Notes on projective structures and Kleinian groups

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1 Introduction

Throughout this paper, $\mathbf{C}$ will denote the complex plane, $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ the number sphere, and $\mathbf{D} = \{z : |z| < 1\} \subset \mathbf{C}$ the unit disk. We use $\mathrm{PSL}(2, \mathbf{C}) = \mathrm{SL}(2, \mathbf{C})/ \pm \mathrm{id}$ for the group of Möbius transformations of $\hat{\mathbf{C}}$. With Γ an arbitrary Fuchsian group, possibly having elliptic elements, let $\mathcal{R}$ be the hyperbolic orbifold $\mathbf{D}/\Gamma$.

A projective structure $P = (M, f)$ on $\mathcal{R}$ is a representation (the *monodromy representation*) $M : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ and a locally univalent holomorphic M-equivariant map (the *developing map*) $f : \mathbf{D} \to \hat{\mathbf{C}}$, so that $f \circ \gamma = M(\gamma) \circ f$ for all $\gamma \in \Gamma$ (see [2, ch.9]). Let $M(\Gamma)$ denote the image of Γ by M. The kernel of a projective structure means $\ker(M)$, and a projective structure is called faithful if its kernel is trivial. Two projective structures (M_1, f_1) and (M_2, f_2) are said to be equivalent if and only if there is a $g \in \mathrm{PSL}(2, \mathbf{C})$ so that $f_1 = g \circ f_2$ (and hence $M_1 = gM_2g^{-1}$). It is well known that the space of equivalence classes of projective structures is in one–to–one correspondence with the affine space of holomorphic quadratic differentials on $\mathcal{R}$, $\mathrm{Q}(\mathcal{R})$.

We choose an origin in the space of quadratic differentials by fixing an equivalence class of projective structures. In this case the Fuchsian equivalence class will be denoted by $0 \in \mathrm{Q}(\mathcal{R})$, making $\mathrm{Q}(\mathcal{R})$ a vector space. Now

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the mapping from equivalence classes of projective structures to quadratic differentials is readily expressed: $P = (M, f)$ corresponds to the quadratic differential $S_f \in \mathrm{Q}(\mathcal{R})$, where S_f is the Schwarzian derivative of $f : \mathbf{D} \to \hat{\mathbf{C}}$. This indeed determines a map from the equivalence classes of projective structures to $\mathrm{Q}(\mathcal{R})$; for any $g \in \mathrm{PSL}(2, \mathbf{C})$, $S_{g \circ f} = S_f$, implying that equivalent projective structures correspond to the same quadratic differential. This map is actually a bijection (see either [2, ch.9] or [4, §II.3]), and we will identify equivalent projective structures and use the identification with $\mathrm{Q}(\mathcal{R})$ implicitly. Thus P_Q means a representative of the equivalence class of projective structures corresponding to Q. Note that $\ker(M_Q)$ depends only on Q, not on a choice of representative.

Letting $\rho_{\mathcal{R}}|dz|$ denote the metric of curvature -1 on $\mathcal{R}$ induced by projection of that on $\mathbf{D}$, where $\rho_{\mathbf{D}}(z) = \dfrac{2}{(1 - |z|^2)}$, we norm $\mathrm{Q}(\mathcal{R})$ by $\|Q\| = \sup_{\mathbf{D}} |Q(z) \cdot \rho_{\mathbf{D}}^{-2}(z)| = \sup_{\mathcal{R}} |Q(z) \cdot \rho_{\mathcal{R}}^{-2}(z)|$. We denote by $\mathrm{Q}^\infty(\mathcal{R}) \subset \mathrm{Q}(\mathcal{R})$ the space of norm-bounded quadratic differentials on $\mathcal{R}$, and the corresponding projective structures will be called *bounded projective structures*. Unless otherwise stated, all quadratic differentials considered will be presumed bounded. Nehari [18] showed that if $\|Q\| \le \frac{1}{2}$ then f_Q is univalent, while it follows from a standard theorem of Kraus [12] that $\{Q : f_Q \quad \text{is univalent}\}$ is a closed subset of $\{Q : \|Q\| \le \frac{3}{2}\}$.

The behavior of f_Q when it is not univalent is not well understood. Gunning [3] showed that for compact $\mathcal{R}$ either f_Q maps $\mathbf{D}$ onto $\hat{\mathbf{C}}$ or else f_Q is a covering map of a domain Ω. Kra [8], [9] added that $M_Q(\Gamma)$ acts discontinuously on $\Omega = f_Q(\mathbf{D})$ if and only if f_Q is a covering map of Ω and extended these results to finite area $\mathcal{R}$ in the case when $Q \in \mathrm{Q}^\infty(\mathcal{R})$. Whether or not this is the case clearly depends only on Q.

Given these considerations, for an arbitrary hyperbolic Riemann surface, we define three classes of projective structures on $\mathcal{R}$, listed in order of decreasing size:

1. bounded discrete projective structures (or simply discrete projective structures, if boundedness is either assumed or dropped) — $\mathcal{D}(\mathcal{R}) = \{P_Q : Q \in \mathrm{Q}^\infty(\mathcal{R}), M_Q(\Gamma) \text{ is discrete }\}$,

2. bounded Kleinian projective structures (or Kleinian projective structures) — $\mathcal{K}(\mathcal{R}) = \{P_Q \in \mathcal{D}(\mathcal{R}) : M_Q(\Gamma)$ has a nonempty region of discontinuity in $\hat{\mathbf{C}}\}$,

3. <u>bounded covering projective structures</u> (or covering projective structures) — $\mathcal{S}(\mathcal{R}) = \{P_Q \in \mathcal{K}(\mathcal{R}) : f_Q$ is a covering map of its image and $M_Q(\Gamma)$ acts discontinuously there $\}$.

It follows that the space of faithful covering projective structures corresponds precisely to those $Q \in \mathrm{Q}^\infty(\mathcal{R})$ for which f_Q is univalent. Note that both Maskit [13] and Hejhal [5] have given examples of projective structures where $M_Q(\Gamma)$ is discrete but f_Q is not a covering map. Hence, in general, $\mathcal{S}(\mathcal{R})$ is a proper subset of $\mathcal{K}(\mathcal{R})$. More recently Kra [10], following Hejhal [6], completely classified the geometrically finite isolated points in $\mathcal{S}(\mathcal{R})$.

Assuming for the moment that $\mathcal{R}$ is compact, Kra and Maskit showed in [11] that $\mathcal{S}(\mathcal{R}) \subset \mathrm{Q}^\infty(\mathcal{R})$ is compact. Shiga showed in [20] that $\mathrm{Int}(\mathcal{K}(\mathcal{R})) \cap \mathcal{S}(\mathcal{R})$ coincides with the Bers embedding, centered at $\mathcal{R}$, of the Teichmüller space of $\mathcal{R}$. In theorem 3 and 4, we extend these results in non-compact cases.

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2 Bounded covering projective structures with distinct kernels are separated

Throughout this section we will only be concerned with bounded projective structures.

In personal conversation, Fred Gardiner asked whether or not quadratic differentials whose corresponding classes of projective structures are covering and have distinct kernels are in distinct components of $\mathcal{S}(\mathcal{R})$. We answer this in the positive with theorem 1 of this section, under rather loose conditions on the hyperbolic geometry of $\mathcal{R}$. These conditions include $\mathcal{R}$ having finite hyperbolic area. But first we set the stage for the theorem.

Definition. We say a Fuchsian group Γ is *maximal* if it is not properly contained in any other Fuchsian group. The corresponding $\mathcal{R}$ will also be called maximal.

It is immediate that all maximal Fuchsian groups are of the first kind. The following proposition simplifies considerations of $\mathcal{S}(\mathcal{R})$ for $\mathcal{R}$ maximal.

PROPOSITION 1: *If $\mathcal{R}$ is maximal then for any covering projective structure P_Q on $\mathcal{R}$, $f_Q(\mathbf{D})/M_Q(\Gamma)$ is conformally equivalent to $\mathcal{R}$.*

PROOF: We have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{D} & \longrightarrow & f_Q(\mathbf{D}) \\ \downarrow & & \downarrow \\ \mathcal{R} & \longrightarrow & f_Q(\mathbf{D})/M_Q(\Gamma) \end{array}$$

If $\mathcal{R}$ is maximal then $f_Q(\mathbf{D})/M_Q(\Gamma) \cong \mathcal{R}$. $\square$

LEMMA 1. *If $Q \in \mathcal{S}(\mathcal{R})$, then $M_Q(\Gamma)$ is a non-elementary Kleinian group whenever*

1. *$\mathcal{R}$ has finite hyperbolic area, or*

2. *$\mathcal{R}$ is maximal and has cusps.*

PROOF: In case (1) the proof may be found in [2] or [8, th. 1]. For case (2), by proposition 1 we have $f_Q(\mathbf{D})/M_Q(\Gamma) \cong \mathcal{R}$. Here, if $M_Q(\Gamma)$ is elementary, the planar surface $f_Q(\mathbf{D})$ must have a cusp. The following proposition shows that $|Q(z) \cdot \rho^{-2}(z)|$ is not bounded. Thus $P_Q \notin \mathcal{S}(\mathcal{R})$. $\square$

PROPOSITION 2. *Let Ω be a planar domain with cusps and $f : \mathbf{D} \to \Omega$ a locally univalent covering map. Then $|S_f \cdot \rho_{\mathbf{D}}^{-2}|$ is unbounded on $\mathbf{D}$.*

PROOF: By conjugation we may assume that $0 \in \hat{\mathbf{C}} \setminus \Omega$ is a cusp, that the group

$$\Gamma' = \langle z \mapsto \frac{(1+\frac{i}{\pi})z - 1}{z - (1-\frac{i}{\pi})} \rangle$$

is a subgroup of the deck group under which a horodisk B tangent to the unit circle at 1 is precisely invariant, and that B/Γ' is a neighborhood of the cusp under via f. Then there is a conformal map h on some neighborhood of 0 such that $h(0) = 0$ and $h \circ f(z) = \exp \frac{z+1}{z-1}$ on B.

By the Cayley identity, we have

$$|S_{h\circ f}(z) \cdot \rho_{\mathbf{D}}^{-2}(z)| = |S_h(f(z)) \cdot \rho_{\Omega}^{-2}(f(z)) + S_f(z) \cdot \rho_{\mathbf{D}}^{-2}(z)|.$$

As $z \to 1$ radially in B, the left-hand side of this equality, which is equal to

$$|\frac{2}{(z-1)^4}\frac{(1-|z|^2)^2}{4}|,$$

goes to ∞. At the same time, on the right-hand side, $S_h(f(z))$ is bounded near 0, and

$$\lim_{f(z)\to 0} \rho_{\Omega}^{-2}(f(z)) = 0.$$

Hence we have $|S_f(z)\cdot\rho_{\mathbf{D}}^{-2}(z)| \to \infty$ as $z \to 1$ radially in B, and $|S_f \cdot \rho_{\mathbf{D}}^{-2}|$ is unbounded. $\square$

We now have

THEOREM 1. *Let $\mathcal{R}$ satisfy one of the conditions of lemma 1. If P_{Q_1}, P_{Q_2} are covering projective structures corresponding to Q_1, Q_2, and $\ker(M_{Q_1}) \neq \ker(M_{Q_2})$, then P_{Q_1} and P_{Q_2} are in different components of $\mathcal{S}(\mathcal{R})$.*

The proof requires several facts ...

LEMMA 2. *If $Q \in \mathcal{S}(\mathcal{R})$, and $\alpha \in \Gamma \setminus \ker(M_Q)$ then there is some $\beta \in \Gamma$ such that $M_Q(\alpha)$ and $M_Q(\beta)$ generate a non-elementary discrete group.*

PROOF OF LEMMA 2: For any $\beta \in \Gamma$, the discreteness of the group generated by $M_Q(\alpha)$ and $M_Q(\beta)$ is given, as $Q \in \mathcal{S}(\mathcal{R})$. Since $M_Q(\Gamma)$ is non-elementary (lemma 1), for any $\alpha \in \Gamma$ with $M_Q(\alpha) \neq \text{id}$ there is some $\beta \in \Gamma \setminus \ker(M_Q)$ such that the fixed points of $M_Q(\alpha)$ and of $M_Q(\beta)$ are distinct. For such β, $M_Q(\alpha)$ and $M_Q(\beta)$ generate a non-elementary group (see, for example, [15, p.23]). $\square$

Now, for any $\alpha, \beta \in \Gamma$, let $\mathcal{S}(\mathcal{R}, \alpha, \beta) = \{Q \in \mathcal{S}(\mathcal{R}) : \langle M_Q(\alpha), M_Q(\beta)\rangle$ is not elementary $\}$.

LEMMA 3. *For any $\alpha, \beta \in \Gamma$, $\mathcal{S}(\mathcal{R}, \alpha, \beta)$ is both open and closed in $\mathcal{S}(\mathcal{R})$.*

PROOF OF LEMMA 3: That $\mathcal{S}(\mathcal{R}, \alpha, \beta)$ is open is seen as follows. Let $Q_0 \in \mathcal{S}(\mathcal{R}, \alpha, \beta)$. Since $\langle M_{Q_0}(\alpha), M_{Q_0}(\beta)\rangle$ is non-elementary, it contains a Schottky group of rank 2, *i.e.* there exist $\alpha', \beta' \in \langle \alpha, \beta \rangle \subset \Gamma$ so that $\langle M_{Q_0}(\alpha'), M_{Q_0}(\beta') \rangle = M_{Q_0}(\alpha') * M_{Q_0}(\beta')$ is Schottky. Since small deformations of Schottky groups are also Schottky [15], there is an $\epsilon > 0$ such that if $\|Q - Q_0\| < \epsilon$ then $\langle M_Q(\alpha'), M_Q(\beta') \rangle$ is also a Schottky group of rank 2. Hence when $\|Q - Q_0\| < \epsilon$ and $Q \in \mathcal{S}(\mathcal{R})$ we have $\langle M_Q(\alpha), M_Q(\beta) \rangle$ is non-elementary. This implies that $Q \in \mathcal{S}(\mathcal{R}, \alpha, \beta)$.

To show that $\mathcal{S}(\mathcal{R}, \alpha, \beta)$ is also closed, proving the lemma, we use the ensuing lemma.

LEMMA.[7] *The algebraic limit of a sequence of non-elementary discrete groups with a bounded number of generators is also a non-elementary discrete group.*

Thus we assume $\{Q_n\}$ is a sequence in $\mathcal{S}(\mathcal{R}, \alpha, \beta)$ with $Q_n \to Q$ as $n \to \infty$. Since the $\langle M_{Q_n}(\alpha), M_{Q_n}(\beta) \rangle$ are non-elementary and $\langle M_{Q_n}(\alpha), M_{Q_n}(\beta) \rangle \to \langle M_Q(\alpha), M_Q(\beta) \rangle$ algebraically, we conclude that $\langle M_Q(\alpha), M_Q(\beta) \rangle$ is a non-elementary discrete group, *i.e.* $Q \in \mathcal{S}(\mathcal{R}, \alpha, \beta)$. $\square$

We are now ready to prove our theorem.

PROOF OF THEOREM 1: Suppose P_{Q_1} and P_{Q_2} are discrete projective structures representing Q_1 and Q_2, respectively, such that $\ker(M_{Q_1}) \neq \ker(M_{Q_2})$. Without loss of generality let α, $\beta \in \Gamma$ satisfy the following conditions: $\alpha \in \ker(M_{Q_2}) \setminus \ker(M_{Q_1})$ and $M_{Q_1}(\alpha)$, $M_{Q_1}(\beta)$ generate a non-elementary discrete group. This is always possible by lemma 2 and the hypotheses of our theorem.

By lemma 3, whether $Q \in \mathcal{S}(\mathcal{R}, \alpha, \beta)$ or not is determined on components of $\mathcal{S}(\mathcal{R})$. Since by our choice of α, β we have $Q_1 \in \mathcal{S}(\mathcal{R}, \alpha, \beta)$ while $Q_2 \notin \mathcal{S}(\mathcal{R}, \alpha, \beta)$, it follows that Q_1 and Q_2 are in distinct components of $\mathcal{S}(\mathcal{R})$. $\square$

It is interesting to ask to what extent $\ker(M_Q)$, as a subgroup of Γ, determines the component of $\mathcal{S}(\mathcal{R})$ in which Q lies. We cannot answer this at present, so we instead ask to what extent does $\ker(M_Q)$ determine the conformal equivalence class of $f_Q(\mathbf{D})$. One sees easily that if $Q \in \mathcal{S}(\mathcal{R})$ has trivial kernel then $f_Q(\mathbf{D})$ is simply connected and conformally equivalent to $\mathbf{D}$. More generally we show

THEOREM 2. *If* $\ker(M_Q)$ *is such that* $\mathbf{D}/\ker(M_Q)$ *is conformally a plane do-*

main (i.e. simple closed curves separate), then $f_Q(\mathbf{D})$ *is conformally equivalent to* $\mathbf{D}/\ker(M_Q)$*.*

Proof: We let Γ' be a Fuchsian extension of Γ such that $\mathbf{D}/\Gamma' = \mathcal{R}'$ is conformally the surface $f_Q(\mathbf{D})/M_Q(\Gamma)$, and $\ker'$ the kernel of the monodromy map from Γ'. Let $\hat{\mathcal{R}} = \mathbf{D}/\ker(M_Q)$, and $\hat{\mathcal{R}}' = f_Q(\mathbf{D})$ is conformally $\mathbf{D}/\ker'$. We have that the diagram

$$\begin{array}{ccccc} & & \mathbf{D} & & \\ & \swarrow /\ker & & \searrow /\ker' & \\ \hat{\mathcal{R}} & & \xrightarrow{\hat{f}_Q} & & \hat{\mathcal{R}}' \\ \downarrow/(\Gamma/\ker) & & & & \downarrow/(\Gamma'/\ker') \\ \mathcal{R} & & \xrightarrow{\tilde{f}_Q} & & \mathcal{R}' \end{array}$$

commutes (see [11]).

We will establish the theorem by assuming that $\hat{f}_Q$ is not a conformal equivalence and showing that $\hat{\mathcal{R}}$ must have a nonseparating simple closed curve. To do this we use

Lemma 4. *There exists* $\gamma' \in \ker' \setminus \Gamma$ *corresponding to a simple closed curve on* $\hat{\mathcal{R}}'$*.*

Proof of Lemma 4: The $\gamma' \in \ker'$ corresponding to simple closed curves on $\hat{\mathcal{R}}'$ generate $\ker'$, if such a γ' does not exist then $\ker' < \Gamma$, whence $\ker' = \ker$ and $\hat{f}_Q$ is a conformal equivalence. □

Let Ω_i $(i = 1, 2)$ be two sheets of $\hat{\mathcal{R}}$ over $\hat{\mathcal{R}}'$, with cuts A_i and B_i to be determined. The first criterion for this choice is that A_1^+ (B_1^+) are identified with A_2^- $(B_2^-$, respectively) by the covering, a simple closed curve corresponding to γ' of lemma 4 crosses from A_1^- to A_2^+, and B_i is the image of A_i by some non-trivial element of $\Gamma/\ker$. See figure 1.

We choose the B_i, moreover, so that simple arcs $\alpha_i^\pm$ may be taken from $A_i^\pm$ to $B_i^\pm$ in Ω_i, with the same endpoints on the respective cuts. By monodromy this is always possible. The curve $\alpha_2^+(\alpha_1^-)^{-1}$ is a closed curve in $\hat{\mathcal{R}}$. See figure 2. It does not separate $\hat{\mathcal{R}}$, as we may connect any two points in $\Omega_i \setminus \alpha_i^+$ or $\Omega_i \setminus \alpha_i^-$ by a simple arc in Ω_i since neither α^+ nor α^- disconnects Ω_i, and using the action of the infinite group $\Gamma/\ker$ we may connect any Ω_i to any other Ω_j through a finite number of other Ω_k while staying away from $\alpha_2^+(\alpha_1^-)^{-1}$. □

Figure 1:

Figure 2:

3 Boundedness of $\mathcal{S}(\mathcal{R})$

By considering the boundedness of $\mathcal{S}(\mathcal{R})$ for more general $\mathcal{R}$, we extend the Kra–Maskit result mentioned earlier.

THEOREM 3. *Let $\mathcal{R}$ be maximal. Then $\mathcal{S}(\mathcal{R})$ is bounded in $\mathrm{Q}^\infty(\mathcal{R})$ precisely when the lengths of simple closed geodesics on $\mathcal{R}$ are bounded away from* 0*.*

PROOF: For a covering map $f_Q : \mathbf{D} \to \Omega \subset \hat{\mathbf{C}}$, with $Q \in \mathcal{S}(\mathcal{R})$, let us consider the injectivity radius. Assume that the lengths of simple closed geodesics on $\mathcal{R}$ are bounded away from 0. Then the injectivity radius of the universal covering map $\pi : \mathbf{D} \to \mathcal{R}$ is bounded away from 0 for any $z \in \mathbf{D}$ except in the cusped regions, and so is f_Q. Since $Q \in \mathrm{Q}^\infty(\mathcal{R})$, again by [9, lem. 1], f_Q is univalent on the cusped regions.

Hence we know there exists a positive constant δ such that the injectivity radius of f_Q for any $Q \in \mathcal{S}(\mathcal{R})$ is larger than δ at any $z \in \mathbf{D}$. By the Kra–Maskit lemma [11, lem. 5.1a] we have $\|Q\| \leq 6 \tanh^{-2} \delta$, so that $\mathcal{S}(\mathcal{R})$ is indeed bounded.

Conversely, let $\{\gamma_n\}$ be a sequence of simple closed geodesics on $\mathcal{R}$ whose lengths $\ell(\gamma_n) \to 0$ as $n \to \infty$. For each n, we construct a special Kleinian group G_n. Cutting $\mathcal{R}$ along γ_n and using the combination theorems [15], we construct G_n, with an invariant component Ω_n of the region of discontinuity such that Ω_n is conformally the normal cover of $\mathcal{R}$ corresponding to γ_n and $\mathcal{R} = \Omega_n/G_n$.

Let f_n be the locally univalent covering map $\mathbf{D} \to \Omega_n$ and set $Q_n = S_{f_n}$. They belong to $\mathrm{Q}(\mathcal{R})$. Furthermore, since

$$\inf_{z\in\mathbf{D}}\{\text{ injectivity radius of } f_n \text{ at } z\} = \frac{\ell(\gamma_n)}{2},$$

Q_n is actually in $\mathrm{Q}^\infty(\mathcal{R})$.

Now using [11, lem. 5.1b], $\|Q_n\| \to \infty$ as $n \to \infty$, since $\ell(\gamma_n) \to 0$. □

Remark: The above condition on $\mathcal{R}$ has arisen recently in related settings ([16], [17], and [19], for example).

Corollary. *Compactness of $\mathcal{S}(\mathcal{R})$ is equivalent to $\mathcal{R}$ having finite hyperbolic area. (Recall the topology of $\mathcal{S}(\mathcal{R})$.)*

Proof: Since $\mathcal{S}(\mathcal{R})$ is closed [8], and as the ability of $\mathcal{R}$ to cover only finitely many other Riemann surfaces (see [10]) implies that $\mathcal{S}(\mathcal{R})$ is bounded (of course, finite area is used here), the result follows as $Q^\infty(\mathcal{R})$ is finite dimensional. □

The following proposition shows that the presence of cusps does allow, however, for the existence of unbounded covering projective structures.

Proposition 4. *If $\mathcal{R}$ has cusps there are necessarily unbounded covering projective structures on $\mathcal{R}$.*

Proof: Let $\mathcal{R}$ be conformally $\mathcal{R}' \setminus \{p\}$, where $\mathcal{R}' = \mathcal{U}/\Gamma'$, $\mathcal{U} \subset \hat{\mathbf{C}}$ is a holomorphic universal cover of $\mathcal{R}'$, and Γ' is a subgroup of $\mathrm{PSL}(2,\mathbf{C})$. Now suppose $\mathcal{U} \ni 0 \mapsto p \in \mathcal{R}'$ via the universal covering of $\mathcal{R}'$. In this case the universal covering of $\mathcal{U} \setminus \Gamma'(0)$ by $\mathbf{D}$ is the developing map f of a discrete projective structure on $\mathcal{R}$. The kernel of the monodromy group here is the normalizer of a parabolic element in Γ corresponding to the cusp on $\mathcal{R}$. That $|S_f \cdot \rho^{-2}|$ is unbounded on $\mathbf{D}$ was shown in proposition 2. □

4 The structure of $\mathrm{Int}(\mathcal{D}(\mathcal{R}))$

Shiga [20] studied the structure of $\mathrm{Int}(\mathcal{K}(\mathcal{R}))$ for a compact hyperbolic surface. We extend this by considering the case where $\mathcal{R} = \mathbf{D}/\Gamma$ is of finite type, possibly with cone points, and studying $\mathrm{Int}(\mathcal{D}(\mathcal{R}))$. Though our argument is the same as Shiga's, we present it in full for the reader's convenience.

LEMMA 5. *For $\mathcal{R} = \mathbf{D}/\Gamma$ arbitrary, if $Q \in \mathrm{Int}(\mathcal{D}(\mathcal{R}))$, then M_Q is a type preserving isomorphism.*

PROOF: For $Q \in \mathrm{Q}^\infty(\mathcal{R})$, the homomorphism M_Q preserves parabolic elements and the type of elliptic elements. Thus if there exists some $Q_0 \in \mathrm{Int}(\mathcal{D}(\mathcal{R}))$ such that M_{Q_0} is not a type preserving isomorphism, then there exists a hyperbolic element $\gamma \in \Gamma$ such that $\mathrm{tr}^2(M_Q(\gamma))$ is a non-constant holomorphic function of $Q \in \mathrm{Q}^\infty(\mathcal{R})$ and $M_{Q_0}(\gamma)$ is either elliptic or parabolic. In either case, since $\mathrm{tr}^2(M_Q(\gamma))$ is an open mapping, there is a $Q_1 \in \mathrm{Int}(\mathcal{D}(\mathcal{R}))$ near to Q_0 such that $M_{Q_1}(\gamma)$ is elliptic with infinite order. But a discrete subgroup of $\mathrm{PSL}(2, \mathbf{C})$ cannot have such elements, yielding a contradiction. □

LEMMA 6. *Let $\mathcal{R} = \mathbf{D}/\Gamma$ have finite hyperbolic area. For $Q \in \mathrm{Int}(\mathcal{D}(\mathcal{R}))$, $M_Q(\Gamma)$ has a non-empty region of discontinuity in $\hat{\mathbf{C}}$. Thus $\mathrm{Int}(\mathcal{D}(\mathcal{R})) = \mathrm{Int}(\mathcal{K}(\mathcal{R}))$.*

PROOF: Assume $Q_0 \in \mathrm{Int}(\mathcal{D}(\mathcal{R}))$ is such that $M_{Q_0}(\Gamma)$ is not Kleinian. Since $M_{Q_0}(\Gamma)$ is finitely generated, it is Mostow–Sullivan rigid [21, th. 5]. Take a small ball B with center at Q_0 in $\mathrm{Int}(\mathcal{D}(\mathcal{R}))$. By Lemma 5, we can define a family $\{M_Q \circ M_{Q_0}^{-1}\}$ of type-preserving isomorphisms depending holomorphically on $Q \in B$. Then by Bers [1] (see also Shiga [20, th. 1]), we see $M_{Q_0}(\Gamma)$ and $M_Q(\Gamma)$ are quasiconformally equivalent for each $Q \in B \setminus \{Q_0\}$. But the rigidity of $M_{Q_0}(\Gamma)$ implies that the representations M_Q and M_{Q_0} are actually conformally equivalent. This means that $Q = Q_0$, which is impossible. □

THEOREM 4. *For $\mathcal{R}$ of finite area, $\mathrm{Int}(\mathcal{D}(\mathcal{R})) \cap \mathcal{S}(\mathcal{R}) = \mathbf{T}(\mathcal{R})$, where $\mathbf{T}(\mathcal{R})$ is the Bers embedding, centered at $\mathcal{R}$, of the Teichmüller space of $\mathcal{R}$.*

PROOF: The inclusion $\mathrm{Int}(\mathcal{D}(\mathcal{R})) \cap \mathcal{S}(\mathcal{R}) \supset \mathbf{T}(\mathcal{R})$ is clear. For the other direction, let Q be a point in $\mathrm{Int}(\mathcal{D}(\mathcal{R})) \cap \mathcal{S}(\mathcal{R})$. By the above two lemmas, we know $Q \in \mathrm{Int}(\mathcal{K}(\mathcal{R}))$ and M_Q is a type preserving isomorphism. So by a theorem of Maskit, [14, th. 6], $M_Q(\Gamma)$ is quasi-Fuchsian or totally degenerate without accidental parabolics, and the developing map f_Q is univalent.

By the same reasoning as given by Shiga in [20, th. 2], we see that $M_Q(\Gamma)$ cannot be totally degenerate. Hence $M_Q(\Gamma)$ is quasi-Fuchsian, and $Q \in \mathbf{T}(\mathcal{R})$ as desired. □

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