A comparison of 2-CUSUM stopping rules for quickest detection of two-sided alternatives in a Brownian motion model

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Abstract. This work compares the performance of all existing 2-CUSUM stopping rules used in the problem of sequential detection of a change in the drift of a Brownian motion in the case of two-sided alternatives. As a performance measure an extended Lorden criterion is used. According to this criterion the optimal stopping rule is an equalizer rule. This paper compares the performance of the modified drift harmonic mean 2-CUSUM equalizer rules to the performance of the best classical 2-CUSUM equalizer rule whose threshold parameters are chosen so that equalization is achieved. This comparison is made possible through the derivation of a closed-form formula for the expected value of a general classical 2-CUSUM stopping rule.

Keywords. Quickest Detection, 2-CUSUM.

1 Introduction and Mathematical formulation

In this work we are concerned with the problem of quickest detection of a two-sided change in the drift of a Brownian motion model. To mathematically set up this problem, we sequentially observe a process $\{\xi_t\}$ with the following dynamics:

$$d\xi_t = \begin{cases} dw_t & t \leq \theta \\ \\ \begin{matrix} \mu_1 dt + dw_t \\ \text{or} \\ -\mu_2 dt + dw_t \end{matrix} & t \geq \theta \end{cases}$$

where θ, the time of change, is assumed to be deterministic but unknown; w_t is a standard Brownian motion process; μ_i $(i = 1, 2)$, the possible drifts to which the process can change, are assumed to be known, but the specific drift to which the process is changing is unknown. Both μ_1 and μ_2 are assumed to be positive. The probability triplet consists of $(C[0,\infty], \cup_{t>0}\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{\xi_s, 0 < s \leq t\}$, and the families of probability measures $\{\mathcal{P}_\theta^i\}$, $\theta \in [0,\infty)$ corresponding to each change μ_i $(i = 1,2)$, with $\mathcal{P}_\infty$ the Wiener measure.

Our goal is to detect a change by means of a stopping rule τ adapted to the filtration $\mathcal{F}_t$. As a performance measure for this stopping rule we propose an extended Lorden criterion (see Hadjiliadis & Moustakides(2006))

$$J_L(\tau) = \max\{J_1(\tau), J_2(\tau)\} \tag{1}$$

where $J_i(\tau) = \sup_\theta \text{ essup } E_\theta^i\left[(\tau-\theta)^+|\mathcal{F}_\theta\right]$, $i = 1,2$. This gives rise to the following min-max constrained optimization problem:

$$\inf_\tau J_L(\tau) \text{ subject to } E_\infty[\tau] \geq \gamma, \tag{2}$$

where the constraint specifies the minimum allowable mean time between false alarms. As discussed in Moustakides (1986), in seeking solutions to the above problem, we can restrict our attention to stopping times that achieve the false alarm constraint with equality, i.e. τ for which

$$E_\infty[\tau] = \gamma. \tag{3}$$

This paper is a continuation of the work started in Hadjiliadis & Moustakides (2006), Hadjiliadis (2005) and Hadjiliadis & Poor (2007). In Hadjiliadis & Moustakides (2006) it is conjectured, but not proven, that within the class of harmonic mean rules drift equalizer rules are best. Also, two strong asymptotic optimality results as the mean time between false alarms tends to ∞ are presented both in the symmetric case, where a classical 2-CUSUM harmonic mean rule is proposed, and in the non-symmetric case, where a modified drift 2-CUSUM harmonic mean equalizer rule is proposed. These asymptotic results enhance the 2-CUSUM asymptotic optimality results of Tartakovsky (1994). As seen in Hadjiliadis (2005), within the class of modified drift 2-CUSUM harmonic mean rules the best rules are those for which the drift parameters of the modified drift 2-CUSUM harmonic mean stopping rules λ_1 and λ_2 are chosen so that $\lambda_2 - \lambda_1 = 2(\mu_2 - \mu_1)$, for *any* value of the mean time between false alarms. In Hadjiliadis & Poor (2007) it is proven that the optimal solution to problem (2) has to satisfy

$$J_1(\tau) = J_2(\tau). \tag{4}$$

In the same paper explicit upper and lower bounds for the first moment and the rate of change of the first moment of a general 2-CUSUM stopping rule are derived. By means of these bounds, it is seen that the best amongst the classical 2-CUSUM stopping rules is unique. It is a harmonic mean rule in the case of a symmetric change in the drift. In the non-symmetric case, it is a non-harmonic mean rule with threshold parameters $\nu_1 > \nu_2$ ($\nu_1 < \nu_2$) when $\mu_1 > \mu_2$ ($\mu_1 < \mu_2$) for any value of the mean time between false alarms.

We now proceed to define the modified drift 2-CUSUM stopping rules, a special case of which are the classical 2-CUSUM stoping rules.

Definition 1. Let $\nu_1 > 0$ and $\nu_2 > 0$. Define

1. $u_t^+(\lambda_1) = \xi_t - \frac{1}{2}\lambda_1 t$; $m_t^+(\lambda_1) = \inf_{s\leq t} u_s^+(\lambda_1)$; $y_t^+(\lambda_1) = u_t^+(\lambda_1) - m_t^+(\lambda_1)$; $\tau_1(\lambda_1,\nu_1) = \inf\{t > 0; y_t^+(\lambda_1) \geq \nu_1\}$,
2. $u_t^-(\lambda_2) = -\xi_t - \frac{1}{2}\lambda_2 t$; $m_t^-(\lambda_2) = \inf_{s\leq t} u_s^-(\lambda_2)$; $y_t^-(\lambda_2) = u_t^-(\lambda_2) - m_t^-(\lambda_2)$; $\tau_2(\lambda_2,\nu_2) = \inf\{t > 0; y_t^-(\lambda_2) \geq \nu_2\}$.

The modified drift 2-CUSUM rules are then of the form $\tau(\lambda_1,\lambda_2,\nu_1,\nu_2) = \tau_1(\lambda_1,\nu_1) \wedge \tau_2(\lambda_2,\nu_2)$. In this paper, we concentrate on modified drift 2-CUSUM harmonic mean rules for which $\nu_1 = \nu_2$.

The classical 2-CUSUM stopping rules are special cases of the modified drift rules for $\lambda_i = \mu_i$ $i = 1,2$ and are denoted by $\tau(\nu_1,\nu_2) = \tau_1(\nu_1) \wedge \tau_2(\nu_2)$.

Remark 1 It is useful at this stage to contrast the stopping time $\tau(\nu_1,\nu_2)$ to the one considered in Khan (2007). The latter, using the notation in this paper, is defined as $T = T_1(h_1) \wedge T_2(h_2)$, where $T_1(h_1)$ is the first passage time of the process $\xi_t - \inf_{s\leq t}\xi_s$ reaches h_1, and $T_2(h_2)$ is the first passage time of the process $\sup_{s\leq t}\xi_s - \xi_t$ reaches h_2. Thus T is different from $\tau(\nu_1,\nu_2)$.
In the sequel we will repeatedly use the indices $i,j \in \{1,2\}$ with $i \neq j$, and the function

$$f_\nu(y) = 2\frac{e^{y\nu} - y\nu - 1}{y^2}. \tag{5}$$

The exact first moments of $\tau(\lambda_i,\nu_i)$ and $\tau(\nu_i)$ are given in terms of the above function as seen in Hadjiliadis (2005). In particular, we have

$$E_\infty(\tau(\lambda_i,\nu)) = 2f_\nu(\lambda_i), \tag{6}$$

$$E_0^i(\tau(\lambda_i,\nu)) = 2f_\nu(\lambda_i - 2\mu_i),\ E_0^i(\tau(\lambda_j,\nu)) = 2f_\nu(\lambda_j + 2\mu_i). \tag{7}$$

Moreover, the expected values of $\tau(\nu)$ under each of the above measures are given by the above formulas evaluated at $\lambda_i = \mu_i$ and $\lambda_j = \mu_j$ (see Hadjiliadis & Moustakides (2006)).

2 The first moment of a general 2-CUSUM rule

We begin with our main expression for the first moment of a general classical 2-CUSUM stopping rule $\tau(\nu_1,\nu_2)$. To simplify the expressions that follow we introduce

$$\beta_j(r,\xi) = \exp\{-(r-1)\nu_j\,(r(\mu_1+\mu_2) + \mu_j - \xi)\} \tag{8}$$

$$A_j(\xi) = \sum_{r=1}^{\infty}\left[\Big(r\,[r(\mu_1+\mu_2) + \mu_j - \xi]\Big)\Big(\beta_j(r,\xi) - \beta_j(r+1,\xi)\Big)\right] \tag{9}$$

$$B_i(\xi) = \sum_{r=1}^{\infty}\left[\Big(r(\mu_1+\mu_2) + \frac{1}{2}(\mu_i - \xi)\Big)\exp\{-2(\nu_i - \nu_j)\}\right] \tag{10}$$

Theorem 1. *Let $\tau(\nu_1,\nu_2)=\tau_1(\nu_1)\wedge\tau_2(\nu_2)$ be any 2-CUSUM stopping rule and denote $\tau(\nu_1,\nu_2)$ by τ. Moreover, let f be as in (5). Then, for $\nu_i \geq \nu_j$, $i \neq j$, with $\Delta=\nu_i-\nu_j$, we have*

$$E_0^i\left[\tau\right]=2f_{\nu_j}(\mu_j+2\mu_i)\left[1-\frac{f_{\nu_j}(\mu_j+2\mu_i)}{f_{\nu_j}(-\mu_i)+f_{\nu_j}(\mu_j+2\mu_i)}e^{-2\Delta\cdot B_i(0)}\right], \quad (11)$$

$$E_\infty\left[\tau\right]=2f_{\nu_j}(\mu_j)\left[1-\frac{f_{\nu_j}(\mu_j)}{f_{\nu_j}(\mu_j)+f_{\nu_j}(\mu_i)}e^{-2\Delta\cdot B_i(2\mu_i)}\right], \text{ for } i=2, \quad (12)$$

$$E_0^j\left[\tau\right]=2f_{\nu_j}(-\mu_j)\left[1-\frac{f_{\nu_j}(-\mu_j)}{f_{\nu_j}(-\mu_j)+f_{\nu_j}(\mu_i+2\mu_j)}e^{-\Delta\cdot A_j(0)}\right], \quad (13)$$

$$E_\infty\left[\tau\right]=2f_{\nu_j}(\mu_j)\left[1-\frac{f_{\nu_j}(\mu_j)}{f_{\nu_j}(\mu_j)+f_{\nu_j}(\mu_i)}e^{-\Delta\cdot A_j(2\mu_j)}\right], \text{ for } j=2. \quad (14)$$

We notice that for any τ with $\nu_1=\nu_2$, all of the above expressions reduce to the well-known harmonic mean rule (see Siegmund(1985)). That is, for $\nu_1=\nu_2$, under any measure, we obtain

$$E\left[\tau\right]=\frac{E\left[\tau_1\right]E\left[\tau_2\right]}{E\left[\tau_1\right]+E\left[\tau_2\right]}. \quad (15)$$

Moreover, it can easily be seen from the expressions of Theorem 1 that the harmonic mean rule holds as a lower bound to the first moment of a general classical 2-CUSUM stopping rule (see for example Dragalin (1997)).

3 Classical vs Modified 2-CUSUM stopping rules

In this section we compare the classical 2-CUSUM rules to the modified drift 2-CUSUM harmonic mean rules, all of which satisfy (4), both in the symmetric and the non-symmetric case. We point out that the modified drift 2-CUSUM harmonic mean rules have to satisfy

$$\lambda_2-\lambda_1=2(\mu_2-\mu_1). \quad (16)$$

3.1 Symmetric case $\mu_1=\mu_2$

In this case, the unique classical 2-CUSUM stopping time that satisfies (4) is the one with $\nu_1=\nu_2$ (see Hadjiliadis & Poor (2007)). For the selection of the modified drift 2-CUSUM harmonic mean rule, notice that, since $\mu_1=\mu_2=\mu$, (16) implies $\lambda_1=\lambda_2=\lambda$, where λ is a free parameter. In order to compare the performance of the classical 2-CUSUM to its modified drift 2-CUSUM counterpart, we constrain them to both satisfy (3). For the classical 2-CUSUM this constraint is sufficient to give us a specific value for ν which in turn implies a specific value of $J(\cdot)$ through the functions for the first moment equations that appear at the end of Section 1 and the harmonic mean rule (15). However, for the modified drift parameter 2-CUSUM, the constraint (3) allows for a free choice of parameter for λ, which can then be used to minimize $J(\cdot)$. Thus the modified drift 2-CUSUM stopping time will have a lesser detection delay $J(\cdot)$ than its classical counterpart. The optimal choice of parameter λ however, converges to μ as $\gamma \to \infty$. This is easily seen by noticing that for the modified-drift 2-CUSUM, we have $J(\cdot) = \frac{2\log\gamma}{\lambda(2\mu-\lambda)}(1+o(1))$, which is minimized for the choice $\lambda = \mu$ as $\gamma \to \infty$. The convergence of the optimal choice of λ to μ is faster for higher values of μ as seen in Figure 1. This asserts the asymptotic optimality of the classical 2-CUSUM (see Hadjiliadis & Moustakides (2006)).

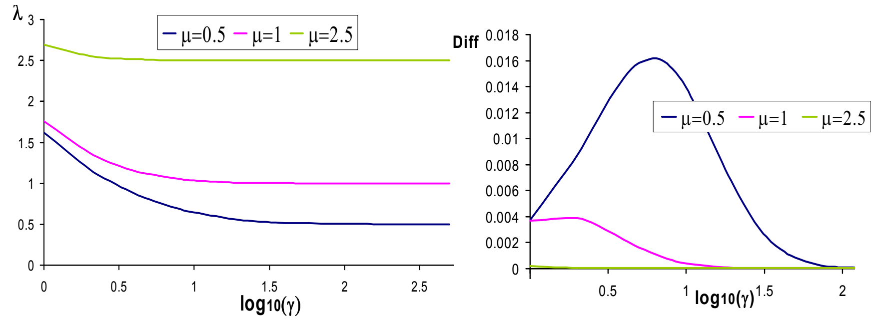

Fig. 1. **(Left)** Convergence of λ as a function of $\log(\gamma)$. The blue curve corresponds to $\mu = 0.5$, the magenta curve to $\mu = 1$, and the green curve to $\mu = 2.5$. **(Right)** Difference between detection delays of the classical and modified drift 2-CUSUM as a function of $\log(\gamma)$. Blue, magenta, and green curves same as in (Left).

3.2 Non-symmetric case $\mu_1 > \mu_2$

In this subsection we consider the case of a non-symmetric change. Without

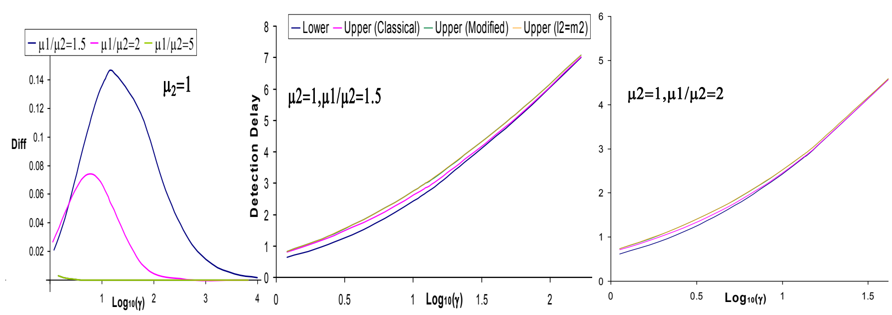

Fig. 2. **(Left)** Case of $\mu_2 = 1$: The difference in detection delay of the modified 2-CUSUM minus the classical 2-CUSUM equalizer rules is displayed as a function of $\log(\gamma)$. The blue curve corresponds to $\frac{\mu_1}{\mu_2} = 1.5$, the magenta curve to $\frac{\mu_1}{\mu_2} = 2$, and the green curve to $\frac{\mu_1}{\mu_2} = 5$. **(Middle)** Case of $\mu_2 = 1$ and $\frac{\mu_1}{\mu_2} = 1.5$: Upper (green and magenta curves correspond to the detection delay of the modified and the classical 2-CUSUM equalizer rules respectively, whereas orange curve is the detection delay of modified when $\lambda_2 = \mu_2$) and lower (blue line) bounds to the detection delay of the unknown optimal stopping rule as a function of $\log(\gamma)$. **(Right)** Case of $\mu_2 = 1$ and $\frac{\mu_1}{\mu_2} = 2$: Same type of graph as the middle.

loss of generality we assume that $\mu_1 > \mu_2$. In Hadjiliadis & Poor (2007) it is seen that the best classical 2-CUSUM stopping rule is unique and satisfies $\nu_1 > \nu_2$. We thus compare the detection delay $J(\cdot)$ of the unique classical 2-CUSUM stopping rule $\tau(\nu_1, \nu_2)$ with $\nu_1 > \nu_2$ that satisfies (4) with the modified drift 2-CUSUM harmonic mean rule with λ_2 a free parameter, over which the detection delay $J(\cdot)$ is minimized. For the modified drift 2-CUSUM harmonic mean rule (16) implies that $\lambda_1 = 2(\mu_1 - \mu_2) + \lambda_2$. Both rules are chosen so as to satisfy the false alarm constraint (3). The best choice of parameter for λ_2 converges to μ_2 as $\gamma \to \infty$ as seen in Hadjiliadis & Moustakides (2006). The detection delay of the unknown optimal stopping rule T^* is bounded above by the detection delay of the modified drift 2-CUSUM equalizer rule and the detection delay of the classical 2-CUSUM equalizer rule. It is also bounded below by the detection delay of $T_2(\nu_2)$ under P_0^2 with ν_2 chosen so that (3) is satisfied. It is seen that both detection delays displayed by the upper bound 2-CUSUM rules converge asymptotically (as $\gamma \to \infty$) to $\frac{2\log(\gamma)}{\mu_2^2}(1 + o(1))$ and thus that both approach the lower bound uniformly as $\gamma \to \infty$. However, for small values of γ the difference in detection delay of the modified drift 2-CUSUM harmonic mean equalizer rule minus the detection delay of the classical 2-CUSUM equalizer rule is positive for all values of μ_1 and μ_2 with $\mu_1 > \mu_2$. For example, in the left graph of Figure 2, this difference reaches the level 0.15 for $\frac{\mu_1}{\mu_2} = 1.5$, $\mu_2 = 1$ and $\log\gamma \approx 1.5$. (For more examples refer to Hadjiliadis et. al.(2008)). Of course, this difference decreases as $\gamma \to \infty$ since both detection delays approach the lower bound (middle and right graphs of Figure 2). This suggests that it is better to equalize 2-CUSUM rules by an appropriate selection of thresholds as opposed to modifying its drift parameters for small values of γ in the case of a non-symmetric change.

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