# One Shot Schemes for Decentralized Quickest Change Detection

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*Abstract*—**This work considers the problem of quickest detection with $N$ distributed sensors that receive sequential observations either in discrete or in continuous time from the environment. These sensors employ cumulative sum (CUSUM) strategies and communicate to a central fusion center by one shot schemes. One shot schemes are schemes in which the sensors communicate with the fusion center only once, via which they signal a detection. The communication is clearly asynchronous and the case is considered in which the fusion center employs a minimal strategy, which means that it declares an alarm when the first communication takes place. It is assumed that the observations received at the sensors are independent and that the time points at which the appearance of a signal can take place are different. Both the cases of the same and different signal distributions across sensors are considered. It is shown that there is no loss of performance of one shot schemes as compared to the centralized case in an extended Lorden min-max sense, since the minimum of $N$ CUSUMs is asymptotically optimal as the mean time between false alarms increases without bound. In the case of different signal distributions the optimal threshold parameters are explicitly computed.**

*Index Terms*—**Cumulative sum (CUSUM), one shot schemes, optimal sensor threshold selection, quickest detection.**

## I. INTRODUCTION

THE problem of decentralized sequential detection with data fusion dates back to the 1980s with the works of [1] and [2]. We are interested in the problem of quickest detection in an $N$-sensor network in which the information available is distributed and decentralized, a problem introduced in [7] and [29]. We consider the situation in which the onset of a signal occurs at different times in the $N$ sensors; that is, the change points are different for each of the $N$ sensors. We also consider the case of equal-strength and unequal-strength signals across sensors, which in discrete-time models corresponds to the cases of the same and different out-of-control distributions. We assume that each sensor runs a cumulative sum (CUSUM) algorithm as suggested in [16], [25]–[28] and communicates with a central fusion center only when it is ready to signal an alarm. In other words, each sensor communicates with the central fusion center through a one shot scheme. We assume that the $N$ sensors receive independent observations, which constitutes an assumption consistent with the fact that the $N$ change points can be different. This setup has numerous applications especially in the detection of structural damage [3]–[5], [9], [11], [12], [18]. So far in the literature of this type of problem (see [16], [25]–[28]) it has been assumed that the change points are the same across sensors. Recently, the case was also considered of change points that propagate in a sensor array [20]. However, in this configuration the propagation of the change points depends on the <u>unknown</u> identity of the first sensor affected. In this paper, we consider the case in which the central fusion center employs a minimal strategy; that is, it reacts when the first communication from the sensors takes place. We demonstrate that, in the situation described above, at least asymptotically, there is no loss of information at the fusion center by employing the minimal one shot scheme. That is, we demonstrate that the minimum of $N$ CUSUMs is asymptotically optimal in detecting the minimum of the $N$ different change points, as the mean time between false alarms tends to $\infty$, with respect to an appropriately extended Lorden criterion [14] that incorporates the possibility of $N$ different change points. It is interesting that the asymptotic optimality obtained in the case of unequal-signal strengths is stronger than that in the case of equal-signal strengths. In particular, it is seen that in the case of equal-strength signals across sensors, the difference in performance of the $N$-CUSUM stopping rule with equal thresholds across sensors and the unknown optimal stopping rule is bounded by a constant as the mean time between false alarms increases without a bound. This constant is inversely proportional to the square of the signal strength and increases logarithmically as the number of sensors increases. On the other hand, in the case of unequal-strength signals across sensors, the difference in performance of the $N$-CUSUM stopping rule tends to $0$ as the mean time between false alarms increases without bound.

The communication structure considered in this paper is summarized in Fig. 1, in which $T_i$ for $i = 1, \ldots, N$ denote stopping rules associated with alarms at sensors $S_i$ $i = 1, \ldots, N$, respectively.

In the next two sections we formulate the problem and state our results on the asymptotic optimality (as the mean time between false alarms tends to $\infty$), in an extended min-max Lorden sense, of the minimum of $N$-CUSUM stopping rules in the case of centralized detection. We then discuss the implications of these results for decentralized detection.

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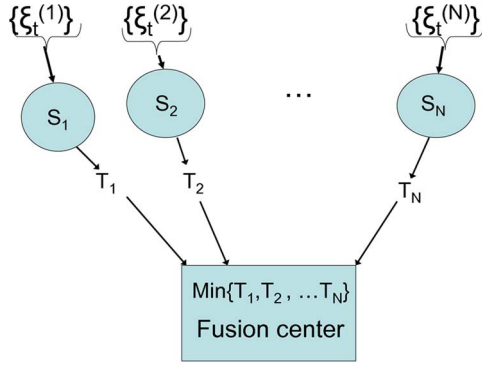
Fig. 1.   One shot communication in a decentralized system of $N$ sensors.

## II. THE CENTRALIZED PROBLEM: THE BROWNIAN MOTION MODEL

In this section, we consider a continuous-time Brownian motion model. We begin with the case of equal-strength signals across sensors and proceed to treat the case of unequal-strength signals across sensors.

### A. Equal-Strength Signals

We sequentially observe the processes $\{\xi_t^{(i)}; t \geq 0\}$ for all $i = 1, \ldots, N$ with the following dynamics:

$$d\xi_t^{(i)} = \begin{cases} dw_t^{(i)}, & t \leq \tau_i \\ \mu \, dt + dw_t^{(i)}, & t > \tau_i \end{cases} \quad (1)$$

where $\mu > 0$ is known and represents the signal strength,[1] $\{w_t^{(i)}\}$ are independent standard Brownian motions, and the $\tau_i$'s are unknown constants, with $\tau_i$ representing the time point of onset of the signal in sensor $S_i$.

An appropriate measurable space is $\Omega = C[0, \infty) \times C[0, \infty) \times \cdots \times C[0, \infty)$ and $\mathcal{F} = \cup_{t>0} \mathcal{F}_t$, where $\{\mathcal{F}_t\}$ is the filtration of the observations with $\mathcal{F}_t = \sigma\{s \leq t; (\xi_s^{(1)}, \ldots, \xi_s^{(N)})\}$. Notice that in the case of centralized detection, the filtration consists of the totality of the observations that have been received up until the specific point in time $t$.

On this space, we have the following family of probability measures $\{P_{\tau_1, \ldots, \tau_N}\}$, where $P_{\tau_1, \ldots, \tau_N}$ corresponds to the measure generated on $\Omega$ by the processes $(\xi_t^{(1)}, \ldots, \xi_t^{(N)})$ when the change in the $N$-tuple process occurs at time point $\tau_i$, $i = 1, \ldots, N$. Notice that the measure $P_{\infty, \ldots, \infty}$ corresponds to the measure generated on $\Omega$ by $N$ independent Brownian motions without drifts.

Our objective is to find a stopping rule $T$ that balances the tradeoff between a small detection delay subject to a lower bound on the mean time between false alarms and will ultimately detect $\min\{\tau_1, \ldots, \tau_N\}$.[2]

As a performance measure we consider the following generalization of Lorden's performance index [14]:

$$J^{(N)}(T) =$$
$$\sup_{\tau_1, \ldots, \tau_N} \text{essup} \, E_{\tau_1, \ldots, \tau_N} \{(T - \tau_1 \wedge \ldots \wedge \tau_N)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N}\} \quad (2)$$

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[1]Due to the symmetry of Brownian motion, without loss of generality, we can assume that $\mu > 0$.

[2]In what follows we will use $\tau_1 \wedge \ldots \wedge \tau_N$ to denote $\min\{\tau_1, \ldots, \tau_N\}$.

where the supremum over $\tau_1, \ldots, \tau_N$ is taken over the set in which $\min\{\tau_1, \ldots, \tau_N\} < \infty$. That is, we consider the worst detection delay over all possible realizations of paths of the $N$-tuple of stochastic processes $(\xi_t^{(1)}, \ldots, \xi_t^{(N)})$ up to $\min\{\tau_1, \ldots, \tau_N\}$ and then consider the worst detection delay over all possible $N$-tuples $\{\tau_1, \ldots, \tau_N\}$ over a set in which at least one of them is forced to take a finite value. This is because $T$ is a stopping rule meant to detect the minimum of the $N$ change points and therefore if one of the $N$ processes undergoes a regime change, any unit of time by which $T$ delays in reacting, should be counted towards the detection delay.

The performance index presented in (2) results in the corresponding stochastic optimization problem of the form

$$\inf_T J^{(N)}(T)$$
$$\text{subject to} \quad E_{\infty, \ldots, \infty} \{T\} \geq \gamma. \quad (3)$$

We notice that the expectation in the above constraint is taken under the measure $P_{\infty, \ldots, \infty}$. This is the measure generated on the space $\Omega$ in the case that none of the $N$ processes $(\xi_t^{(1)}, \ldots, \xi_t^{(N)})$ changes regime. Therefore, $E_{\infty, \ldots, \infty} \{T\}$ is the mean time between false alarms, and $\gamma$ is the minimal acceptable value for this quantity.

The optimal solution to (3), $T^*$, must be an equalizer rule. That is, it must display the same detection delay regardless of which of the processes $\{\xi_t^{(i)}; t \geq 0\}$, $i = 1, \ldots, N$ undergoes a change first.

More specifically, let

$$J_i^{(N)}(T) = \sup_{\tau_i \leq \tau_j, j \neq i} \text{essup} \, E_{\tau_1, \ldots, \tau_N} \{(T - \tau_i)^+ | \mathcal{F}_t\}$$

for $i = 1, \ldots, N$. That is, $J_i^{(N)}(T)$ is the detection delay of the stopping rule $T$ when $\tau_i \leq \min_{j \neq i}\{\tau_j\}$. Then

$$J^{(N)}(T) = \max \left\{ J_1^{(N)}(T), J_2^{(N)}(T), \ldots, J_N^{(N)}(T) \right\}. \quad (4)$$

The optimal solution to (3), $T^*$, satisfies

$$J_1^{(N)}(T^*) = J_2^{(N)}(T^*) = \ldots = J_N^{(N)}(T^*). \quad (5)$$

To see this, let us consider the case when $N = 2$. Let $T$ be a stopping rule such that $J_1^{(2)}(T) < J_2^{(2)}(T)$. Consider another stopping rule $S$, which stops as $T$ does, but observes $\xi_t^{(2)}$ in place of $\xi_t^{(1)}$ and $\xi_t^{(1)}$ in place of $\xi_t^{(2)}$. It follows that

$$J_1^{(2)}(S) = J_2^{(2)}(T) \quad \text{and} \quad J_2^{(2)}(S) = J_1^{(2)}(T).$$

We trivially also have that

$$E_{\infty, \infty}\{S\} = E_{\infty, \infty}\{T\}.$$

Now let us use a binary random variable $X \in \{0, 1\}$, which is independent of $\{\mathcal{F}_t\}$, to construct a randomized stopping rule adapted to $\hat{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(X)$

$$\hat{T} = XT + (1 - X)S. \quad (6)$$

It is easy to observe that

$$E_{\infty, \infty}\{\hat{T}\} = E_{\infty, \infty}\{T\}$$

and

$$J_1^{(2)}(\hat{T}) = J_2^{(2)}(\hat{T})$$
$$= \frac{1}{2}\left[J_1^{(2)}(T) + J_2^{(2)}(T)\right] < J_2^{(2)}(T)$$

which, implies

$$J^{(2)}(\hat{T}) < J^{(2)}(T)$$

by (4). Therefore, the optimal solution to (3) must satisfy (5).[3]

In the case of only a single observation process (say $\{\xi_t^{(1)}\}$), the problem becomes one of detecting a one-sided change in a sequence of Brownian observations, whose optimal solution was found in [6] and [21]. The optimal solution is the continuous-time version of Page's CUSUM stopping rule, namely, the first passage time of the process

$$y_t^{(1)} = \sup_{0 \le \tau_1 \le t} \log \frac{dP_{\tau_1}}{dP_\infty}\Big|_{\mathcal{F}_t} = u_t^{(1)} - m_t^{(1)} \quad \text{where} \quad (7)$$

$$u_t^{(1)} = \mu \xi_t^{(1)} - \frac{1}{2}\mu^2 t \tag{8}$$

and

$$m_t^{(1)} = \inf_{0 \le s \le t} u_s^{(1)}. \tag{9}$$

The CUSUM stopping rule is thus

$$T_\nu = \inf\left\{t \ge 0; y_t^{(1)} \ge \nu\right\} \tag{10}$$

where $\nu$ is chosen so that $E_\infty\{T_\nu\} \equiv \frac{2}{\mu^2}f(\nu) = \gamma$, with $f(\nu) = e^\nu - \nu - 1$ (see, for example, [10]) and

$$J^{(1)}(T_\nu) \equiv E_0\{T_\nu\} = \frac{2}{\mu^2}f(-\nu). \tag{11}$$

The fact that the worst detection delay is the same as that incurred in the case in which the change point is exactly 0 is a consequence of the nonnegativity of the CUSUM process, from which it follows that the worst detection delay occurs when the CUSUM process at the time of the change is at 0 [10].

We remark here that if the $N$ change points were the same then the problem (3) would be equivalent to observing only one stochastic process which is now $N$-dimensional. Thus, in this case, the solution is the same as that given in the above paragraph with $y_t^{(1)}$ replaced by the projection of $(y_t^{(1)}, \ldots, y_t^{(N)})$ onto the $N$-vector of all 1's.

Returning to problem (3), it is easily seen that in seeking solutions to this problem, we can restrict our attention to stopping rules that achieve the false alarm constraint with equality [17]. The optimality of the CUSUM stopping rule in the presence of only one observation process suggests that a CUSUM type of stopping rule might display similar optimality properties in the case of multiple observation processes. In particular, an intuitively appealing rule, when the detection of $\min\{\tau_1, \ldots, \tau_N\}$ is of interest, is $T_h = T_h^1 \wedge \ldots \wedge T_h^N$, where $T_h^i$ is the CUSUM stopping rule for the process $\{\xi_t^{(i)}; t \ge 0\}$ for $i = 1, \ldots, N$. That is, we use what is known as a multi-chart CUSUM stopping rule [23], which can be written as

$$T_h = \inf\left\{t \ge 0; \max\left\{y_t^{(1)}, \ldots, y_t^{(N)}\right\} \ge h\right\} \tag{12}$$

[3]Although $\hat{T}$ of (6) is measurable with respect to the enlarged filtration $\{\hat{\mathcal{F}}_t\}$, the optimal solution to (3) must be adapted to the original filtration $\{\mathcal{F}_t\}$.

where

$$y_t^{(i)} = \sup_{0 \le \tau_i \le t} \log \frac{dP_{\tau_i}}{dP_\infty}\Big|_{\mathcal{F}_t}$$
$$= \mu \xi_t^{(i)} - \frac{1}{2}\mu^2 t - \inf_{s \le t}\left(\mu \xi_s^{(i)} - \frac{1}{2}\mu^2 s\right)$$

and the $P_{\tau_i}$ are the respective restrictions of the measure $P_{\tau_1,\ldots,\tau_N}$ to $C[0, \infty)$.

It is easily seen that

$$J^{(N)}(T_h) = E_{0,\infty,\ldots,\infty}\{T_h\} = E_{\infty,0,\infty,\ldots,\infty}\{T_h\}$$
$$= \ldots$$
$$= E_{\infty,\ldots,\infty,0}\{T_h\}. \tag{13}$$

This is because the worst detection delay occurs when only one of the $N$ processes changes regime. The reason for this lies in the fact that the CUSUM process is a monotone function of $\mu$, resulting in a longer on average passage time if $\mu = 0$ [19]. Thus, the worst detection delay will occur when none of the other processes changes regime, and due to the nonnegativity of the CUSUM process the worst detection delay will occur when the CUSUM process of the remaining one process is at 0. We also point out that the proposed rule (12) also satisfies (5). That is, it is an equalizer rule.

Notice that the threshold $h$ is used for the multi-chart CUSUM stopping rule (12) in order to distinguish it from $\nu$, the threshold used for the one sided CUSUM stopping rule (10).

In what follows, we will demonstrate the asymptotic optimality of (12) as $\gamma \to \infty$. In view of the discussion in the previous paragraph, in order to assess the optimality properties of the multi-chart CUSUM rule (12) we will thus need to begin by evaluating $E_{0,\infty,\ldots,\infty}\{T_h\}$ and $E_{\infty,\ldots,\infty}\{T_h\}$.

Since the processes $\{\xi_t^{(i)}\}$, $i = 1, \ldots, N$, are independent, it is possible to obtain a closed-form expression through the formula

$$E_{0,\infty,\ldots,\infty}\{T_h\}$$
$$= \int_0^\infty P_{0,\infty,\ldots,\infty}(T_h > t)$$
$$= \int_0^\infty P_{0,\infty,\ldots,\infty}\left(\{T_h^1 > t\} \cap \ldots \cap \{T_h^N > t\}\right) dt$$
$$= \int_0^\infty P_0(T_h^1 > t)\left[P_\infty\left(T_h^1 > t\right)\right]^{N-1} dt. \tag{14}$$

Similarly

$$E_{\infty,\ldots,\infty}\{T_h\} = \int_0^\infty \left[P_\infty\left(T_h^1 > t\right)\right]^N dt \tag{15}$$

where $\{T_h^i > t\} = \{\sup_{0 \le s \le t} y_s^{(i)} < h\}$. In other words, the evaluation of (14) and (15) is possible through the probability density function of the random variable $\sup_{0 \le s \le t} y_s^{(i)}$ for arbitrary fixed $t$, which appears in [15].

In order to demonstrate the asymptotic optimality of (12) we bound the detection delay $J^{(N)}$ of the unknown optimal stopping rule $T^*$ by

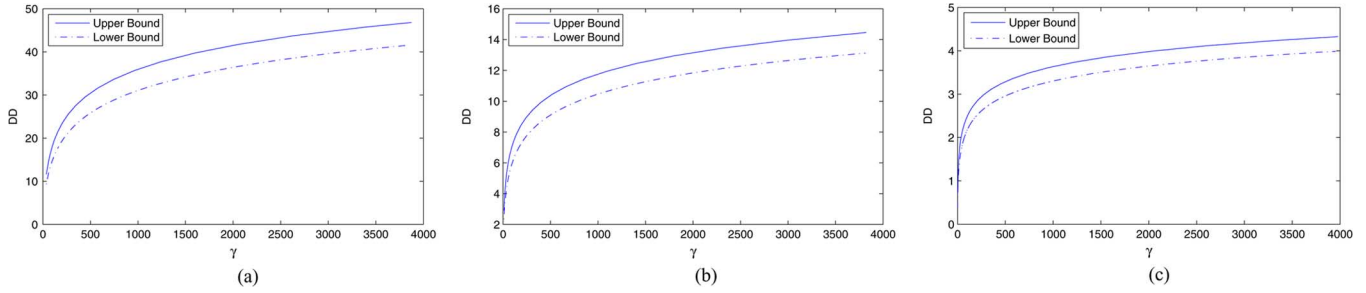$$E_{0,\infty,\ldots,\infty}\{T_h\} > J^{(N)}(T^*), \tag{16}$$

Fig. 2. **(Left)** Case of $\mu = 0.5$. **(Middle)** Case of $\mu = 1$. **(Right)** Case of $\mu = 2$. (Note that the differences between the upper and the lower bounds are all bounded as $\gamma$ increases). (a) $\mu = 0.5$ (b) $\mu = 1$ (c) $\mu = 2$.

where $h$ is chosen so that

$$E_{\infty,\ldots,\infty}\{T_h\} = \gamma. \tag{17}$$

It is also obvious that $J^{(N)}(T^*)$ is bounded from below by the detection delay of the one CUSUM when there is only one observation process, say only the first one, in view of the fact that

$$\sup_{\tau_1,\ldots,\tau_N} \operatorname{essup} E_{\tau_1,\ldots,\tau_N}\{(T - \tau_1 \wedge \ldots \wedge \tau_N)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N}\}$$
$$\geq \sup_{\tau_1} \operatorname{essup} E_{\tau_1}\left\{(T - \tau_1)^+ | \mathcal{F}_{\tau_1}^{(1)}\right\}$$

where $\mathcal{F}_{\tau_1}^{(1)} = \sigma\{s \leq \tau_1; \xi_s^{(1)}\}$. Notice that the above inequality holds for all $T$ adapted to the filtration $\{\mathcal{F}_t^{(1)}\}$. The stopping rule that minimizes $\sup_{\tau_1} \operatorname{essup} E_{\tau_1}\{(T - \tau_1)^+ | \mathcal{F}_{\tau_1}^{(1)}\}$ is the CUSUM stopping rule $T_\nu$ of (10), with $\nu$ chosen so as to satisfy

$$E_\infty\{T_\nu\} = \gamma. \tag{18}$$

We will demonstrate that the difference between the upper and the lower bounds

$$E_{0,\infty,\ldots,\infty}\{T_h\} > J^{(N)}(T^*) > E_0\{T_\nu\} \tag{19}$$

is bounded by a constant as $\gamma \to \infty$, with $h$ and $\nu$ satisfying (17) and (18), respectively.

*Lemma 1:* We have

$$E_{0,\infty,\ldots,\infty}\{T_h\} = \frac{2}{\mu^2}\left[\log\gamma + \log\frac{N\mu^2}{2} - 1 + o(1)\right] \tag{20}$$

as $\gamma \to \infty$,

*Proof:* Please refer to Appendix A for the proof. □

Moreover, it is easily seen from (11) that

$$E_0\{T_\nu\} = \frac{2}{\mu^2}\left[\log\gamma + \log\frac{\mu^2}{2} - 1 + o(1)\right]. \tag{21}$$

Thus, we have the following result.

*Theorem 1:* The difference in detection delay $J^{(N)}$ of the unknown optimal stopping rule $T^*$ and the detection delay of $T_h$ of (12) with $h$ satisfying (17) is bounded above by

$$\frac{2}{\mu^2}\log N$$

as $\gamma \to \infty$.

*Proof:* The proof follows from Lemma 1 and (21). □

*Remark:* Since $J^{(N)}(T_h)$ increases without bound as $\gamma \to \infty$, Theorem 1 asserts the asymptotic optimality of $T_h$.

The upper and the lower bounds on detection delay for the optimal stopping rule, when $\mu$ is 0.5, 1, and 2, in the case that $N = 2$ are shown in Fig. 2.

We now proceed to treat the case of unequal-strength signals across sensors.

### B. Unequal-Strength Signals

In this subsection, we consider the case in which the signal strengths can be different across sensors. That is, we sequentially observe the processes $\{\xi_t^{(i)}; t \geq 0\}$ for all $i = 1, \ldots, N$ with the following dynamics:

$$d\xi_t^{(i)} = \begin{cases} dw_t^{(i)}, & t \leq \tau_i \\ \mu_i\, dt + dw_t^{(i)}, & t > \tau_i \end{cases} \tag{22}$$

where $\{\mu_i\}_{i=1}^N$ are known, with

$$\max_i\{\mu_i\} > \min_i\{\mu_i\} > 0;$$

and $\{w_t^{(i)}\}$ and $\tau_i$'s are as before.

In order to incorporate the fact that we may have different signal strengths after the onset of a signal, we employ a general-threshold $N$-CUSUM stopping rule

$$T_\hbar = \inf\left\{t \geq 0; \max\left\{\frac{y_t^{(1)}}{h_1}, \ldots, \frac{y_t^{(N)}}{h_N}\right\} \geq 1\right\} \tag{23}$$

where $\hbar = (h_1, h_2, \ldots, h_N)$ is used to denote the vector of thresholds.

The constraint (17), namely

$$E_{\infty,\ldots,\infty}\{T_\hbar\} = \gamma, \tag{24}$$

does not uniquely determine the vector $\hbar$. However, (13) implies that the optimal choice of thresholds satisfies

$$E_{0,\infty,\ldots,\infty}\{T_\hbar\} = E_{\infty,0,\ldots,\infty}\{T_\hbar\} = \ldots = E_{\infty,\ldots,\infty,0}\{T_\hbar\} \tag{25}$$

We provide an explicit condition on thresholds such that (25) holds.

*Lemma 2:* For $\hbar = (h_1, h_2, \ldots, h_N)$ such that

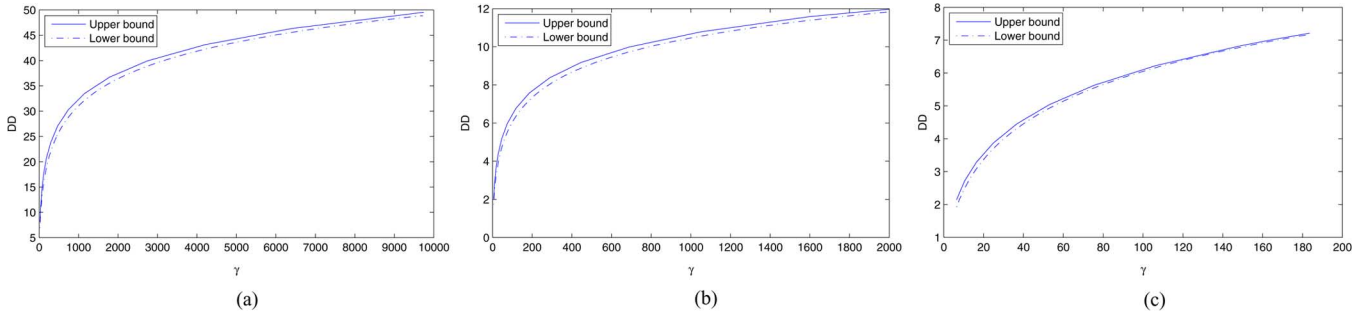$$\frac{1}{\mu_1^2}(h_1 - 1) = \frac{1}{\mu_2^2}(h_2 - 1) = \ldots = \frac{1}{\mu_N^2}(h_N - 1) \tag{26}$$

Fig. 3. **(Left)** Case of $\mu_1 = 0.5, \mu_2 = 1.2\mu_1$. **(Middle)** Case of $\mu_1 = 1, \mu_2 = 1.2\mu_1$. **(Right)** Case of $\mu_1 = 1, \mu_2 = 1.5\mu_1$. (Note that the differences between the upper and the lower bounds converge to zero as $\gamma$ increases). (a) $\mu_1 = 0.5, \mu_2 = 1.2\mu_1$. (b) $\mu_1 = 1, \mu_2 = 1.2\mu_1$. (c) $\mu_1 = 1, \mu_2 = 1.5\mu_1$.

(25) holds asymptotically, and

$$J^{(N)}(T_\hbar) = \frac{2}{\mu_1^2}(h_1 - 1) + o(1) \qquad (27)$$

as $h_1 \to \infty$.

*Proof:* Please refer to Appendix A for the proof. □

Not surprisingly, Lemma 2 suggests common thresholds across sensors in the case of equal-strength signals. In the case of unequal-strength signals, we discuss the optimality of the $N$-CUSUM with thresholds determined by (24) and (26).

Without loss of generality, let $\mu_1 = \min_i\{\mu_i\}$. We will demonstrate the asymptotic optimality of (23) when $\mu_1 < \min_{i>1}\{\mu_i\}$.

We begin by bounding the detection delay $J^{(N)}$ of the unknown optimal stopping rule $T^*$ both above and below by

$$J^{(N)}(T_\hbar) > J^{(N)}(T^*) > \max_{1 \le i \le N}\{E_0\{T^i_{\nu_i}\}\} \qquad (28)$$

where $\{\nu_i\}_{i=1}^N$ are chosen so that

$$E_\infty\{T^i_{\nu_i}\} = \gamma, \quad i = 1, \ldots, N. \qquad (29)$$

We will demonstrate that the difference between the upper and the lower bounds tends to zero as $\gamma \to \infty$, with $\hbar$ and $\nu_i$ satisfying (24), (26), and (29).

*Lemma 3:* For $\hbar = (h_1, h_2, \ldots, h_N)$ satisfying (24) and (26)

$$J^{(N)}(T_\hbar) = \frac{2}{\mu_1^2}\left[\log\gamma + \log\frac{\mu_1^2}{2} - 1 + o(1)\right] \qquad (30)$$

as $\gamma \to \infty$.

*Proof:* Please refer to Appendix A for the proof. □

It is worth pointing out that Lemma 3 justifies us in ignoring signals with stronger strength as long as only asymptotic behavior is concerned. Comparing the result of Lemma 3 with (21) for $\mu = \mu_1$, we have the following result.

*Theorem 2:* The difference in detection delay $J^{(N)}$ of the unknown optimal stopping rule $T^*$ and the detection delay of $T_\hbar$ of (23) with $\hbar$ satisfying (24) and (26) converges to zero as $\gamma \to \infty$.

*Proof:* Clearly, the asymptotic lower bound in (28) is $E_0\{T^1_{\nu_1}\}$. From Lemma 3 and (21) we obtain

$$J^{(N)}(T_\hbar) - J^{(N)}(T^*) \le J^{(N)}(T_\hbar) - E_0\{T^1_{\nu_1}\} = o(1)$$

as $\gamma \to \infty$. □

The upper and the lower bounds on detection delay for the optimal stopping rule, when $\mu_1 = 0.5$ and $\mu_2 = 1.2\mu_2$, $\mu_1 = 1$ and $\mu_2 = 1.2\mu_1$, $\mu_1 = 1$ and $\mu_2 = 1.5\mu_1$, for the case $N = 2$ are shown in Fig. 3. An important observation is that the convergence of the upper and the lower bounds is faster for stronger signal strength, and for larger ratio between the stronger signal strength and weaker signal strength.

We now treat more general cases in which

$$\mu_1 = \mu_2 = \ldots = \mu_k < \min_{i>k}\{\mu_i\} \qquad (31)$$

with $1 < k < N$.[4] In such cases, the $N$-CUSUM with thresholds chosen by (26) behaves asymptotically like the $k$-CUSUM with equal thresholds. This is because, as far as the asymptotic behavior is concerned, only the first $k$ processes with weakest signal strengths need consideration and all the other $N - k$ processes with stronger signal strengths can be ignored.

More specifically, we have.

*Lemma 4:* Under (31), for $\hbar = (h_1, h_2, \ldots, h_N)$ satisfying (24) and (26)

$$J^{(N)}(T_\hbar) = \frac{2}{\mu_1^2}\left[\log\gamma + \log\frac{k\mu_1^2}{2} - 1 + o(1)\right] \qquad (32)$$

as $\gamma \to \infty$.

*Proof:* Please refer to Appendix A for the proof. □

By examining the asymptotic difference of the upper and the lower bounds in (28), we obtain the following.

*Theorem 3:* When the number of signals with weakest strengths is $k$, the difference in detection delay $J^{(N)}$ of the unknown optimal stopping rule $T^*$ and the detection delay of $T_\hbar$ of (23) with $\hbar$ satisfying (24) and (26) is bounded above by

$$\frac{2}{\mu_1^2}\log k$$

as $\gamma \to \infty$.

*Proof:* The asymptotic lower bound in (28) is $E_0\{T^1_{\nu_1}\}$. From (21) and Lemma 4, we obtain

$$J^{(N)}(T_\hbar) - J^{(N)}(T^*) \le J^{(N)}(T_\hbar) - E_0\{T^1_{\nu_1}\}$$
$$\le \frac{2}{\mu_1^2}\log k + o(1)$$

as $\gamma \to \infty$. □

[4]The case of $k = N$ and that of $k = 1$ are already treated in Theorem 1 and Theorem 2, respectively.

The consequence of Theorems 1, 2, and 3 is the asymptotic optimality of (23) in the case in which all of the information becomes directly available through the filtration $\{\mathcal{F}_t\}$ at the fusion center. We notice, however, that this asymptotic optimality holds for any finite number of sensors $N$. Moreover, the more diverse the signal strengths are, the better the asymptotic optimality we achieve.

We now discuss the results under discrete observation.

## III. THE CENTRALIZED PROBLEM: THE DISCRETE-TIME MODEL

In this section, we consider a discrete-time model. It is assumed that the in-control distributions of the observations are the same across sensors. We then treat separately the cases in which the out-of-control distributions are the same and different across sensors.

### A. Common Out-of-Control Distributions

We sequentially observe $N$ mutually independent processes $\{\xi_n^{(i)}; n \geq 1\}$, $i = 1, \ldots, N$, with the following probability density functions (with respect to a $\sigma$-finite measure $\lambda$):

$$\xi_n^{(i)} \sim \begin{cases} g_\infty(x), & n < \tau_i \\ g_0^{(i)}(x), & n \geq \tau_i \end{cases} \tag{33}$$

where $g_\infty(x)$ and $g_0^{(i)}(x) = g_0(x)$ for all $i = 1, \ldots, N$, represent the distributions of the observations before and after the onset of the change in sensor $S_i$, and the $\tau_i$'s are unknown positive integers, with $\tau_i$ representing the time point of onset of the change in sensor $S_i$.

An appropriate measurable space is $\Omega = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \ldots \mathbb{R}^\infty$ and $\mathcal{F} = \cup_{n \geq 1} \mathcal{F}_n$, where $\{\mathcal{F}_n\}$ is the filtration of the observations with $\mathcal{F}_n = \sigma\{k \leq n; (\xi_k^{(1)}, \ldots, \xi_k^{(N)})\}$. Notice that in the case of centralized detection, the filtration consists of the totality of the observations that have been received up until the specific point in time $n$.

Analogously to the Brownian motion observation model, on this space, we can define the family of probability measures $\{P_{\tau_1, \ldots, \tau_N}\}$ as before. In order to appropriately formulate this problem in discrete time we need to specify assumptions regarding the probability density functions $g_0(x)$ and $g_\infty(x)$. To this effect, let us consider the projection of $P_{\tau_1, \ldots, \tau_N}$ on the $i$th component of $\Omega$, with special attention to $P_1^{(i)}$ and $P_\infty$, for all $i = 1, \ldots, N$. Let us also define the log-likelihood ratio

$$Z_n^{(i)} = \log \frac{g_0^{(i)}\left(\xi_n^{(i)}\right)}{g_\infty\left(\xi_n^{(i)}\right)} \tag{34}$$

for which we assume that for all $i = 1, \ldots, N$

$$-\infty < E_\infty\left\{Z_n^{(i)}\right\} < 0 < E_1^{(i)}\left\{Z_n^{(i)}\right\} < \infty \tag{35}$$

$$E_1^{(i)}\left\{\left|Z_n^{(i)}\right|^2\right\} < \infty \tag{36}$$

and that the $Z_n^{(i)}$'s are nonarithmetic with respect to $P_1^{(i)}$ and $P_\infty$. We note that $E_1^{(i)}\{Z_n^{(i)}\}$ is the Kullback–Leibler divergence $D(g_0^{(i)}\|g_\infty)$, which can also be written as

$$I_{g_0}^{(i)} = D\left(g_0^{(i)}\|g_\infty\right) = \int \log \frac{g_0^{(i)}(x)}{g_\infty(x)} g_0^{(i)}(x)\lambda(dx). \tag{37}$$

Our objective is to find a stopping rule $T$ that balances the tradeoff between a small detection delay subject to a lower bound on the mean time between false alarms and will ultimately detect $\min\{\tau_1, \ldots, \tau_N\}$.

As a performance measure we consider the following generalization of Lorden's performance index [14]:

$$J_D^{(N)}(T) =$$
$$\sup_{\tau_1, \ldots, \tau_N} \operatorname{essup} E_{\tau_1, \ldots, \tau_N}\{(T - \tau_1 \wedge \ldots \wedge \tau_N + 1)^+ | \mathcal{F}_{\tau_1 \wedge \ldots \wedge \tau_N}\} \tag{38}$$

where the supremum over $\tau_1, \ldots, \tau_N$ is taken over the set in which $\min\{\tau_1, \ldots, \tau_N\} < \infty$. The performance index presented in (38) results in the corresponding stochastic optimization problem of the form

$$\inf_T J_D^{(N)}(T)$$
$$\text{subject to} \quad E_{\infty, \ldots, \infty}\{T\} \geq \gamma. \tag{39}$$

Then similar arguments as before apply. In particular, the optimal solution to (39), $T^*$, still satisfies (5).

In the case of only a single observation process (say $\{\xi_n^{(1)}\}$), the problem becomes one of detecting a one-sided change in the distribution of a sequence of discrete observations, whose optimal solution was found in [17]. The optimal solution is Page's CUSUM stopping rule, namely, the first passage time of the process

$$y_n^{(1)} = \sup_{1 \leq \tau_1 \leq n} \log \frac{dP_{\tau_1}^{(1)}}{dP_\infty}\bigg|_{\mathcal{F}_n} = u_n^{(1)} - m_n^{(1)} \quad \text{where} \tag{40}$$

$$u_n^{(1)} = \sum_{k=1}^n Z_k^{(1)}, \tag{41}$$

and

$$m_n^{(1)} = \min_{1 \leq k \leq n} u_n^{(1)}. \tag{42}$$

The CUSUM stopping rule is thus

$$T_\nu^1 = \inf\left\{n \geq 1; y_n^{(1)} \geq \nu\right\} \tag{43}$$

where $\nu$ is chosen so that $E_\infty\{T_\nu^1\} = \gamma$. Then $J_D^{(1)}(T_\nu^1) = E_1^{(1)}\{T_\nu^1\}$. Stopping rules involving likelihood ratios of discrete time models of the type described in (33), are characterized by overshoot of the threshold $\nu$. For this reason we define[5]

$$\kappa_i = \lim_{\nu \to \infty} E_1^{(i)}\left\{y_{T_\nu^i} - \nu\right\} \tag{44}$$

$$\beta_i = E_1^{(i)}\left\{m_\infty^{(i)}\right\} \tag{45}$$

and

---

[5]$\kappa_i$ is also the limiting expectation of overshoots of the one-sided sequential probability ratio test (SPRT), i.e., $\kappa_i = \lim_{\nu \to \infty} E_1^{(i)}\{u_{n_\nu^i}^{(i)} - \nu\}$; see [23, p. 323] and [30, Theorem 4.1] for details.

$$R_i = \lim_{\nu \to \infty} E_1^{(i)} \left\{ \exp \left[ - \left( u_{\eta_\nu^i}^{(i)} - \nu \right) \right] \right\} \quad (46)$$

where $\eta_\nu^i = \inf\{n \geq 1; u_n^{(i)} \geq \nu\}$. From Lemma 1 of [23] (or Theorem 3 of [13]) we have that as $\nu \to \infty$

$$E_\infty \{T_\nu^i\} = E_\infty \{T_\nu^1\} = \frac{1}{I_{g_0}^{(1)}(R_1)^2} e^\nu [1 + o(1)]; \quad (47)$$

and using similar arguments to the ones in [23, p. 323], it follows that

$$E_1^{(i)} \{T_\nu^i\} = E_1^{(1)} \{T_\nu^1\} = \frac{1}{I_{g_0}^{(1)}} (\nu + \beta_1 + \kappa_1) + o(1). \quad (48)$$

Returning to problem (39), we will focus on the performance of the $N$-CUSUM stopping rule (12). Just as in the Brownian motion case, we have

$$
\begin{aligned}
J_D^{(N)}(T_h) = E_{1,\infty,\dots,\infty}\{T_h\} &= E_{\infty,1,\infty,\dots,\infty}\{T_h\} \\
&= \dots \\
&= E_{\infty,\dots,\infty,1}\{T_h\}. \quad (49)
\end{aligned}
$$

Moreover

$$E_{1,\infty,\dots,\infty}\{T_h\} > J_D^{(N)}(T^*) > E_1^{(1)}\{T_\nu^1\} \quad (50)$$

where $h$ and $\nu$ satisfy (17) and (18), respectively. We will demonstrate that the difference between the upper and the lower bounds is bounded by a constant as $\gamma \to \infty$.

*Lemma 5:* We have

$$
\begin{aligned}
&E_{1,\infty,\dots,\infty}\{T_h\} \\
&= \frac{1}{I_{g_0}^{(1)}} \left[ \log \gamma + \log \left( N I_{g_0}^{(1)}(R_1)^2 \right) + \beta_1 + \kappa_1 \right] + o(1)
\end{aligned}
$$
$$(51)$$

as $\gamma \to \infty$.

*Proof:* Please refer to Appendix B for the proof.    □

Moreover, it is easily seen from (47) and (48) that

$$
\begin{aligned}
&E_1^{(1)}\{T_\nu^1\} \\
&= \frac{1}{I_{g_0}^{(1)}} \left[ \log \gamma + \log \left( I_{g_0}^{(1)}(R_1)^2 \right) + \beta_1 + \kappa_1 \right] + o(1)
\end{aligned}
$$
$$(52)$$

as $\gamma \to \infty$. Thus, we have the following result.

*Theorem 4:* The difference in detection delay $J_D^{(N)}$ of the unknown optimal stopping rule $T^*$ and the detection delay of $T_h$ of (12) with $h$ satisfying (17) is bounded above by

$$\frac{1}{I_{g_0}^{(1)}} \log N,$$

as $\gamma \to \infty$.

*Proof:* The proof follows from Lemma 5 and (52).    □

Since $J_D^N(T_h)$ increases without bound as $\gamma \to \infty$, Theorem 4 asserts the asymptotic optimality of $T_h$.

We now proceed to treat the case of different out-of-control distributions across sensors.

### B. Different Out-of-Control Distributions

In this subsection, we consider the case in which the out-of-control distributions can be different across sensors. That is, we sequentially observe $N$ mutually independent processes $\{\xi_n^{(i)}; n \geq 1\}$, $i = 1, \dots, N$, with the following probability density functions (with respect to a $\sigma$-finite measure $\lambda$):

$$\xi_n^{(i)} \sim \begin{cases} g_\infty(x), & n < \tau_i \\ g_0^{(i)}(x), & n \geq \tau_i, \end{cases} \quad (53)$$

where $g_\infty(x)$ and $g_0^{(i)}(x)$ represent the distributions of the observations before and after the onset of the signal in sensor $S_i$, with

$$\infty > \max_i \left\{ I_{g_0}^{(i)} \right\} > \min_i \left\{ I_{g_0}^{(i)} \right\} > 0;$$

and the $\tau_i$'s are as before.

We will focus on the performance of the $N$-CUSUM stopping rule (23) with $\hbar = (h_1, h_2, \dots, h_N)$ satisfying (24) and

$$E_{1,\infty,\dots,\infty}\{T_\hbar\} = E_{\infty,1,\dots,\infty}\{T_\hbar\} = \dots = E_{\infty,\dots,\infty,1}\{T_\hbar\}. \quad (54)$$

We provide an explicit condition on thresholds such that (54) holds.

*Lemma 6:* For $\hbar = (h_1, h_2, \dots, h_N)$ such that

$$
\begin{aligned}
\frac{1}{I_{g_0}^{(1)}}(h_1 + \beta_1 + \kappa_1) &= \frac{1}{I_{g_0}^{(2)}}(h_2 + \beta_2 + \kappa_2) \\
&= \dots \\
&= \frac{1}{I_{g_0}^{(N)}}(h_N + \beta_N + \kappa_N) \quad (55)
\end{aligned}
$$

(54) holds asymptotically, and

$$J_D^{(N)}(T_\hbar) = \frac{1}{I_{g_0}^{(1)}}(h_1 + \beta_1 + \kappa_1) + o(1) \quad (56)$$

as $h_1 \to \infty$.

*Proof:* Please refer to Appendix B for the proof.    □

It is easily seen that Lemma 6 suggests common thresholds across sensors in the case of common out-of-control distributions. In the case of different out-of-control distributions, we discuss the optimality of the $N$-CUSUM with thresholds determined by (24) and (55).

Without loss of generality, let $I_{g_0}^{(1)} = \min_i\{I_{g_0}^{(i)}\}$. We will demonstrate the asymptotic optimality of (23) when $I_{g_0}^{(1)} < \min_{i>1}\{I_{g_0}^{(i)}\}$.

Just as in the Brownian motion case, we have

$$J_D^{(N)}(T_\hbar) > J_D^{(N)}(T^*) > \max_{1 \leq i \leq N} \left\{ E_1^{(1)}\{T_{\nu_i}^i\} \right\} \quad (57)$$

where $\{\nu_i\}_{i=1}^N$ are chosen according to (29). We will demonstrate that the difference between the upper and the lower

bounds tends to zero as $\gamma \to \infty$, with $\hbar$ and $\nu_i$ satisfying (24), (55), and (29).

*Lemma 7:* For $\hbar = (h_1, h_2, \ldots, h_N)$ satisfying (24) and (55)

$$
\begin{aligned}
&J_D^{(N)}(T_\hbar) \\
&= \frac{1}{I_{g_0}^{(1)}} \left[ \log \gamma + \log \left( I_{g_0}^{(1)}(R_1)^2 \right) + \beta_1 + \kappa_1 \right] + o(1)
\end{aligned} \quad (58)
$$

as $\gamma \to \infty$.

*Proof:* Please refer to Appendix B for the proof. $\square$

It is worth pointing out that Lemma 7 justifies us in ignoring changes with larger Kullback–Leibler divergences as long as only asymptotic behavior is considered. Comparing the result of Lemma 7 with (52), we have the following result.

*Theorem 5:* The difference in detection delay $J_D^{(N)}$ of the unknown optimal stopping rule $T^*$ and the detection delay of $T_\hbar$ of (23) with $\hbar$ satisfying (24) and (55) converges to zero as $\gamma \to \infty$.

*Proof:* Clearly, the asymptotic lower bound in (57) is $E_1^{(1)}\{T_{\nu_1}^1\}$. From Lemma 7 and (52) we obtain

$$
J_D^{(N)}(T_\hbar) - J_D^{(N)}(T^*) \leq J_D^{(N)}(T_\hbar) - E_1\{T_{\nu_1}^1\} = o(1)
$$

as $\gamma \to \infty$. $\square$

We now treat more general cases in which

$$
I_{g_0}^{(1)} = I_{g_0}^{(2)} = \ldots = I_{g_0}^{(k)} < \min_{i>k}\{I_{g_0}^{(i)}\} \quad (59)
$$

with $1 < k \leq N$.[6] Without loss of generality, we also assume that

$$
(R_1)^2 e^{\beta_1 + \kappa_1} = \max_{1 \leq i \leq k}\{(R_i)^2 e^{\beta_i + \kappa_i}\}. \quad (60)
$$

Thus, by (52)

$$
\max_{1 \leq i \leq N}\left\{E_1^{(i)}\{T_{\nu_i}^i\}\right\} = \max_{1 \leq i \leq k}\left\{E_1^{(i)}\{T_{\nu_i}^i\}\right\} = E_1^{(1)}\{T_{\nu_1}^1\}. \quad (61)
$$

In such cases, we have the following.

*Lemma 8:* Under (59) and (60), for $\hbar = (h_1, h_2, \ldots, h_N)$ satisfying (24) and (55), as $\gamma \to \infty$

$$
\begin{aligned}
&J^{(N)}(T_\hbar) \\
&= \frac{1}{I_{g_0}^{(1)}} \left[ \log \gamma + \log \left( I_{g_0}^{(1)} \sum_{i=1}^{k}(R_i)^2 r_i \right) + \beta_1 + \kappa_1 \right] + o(1)
\end{aligned}
$$

$$
(62)
$$

where

$$
r_i = e^{(\beta_i - \beta_1) + (\kappa_i - \kappa_1)}.
$$

*Proof:* Please refer to Appendix B for the proof. $\square$

By examining the asymptotic difference of the upper and the lower bounds in (57), we obtain the following.

[6]The case of $k = 1$ is already treated in Theorem 5.

*Theorem 6:* When (59) and (60) hold, the difference in detection delay $J_D^{(N)}$ of the unknown optimal stopping rule $T^*$ and the detection delay of $T_\hbar$ of (23) with $\hbar$ satisfying (24) and (55) is bounded above by

$$
\frac{1}{I_{g_0}^{(1)}} \log \left[ \sum_{i=1}^{k}\left(\frac{R_i}{R_1}\right)^2 r_i \right]
$$

as $\gamma \to \infty$.

*Proof:* The asymptotic lower bound in (57) is $E_1^{(1)}\{T_{\nu_1}^1\}$. From (52) and Lemma 8 we obtain

$$
\begin{aligned}
J_D^{(N)}(T_\hbar) - J_D^{(N)}(T^*) &\leq J^{(N)}(T_\hbar) - E_1^{(1)}\{T_{\nu_1}^1\} \\
&\leq \frac{1}{I_{g_0}^{(1)}} \log \left[ \sum_{j=1}^{k}\left(\frac{R_j}{R_1}\right)^2 r_j \right] + o(1)
\end{aligned}
$$

as $\gamma \to \infty$. $\square$

The consequence of Theorems 4–6 is the asymptotic optimality of (23) in the discrete-time models described in (53). We notice, however, that this asymptotic optimality holds for any finite number of sensors $N$. Moreover, the more diverse the out-of-control distributions are, the better the asymptotic optimality we achieve.

We now discuss the implications of the above results for decentralized detection in the case of one shot schemes.

## IV. DECENTRALIZED DETECTION

Let us now suppose that each of the observation processes $\{\xi_t^{(i)}\}$ become sequentially available at its corresponding sensor $S_i$, which then employs an asynchronous communication scheme to the central fusion center. In particular, sensor $S_i$ communicates to the central fusion center only when it wants to signal an alarm, which is elicited according to a CUSUM rule $T_{h_i}^i$ as in (10). Once again the observations received at the $N$ sensors are independent and can change dynamics at distinct unknown points $\tau_i$. An example of such a case is described in [5], where the motivation suggested arises in the health monitoring of mechanical, civil, and aeronautic structures. In this treatment, the vibration-based and health-monitoring problems translate into the identification and monitoring of the eigenstructure of a state transition matrix of a linear dynamical state–space system excited by noise [9], [11], [12], [18]. This is achieved in practice by detecting a change in an associated residual vector. In [5] it is characteristically pointed out that the individual subspace-based tests, monitoring each residual-vector component, appear to behave in a reasonably decoupled manner and to perform a correct isolation of the components of the vector parameter that has changed. This setup the distinct change points $\tau_i$ correspond to the change points of the value of each residual-vector component. The decoupled manner in which each residual-vector component behaves corresponds to the fact that there is absence of across-sensor correlations. The fusion center, whose objective is to detect the first time when there is a change in at least one of the residual-vector components, devises a minimal strategy; that is,

it declares that a change has occurred at the first instance when one of the sensors communicates an alarm. The implication of Theorems 1–6 is that in fact this strategy is the best, at least asymptotically, that the fusion center can devise, and that there is no loss in performance between the case in which the fusion center receives the raw data $\{\xi_t^{(1)}, \ldots \xi_t^{(N)}\}$ directly and the case in which the communication that takes place is limited to the one shown in Fig. 1. To see this, consider the general case in which the first $k$ out of $N$ sensors receive the same signal strength after the onset of a signal or, equivalently, in discrete time the case in which $k$ out of the $N$ out-of-control distributions are the same. Then the rule suggested by Theorem 3 is

$$T_\hbar = T_h^1 \wedge \ldots \wedge T_h^k \wedge \ldots \wedge T_{h_{k+1}}^{k+1} \wedge \ldots \wedge T_{h_N}^N$$

with $\hbar = (h, \ldots, h, h_{k+1}, \ldots, h_N)$ so that, at least asymptotically, (25) holds. Thus, the detection delay of $T_\hbar$ is the same, at least asymptotically, regardless of which of the sensors $S_i$ draws the alarm of detection first. The mean time between false alarms for the fusion center that uses the rule $T_\hbar$ is thus $E_{\infty,\ldots,\infty}\{T_\hbar\}$. But Theorems 1–6 assert that this rule, namely $T_\hbar$, is asymptotically optimal as the mean time between false alarms tends to $\infty$ in the centralized case for any finite $N$. In other words, the CUSUM stopping rule $T_\hbar$ is a sufficient statistic (at least asymptotically) of the minimum $N$ possibly distinct change points. That is, the stopping rule $T_\hbar$ is an asymptotically optimal solution to the problems of quickest detection presented in (3) and (39).

## V. Summary

The main contribution of this paper is that it shows that one can distribute most of the work of change detection in sensor network to the sensors without any loss of performance, at least asymptotically, both in the case of continuous-time models and in the case of discrete-time models. The applications of this setup are numerous, and we have noted in particular the detection of the individual components in a vector parameter corresponding to the eigenstructure of linear dynamical state–space models. Such models have been extensively used to describe for monitoring the health of mechanical, civil, and aeronautical structures [9], [11], [12], [18]. The assumption of across-sensor independence is realistic at least in the particular examples which are described in detail in [5]. Moreover, the setup treated in this paper is also relevant to the case in which the change points propagate in a sensor array [20]. This is because even in this configuration the propagation of the change points depends on the unknown identity of the first sensor affected. In our paper, we give explicit formulas for the optimal sensor threshold selection which becomes particularly relevant in the general case in which the observation out-of-control distributions or the signal strengths are different across sensors.

## Appendix A
### The Continuous-Time Brownian Motion Model

As an illustration for the general case, let us prove the results for $N = 2$. The general case for $N \geq 2$ will be discussed afterwards.

We begin by writing down the probability distributions of CUSUM stopping rule for single observation process appearing in [15]. For $h_i > 2$, $i = 1, 2$, we have

$$P_0\left(T_{h_i}^i > t\right) = 2e^{\frac{h_i}{2}} \sum_{n \geq 1} u\left(\phi_n^{(i)}\right) e^{-\frac{\mu_i^2 t}{8 \cos^2 \phi_n^{(i)}}},$$

and

$$P_\infty(T_{h_i}^i > t) = 2e^{-\frac{h_i}{2}} \sum_{n \geq 1} u\left(\theta_n^{(i)}\right) e^{-\frac{\mu_i^2 t}{8 \cos^2 \theta_n^{(i)}}}$$
$$+ 2e^{-\frac{h_i}{2}} v\left(\eta^{(i)}\right) e^{-\frac{\mu_i^2 t}{8 \cosh^2 \eta^{(i)}}}$$
$$= A(h_i, t) + B(h_i) e^{-\frac{\mu_i^2 t}{8 \cosh^2 \eta^{(i)}}}$$

where

$$u(x) = \frac{\sin^3 x}{x - \sin x \cos x}$$
$$v(x) = \frac{\sinh^3 x}{\sinh x \cosh x - x}$$

and

$$\tan \phi_n^{(i)} = -\frac{2}{h_i} \phi_n^{(i)} < 0$$
$$\tan \theta_n^{(i)} = \frac{2}{h_i} \theta_n^{(i)} > 0$$

and

$$\tanh \eta^{(i)} = \frac{2}{h_i} \eta^{(i)} > 0.$$

Using the above notation, we can use (14) and (15) to derive expressions for $E_{0,\infty}\{T_\hbar\}$, $E_{\infty,0}\{T_\hbar\}$, and $E_{\infty,\infty}\{T_\hbar\}$, where $\hbar = (h_1, h_2)$. In particular, we have

$$E_{0,\infty}\{T_\hbar\} = \int_0^\infty P_0\left(T_{h_1}^1 > t\right) P_\infty\left(T_{h_2}^2 > t\right) dt$$
$$= \int_0^\infty P_0\left(T_{h_1}^1 > t\right)$$
$$\times \left[A(h_2, t) + B(h_2) e^{-\frac{\mu_2^2 t}{8 \cosh^2 \eta^{(2)}}}\right] dt$$
$$= I_1(h_1, h_2) + I_2(h_1, h_2)$$
$$E_{\infty,0}\{T_\hbar\} = \int_0^\infty P_\infty\left(T_{h_1}^1 > t\right) P_0\left(T_{h_2}^2 > t\right) dt$$
$$= \int_0^\infty \left[A(h_1, t) + B(h_1) e^{-\frac{\mu_1^2 t}{8 \cosh^2 \eta^{(1)}}}\right]$$
$$\times P_0\left(T_{h_2}^2 > t\right) dt$$
$$= I_1(h_2, h_1) + I_2(h_2, h_1).$$

and

$$E_{\infty,\infty}\{T_\hbar\} = \int_0^\infty P_\infty\left(T_{h_1}^1 > t\right) P_\infty\left(T_{h_2}^2 > t\right) dt$$
$$= \int_0^\infty \left[A(h_1, t)A(h_2, t)\right.$$
$$\left. + B(h_2)A(h_1, t) e^{-\frac{\mu_2^2 t}{8 \cosh^2 \eta^{(2)}}}\right] dt$$

$$+ \int_0^\infty B(h_1) A(h_2, t) e^{-\frac{\mu_1^2 t}{8 \cosh^2 \eta^{(1)}}} dt$$

$$+ B(h_1) B(h_2) \int_0^\infty e^{-\frac{\mu_1^2 t}{8 \cosh^2 \eta^{(1)}} - \frac{\mu_1^2 t}{8 \cosh^2 \eta^{(1)}}} dt$$

$$= I_3(h_1, h_2) + I_4(h_1, h_2) + I_4(h_2, h_1) + I_5(h_1, h_2).$$

Let us examine the asymptotic behavior of $I_1(h_1, h_2)$ through $I_5(h_1, h_2)$ as $h_1, h_2 \to \infty$. We have four preliminary results to help us:

*Result 1:*

$$\left| \sum_{m,n \geq 1} u\left(\theta_m^{(1)}\right) u\left(\theta_n^{(2)}\right) \frac{\cos^2 \theta_m^{(1)} \cos^2 \theta_n^{(2)}}{\mu_1^2 \cos^2 \theta_n^{(2)} + \mu_2^2 \cos^2 \theta_m^{(1)}} \right| \leq C \tag{63}$$

where

$$C = \int_0^\infty \int_0^\infty \frac{\pi^{-2} dx \, dy}{\sqrt{(1+x^2)(1+y^2)} \left(\mu_1^2 + \mu_2^2 + \mu_2^2 x^2 + \mu_1^2 y^2\right)}.$$

*Result 2:* For $\alpha_n^{(i)} = \theta_n^{(i)}$ ($\alpha_n^{(i)} = \phi_n^{(i)}$, resp.), $i = 1, 2$,

$$\lim_{h_i \to \infty} \left| \sum_{n \geq 1} u\left(\alpha_n^{(i)}\right) \cos^2 \alpha_n^{(i)} \right| \leq \frac{1}{\pi}. \tag{64}$$

*Result 3:* Asymptotically

$$e^{2\eta^{(i)} - h_i} = 1 - 4\eta^{(i)} e^{-2\eta^{(i)}} + o\left(e^{-3\eta^{(i)}}\right) \tag{65}$$

and

$$B(h_i) = 1 + 2\eta^{(i)} e^{-2\eta^{(i)}} - 3e^{-2\eta^{(i)}} + O\left(e^{-2\eta^{(i)}}\right) \tag{66}$$

as $h_i \to \infty$.

*Result 4:* If there exists an $\alpha > 0$ such that $h_1 - \alpha h_2 = O(1)$ holds asymptotically as $h_1, h_2 \to \infty$, then we have

$$\lim_{h_1, h_2 \to \infty} I_1(h_1, h_2) = \lim_{h_1, h_2 \to \infty} I_1(h_2, h_1) = 0 \tag{67}$$

$$\lim_{h_1, h_2 \to \infty} \left[ I_2(h_1, h_2) - \frac{2}{\mu_1^2}(h_1 - 1) \right] = 0 \tag{68}$$

and

$$\lim_{h_1, h_2 \to \infty} \left[ I_2(h_2, h_1) - \frac{2}{\mu_2^2}(h_2 - 1) \right] = 0. \tag{69}$$

In the following paragraph we shall prove our lemmas for $N = 2$ in the order: Lemma 2 $\to$ Lemma 3 $\to$ Lemma 1. Then we discuss the asymptotic behavior of the $N$-CUSUM for $N \geq 2$, and prove Lemma 4 at the end.

*Proof of Lemma 2:* By *Result 4*, we have under the constraint (26) that

$$J^{(2)}(T_\hbar) = E_{0,\infty}\{T_\hbar\} + o(1)$$
$$= E_{\infty,0}\{T_\hbar\} + o(1)$$
$$= \frac{2}{\mu_1^2}(h_1 - 1) + o(1) \tag{70}$$

as $h_1, h_2 \to \infty$. So Lemma 2 is proven for $N = 2$. $\square$

*Proof of Lemma 3:* We will show that $I_3(h_1, h_2)$, $I_4(h_1, h_2)$, and $I_4(h_2, h_1)$ all converge to zero as $h_1, h_2 \to \infty$

without any constraint on dependence of thresholds, and then examine how $I_5(h_1, h_2)$ behaves as $h_1, h_2 \to \infty$ under constraint (26).

First, *Result 1* implies that

$$|I_3(h_1, h_2)| = O\left(e^{-\frac{h_1 + h_2}{2}}\right), \quad \text{as } h_1, h_2 \to \infty; \tag{71}$$

*Result 2*, as well as (66) in *Result 3* imply that

$$|I_4(h_1, h_2)| \leq 16 e^{-\frac{h_1}{2}} B(h_2) \sum_{n \geq 1} \left| u\left(\theta_n^{(1)}\right) \right|$$

$$\times \frac{\cos^2 \theta_n^{(1)} \cosh^2 \eta^{(2)}}{\mu_1^2 \cosh^2 \eta^{(2)} + \mu_2^2 \cos^2 \theta_n^{(1)}}$$

$$\leq \frac{16}{\mu_1^2} e^{-\frac{h_1}{2}} B(h_2) \sum_{n \geq 1} \left| u\left(\theta_n^{(1)}\right) \right| \cos^2 \theta_n^{(1)}$$

$$= O\left(e^{-\frac{h_1}{2}}\right) \tag{72}$$

as $h_1, h_2 \to \infty$. Similarly

$$|I_4(h_2, h_1)| = O\left(e^{-\frac{h_2}{2}}\right), \quad \text{as } h_1, h_2 \to \infty. \tag{73}$$

Now let us assume $\mu_1 < \mu_2$ and we choose $h_1, h_2$ according to (26). *By Result 3*

$$I_5(h_1, h_2) = \frac{8 B(h_1) B(h_2)}{\mu_1^2 / \cosh^2 \eta^{(1)} + \mu_2^2 / \cosh^2 \eta^{(2)}}$$

$$= \left[ \left(1 + e^{-2\eta^{(1)}}\right)^{-2} \right.$$

$$\left. + \frac{\mu_2^2}{\mu_1^2} e^{-2(\eta^{(1)} - \eta^{(2)})} \left(1 + e^{-2\eta^{(2)}}\right)^{-2} \right]^{-1}$$

$$\times \frac{2 B(h_1) B(h_2)}{\mu_1^2} e^{h_1} e^{2\eta^{(1)} - h_1}$$

$$= \frac{2}{\mu_1^2} [e^{h_1} + \text{``lower exponents''}] \tag{74}$$

as $h_1, h_2 \to \infty$.

Formulas (70)–(74) imply the asymptotic formula in Lemma 3 for $N = 2$. $\square$

*Proof of Lemma 1:* We need only to change the computation of $I_5(h_1, h_2)$ in Lemma 3 to get Lemma 1. By *Result 3*

$$I_5(h_1, h_2) = \frac{4}{\mu_1^2} [B(h_1)]^2 \cosh^2 \eta^{(1)}$$

$$= \frac{4}{\mu_1^2} e^{h_1} \left[ 1 + 2\eta^{(1)} e^{-2\eta^{(1)}} \right.$$

$$\left. -3e^{-2\eta^{(1)}} + o\left(e^{-3\eta^{(1)}}\right) \right]^2$$

$$\times e^{2\eta^{(1)} - h_1} \left(1 + e^{-2\eta^{(1)}}\right)^2$$

$$= \frac{1}{\mu_1^2} e^{h_1} \left[ 1 + 4\eta^{(1)} e^{-2\eta^{(1)}} \right.$$

$$\left. -6e^{-2\eta^{(1)}} + o\left(e^{-3\eta^{(1)}}\right) \right]$$

$$\times \left[ 1 - 4\eta^{(1)} e^{-2\eta^{(1)}} + o\left(e^{-3\eta^{(1)}}\right) \right] \left(1 + e^{-2\eta^{(1)}}\right)^2$$

$$= \frac{1}{\mu_1^2} e^{h_1} \left[ 1 - 4e^{-2\eta^{(1)}} + o\left(e^{-3\eta^{(1)}}\right) \right]$$

$$= \frac{1}{\mu_1^2}\left[e^{h_1} - 4 + o\left(e^{-\frac{h_1}{2}}\right)\right], \tag{75}$$

as $h_1, h_2 \to \infty$.

Formulas (70)–(73) and (75) imply the asymptotic formula in Lemma 1 for $N = 2$. $\qquad \square$

Let us prove the four preliminary results we have just now used.

*Proof of Result 1:* To simplify notation, let us denote $p_i = 2/h_i$, $i = 1, 2$. Then

$$\left| \sum_{m,n\geq 1} u\left(\theta_m^{(1)}\right) u\left(\theta_n^{(2)}\right) \frac{\cos^2\theta_m^{(1)}\cos^2\theta_n^{(2)}}{\mu_1^2\cos^2\theta_n^{(2)} + \mu_2^2\cos^2\theta_m^{(1)}} \right|$$

$$\leq \sum_{m,n\geq 1} \left| u\left(\theta_m^{(1)}\right) u\left(\theta_n^{(2)}\right) \right| \frac{\cos^2\theta_m^{(1)}\cos^2\theta_n^{(2)}}{\mu_1^2\cos^2\theta_n^{(2)} + \mu_2^2\cos^2\theta_m^{(1)}}$$

$$\leq \sum_{m,n\geq 1} w_1\left(p_1\theta_m^{(1)}, p_2\theta_n^{(2)}\right) p_1 p_2,$$

where

$$w_1(x,y) = \frac{1}{\sqrt{(1+x^2)(1+y^2)}\,(\mu_1^2 + \mu_2^2 + \mu_2^2 x^2 + \mu_1^2 y^2)}.$$

Since $(p_1\theta_m^{(1)}, p_2\theta_n^{(2)}) \in ((2m-1)\frac{p_1\pi}{2}, (2m+1)\frac{p_1\pi}{2}) \times ((2n-1)\frac{p_2\pi}{2}, (2n+1)\frac{p_2\pi}{2})$, by monotone decreasing property of $w_1$ in both variables in the first quadrant, we have

$$\sum_{m,n\geq 1} w_1\left(p_1\theta_m^{(1)}, p_2\theta_n^{(2)}\right) p_1 p_2 \leq \frac{1}{\pi^2}\int_0^\infty\int_0^\infty w_1(x,y)\,dx\,dy.$$
$$\square$$

*Proof of Result 2:* Let us denote $p_i = 2/h_i$, $i = 1, 2$. Then

$$\left| \sum_{n\geq 1} u\left(\alpha_n^{(i)}\right)\cos^2\alpha_n^{(i)} \right| \leq \sum_{n\geq 1} \left| u\left(\alpha_n^{(i)}\right) \right| \cos^2\alpha_n^{(i)}$$

$$\leq \sum_{n\geq 1} w_2\left(p_2\alpha_n^{(i)}\right) p_i$$

where

$$w_2(x) = \frac{1}{(1+x^2)^{3/2}}.$$

Because $p_i\alpha_n^{(i)} \in ((2n-1)\frac{p_i\pi}{2}, (2n+1)\frac{p_i\pi}{2})$, and $w_2$ is decreasing on the positive half axis, we have

$$\sum_{m,n\geq 1} w_2\left(p_2\alpha_n^{(i)}\right) p_i \leq \frac{1}{\pi}\int_{-\frac{p_i\pi}{2}}^\infty w_2(x)\,dx \to \frac{1}{\pi}$$

as $p_i \to 0^+$. $\qquad \square$

*Proof of Result 3:* Equation (65) is easily verified. By (65)

$$B(h_i) = 2e^{-\frac{h_i}{2}}\frac{\sinh^2\eta^{(i)}}{\cosh\eta^{(i)}}\left(1 - \frac{\eta^{(i)}}{\sinh\eta^{(i)}\cosh\eta^{(i)}}\right)^{-1}$$

$$= e^{\eta^{(i)} - \frac{h_i}{2}}\frac{\left(1 - e^{-2\eta^{(i)}}\right)^2}{1 + e^{-2\eta^{(i)}}}\left(1 - \frac{4\eta^{(i)}e^{-2\eta^{(i)}}}{1 - e^{-4\eta^{(i)}}}\right)^{-1}$$

$$= 1 + 2\eta^{(i)}e^{-2\eta^{(i)}} - 3e^{-2\eta^{(i)}} + O\left(e^{-2\eta^{(i)}}\right). \tag{76}$$

*Proof of Result 4:* Applying the Schwarz inequality to $I_1(h_1, h_2)$, we have

$$|I_1(h_1, h_2)|$$

$$\leq \sqrt{\int_0^\infty \left[P_0\left(T_{h_1}^1 > t\right)\right]^2 dt}\sqrt{\int_0^\infty [A(h_2, t)]^2 dt}$$

$$\leq \sqrt{\int_0^\infty P_0\left(T_{h_1}^1 > t\right) dt}\sqrt{\int_0^\infty [A(h_2, t)]^2 dt}$$

$$\leq \sqrt{E_0\left\{T_{h_1}^1\right\}}\sqrt{\frac{32}{\mu_2^2}e^{-h_2}\sum_{m,n\geq 1}\frac{u(\theta_m^{(2)})u(\theta_n^{(2)})}{\sec^2\theta_m^{(2)} + \sec^2\theta_n^{(2)}}}$$

$$\leq \frac{8}{\mu_1\mu_2}\sqrt{e^{-h_2}[h_1 + e^{-h_1} - 1]\cdot C}$$

where we used (11) and *Result 1* in the last line. Clearly, with linear dependence between $h_1$ and $h_2$

$$|I_1(h_1, h_2)| = o(1), \quad \text{as } h_1, h_2 \to \infty.$$

So (67) is done.

To prove (68), note that

$$\left| I_2(h_1, h_2) - E_0\left\{T_{h_1}^1\right\} \right| \leq E_0\left\{T_{h_1}^1\right\}|B(h_2) - 1|$$

$$+ B(h_2)\left|\int_0^\infty P_0\left(T_{h_1}^1 > t\right)\left(e^{-\frac{-\mu_2^2 t}{8\cosh^2\eta^{(2)}}} - 1\right) dt\right|. \tag{77}$$

By (11) and (66), the first term in (77) converges to zero as $h_1, h_2 \to \infty$. We need to show the integral in the second absolute value tends to zero as $h_1, h_2 \to \infty$. We have

$$0 \leq \int_0^\infty P_0\left(T_{h_1}^1 > t\right)\left(1 - e^{-\frac{\mu_2^2 t}{8\cosh^2\eta^{(2)}}}\right) dt$$

$$= \int_0^{\frac{8}{\mu_1^2}h_1} P_0\left(T_{h_1}^1 > t\right)\left(1 - e^{-\frac{\mu_2^2 t}{8\cosh^2\eta^{(2)}}}\right) dt$$

$$+ \int_{\frac{8}{\mu_1^2}h_1}^\infty P_0\left(T_{h_1}^1 > t\right)\left(1 - e^{-\frac{\mu_2^2 t}{8\cosh^2\eta^{(2)}}}\right) dt$$

$$= H(h_1, h_2) + T(h_1, h_2).$$

By using the fact that $1 - e^{-x} \leq x$, $H(h_1, h_2)$ can be bounded as follows:

$$0 \leq H(h_1, h_2)$$

$$\leq \int_0^{\frac{8}{\mu_1^2}h_1} P_0\left(T_{h_1}^1 > t\right)\cdot\frac{\mu_2^2}{\mu_1^2}\frac{h_1}{\cosh^2\eta^{(2)}} dt$$

$$\leq \frac{\mu_2^2}{\mu_1^2}\frac{h_1}{\cosh^2\eta^{(2)}}\int_0^\infty P_0\left(T_{h_1}^1 > t\right) dt$$

$$= \frac{\mu_2^2}{\mu_1^2}\frac{h_1}{\cosh^2\eta^{(2)}}E_0\left\{T_{h_1}^1\right\}$$

$$= \frac{\mu_2^2}{\mu_1^4}\frac{h_1(h_1 + e^{-h_1} - 1)}{\cosh^2\eta^{(2)}} \leq \frac{4\mu_2^2}{\mu_1^4}h_1^2 e^{-2\eta^{(2)}}$$

which goes to zero as $h_1, h_2 \to \infty$ due to (65).

Moreover

$$
0 \le T(h_1, h_2) \le \int_{\frac{8}{\mu_1^2} h_1}^{\infty} P_0 \left( T_{h_1}^1 > t \right) dt
$$

$$
= \frac{16}{\mu_1^2} \sum_{n \ge 1} u \left( \phi_n^{(1)} \right) \cos^2 \phi_n^{(1)} e^{-h_1 \left( \sec^2 \phi_n^{(1)} - \frac{1}{2} \right)}
$$

$$
\le \frac{16}{\mu_1^2} e^{-\frac{h_1}{2}} \sum_{n \ge 1} \left| u \left( \phi_n^{(1)} \right) \right| \cos^2 \phi_n^{(1)} = O \left( e^{-\frac{h_1}{2}} \right)
$$

where the last line is because of *Result 2*. So (68) and (69) (by similar argument) are done. $\square$

Now let us consider the $N$-CUSUM with $N \ge 2$. With similar derivation as above, we can extend our *Result 1*, *Result 2*, and *Result 4* to deal with the general case. In this manner, we can determine the asymptotic formula for the detection delay $J^{(N)}$ (Lemma 2 for $N \ge 2$) to be

$$
J^{(N)}(T_\hbar) = \frac{2}{\mu_1^2} (h_1 - 1) + o(1) \tag{78}
$$

and the mean time between false alarm to be

$$
E_{\infty,\dots,\infty}\{T_\hbar\}
$$
$$
= \frac{8 B(h_1) \dots B(h_N)}{\mu_1^2 / \cosh^2 \eta^{(1)} + \dots + \mu_N^2 / \cosh^2 \eta^{(N)}} + o(1). \tag{79}
$$

By using *Result 3*, we can compare $h_i$ with $\eta^{(i)}$ and obtain the asymptotic formulas in Lemma 1 and Lemma 3 for any $N \ge 2$.

Finally, let us prove Lemma 4.

*Proof of Lemma 4:* From the preceding discussion we need only to get the asymptotic formula of (79).

This can be seen as shown in (80) at the bottom of the page. Equations (78)–(80) imply the asymptotic formula in Lemma 4 and finish the proof. $\square$

## APPENDIX B
## THE DISCRETE-TIME MODEL

As before, we prove the results for $N = 2$. The general case for $N \ge 2$ will be discussed afterwards. We have the following preliminary result to help us.

*Result 5:* If there exists an $\alpha > 0$ such that $h_1 - \alpha h_2 = O(1)$ holds asymptotically as $h_1, h_2 \to \infty$, then we have

$$
\lim_{h_1, h_2 \to \infty} \left[ E_{1,\infty}\{T_\hbar\} - E_1^{(1)} \{T_{h_1}^1\} \right] = 0 \tag{81}
$$

$$
\lim_{h_1, h_2 \to \infty} \left[ E_{\infty,1}\{T_\hbar\} - E_1^{(2)} \{T_{h_2}^1\} \right] = 0. \tag{82}
$$

In the following paragraph we shall prove our lemmas for $N = 2$ in the order: Lemma 6 $\to$ Lemma 7 $\to$ Lemma 5. Then we discuss the asymptotic behavior of the $N$-CUSUM for $N \ge 2$, and prove Lemma 8 at the end.

*Proof of Lemma 6:* From *Result 5* and (48), we have under the constraint (55) that

$$
J_D^{(2)}(T_\hbar) = E_{1,\infty}\{T_\hbar\} + o(1)
$$
$$
= E_{\infty,1}\{T_\hbar\} + o(1)
$$
$$
= \frac{1}{I_{g_0}^{(1)}} (h_1 + \beta_1 + \kappa_1) + o(1) \tag{83}
$$

as $h_1, h_2 \to \infty$. So Lemma 6 is proven for $N = 2$.

*Proof of Lemma 7:* We begin by using Lemma 1 of [23] (or Theorem 3 of [13]) to obtain

$$
E_{\infty,\infty}\{T_\hbar\} = \frac{1}{I_{g_0}^{(1)}(R_1)^2 e^{-h_1} + I_{g_0}^{(2)}(R_2)^2 e^{-h_2}} [1 + o(1)] \tag{84}
$$

as $h_1, h_2 \to \infty$. Now let us assume $I_{g_0}^{(1)} < I_{g_0}^{(2)}$, and choose $h_1$ and $h_2$ according to (55). Then

$$
E_{\infty,\infty}\{T_\hbar\} = \frac{1}{I_{g_0}^{(1)}(R_1)^2} e^{h_1} [1 + o(1)] \tag{85}
$$

as $h_1, h_2 \to \infty$.

Formulas (83) and (85) imply the asymptotic formula in Lemma 7 for $N = 2$. $\square$

*Proof of Lemma 5:* We just need to let $I_{g_0}^{(1)} = I_{g_0}^{(2)}$, $R_1 = R_2$, and $h_1 = h_2$ in (84) to obtain

$$
E_{\infty,\infty}\{T_\hbar\} = \frac{1}{2 I_{g_0}^{(1)}(R_1)^2} e^{h_1} [1 + o(1)] \tag{86}
$$

as $h_1, h_2 \to \infty$.

Formulas (83) and (86) imply the asymptotic formula in Lemma 5 for $N = 2$. $\square$

Let us prove the preliminary result we have just now used.

*Proof of Result 5:* Without loss of generality we will only give the proof of (81). We observe that[7]

$$
E_{1,\infty}\{T_\hbar\} = e^{h_2} E_{1,\infty} \left\{ \frac{T_{h_1}^1}{e^{h_2}} \wedge \frac{T_{h_2}^2}{e^{h_2}} \right\}
$$
$$
= e^{h_2} \int_0^{\infty} P_1^{(1)} \left( \frac{T_{h_1}^1}{e^{h_2}} \ge t \right) P_{\infty} \left( \frac{T_{h_2}^2}{e^{h_2}} \ge t \right) dt
$$

[7]The integral representation is used for convenience. However, it should be realized that every integral is actually a summation.

$$
\frac{8 B(h_1) \dots B(h_N)}{\mu_1^2 / \cosh^2 \eta^{(1)} + \dots + \mu_N^2 / \cosh^2 \eta^{(N)}}
$$
$$
= \frac{8 B(h_1) \dots B(h_N)}{k \mu_1^2 / \cosh^2 \eta^{(1)} + \mu_{k+1}^2 / \cosh^2 \eta^{(k+1)} + \dots + \mu_N^2 / \cosh^2 \eta^{(N)}}
$$
$$
= \frac{2}{k \mu_1^2} [e^{h_1} + \text{``lower exponents''}]. \tag{80}
$$

$$= e^{h_2} \int_0^\infty P_1^{(1)} \left( \frac{T_{h_1}^1}{e^{h_2}} \geq t \right) dt$$

$$- e^{h_2} \int_0^\infty P_1^{(1)} \left( \frac{T_{h_1}^1}{e^{h_2}} \geq t \right)$$

$$\times \left[ 1 - P_\infty \left( \frac{T_{h_2}^2}{e^{h_2}} \geq t \right) \right] dt$$

$$= \int_0^\infty P_1^{(1)} \left( T_{h_1}^1 \geq u \right) du$$

$$- e^{h_2} \int_0^\infty P_1^{(1)} \left( \frac{T_{h_1}^1}{e^{h_2}} \geq t \right)$$

$$\times \left[ 1 - P_\infty \left( \frac{T_{h_2}^2}{e^{h_2}} \geq t \right) \right] dt$$

$$= E_1^{(1)} \left\{ T_{h_1}^1 \right\} - I_6(h_1, h_2).$$

To prove (81), it suffices to show $I_6(h_1, h_2)$ tends to zero as $h_1, h_2 \to \infty$. By using Lemma 1 of [23] (or Theorem 3 of [13]), we have for large $h_2$

$$I_6(h_1, h_2) = e^{h_2} \int_0^\infty P_1^{(1)} \left( \frac{T_{h_1}^1}{e^{h_2}} \geq t \right) [1 - e^{-t}[1 + o(1)]] dt$$

$$= [1 + o(1)] \int_0^\infty P_1^{(1)} \left( T_{h_1}^1 \geq u \right) \left( 1 - e^{-\frac{u}{e^{h_2}}} \right) du.$$

By using the fact that $1 - e^{-x} \leq x$, we further have

$$0 \leq I_6(h_1, h_2) \leq [1 + o(1)] \int_0^\infty P_1^{(1)} \left( T_{h_1}^1 \geq u \right) \frac{u}{e^{h_2}} du$$

$$= [1 + o(1)] e^{-h_2} \int_0^\infty u P_1^{(1)} \left( T_{h_1}^1 \geq u \right) du$$

$$= \frac{1 + o(1)}{2} e^{-h_2} E_1^{(1)} \left\{ \left( T_{h_1}^1 \right)^2 \right\}.$$

However, it is easily seen from the proof of Theorem 1 of [23] (also Theorem 4.1 of [8]) that

$$E_1^{(1)} \left\{ \left( T_{h_1}^1 \right)^2 \right\} = O((h_1)^2).$$

Therefore

$$0 \leq I_6(h_1, h_2) = O(e^{-h_2}(h_1)^2) \to 0$$

as $h_1, h_2 \to \infty$. This completes the proof of *Result 5*. $\square$

For the $N$-CUSUM with $N \geq 2$, we can easily extend *Result 5* to address the general case. And by using Lemma 1 of [23] (or Theorem 3 of [13]), (84) becomes

$$E_{\infty,\dots,\infty}\{T_\hbar\} = \left( \sum_{i=1}^N I_{g_0}^{(i)} R_i^2 e^{-h_i} \right)^{-1} [1 + o(1)] \qquad (87)$$

as $h_i \to \infty$, $i = 1, \dots, N$. Then Lemmas 5–7 are proven for any $n \geq 2$.

Finally, let us prove Lemma 8.

*Proof of Lemma 8:* We just need to get the asymptotic formula of (87) when $\hbar$ satisfies (55). This can be seen as follows:

$$\left( \sum_{i=1}^N I_{g_0}^{(i)}(R_i)^2 e^{-h_i} \right)^{-1}$$

$$= \left( \sum_{i=1}^k I_{g_0}^{(i)}(R_i)^2 e^{-h_i} + \sum_{i=k+1}^N I_{g_0}^{(i)}(R_i)^2 e^{-h_i} \right)^{-1}$$

$$= \left( \sum_{i=1}^k I_{g_0}^{(i)}(R_i)^2 e^{-h_i} \right)^{-1} [1 + o(1)]^{-1}$$

$$= \frac{e^{h_1}}{I_{g_0}^{(1)} \sum_{i=1}^k (R_i)^2 e^{h_1 - h_i}} [1 + o(1)]$$

$$= \frac{e^{h_1}}{I_{g_0}^{(1)} \sum_{i=1}^k (R_i)^2 e^{(\beta_i - \beta_1) + (\kappa_i - \kappa_1)}} [1 + o(1)]$$

$$= \frac{e^{h_1}}{I_{g_0}^{(1)} \sum_{i=1}^k (R_i)^2 r_i} [1 + o(1)] \qquad (88)$$

as $h_i \to \infty$, $i = 1, \dots, N$. Formulas (83) and (88) imply the asymptotic formula in Lemma 8 and complete the proof. $\square$

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