The beginning of quantum statistics

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TRANSLATORS' PREFACE

Satyendranath Bose, the author of a four-page paper[1] which appeared in German more than fifty years ago and marked the beginning of quantum statistics, died on 4 February 1974. Professor Sudarshan in a recent issue of this Journal,[2] under the title "A World of Bose Particles," has reviewed some of the important developments in particle physics which followed from the fundamental insight contained in Bose's paper. As a tribute to the great physicist and legendary figure in India, we furnish below for both physicists and historians of science an English translation of that paper. It is of interest to note that the paper was originally written by Bose in English and submitted late in 1923 to the *Philosophical Magazine*, but was rejected.[3] It was Einstein who recognized the worth of that paper when he received the manuscript with the following note from Bose[3]:

Respected Sir:

I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it. You will see that I have tried to deduce the coefficient $8\pi \nu^2/c^3$ in Planck's Law independent of the classical electrodynamics, only assuming that the ultimate elementary region in the phase-space has the content h^3. I do not know sufficient German to translate the paper. If you think the paper worth publication I shall be grateful if you arrange for its publication in *Zeitschrift fur Physik*. Though a complete stranger to you, I do not hesitate in making such a request. Because we are all your pupils though profiting only from your teachings through your writings. . . .

Einstein himself translated the paper and in early July 1924 submitted it to the *Zeitschrift* in Bose's name with the comment, "In my opinion Bose's derivation of the Planck formula signifies an important advance. The method used also yields the quantum theory of the ideal gas, as I will work out in detail elsewhere." It is remarkable that the notation used in modern texts on quantum statistics is almost the same as that employed by Bose.

Planck's law and the light quantum hypothesis

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The phase space of a light quantum in a given volume is subdivided into "cells" of magnitude h^3. The number of possible distributions of the light quanta of a macroscopically defined radiation over these cells gives the entropy and with it all thermodynamic properties of the radiation.

Planck's formula for the distribution of energy in blackbody radiation forms the starting point for the quantum theory which has been developed during the past twenty years and has yielded rich harvests in all fields of physics. Since its publication in the year 1901 many types of derivations of this law have been suggested. It is acknowledged that the fundamental assumptions of the quantum theory are inconsistent with the laws of classical electrodynamics. All existing derivations make use of the relation

$$\rho_\nu \, d\nu = (8\pi \nu^2 \, d\nu/c^3)E,$$

representing the relation between the radiation density and the mean energy of an oscillator, and they make assumptions concerning the number of degrees of freedoms of the ether as exemplified in the above equation (the first factor on the right-hand side). This factor, however, could be deduced only from the classical theory. This is the unsatisfactory point in all derivations, and it is not surprising that again and again efforts are made which try to give a derivation free of this logical deficiency.

A remarkably elegant derivation has been given by Einstein. Recognizing the logical defect in the existing derivations, he attempted to deduce the formula independently of any classical theory. Starting with very simple assumptions about the energy exchange between molecules and the radiation field, he finds the relation

$$\rho_\nu = \frac{\alpha_{mn}}{\exp[(\epsilon_m - \epsilon_n)/kT] - 1}.$$

However, in order to make this formula agree with that of Planck, he has to make use of Wien's displacement law and Bohr's correspondence principle. Wien's law is based on the classical theory; the correspondence principle assumes that the quantum theory agrees asymptotically with the classical theory in certain limiting cases.

In all cases it appears to me that the derivations have insufficient logical foundation. In contrast, the combining

of the light quanta hypothesis with statistical mechanics in the form adjusted by Planck to the needs of the quantum theory does appear to be sufficient for the derivation of the law, independent of any classical theory. In the following I wish to sketch briefly the new method.

Let the radiation be enclosed in a volume V and its total energy be E. Let there be different species of quanta each characterized by the number N_s and energy $h\nu_s$ ($s = 0$ to $s = \infty$). The total energy E is then

$$E = \sum_s N_s h\nu_s = V\int \rho_\nu \, d\nu. \tag{1}$$

The solution of our problem requires then the determination of the numbers N_s which determine ρ_ν. If we can state the probability for any distribution characterized by an arbitrary set of N_s, then the solution is determined by the requirement that the probability be a maximum provided the auxiliary condition (1) is satisfied. It is this probability which we now intend to find.

The quantum has a moment of magnitude $h\nu_s/c$ in the direction of its forward motion. The instantaneous state of the quantum is characterized by its coordinates x, y, z, and the associated momenta p_x, p_y, p_z. These six quantities can be interpreted as point coordinates in a six-dimensional space; they satisfy the relation

$$p_x{}^2 + p_y{}^2 + p_z{}^2 = h^2\nu^2/c^2,$$

by virtue of which the above-mentioned point is forced to remain on a cylindrical surface which is determined by the frequency of the quantum. In this sense the frequency domain $d\nu_s$ is associated with the phase space domain

$$\int dx \, dy \, dz \, dp_x \, dp_y \, dp_z$$
$$= V4\pi(h\nu/c)^2 \, h \, d\nu/c = 4\pi(h^3\nu^2/c^3)V \, d\nu.$$

If we subdivide the total phase space volume into cells of magnitude h^3, then the number of cells belonging to the frequency domain $d\nu$ is $4\pi V(\nu^2/c^3) \, d\nu$. Concerning the kind of subdivision of this type, nothing definitive can be said. However, the total number of cells must be interpreted as the number of the possible arrangements of one quantum in the given volume. In order to take into account the polarization, it appears mandatory to multiply this number by the factor 2 so that the number of cells belonging to an interval $d\nu$ becomes $8\pi V(\nu^2 \, d\nu/c^3)$.

It is now very simple to calculate the thermodynamic probability of a macroscopically defined state. Let N^s be the number of quanta belonging to the frequency domain $d\nu^s$. In how many different ways can we distribute these quanta over those cells which belong to the frequency interval $d\nu^s$? Let $p_0{}^s$ be the number of vacant cells, $p_1{}^s$ the number of those cells which contain one quantum, $p_2{}^s$ the number of cells containing two quanta, etc.; then the number of different distributions is

$$\frac{A^s!}{p_0{}^s! p_1{}^s! \cdots},$$

where

$$A^s = (8\pi\nu^2/c^3) \, d\nu^s,$$

and

$$N^s = 0p_0{}^s + 1p_1{}^s + 2p_2{}^s + \cdots$$

is the number of quanta belonging to the interval $d\nu^s$.

The probability of the state which is defined by all the $p_r{}^s$ is obviously

$$\prod_s \frac{A^s!}{p_0{}^s! p_1{}^s! \cdots}.$$

In view of the fact that we can look at the $p_r{}^s$ as large numbers, we have

$$\ln W = \sum_s A^s \ln A^s - \sum_s \sum_r p_r{}^s \ln p_r{}^s,$$

where

$$A^s = \sum_r p_r{}^s.$$

This expression should be maximum satisfying the auxiliary condition

$$E = \sum_s N^s h\nu^s; \quad N^s = \sum_r r p_r{}^s.$$

Carrying out the variation gives the condition

$$\sum_s \sum_r \delta p_r{}^s(1 + \ln p_r{}^s) = 0, \quad \sum_s \delta N^s h\nu^s = 0,$$

$$\sum_r \delta p_r{}^s = 0, \quad \delta N^s = \sum_r r\delta p_r{}^s.$$

It follows that

$$\sum_s \sum_r \delta p_r{}^s(1 + \ln p_r{}^s + \lambda^s) + \frac{1}{\beta}\sum_s h\nu^s \sum_r r\delta p_r{}^s = 0.$$

From this we get as the next step

$$p_r{}^s = B^s \exp(-rh\nu^s/\beta).$$

However, since

$$A^s = \sum_r B^s \exp\left(-\frac{rh\nu^s}{\beta}\right)$$
$$= B^s\left[1 - \exp\left(-\frac{h\nu^s}{\beta}\right)\right]^{-1},$$

we have

$$B_s = A^s[1 - \exp(-h\nu^s/\beta)].$$

Furthermore, we have the relation

$$N^s = \sum_r r p_r{}^s$$
$$= \sum_r rA^s\left[1 - \exp\left(-\frac{h\nu^s}{\beta}\right)\right]\exp\left(-\frac{rh\nu^s}{\beta}\right)$$
$$= \frac{A^s \exp(-h\nu^s/\beta)}{1 - \exp(-h\nu^s/\beta)}.$$

Because of the above stated value of A^s, it is also true that

$$E = \sum_s \frac{8\pi h\nu^{s3} \, d\nu^s}{c^3} V \frac{\exp(-h\nu^s/\beta)}{1 - \exp(-h\nu^s/\beta)}.$$

Using the preceding results, one finds also that

$$S = k\left\{\frac{E}{\beta} - \sum_s A^s \ln\left[1 - \exp\left(\frac{h\nu^s}{\beta}\right)\right]\right\};$$

from this it follows that $\beta = kT$, because of the condition $\partial S/\partial E = 1/T$. Substituting kT for β in the above equation for E, one obtains

$$E = \sum_s \frac{8\pi h\nu^{s3}}{c^3} V\left[\exp\left(\frac{h\nu^s}{kT}\right) - 1\right]^{-1} d\nu^s,$$

which is equivalent to Planck's formula.

[1] S. N. Bose, Z. Phys. 26, 178 (1924).
[2] E. C. G. Sudarshan, Am. J. Phys. 43, 69 (1975).
[3] See, for example, W. A. Blanpied, Am. J. Phys. 40, 1212 (1972).