before an alarm indicating that the regime switched to (29) given that in fact (29) is the true regime. Thus (31) can be seen as the probability of a false regime identification. Moreover, in the case that the density of the random variable X_τ admits a closed-form representation, we can also compute

$$\int \mathbb{P}_y^{\tau,(1)}[T(a) = T_D(a)] f_{X_\tau}(y|x)dy = \int \mathbb{P}_y^{\tau,(1)}[T_D(a) \leq T_U(a)] f_{X_\tau}(y|x)dy,$$

which can be seen as the aggregate probability (or unconditional probability) of a false identification for any given change point τ.

6. Conclusion

In this paper, we discussed properties of a diffusion process stopped at time $T_D(a)$, the first time when the process drops by amount a from its running maximum. We derived the joint distribution of the running minimum and the running maximum stopped at time $T_D(a)$. This allowed us to obtain a formula for the probability that a drawdown of size a precedes a drawup of size b.

A possible extension of our work is the calculation of the probability that a drawup precedes a drawdown in a finite time horizon. This would require a combination of our results with the distributions of times $T_D(a)$ and $T_U(b)$. We do not expect that this would lead to a closed-form solution.

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