Mapping the disk to convex subregions

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§1. Introduction.

Univalent maps $f : \mathbf{D} \to \mathbf{D}$ have been much studied by many authors. See [1], [2], [5], for just a few references. A quick scan of the content of these references shows that such mappings occur in a number of different settings for a variety of reasons. We will consider here whether or not such mappings exist satisfying certain rather restrictive boundary conditions.

To state our problem, and its partial solution, precisely, it will be helpful to summarize several known results. These may be found in [3] and [4].

Theorem A: *Given two collections of distinct points, $z_1, \ldots, z_n \in \mathbf{S}^1$, and $w_1, \ldots, w_n \in \mathbf{S}^1$, there exists a locally univalent $f : \mathbf{D} \to \mathbf{D}$, continuous up to $\mathbf{S}^1$, such that $f(z_i) = w_i$ and if $z \in \mathbf{S}^1 \setminus \{z_1, \ldots z_n\}$ then $|f(z)| < 1$.*

Similar in flavor is

Theorem B: *If $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ from theorem A are ordered cyclically then there exists a univalent f satisfying the conclusion of theorem A.*

The modulus of the rectangle with vertices at $\pm a \pm i$ in the complex plane is $a > 0$. Any convex proper subdomain of $\mathbf{C}$ with four distinct boundary points $z_1, \ldots, z_4$ marked cyclically is conformally equivalent to a unique such domain by a conformal map satisfying $z_1 \mapsto a - i$, $z_2 \mapsto a + i$, $z_3 \mapsto -a + i$, $z_4 \mapsto -a - i$. Thus we refer to a as the modulus of such a marked domain.

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Given four points $z_1, \ldots, z_4 \in \mathbf{S}^1$ ordered cyclically, we let $\Omega(z_1, \ldots, z_4)$ denote a convex domain contained in $\mathbf{D}$ with $z_1, \ldots, z_4$ in its boundary. Let $\operatorname{mod}(\Omega(z_1, \ldots, z_4))$ denote its modulus as a quadrilateral with vertices $z_1, \ldots, z_4$. If in this paper the z_j are understood, Ω and $\operatorname{mod}(\Omega)$ will be used. We have

THEOREM 1: *For fixed $z_1, \ldots, z_4$ cyclically ordered in $\mathbf{S}^1$, $\inf_\Omega \operatorname{mod}(\Omega)$ is realized uniquely by the convex domain having linear segments from z_1 to z_2 and from z_3 to z_4, and circular arcs in $\mathbf{S}^1$ from z_2 to z_3 and from z_4 to z_1. Similarly $\sup_\Omega \operatorname{mod}(\Omega)$ is realized uniquely by the convex domain having circular arcs in $\mathbf{S}^1$ from z_1 to z_2 and from z_3 to z_4, and linear segments from z_2 to z_3 and from z_4 to z_1.*

If $z_1, \ldots, z_4 \in \mathbf{S}^1$ are ordered cyclically, let $Q(z_1, \ldots, z_4)$ denote the convex domain which is the circle with the four points marked. We will show

THEOREM 2: *Let z_j, w_k be cyclically ordered 4-tuples in $\mathbf{S}^1$. There exists univalent $f : \mathbf{D} \to \mathbf{D}$, continuous to $\mathbf{S}^1$, with $f(z_i) = w_i$, $|f(z)| < 1$ for $z \in \mathbf{S}^1 \setminus \{z_1, \ldots, z_4\}$, and $f(\mathbf{D})$ convex, if and only if the inequalities*

$$\inf_{\Omega(w_1,\ldots,w_4)} \operatorname{mod}(\Omega(w_1, \ldots, w_4)) < \operatorname{mod}(Q(z_1, \ldots, z_4))$$

and

$$\operatorname{mod}(Q(z_1, \ldots, z_4)) < \sup_{\Omega(w_1,\ldots,w_4)} \operatorname{mod}(\Omega(w_1, \ldots, w_4))$$

hold, where $Q(z_1, \ldots, z_4)$ is $\mathbf{D}$ marked at the four points $z_1, \ldots, z_4$.

We give the proofs of these two theorems in §2. It is easy to see that similar results hold when we consider convexity with respect to the Poincaré hyperbolic metric on $\mathbf{D}$.

In greater generality, suppose we ask whether a result like that of theorem 2 above holds when we are given two cyclically ordered n-tuples of points in $\mathbf{S}^1$. To be precise, given $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ cyclically ordered n-tuples of points in $\mathbf{S}^1$, does there exist a univalent map $f : \mathbf{D} \to \mathbf{D}$ extending continuously to $\mathbf{S}^1$ such that $f(z_k) = w_k$, $f(\mathbf{D})$ is convex, and $|f(z)| < 1$ if $z \in \mathbf{S}^1 \setminus \{z_1, \ldots, z_n\}$? By comparing moduli of a finite number of rectangles, we obtain necessary conditions for the existence of such a function. This is explained in §3. Again, similar conditions hold when considering the Poincaré hyperbolic metric on **D**. Whether or not these conditions are sufficient is not clear to this author.

§2. Proofs of theorems 1 and 2.

Let $\Omega(z_1, \ldots, z_4)$ be a convex quadrilateral, and label the four boundary components z_1 to z_2 by R, z_2 to z_3 by T, z_3 to z_4 by L, and z_4 to z_1 by B. Theorems 1 and 2 depend on the following standard lemma. (See the version of Löwner's theorem in [6], for example.)

LEMMA: *If $\Omega_1 \subset \Omega_2$ are two convex quadrilaterals sharing opposite sides T and B (L and R, respectively), then*

$$\mathrm{mod}(\Omega_1) \leq \mathrm{mod}(\Omega_2)$$
$$(\mathrm{mod}(\Omega_1) \geq \mathrm{mod}(\Omega_2)).$$

Equality holds if and only if $\Omega_2 = \Omega_1$.

To see theorem 1, if $z_1, \ldots, z_4$ are cyclically ordered, let Ω_1 and Ω_2 be the convex regions described in theorem 1 as yielding the minimum and maximum moduli. If $\Omega \subset \mathbf{D}$ is a convex quadrilateral with corners at $z_1, \ldots, z_4$ then $\Omega_1 \cap \Omega$ and $\Omega_2 \cap \Omega$ are both convex regions sharing opposite sides with Ω. By the above lemma we have

$$\mathrm{mod}(\Omega_1) \leq \mathrm{mod}(\Omega_1 \cap \Omega) \leq \mathrm{mod}(\Omega) \leq \mathrm{mod}(\Omega_2 \cap \Omega) \leq \mathrm{mod}(\Omega_2),$$

and theorem 1 follows.

The necessity of the condition in theorem 2 now follows, as conformal univalent maps preserve modulus. The sufficiency also follows readily, as one may consider regions bounded by circular arcs of decreasing curvature through w_1, w_2, etc.

We see that if the condition of theorem 2 holds then $Q(z_1, \ldots, z_4)$ maps to an uncountable number of convex $\Omega(w_1, \ldots, w_4)$.

§3. A necessary condition for the general problem.

The general problem of mapping the disk to a convex region with prescribed boundary points in $\mathbf{S}^1$ seems to some extent accessable via consideration of moduli of rectangles. If we have an n-tuple $z_1, \ldots, z_n$ of points in $\mathbf{S}^1$ ordered cyclically we may consider convex subdomains of **D** which contain all of the z_k in their boundary. For any choice of four of the z_k ordered cyclically, we can identify among the convex domains here considered, as in theorem 1, the unique such domains for which the modulus is maximized and minimized. In the following extension of theorem 1, Ω will denote such a convex domain with four specified boundary points.

THEOREM 1′: *For fixed* $z_1, \ldots, z_n$ *cyclically ordered in* $\mathbf{S}^1$, *let* $z_{k_1}, \ldots, z_{k_4}$ *denote four of the* z_k *ordered cyclically. In the collection of convex subdomains of* $\mathbf{S}^1$ *having all* z_k *in the boundary and marked by the choice of* z_{k_i} *as vertices,* $\inf_\Omega \operatorname{mod}(\Omega)$ *is realized uniquely by the convex domain having linear segments from* z_{k_1} *to* z_{k_1+1}, z_{k_1+1} *to* z_{k_1+2}, $\ldots$, z_{k_2-1} *to* z_{k_2}, *and from* z_{k_3} *to* z_{k_3+1}, z_{k_3+1} *to* z_{k_3+2}, $\ldots$, z_{k_4-1} *to* z_{k_4}, *and circular arcs in* $\mathbf{S}^1$ *from* z_{k_2} *to* z_{k_3} *and from* z_{k_4} *to* z_{k_1}. *Similarly* $\sup_\Omega \operatorname{mod}(\Omega)$ *is realized uniquely by the convex domain having circular arcs in* $\mathbf{S}^1$ *from* z_{k_1} *to* z_{k_2} *and from* z_{k_3} *to* z_{k_4}, *and linear segments from* z_{k_2} *to* z_{k_2+1}, z_{k_2+1} *to* z_{k_2+2}, $\ldots$, z_{k_3-1} *to* z_{k_3}, *and from* z_{k_4} *to* z_{k_4+1}, z_{k_4+1} *to* z_{k_4+2}, $\ldots$, z_{k_1-1} *to* z_{k_1}.

The proof of this is the same as that of theorem 1.

Again, since conformal maps preserve modulus, the following necessary conditions on moduli are immediate.

THEOREM 2′: *Let* z_j, w_k *be cyclically ordered* n*-tuples in* $\mathbf{S}^1$. *For there to exist a univalent* $f : \mathbf{D} \to \mathbf{D}$, *continuous to* $\mathbf{S}^1$, *with* $f(z_i) = w_i$, $|f(z)| < 1$ *for* $z \in \mathbf{S}^1 \setminus \{z_1, \ldots, z_n\}$, *and* $f(\mathbf{D})$ *convex, we must have, for every choice of four cyclically ordered points* $z_{k_1}, \ldots, z_{k_4}$, *the satisfaction of the inequalities*

$$\inf_{\Omega(w_{k_1}, \ldots, w_{k_4})} \operatorname{mod}(\Omega(w_{k_1}, \ldots, w_{k_4})) < \operatorname{mod}(Q(z_{k_1}, \ldots, z_{k_4}))$$

and

$$\operatorname{mod}(Q(z_{k_1}, \ldots, z_{k_4})) < \sup_{\Omega(w_{k_1}, \ldots, w_{k_4})} \operatorname{mod}(\Omega(w_{k_1}, \ldots, w_{k_4}))$$

where $Q(z_{k_1}, \ldots, z_{k_4})$ *is* $\mathbf{D}$ *marked at the* z_{k_i}.

As mentioned in §1, whether or not these conditions suffice is not known to this author.

References

[1] Ahlfors, L., *Conformal Invariants*, McGraw–Hill, New York (1973).

[2] Caratheodory, C., *Theory of Functions*, vol.II, Chelsea, New York (1954).

[3] Clunie, J.; Hallenbeck, D. J. and MacGregor, T. H., A peaking and interpolation problem for univalent functions, *J. Math. Anal. Appl.* **111** (1985) 559–570.

[4] MacGregor, T. H. and Tepper, D. E., Finite interpolation by univalent functions, *J. Approx. Theory* **52** (1988) 315–321.

[5] Shapiro, J., *Composition Operators*, Springer–Verlag, New York (1993).

[6] Velling, J., The uniformization of rectangles: an exercise in Schwarz's lemma, *Amer. Math. Monthly* **99** (1992) 112–115.