Existence and uniqueness of complete constant mean curvature surfaces at infinity of $\mathbf{H}^3$ *

John A. Velling †

December 1992

Abstract

A set of conditions are given, each equivalent to the constancy of mean curvature of a surface in $\mathbf{H}^3$. It is shown that analogs of these equivalences exist for surfaces in $\mathbf{S}^2_\infty$, the bounding ideal sphere of $\mathbf{H}^3$, leading to a notion of *constant mean curvature at infinity of* $\mathbf{H}^3$. A parametrization of all complete constant mean curvature surfaces at infinity of $\mathbf{H}^3$ is given by holomorphic quadratic differentials on $\hat{\mathbf{C}}$, $\mathbf{C}$, and $\mathbf{D}$.

1 Introduction

Let $\mathbf{H}^3$ denote the usual hyperbolic 3-space, with the upper half-space model $\{(X, Y, T) \in \mathbf{R}^2 \times \mathbf{R}_+\}$ or $\{(Z, T) \in \mathbf{C} \times \mathbf{R}_+\}$ and metric line element given by the expression

$$ds^2 = \frac{dX^2 + dY^2 + dT^2}{T^2} = \frac{|dZ|^2 + dT^2}{T^2}. \tag{1.1}$$

We will denote the corresponding inner product by $\langle \cdot, \cdot \rangle$, and note that $\mathbf{H}^3$ is a symmetric space with constant sectional curvature -1. There are other

*Key words and phrases: Hyperbolic space, Constant mean curvature, Quasiconformal deformations

*Mathematics Subject Classification: 53C42, 30C62

†Partially supported by PSC-CUNY grant #661365.

models of $\mathbf{H}^3$ which may facilitate calculations or geometric observations, but for convenience the upper half-space model will be used exclusively.

In various computations throughout the paper we will use the summation convention implicitly, so that if g_{ij}, $h^{kl} \in \mathbf{C}$ for $i, j, k, l \in 1, \cdots, n$ then $g_{ij}h^{jk}$ means $\sum_{j=1}^{n} g_{ij}h^{jk}$.

This paper concerns an apparently new use of quasiconformal analysis in surface theory, and a criterion the satisfaction of which is equivalent to a surface $\Sigma_0 \hookrightarrow \mathbf{H}^3$ having constant mean curvature, $H(\Sigma_0) = c$. Briefly:

For a smooth surface $\Sigma_0 \hookrightarrow \mathbf{H}^3$, consider unit speed normal geodesic flow. This gives a deformation of Σ_0 in $\mathbf{H}^3$, with the image of Σ_0 under the flow denoted by Σ_t after time t. Let $N_t : \Sigma_0 \to \Sigma_t$ denote this map. For the moment let us assume that Σ_t is smooth. In this case both Σ_0 and Σ_t inherit metrics from $\mathbf{H}^3$, and therewith conformal structures. The conformal deformation by the map N_t is measured from a conformal parameter Z on Σ_0 by the Beltrami differential

$$\mu_t = \frac{\partial_{\bar{Z}} N_t}{\partial_Z N_t} \tag{1.2}$$

on Σ_0.

The space of Beltrami differentials on Σ_0 is a vector space, and μ_t is a point in this space. Thus there is natural a map $[-\infty, +\infty] \to$ (Beltrami differentials) $\cup\{\infty\}$ on Σ_0, where $t \mapsto \mu_t$. (The images of $\pm\infty$ are obtained from the hyperbolic Gauss maps $G_\pm : \Sigma_0 \to \mathbf{S}^2_\infty = \partial\mathbf{H}^3$.) As $N_0 = id$, $\mu_0 \equiv 0$, and the curve of realized Beltrami differentials passes through the origin. It turns out that $H(\Sigma_0) = c$ if and only if this curve is a radial segment, i.e. if and only if there is some Beltrami differential μ' so that $\mu_t = f(t)\mu'$ with $f : [-\infty, +\infty] \to \mathbf{R} \cup \{\infty\}$ injective.

This is the first point of the paper, and is presented in §5. In §2–4 background is presented to make this precise: normal flow in $\mathbf{H}^3$, harmonic maps between surfaces, the hyperbolic Gauss maps, and several related topics. Much of the background material is a recapitulation of unpublished results of C. L. Epstein. One consequence of this development is that several apparently new characterizations of minimal surfaces are given in the main theorem (theorem 1) of §5. One example: Σ_0 is minimal if and only if this radial segment is symmetric with respect to 0. Though much of the development herein holds for $\mathbf{R}^3$ or $\mathbf{S}^3$, the treatment is restricted to $\mathbf{H}^3$.

The second major point of the paper is that the characterizations of constant mean curvature surfaces in $\mathbf{H}^3$ (from §5) can be used to give a notion of constant mean curvature surfaces immersed in $\mathbf{S}^2_\infty$. A surface $\Sigma_{+\infty}$ immersed in $\mathbf{S}^2_\infty$ with constant mean curvature must necessarily satisfy $H(\Sigma_{+\infty}) = \pm 1$. This is expected: if one considers spheres concentric about $\mathbf{x} \in \mathbf{H}^3$, $S_\rho(\mathbf{x})$ of radius ρ, their mean curvatures are $\pm \coth \rho$, which tend to ± 1 as $\rho \to \infty$. Similarly, all horospheres $\Sigma_0 \subset \mathbf{H}^3$ have $H(\Sigma_0) = \pm 1$. The development of the local theory of constant mean curvature surfaces at infinity is found in §6, the point (theorem 4) of which is to show that theorem 1 holds in this boundary case. When this boundary equivalence is satisfied we say that $\Sigma_{+\infty}$ has constant mean curvature at infinity. One example of this equivalence: considering a family $\{\Sigma_t\}_{t\in\mathbf{R}}$ of parallel surfaces in $\mathbf{H}^3$, as above, and letting $N^t_{+\infty} : \Sigma_t \to \Sigma_{+\infty}$ denote the forward Gauss map from Σ_t, let $N_t^{+\infty} = (N^t_{+\infty})^{-1} : \Sigma_{+\infty} \to \Sigma_t$. This is a potentially multivalued map. If the maps $N^t_{+\infty}$ are sufficiently nice, we can give $\Sigma_{+\infty}$ a conformal structure (the limit of the structures on the Σ_t as $t \to \infty$). Letting Z now be a complex parameter on $\Sigma_{+\infty}$, we look at

$$\mu_t^{+\infty} = \frac{\partial_{\bar{Z}} N_t^{+\infty}}{\partial_Z N_t^{+\infty}} \tag{1.3}$$

as a Beltrami differential on $\Sigma_{+\infty}$. The constancy of mean curvature for $\Sigma_{+\infty}$is equivalent to the map $t \mapsto \mu_t^{+\infty}$, given by $[-\infty, +\infty] \to$ (Beltrami differentials on $\Sigma_{+\infty} \cup \{\infty\}$), having image on a radial line.

Examples of such surfaces are known. They are precisely the images of $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ by locally univalent maps, up to Möbius transformation, with the infinitesimal form of the metric on each sheet being the push-forward of the Poincaré metric on $\mathbf{D}$. The corresponding family of parallel surfaces Σ_t interpolate between this possibly multisheeted conformal image of $\mathbf{D}$ and a generally not conformal (and often not even smooth) image of $\mathbf{D}$ obtained by a construction of Ahlfors and Weill. These are presented briefly in §7. In §8 an apparently new class of such surfaces is given, generalizing to flat structures the above interpolation and the Ahlfors–Weill construction.

Now if Σ_0 is a complete surface of constant mean curvature, its second fundamental form pulls back to its universal cover $\tilde{\Sigma}_0$, furnished with a complete metric. This form splits and gives a holomorphic quadratic differential on $\tilde{\Sigma}_0$ which is invariant under the deck transformations of the covering. It is natural to ask to what extent this quadratic differential, plus the completeness condition, characterize the surface Σ_0. This has been considered by other authors, and is discussed in the beginning of §9. Much of the motivation for this paper comes from this question. The essential existence/uniqueness issues in this setting remain unresolved.

It turns out that for constant mean curvature surfaces at infinity the second fundamental form again splits and gives a holomorphic quadratic differential. With notions of both completeness and second fundamental form for surfaces immersed in $\mathbf{S}^2_\infty$ (given in §4), the same sorts of existence and uniqueness questions as were asked above may be asked in this case. Here much more satisfactory answers are obtained. A full classification of complete constant mean curvature surfaces at infinity of $\mathbf{H}^3$ is established.

To put it simply, the space of complete constant mean curvature surfaces at infinity of $\mathbf{H}^3$, up to ambient isometry, is parametrized by the holomorphic quadratic differentials on one of $\mathbf{S}^2 = \hat{\mathbf{C}}$, $\mathbf{C}$, or $\mathbf{D}$. The parametrization is given by solving the Schwarzian differential equation arising on one of the given spaces from a given quadratic differential, and one can recover the quadratic differential by reading off the holomorphic part of the second fundamental form of our surface at infinity (sense must be made of this). In this way it is seen that the examples of §7 and 8, plus $\mathbf{S}^2_\infty$ as the limit of a family of concentric spheres in $\mathbf{H}^3$, are precisely the complete examples arising. This terminates the mathematical content of the paper, which is intended to be relatively self-contained and as such includes some proofs of known results.

The author thanks Dennis Sullivan for his interest and support as this paper developed. IHES is also thanked for their wonderful hospitality while the author visited Bures–sur–Yvette. This work also benefitted from the regular comments of Isaac Chavel and Józef Dodziuk.

In closing, a special word of thanks is extended to the author's thesis adviser, Newton S. Hawley, who died of cancer in early 1988. He was irascible and witty, feisty and gentle. He was a spellbinding instructor who breathed life into the content of his lectures. He pursued his interests with unbounded childlike energy, pulling those who knew him along with him. But most of all he was the author's dear friend. This paper is dedicated to his memory.

2 Normal flow

Let $\Sigma_0 \subset \mathbf{H}^3$ be a smooth, embedded oriented surface patch parametrized by $\Omega \subset \mathbf{R}^2$. Let $\vec{n}$ be a continuous unit normal vector field on Σ_0. This induces a directed geodesic field on Σ_0, with γ_p the geodesic in $\mathbf{H}^3$ passing through p with $\vec{n}(p)$ a unit tangent to γ_p at p. Define a positive direction for each γ_p using $\vec{n}(p)$, and define the maps $N_t : \Sigma_0 \to \mathbf{H}^3$ by

$$N_t(p) = \exp_p(t\vec{n}(p)), \tag{2.1}$$

for $p \in \Sigma_0$.

Denote by Σ_t the image $N_t(\Sigma_0)$, and henceforth consider N_t as a map from Σ_0 to Σ_t. Note that although the Σ_t may not always be manifolds the singularities occurring in this deformation are rather well understood [14]. In an unpublished document [7], C. Epstein studied the geometry of the Σ_t in terms of the geometry of Σ_0. The family of $\{\Sigma_t\}_{t\in\mathbf{R}}$ will be called *a family of parallel surfaces*, even though some of the Σ_t may be singular or degenerate. This paper makes liberal use of Epstein's treatment and, as [7] has not seen print, some of that work will first be reviewed.

If X_1, X_2 are standard positively oriented smooth coordinates on Ω, and ∂_1, ∂_2 the corresponding unit vector fields on Σ_0, the metric on Σ_0 induced by its embedding is denoted by $(g_0)_{ij}$, with inverse $(g_0)^{ij} = (g_0)^{-1}_{ij}$. Let ∇_0 denote covariant differentiation on Σ_0, and $\tilde{\nabla}$ that on $\mathbf{H}^3$. The Weingarten map on Σ_0 is thus

$$(II_0)^i_j = (g_0)^{ik}(II_0)_{kj} \tag{2.2}$$

where

$$(II_0)_{ij} = \langle \tilde{\nabla}_j \partial_i, \vec{n} \rangle \qquad (\tilde{\nabla}_i = \tilde{\nabla}_{\partial_i}). \tag{2.3}$$

is the second fundamental form.

Note in passing that if $k_0^1(p)$, $k_0^2(p)$ are the two eigenvalues of $(II_0)^i_j(p)$, i.e. the principal curvatures of Σ_0 at p, the Gauss and mean curvatures, $K(p)$ and $H(p)$, are defined by

$$\begin{aligned} K(p) &= k_0^1(p)k_0^2(p) - 1 \\ H(p) &= \tfrac{1}{2}(k_0^1(p) + k_0^2(p)) = \tfrac{1}{2}(II_0)^i_i. \end{aligned} \tag{2.4}$$

Using N_t we define the forms $(g_t)_{ij}(p)$, $(II_t)_{ij}(p)$, and $(II_t)^i_j(p)$ on Σ_0:

$$(g_t)_{ij}(p) = \langle (N_t)_*(\partial_i(p)), (N_t)_*(\partial_j(p)) \rangle,$$

$$\begin{aligned}
(II_t)_{ij}(p) &= (g_t)(\partial_i(p), \tilde{\nabla}_{(\partial_j(p))}\vec{n}(p)), \\
(II_t)^i_j(p) &= (g_t)^{ik}(p)(II_t)_{kj}(p).
\end{aligned}$$

These are the pullbacks via N_t to Σ_0 of the metric, second fundamental form, and Weingarten map on Σ_t, and they evolve via the differential equations

$$\begin{aligned}
\frac{d(g_t)_{ij}}{dt} &= -2(II_t)_{ij}, \\
\frac{d(II_t)^i_j}{dt} &= (II_t)^i_k(II_t)^k_j - \delta^i_j, \\
\frac{d^2(II_t)_{ij}}{dt^2} &= 4(II_t)_{ij}
\end{aligned} \tag{2.5}$$

which one solves explicitly by

$$\begin{aligned}
(g_t)_{ij} &= (g_0)_{ik} \cdot \left[(II_0)^k_l \sinh t - \delta^k_l \cosh t\right] \\
&\qquad \cdot \left[(II_0)^l_j \sinh t - \delta^l_j \cosh t\right],
\end{aligned} \tag{2.6}$$

etc. We see that both $(g_t)_{ij}$ and $(II_t)_{ij}$ are finite for all time, with $(g_t)_{ij}$ always non-negative definite. Furthermore, Epstein shows that the only singularities in $(II_t)^i_j$ arise when $(g_t)_{ij}$ is degenerate. In doing so, he considers the vector fields $\tilde{\partial}_i$ obtained from parallel translation of $\partial_i(0)$ along the geodesic field, i.e. $\tilde{\partial}_i(0) = \partial_i(0)$ and $\tilde{\nabla}_{\vec{n}}\tilde{\partial}_i = 0$. Then if $\partial_i(0)$ is a characteristic vector of $II^l_m(0)$ corresponding to the i-th characteristic value k^i_0, *i.e.* $\partial_i(0)$ is one of the directions of principal curvature, then so is ∂_i for all $t \in \mathbf{R}$ and

$$\partial_i(t) = \frac{1}{2}[(1 - k^i_0)e^t + (1 + k^i_0)e^{-t}]\tilde{\partial}_i. \tag{2.7}$$

It is this equation which tells us how lengths deform under N_t, hence how the geometries of the Σ_t evolve.

Note 1. How the Σ_t's may degenerate can be seen from (2.5). This has been analyzed thoroughly by Epstein. In particular, if Σ_0 is a smooth surface patch with directional curvatures k^i_0 satisfying $|k^i_0| \leq 1$ then Σ_t is smooth for all $t \in \mathbf{R}$. If, however, $p \in \Sigma_0$ with some directional curvature $k^i_0(p) > 1$ then for $t > 0$, Σ_t is degenerate at $\gamma_p(t)$ only for $t = \coth^{-1} k^i_0$, and similarly if $k^i_0(p) < -1$. Thus Σ_t can degenerate at $\gamma_p(t)$ only for at most two values of t. It follows that if in a sufficiently small neighborhood of p the directional curvatures of Σ_0 are bounded away from ± 1 then Σ_t are smooth for $|t|$ sufficiently large.

At any rate, it will be useful to have at our avail the corresponding expression in complex notation. Thus if $Z = X_1 + iX_2$, and $\bar{Z} = X_1 - iX_2$, we will use the standard notation for complex derivatives

$$\begin{aligned}\partial_Z &= \frac{1}{2}(\partial_1 - i\partial_2),\\ \partial_{\bar{Z}} &= \frac{1}{2}(\partial_1 + i\partial_2).\end{aligned}$$

Similarly will ∇_Z and $\nabla_{\bar{Z}}$ denote $\frac{1}{2}(\nabla_1 - i\nabla_2)$, and $\frac{1}{2}(\nabla_1 + i\nabla_2)$, respectively, *etc.* These live in the complexified tangent bundle of Σ_0 and if necessary in the complexified tangent bundle of either Σ_t or $\mathbf{H}^3$.

At times it will be convenient to write these forms and other geometric entities on Σ_t in terms of these complex parameters. To this end we make a couple of purely formal observations.

If $E\,dX^2 + 2F\,dX\,dY + G\,dY^2$ denotes the symmetric quadratic form

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

on Σ_t, then the corresponding hermitian form is $A\,dZ^2 + 2B\,dZ\,d\bar{Z} + \bar{A}\,d\bar{Z}^2$, *i.e.*

$$\begin{pmatrix} B & A \\ \bar{A} & B \end{pmatrix}$$

where $A = \frac{1}{4}(E - 2iF - G)$, and $B = \frac{1}{4}(E + G)$. Thus, for example, $4(B^2 - A\bar{A}) = EG - F^2$ tells us how determinants of the real and hermitian forms are related. Similarly, if the quadratic form is a square, *i.e.*

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^2$$

in real coordinates, in complex notation it becomes

$$2\begin{pmatrix} B & A \\ \bar{A} & B \end{pmatrix}^2$$

with A and B as above. In the same fashion the form $E\,dX\,\partial_X + 2F\,(dX\,\partial_Y + dY\,\partial_X) + G\,dY\,\partial_Y$ becomes, in complex notation, $2A\,dZ\,\partial_Z + 2B\,(dZ\,\partial_{\bar{Z}} + d\bar{Z}\,\partial_Z) + 2\bar{A}\,d\bar{Z}\,\partial_{\bar{Z}}$ with A and B as above.

We will denote by h_t, $(\Pi_t)^i_j$ and $(\Pi_t)_{ij}$ the hermitian forms of g_t, $(II_t)^i_j$ and $(II_t)_{ij}$ respectively. If we have $h = A\,dZ^2 + 2B\,dZ\,d\bar{Z} + \bar{A}\,d\bar{Z}^2$, then we will say $A = h_{ZZ}$, $B = h_{Z\bar{Z}}$, *etc.*, so that

$$h = \begin{pmatrix} h_{Z\bar{Z}} & h_{ZZ} \\ h_{\bar{Z}\bar{Z}} & h_{Z\bar{Z}} \end{pmatrix}.$$

Finally we note that $\Sigma_0 \hookrightarrow \mathbf{H}^3$ inherits a conformal structure, *i.e.* a notion of an infinitesimal circle, via the immersion. In particular, the Gauss equations for isothermal coordinates (conformal coordinates) on Σ_0 are

$$\begin{aligned} d\xi &= \frac{1}{\lambda\sqrt{g_{11}}}\,(g_{11}\,dx_1 + g_{12}\,dx_2) \\ d\eta &= \frac{\sqrt{g_{11}g_{22} - g_{12}^2}}{\lambda\sqrt{g_{11}}}\,dx_2 \end{aligned}$$

as is well-known [10, §17]. In ξ, η–coordinates the metric is $ds^2 = \lambda^2(d\xi^2 + d\eta^2)$. If $\zeta = \xi + i\eta$, then the hermitian form h is, with respect to this complex parameter, $\lambda^2\,d\zeta\,d\bar{\zeta}$. These coordinates will be used presently.

3 Harmonic maps and the deformation of surfaces

This section reviews several phenomena associated with differentiable maps from a smooth surface (with either a Riemannian or a conformal structure) into Riemannian manifolds. Most of this is well known, but we recount it here for the sake of completeness. It will be useful but not required herein to keep in mind the example of normal flow of surfaces discussed in the preceding section.

We consider

1. harmonicity of maps from a surface,

2. harmonicity of quadratic differentials, and

3. quasiconformal deformation and surface maps.

Each of these will be considered in turn.

(1) Harmonic maps. Given Riemannian manifolds (M, g_M) and (N, g_N) and a smooth map

$$f : (M, g_M) \to (N, g_N),$$

suppose X_i are coordinates on M. By $f^*(g_N)$ ($f^*(h_N)$ respectively) will be meant the quadratic form (hermitian form) which is the pull-back by f of g_N (h_N - the Hermitian form corresponding to g_N). When f is understood we will use $(g_N)^*$ ($(h_N)^*$) for $f^*(g_N)$ ($f^*(h_N)$ respectively). We let ∇ denote the metric covariant differentiation on M. This is defined for quadratic forms [15, ch.6].

The map f is said to be harmonic if and only if

$$-\nabla_k (g_M)^{ij} (g_N)^*_{ij} + 2(g_M)^{ij} \nabla_i (g_N)^*_{jk} = 0$$

for all k [6, §3]. Since $\nabla_i (g_M)^{ij} = 0$ we may rewrite this as

$$(g_M)^{ij}(-\nabla_k (g_N)^*_{ij} + 2\nabla_i (g_N)^*_{jk}) = 0.$$

In the case when M is a surface and g_M is given in conformal parameters, we have $(g_M)_{ij} = \lambda^2 \delta_{ij}$ and the equations for harmonicity become

$$-\nabla_k (g_N)^*_{ii} + 2\nabla_i (g_N)^*_{ik} = 0 \tag{3.1}$$

for $k = 1, 2$.

If we rewrite these as

$$\begin{aligned} \nabla_1((g_N)^*_{11} - (g_N)^*_{22}) + 2\nabla_2 (g_N)^*_{12} &= 0 \\ \nabla_2((g_N)^*_{22} - (g_N)^*_{11}) + 2\nabla_1 (g_N)^*_{12} &= 0 \end{aligned}$$

and let $Z = X_1 + iX_2$, etc., we find that these are the Cauchy–Riemann equations for $(h_N)^*_{ZZ}$, *i.e.* the above equations are the real and negative imaginary parts of the expression

$$\nabla_{\bar{Z}} (h_N)^*_{ZZ} = 0.$$

Thus we have the following standard.

Lemma 1. *If M is a surface, the map $f : (M, g_M) \to (N, g_N)$ is harmonic only if the ZZ–part of the induced hermitian form $(h_N)^*$ is holomorphic.*

(2) Harmonic Quadratic Differentials. There are various notions of harmonicity for tensor fields over manifolds with certain features – like a Riemannian metric or a Kähler structure, etc. Here we have a conformal metric and call a symmetric 2–tensor $Q = A\,dZ^2 + 2B\,dZ\,d\bar{Z} + \bar{A}\,d\bar{Z}^2$, written in conformal coordinates, a quadratic differential. We will follow Kodaira and say that Q is harmonic if and only if

$$\nabla_Z \nabla_{\bar{Z}}(A\,dZ^2) + (\nabla_Z \nabla_{\bar{Z}} + \nabla_Z \nabla_{\bar{Z}})(B\,dZ\,d\bar{Z}) + \nabla_{\bar{Z}} \nabla_Z(\bar{A}\,d\bar{Z}^2) = 0.$$

The explicit covariant derivatives of quadratic differentials in conformal coordinates are precisely

$$\begin{aligned} \nabla_{\bar{Z}}(A\,dZ^2) &= \partial_{\bar{Z}} A\,dZ^2 \\ \nabla_Z \nabla_{\bar{Z}}(A\,dZ^2) &= \lambda^4\,\partial_Z(\lambda^{-4}\partial_{\bar{Z}} A)\,dZ^2 \\ \nabla_Z(B\,dZ\,d\bar{Z}) &= \lambda^2\,\partial_Z(B\lambda^{-2})\,dZ\,d\bar{Z} \end{aligned} \tag{3.2}$$

and their conjugates. Thus $A\,dZ^2 + 2B\,dZ\,d\bar{Z} + \bar{A}\,d\bar{Z}^2$ is harmonic if and only if

$$\partial_Z(\lambda^{-4}(\partial_{\bar{Z}} A)) = \partial_Z \partial_{\bar{Z}}(\lambda^{-2} B) = 0.$$

We sum this up in lemma 2.

Lemma 2. *If $Q = A\,dZ^2 + 2B\,dZ\,d\bar{Z} + \bar{A}\,d\bar{Z}^2$ is a harmonic quadratic differential on (M, g) in conformal coordinates, with the hermitian form of the metric h given by $h = \lambda^2 |dZ|^2$, then $\partial_Z(\lambda^{-4}\,\partial_{\bar{Z}} A) = 0$ and $B\lambda^{-2}$ is a harmonic function on M. In particular, if A is holomorphic and B is a constant multiple of the metric, the quadratic differential is harmonic.*

Lemma 3. *Given Q, (M, g_M) as in the preceding lemma, with Q the hermitian form of the Weingarten map for some isometric immersion $M \overset{\iota}{\hookrightarrow} (N, g_N)$ into a Riemannian manifold N of dimension ≥ 3 then Q is harmonic if and only if $B\lambda^{-2}$ is constant, or equivalently if and only if A is holomorphic.*

Proof: The Codazzi–Mainardi equations for the image of M are $\nabla_{\bar{Z}}(A\,dZ^2) = \nabla_Z(B\,dZ\,d\bar{Z})$ and its conjugate. Thus the holomorphicity of A is equivalent to the constancy of $B\lambda^{-2}$ and, as noted in the preceding lemma, both imply the harmonicity of Q.

Now assume Q is harmonic. Covariant differentiation of the first Codazzi–Mainardi equation with respect to Z plus the fact that $\nabla_{\bar{Z}}\nabla_Z(B\,dZ\,d\bar{Z}) = 0$ gives that $\nabla_Z(B\,dZ\,d\bar{Z}) = c\lambda^2\,dZ\,d\bar{Z}$ where c is globally constant. From (3.2) we deduce that $\partial_Z(B\lambda^{-2}) = c$. Conjugation gives the conclusion that $\partial_{\bar{Z}}(B\lambda^{-2}) = \bar{c}$. Thus $B\lambda^{-2} = cZ + \bar{c}\bar{Z} + d$, *i.e.* is real affine, for *any* choice of conformal coordinate, whence $c = 0$ and $\nabla_{\bar{Z}}(A\,dZ^2) = \nabla_Z(B\,dZ\,d\bar{Z}) = 0$. The lemma follows. □

(3) Quasiconformal Deformations. Let M again be a surface with Riemannian structure and assume we have conformal coordinates thereon. Let $f : (M, g_M) \to (N, g_N)$ be a map to a general Riemannian manifold N. If the hermitian form induced by f has the matrix form

$$(h_N)^* = \begin{pmatrix} \beta & \alpha \\ \bar{\alpha} & \beta \end{pmatrix}^2 \tag{3.3}$$

then the Beltrami differential of f is easily expressed by the entries of this matrix. The proof is a straightforward and formal computation.

LEMMA 4. *The Beltrami differential of f is $\mu = \dfrac{\bar{\alpha}}{\beta}$.*

Assuming (3.3), we make the following definition.

DEFINITION: The Beltrami differential μ of f is given by

$$\mu = \frac{\bar{\alpha}}{\beta}.$$

We will sometimes call this the "conformal deformation" of M by f.

DEFINITION: The space of Beltrami differentials on M will be denoted by $B(M) = \mathcal{F}(M) \otimes T^{0,1}(M) \otimes T^{-1,0}(M)$, where $\mathcal{F}(M)$ is the vector space of measurable functions on M. Similarly we define $B^\infty(M) = L^\infty(M) \otimes T^{0,1}(M) \otimes T^{-1,0}(M)$. We give $B^\infty(M)$ the topology inherited from $L^\infty(M)$.

If g_M is not given in terms of the conformal structure, one lets

$$h_M = \begin{pmatrix} \beta_M & \alpha_M \\ \bar{\alpha}_M & \beta_M \end{pmatrix}^2$$

be the metric hermitian form on M and $(h_N)^*$ the induced hermitian form from f of (3.3). It is readily checked that if

$$\begin{pmatrix} \beta & \alpha \\ \bar{\alpha} & \beta \end{pmatrix}^2 = \begin{pmatrix} \beta_M & \alpha_M \\ \bar{\alpha}_M & \beta_M \end{pmatrix}^{-1} \begin{pmatrix} \beta_N & \alpha_N \\ \bar{\alpha}_N & \beta_N \end{pmatrix}^2 \begin{pmatrix} \beta_M & \alpha_M \\ \bar{\alpha}_M & \beta_M \end{pmatrix}^{-1}$$

then

$$\mu = \frac{\bar{\alpha}}{\beta}.$$

Note 2. It is important that lemmas 1 and 4 make sense even if N is not a manifold. All that is necessary is that N carries a symmetric quadratic form g_N so that the expression $(g_N)^*$ makes sense. We thus make the following definition.

DEFINITION: The map $f : (M, g_M) \to (N, g_N)$ is said to be harmonic if and only if $\nabla_{\bar{Z}}(h_N)^*_{ZZ} = 0$. The Beltrami differential of f is given by $\mu = \frac{\bar{\alpha}}{\beta}$.

Note 3. The notions of harmonic maps, holomorphic differentials and conformal deformation are determined by the conformal structure on M, and up to scalar multiplication of the appropriate quadratic forms.

Lemma 1 and the above definition can be generalized a bit if N is also a surface.

PROPOSITION 1. *Let (M, g_M) and (N, g_N) be surfaces. Suppose that $f : M \to N$ is a local diffeomorphism. The map f is harmonic if and only if the ZZ–part of $(g_N)^*$ is holomorphic on M.*

PROOF: Let Z (W) be a conformal parameter on M (N) such that at $\mathbf{x} \in M$ $(f(\mathbf{x}) \in N)$ the metric g_M (g_N) is given by $|dZ|^2$ $(|dW|^2)$, and the metric connection vanishes. The rest of the proof now takes place at $\mathbf{x} \in M$. The form $|dW|^2$ on N pulls back to $|W_Z\, dZ + W_{\bar{Z}}\, d\bar{Z}|^2 = W_Z \overline{W}_Z\, dZ^2 + (W_Z \bar{W}_{\bar{Z}} + W_{\bar{Z}} \overline{W}_Z)\, dZ\, d\bar{Z} + W_{\bar{Z}} \overline{W}_{\bar{Z}}\, d\bar{Z}^2$ on M. The ZZ–part of this is $W_Z \overline{W}_Z$.

Now if f is harmonic then $W_{Z\bar{Z}} = \overline{W}_{Z\bar{Z}} = 0$ so that $\partial_{\bar{Z}}(W_Z \overline{W}_Z) = 0$ as desired. In the other direction, if $\partial_{\bar{Z}}(W_Z \overline{W}_Z) = 0$ we suppose $W_{Z\bar{Z}} \neq 0$. Since f is a local diffeomorphism at every point either W_Z or $W_{\bar{Z}}$ is nonzero.

We'll assume that $W_Z \neq 0$, as if $W_{\bar{Z}} \neq 0$ the argument will be identical. As $\overline{W}_{Z\bar{Z}} = \overline{(W_{Z\bar{Z}})}$ we have $W_{Z\bar{Z}}\overline{W}_Z + W_Z\overline{W}_{Z\bar{Z}} = 0$ implies

$$\frac{\overline{W}_Z}{W_Z} = -\frac{\overline{W}_{Z\bar{Z}}}{W_{Z\bar{Z}}}$$

is of unit modulus. The Jacobian of f, $|W_Z|^2 - |W_{\bar{Z}}|^2$, must therefore vanish, contradicting the fact that f is a local diffeomorphism. Thus indeed $W_{Z\bar{Z}} = 0$. $\square$

This argument is due to Newton Hawley [11].

4 Flow to infinity and envelopes of horospheres

In §2 we introduced for each $t \in (-\infty, +\infty)$ a nonnegative definite quadratic form $(g_t)_{ij}$ (with inverse $(g_t)^{ij}$ when extant) such that

$$(g_t)_{ij} = (g_0)_{ik}[II^k_l(0)II^l_j(0)\sinh^2 t - 2II^k_j(0)\sinh t \cosh t + \delta^k_j \cosh^2 t].$$

This is the pullback by N_t of the metric on the possibly singular variety Σ_t. As such we will use g_t both for the metric on Σ_t and its pullback to Σ_0. We may for all $t \in (-\infty, +\infty)$ determine whether or not the map N_t is harmonic. As was mentioned in note 3, the harmonicity of N_t is determined by the definition of g_t up to a multiplicative constant. Thus if we give Σ_t the metric $\hat{g}_t = 4e^{-2t}g_t$, the map $\hat{N}_t : \Sigma_0 \to (\Sigma_t, \hat{g}_t)$, agreeing pointwise with N_t, is harmonic precisely when N_t is.

We define the maps $N_{\pm\infty}$ as

$$N_{+\infty}(p) = \lim_{t\to+\infty} \exp_p(t\vec{n}(p)) \in \mathbf{S}^2_\infty,$$

with $\vec{n}(p)$ as in §2. Similarly for $N_{-\infty}$. The metrics g_t diverge as $t \to \pm\infty$. Thus to consider whether the map $N_{+\infty}$ is harmonic or not it is more convenient to note that the $\hat{g}_t$ to converge to a limiting quadratic form $\hat{g}_{+\infty}$ as $t \to +\infty$.

Epstein showed [7] that the limiting metric $\hat{g}_{+\infty}$ is nondegenerate if and only if the map $N_{+\infty}$ is a local diffeomorphism to $\mathbf{S}^2_\infty$, or equivalently if and only if the Σ_t converge at $+\infty$ to a surface immersed in $\mathbf{S}^2_\infty$. Whether or not

$\hat{g}_{+\infty}$ is degenerate, the (possibly degenerate) surface on which it lives will be called $\Sigma_{+\infty}$, and we note that

$$(\hat{g}_{+\infty})_{ij} = \lim_{t\to+\infty} 4e^{-2t}(g_t)_{ij} = (g_0)_{ik}[(II_0)^k_l(II_0)^l_j - 2(II_0)^k_j + \delta^k_j].$$

We make a similar definition when $t \to -\infty$.

This suggests a definition.

DEFINITION: We say that $N_{+\infty}$ is harmonic if and only if the map $\hat{N}_{+\infty}$: $\Sigma_0 \to (\Sigma_{+\infty}, \hat{g}_{+\infty})$, agreeing pointwise with $N_{+\infty}$, is harmonic. Similarly for $N_{-\infty}$.

Note 4. Epstein [7] has shown that where $\Sigma_{+\infty}$ is nondegenerate, its conformal structure is precisely that inherited from $\mathbf{S}^2_\infty$. This is intuitively clear as if X_1, X_2 form an orthonormal basis for $T_p\Sigma_0$ then parallel translation along γ_p to $N_t(p)$ gives an orthonormal basis for $T_{N_t(p)}\Sigma_t$. However if we orient such an orthonormal basis and project out to $\mathbf{S}^2_\infty$, one of the orientations is the same as that of $\mathbf{S}^2_\infty$, while the other is different. Our convention will be to give the Σ_t orientations opposite those induced by $\mathbf{H}^3$ with normal $\vec{n}(p,t)$ on Σ_t (see §2). Thus $\Sigma_{+\infty}$ is given an inward pointing normal so that its orientation is that induced by $\mathbf{S}^2_\infty = \hat{\mathbf{C}}$. In the same fashion $\Sigma_{-\infty}$ inherits an outward pointing normal so that its orientation is opposite that induced by $\mathbf{S}^2_\infty = \hat{\mathbf{C}}$.

Note 5. In a family of parallel surfaces, there is no natural choice of Σ_0. These families are naturally parametrized by $\mathbf{R}$ as a metric space with its Euclidean metric, not as a vector space. Thus for a different parametrization of the family $\tau = t - t_0$ we have

$$(\hat{g}_{\tau=+\infty})_{ij} = e^{2t_0}(\hat{g}_{t=+\infty})_{ij}.$$

We see then that the quadratic form on $\Sigma_{+\infty}$ is determined up to multiplication by a positive constant by the family of parallel surfaces. Hence whether or not the maps $N^{t_1}_{t_2} = N_{t_2} \circ (N_{t_1})^{-1} : \Sigma_{t_1} \to \Sigma_{t_2}$ (for any choice of Σ_0) are harmonic is intrinsic to the family of surfaces, even when the t_i are $\pm\infty$.

Furthermore, the completeness of a Riemannian manifold (M, g) is equivalent to the completeness of (M, cg) for any $c \in \mathbf{R}_+$. Thus we are led to the following definition.

DEFINITION: We say that $\Sigma_{+\infty}$ is complete if $\hat{g}_{+\infty}$ is positive definite and the Riemannian manifold $(\Sigma_{+\infty}, \hat{g}_{+\infty})$ is complete. Since $\hat{g}_{+\infty}$ is determined up to a constant, it follows that the completeness of $\Sigma_{+\infty}$ is determined by our family of parallel surfaces.

We can represent geometric information on $\Sigma_{+\infty}$ in terms of that on Σ_0. It is not far fetched to suppose we can reverse this. We can construct each Σ_t from Σ_0 by considering the envelope of spheres of radius t about points on Σ_0, we construct Σ_0 from Σ_t in the same fashion. Now let us assume that $\Sigma_{+\infty}$ is immersed in $\mathbf{S}^2_\infty$. Heuristically, letting $t \to +\infty$, we construct Σ_0 from $\Sigma_{+\infty}$ as an envelope of horospheres with centers on $\Sigma_{+\infty}$. Thus, we ask, given a domain $\Omega \subset \mathbf{S}^2_\infty$ and a choice of horosphere at each point $\theta \in \Omega$, how do we reconstruct Σ_0; that is, what sort of conditions on the family of horospheres allow us to integrate to get Σ_0, and, given that we can indeed integrate, how do we extract geometric information on Σ_0 from the family of horospheres?

These questions have also been answered by Epstein [7]. Picking a base-point $\mathbf{x} \in \mathbf{H}^3$ and if $\theta \in \mathbf{S}^2_\infty$ is the center of the horosphere ν_θ, we say that ν_θ has radius $\rho \in \mathbf{R}$ with respect to $\mathbf{x}$ if $\pm d_{\mathbf{H}}(\mathbf{x}, \nu_\theta) = \rho$, where we take $-$ if $\mathbf{x}$ is interior to ν_θ and $+$ if not. Thus if we have a distribution of horospheres ν_θ on $\Omega \subset \mathbf{S}^2_\infty$, on choosing a base point $\mathbf{x} \in \mathbf{H}^3$ we can give criteria for the integrability of ν_θ in terms of the radius function $\rho(\theta)$. In fact Epstein shows that this distribution is integrable if ρ is C^1. Clearly this does not depend on the choice of $\mathbf{x}$. Of course to extract geometric information (metric form, curvature, etc.) on integral surfaces from knowledge of ρ we need more derivatives. Let us assume as many derivatives as necessary to make the computations work. In general ρ being C^4 suffices.

Fixing $\mathbf{x} \in \mathbf{H}^3$, we have a well defined metric on $\mathbf{S}^2_\infty$ with constant curvature $\equiv +1$ by visual identification with the sphere of unit tangent vectors at $\mathbf{x}$. Identifying $\mathbf{S}^2_\infty$ with $\hat{\mathbf{C}}$ via stereographic projection, so that $\theta \in \mathbf{S}^2_\infty$ is $0 \in \mathbf{C}$, and taking z to be a standard planar coordinate in a neighborhood of $0 \in \Omega$, we can express the metric and second fundamental form on Σ_t in a rather straightforward fashion.

The hermitian form of the metric on Σ_t is given by Epstein [7] as

$$h_t|_{z=0} = \frac{1}{2}e^{2(\rho+t)} \begin{pmatrix} (\partial\bar{\partial}\rho - 1)e^{-2(\rho+t)} + 1 & (\partial^2\rho - (\partial\rho)^2)e^{-2(\rho+t)} \\ (\bar{\partial}^2\rho - (\bar{\partial}\rho)^2)e^{-2(\rho+t)} & (\partial\bar{\partial}\rho - 1)e^{-2(\rho+t)} + 1 \end{pmatrix}^2 \tag{4.1}$$

while

$$(\Pi_t)^i_j|_{z=0} = D^{-1}\left[\begin{pmatrix} 0 & 2A \\ 2\bar{A} & 0 \end{pmatrix} + \omega \mathrm{Id}\right] \tag{4.2}$$

where

$$\begin{aligned} A &= \partial^2\rho - (\partial\rho)^2 \\ D &= e^{2(\rho+t)}\left[|1 + (\partial\bar{\partial}\rho - 1)e^{-2(\rho+t)}|^2 - |Ae^{-2(\rho+t)}|^2\right] \\ \omega &= e^{2(\rho+t)}\left[((\partial\bar{\partial}\rho - 1)e^{-2(\rho+t)})^2 - |Ae^{-2(\rho+t)}|^2 - 1\right]. \end{aligned} \tag{4.3}$$

Evidently at $z = 0$ we get $\hat{h}_{+\infty} = \lim_{t\to+\infty} e^{-2t}h_t$ to be

$$\hat{h}_{+\infty}|_{z=0} = \frac{1}{2}e^{2\rho}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2.$$

Thus at the center of our stereographic coordinate system, $z = 0$ on $\Sigma_{+\infty}$, the length element is $e^{2\rho}|dz|^2$. Since the surface Σ_0 determines the metric $\hat{h}_{+\infty}$, we can read off from h_0 and Π_0 the support function ρ. This leads to the following theorem.

THEOREM. [7] *The support function ρ for Σ_0 on $\Sigma_{+\infty}$ determines the metrics h_t (or $\hat{h}_t = e^{-2t}h_t$) for all $t \in [-\infty, +\infty]$. Conversely, any of the h_t determine the support function ρ.*

Also at $z = 0$ we get $(\Pi_{\pm\infty})^i_j = \mp\mathrm{id}$. Thus the mean curvatures of the surfaces Σ_t approach ∓ 1 pointwise as $t \to \pm\infty$. However, it appears that the off diagonal parts of $(\Pi_{\pm\infty})^i_j$ vanish.We wish to note that infinitesimally these terms persist in $(\Pi_t)^i_j$ as $t \to \pm\infty$. We do this by letting $\tau = e^{-2t}$ and taking the 1–jet at $\tau = 0$. Since

$$\lim_{t\to+\infty} \frac{\left[(zz\text{–term of } (\Pi_t)^i_j)\right]}{2A\tau} = e^{-2\rho},$$

for example, we use the following definition.

DEFINITION: The second fundamental form of $\Sigma_{+\infty}$ is defined by

$$(\Pi_{+\infty})^i_j|_{z=0} = \begin{pmatrix} -1 + 2(\partial\bar{\partial}\rho - 1)e^{-2\rho}\,d\tau & 2Ae^{-2\rho}\,d\tau \\ 2\bar{A}e^{-2\rho}\,d\tau & -1 + 2(\partial\bar{\partial}\rho - 1)e^{-2\rho}\,d\tau \end{pmatrix}. \tag{4.4}$$

The zz-term of $(\Pi_{+\infty})^i_j$ is now defined as the infinitesimal Beltrami parameter $2A\,d\tau$. In the same way we may obtain the expression for $(\Pi_{-\infty})^i_j|_{z=0}$.

The intrinsic curvature of Σ_t is readily computed from $(\Pi_t)^i_j$. One checks that $\det(\Pi_t)^i_j = \det(II_t)^i_j$ so that

$$K_t = \det(\Pi_t)^i_j - 1.$$

Thus to compute the intrinsic curvature of $(\Sigma_{+\infty}, \hat{h}_{+\infty})$ we use the fact that $K(cg) = c^{-1}K(g)$ for any metric g, and obtain

$$\begin{aligned} \hat{K}_{+\infty}|_{z=0} &= K(\hat{h}_{+\infty})|_{z=0} \\ &= \lim_{t\to+\infty} \frac{e^{2t}}{4}(\det(\Pi_t)^i_j - 1) \\ &= (1 - \partial\bar{\partial}\rho)e^{-2\rho} \end{aligned} \tag{4.5}$$

We record this as the following proposition.

Proposition 2. *For a family of parallel surfaces Σ_t $(t \in [-\infty, +\infty])$ such that Σ_0 has support function ρ from $\Sigma_{+\infty}$ for the choice of origin $\mathbf{x} \in \mathbf{H}^3$, the curvature of $(\Sigma_{+\infty}, \hat{h}_{+\infty})$ is precisely $\hat{K}_{+\infty} = (1 - \Delta_{\mathbf{S}^2}\rho)e^{-2\rho}$.*

This entire discussion can be made independent of basepoint on $\Sigma_{+\infty}$. We allow for fractional linear transformations by letting $\rho' = \rho - \log(1 + |z|^2)$ in (4.1) – (4.4). For example, we find that

$$h_t = \frac{1}{2}e^{2(\rho'+t)} \begin{pmatrix} (\partial\bar{\partial}\rho')e^{-2(\rho'+t)} + 1 & (\partial^2\rho' - (\partial\rho')^2)e^{-2(\rho'+t)} \\ (\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2(\rho'+t)} & (\partial\bar{\partial}\rho')e^{-2(\rho'+t)} + 1 \end{pmatrix}^2. \tag{4.6}$$

At $t = +\infty$ we get

$$\hat{h}^{+\infty} = \frac{1}{2}e^{2\rho'} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 \tag{4.7}$$

and

$$(\Pi_{+\infty})^i_j = \begin{pmatrix} -1 + 2(\partial\bar{\partial}\rho')e^{-2\rho'}\,d\tau & 2Ae^{-2\rho'}\,d\tau \\ 2\bar{A}e^{-2\rho'}\,d\tau & -1 + 2(\partial\bar{\partial}\rho')e^{-2\rho'}\,d\tau \end{pmatrix}. \tag{4.8}$$

with $\hat{K}_{+\infty} = -e^{-2\rho'}(\partial\bar{\partial}\rho')$. We repose proposition 1 in these terms as follows.

PROPOSITION 3. *For a family of parallel surfaces Σ_t ($t \in [-\infty, +\infty]$) such that Σ_0 has support function ρ from $\Sigma_{+\infty}$ for the choice of origin $\mathbf{x} \in \mathbf{H}^3$, the hermitian metric $\hat{h}_{+\infty}$ is given by $e^{\rho'}|dz|^2$ with the curvature $\hat{K}_{+\infty} = -e^{-2\rho'}(\partial\bar{\partial}\rho')$.*

It is nice to note that the Codazzi–Mainardi equations hold in this boundary setting.

PROPOSITION 4. *The Codazzi–Mainardi equations hold for $(\Pi_{+\infty})_{ij}$, i.e. if*

$$\begin{aligned}(\Pi_{+\infty})_{ij} &= \begin{pmatrix} e^{2\rho'}(-1+2(\partial\bar{\partial}\rho')e^{-2\rho'}\,d\tau) & 2A\,d\tau \\ 2\bar{A}\,d\tau & e^{2\rho'}(-1+2(\partial\bar{\partial}\rho')e^{-2\rho'}\,d\tau) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{B} & \mathcal{A} \\ \bar{\mathcal{A}} & \mathcal{B} \end{pmatrix}\end{aligned}$$

then

$$\nabla_{\bar{Z}}(\mathcal{A}\,dZ^2) = \nabla_Z(\mathcal{B}\,dZ\,d\bar{Z}).$$

PROOF: From (3.2) we must establish the equality

$$\bar{\partial}\mathcal{A} = e^{2\rho'}\partial(\mathcal{B}e^{-2\rho'}).$$

But this follows from the equality

$$\bar{\partial}(\partial^2\rho' - (\partial\rho')^2) = e^{2\rho'}\partial((\partial\bar{\partial}\rho')e^{-2\rho'}). \qquad \square$$

5 Constant mean curvature.

We now have the following theorem.

THEOREM 1. *Let $\Sigma_0 \subset \mathbf{H}^3$ be an oriented smooth surface with metric $\lambda^2 dZd\bar{Z}$. The following are equivalent:*

1. *Σ_0 has constant mean curvature;*

2. *$(\Pi_0)_{ij}$ is harmonic;*

3. *the ZZ–part of $(\Pi_0)_{ij}$ is holomorphic on Σ_0;*

4. *$N_t : \Sigma_0 \to \Sigma_t$ is harmonic for all $t \in [-\infty, +\infty]$;*

5. *there exists $t \in [-\infty, 0) \cup (0, +\infty]$ such that $N_t : \Sigma_0 \to \Sigma_t$ is harmonic;*

6. *$\mu_t(p) = f(t) \cdot g(p)$ where $f : [-\infty, +\infty] \to \mathbf{R} \cup \{\infty\}$ is injective and $g \in B(\Sigma_0)$;*

7. *$\mu_t(p) = f(t) \cdot g(p)$ where $f(t) = \dfrac{\tanh t}{1 - c\tanh t}$, with c a constant ($c = H(\Sigma_0)$) and $g \in \overline{Q(\Sigma_0)} \times \lambda^{-2}$.*

Proof: The equivalence of (1), (2) and (3) is immediate from lemma 3.

That (1) is equivalent to (4) and (5) is straightforward, though we give a sketch for the sake of completeness. First we discuss harmonicity of surface maps in our setting of normal flow.

As for fixed t

$$(g_t)_{ij} = (g_0)_{il}[(II_0)^l_m(II_0)^m_j \sinh^2 t - 2(II_0)^l_j \sinh t \cosh t + \delta^l_j \cosh^2 t],$$

we use the definition of harmonicity given by (3.1). It follows that

$$\begin{aligned}
-\nabla_k(g_0)^{ij}(g_t)_{ij} \quad &+ \quad 2(g_0)^{ij}\nabla_i(g_t)_{jk} \\
&= \quad -\nabla_k\Big((g_0)^{ij}(g_0)_{ip}[(II_0)^p_m(II_0)^m_j \sinh^2 t - 2(II_0)^p_j \sinh t\cosh t]\Big) \\
&\quad\ +2(g_0)^{ij}\nabla_i\Big((g_0)_{jp}[(II_0)^p_m(II_0)^m_k \sinh^2 t - 2(II_0)^p_k \sinh t\cosh t]\Big) \\
&\quad\ -\delta^j_p\Big[((II_0)^p_{m;k}(II_0)^m_j + (II_0)^p_m(II_0)^m_{j;k})\sinh^2 t - 2(II_0)^p_{j;k}\sinh t\cosh t\Big] \\
&\quad\ +2\delta^i_p\Big[((II_0)^p_{m;i}(II_0)^m_k + (II_0)^p_m(II_0)^m_{k;i})\sinh^2 t - 2(II_0)^p_{k;i}\sinh t\cosh t\Big] \\
&= \quad -((II_0)^j_{m;k}(II_0)^m_j + (II_0)^j_m(II_0)^m_{j;k})\sinh^2 t + 2(II_0)^j_{j;k}\sinh t\cosh t \\
&\quad\ +2((II_0)^i_{m;i}(II_0)^m_k + (II_0)^i_m(II_0)^m_{k;i})\sinh^2 t - 4(II_0)^i_{k;i}\sinh t\cosh t \\
&= \quad 0,
\end{aligned}$$

and by using the Codazzi–Mainardi equations, *i.e.* $(II_0)^i_{j;k} = (II_0)^i_{k;j}$, we are able to reduce this to

$$2\sinh t\,[(II_0)^p_k \sinh t - \delta^p_k \cosh t]\,(II_0)^i_{i;p} = 0.$$

We consider this as a system of equations in $(II_0)^i_{i;p}$ for $p=1,2$, *i.e.*

$$\text{(5.1)}\qquad \left\{\begin{array}{rcl} [(II_0)^1_1\sinh t-\cosh t]\,(II_0)^i_{i;1}+(II_0)^2_1\sinh t\,(II_0)^i_{i;2} &=& 0\\ (II_0)^1_2\sinh t\,(II_0)^i_{i;1}+[(II_0)^2_2\sinh t-\cosh t]\,(II_0)^i_{i;2} &=& 0. \end{array}\right.$$

We wish to show that this system has only the trivial solution. To have non-trivial solutions it is necessary that the discriminant is zero, *i.e.* that

$$((II_0)^1_1(II_0)^2_2-(II_0)^1_2(II_0)^2_1)\sinh^2 t-((II_0)^1_1+(II_0)^2_2)\sinh t\cosh t \\ +\cosh^2 t=0.$$

This means that $\coth t$ is everywhere an eigenvalue of $(II_0)^i_j$. Hence at least one of the principal curvatures is fixed if (5.1) is to have non-trivial solutions. We will now see that the other is also fixed, so that indeed no non-trivial solutions can exist.

We may assume $(II_0)^1_1=\coth t$, $(II_0)^2_2\neq\coth t$ and $(II_0)^i_j$ is diagonal, i.e. $k_1\neq k_2$. We wish to show that k_2 is constant, whence $(II_0)^i_i$ is constant. From (5.1) we have

$$\text{(5.2)}\qquad [(II_0)^2_2-\coth t](II_0)^i_{i;2}=0,$$

whence $(II_0)^i_{i;2}=0$. But we already have $(II_0)^1_{1;2}=0$, so that $(II_0)^2_{2;2}=0$. We also have $(II_0)^2_{2;1}=(II_0)^2_{1;2}=0$. Thus indeed $(II_0)^2_2$ is constant, whence $(II_0)^i_i$ is also constant, and (5) implies (1).

That (1) implies (4) is evident from (5.1), and of course (4) implies (5).

To establish (6) and (7), we will prove (6) and note that $f:[-\infty,+\infty]\to[-\infty,+\infty]$ and $g\in L^\infty\otimes T^{1,-1}(\Sigma_0)$ determine the constant mean curvature (up to sign) and are determined thereby (up to multiplicative constants). Furthermore we will show that g must be anti-holomorphic, establishing the theorem.

As mentioned earlier, the metric on Σ_t is given in terms of the metric, $(g_0)_{ij}$, and the Weingarten map, $(II_0)_{ij}$, on Σ_0. If we have a conformal coordinate $Z=X_1+iX_2$ on Σ_0, so that with respect to X_1 and X_2 we have

$$(g_0)_{ij}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}^2 \qquad\text{and}\qquad (II_0)_{ij}=\begin{pmatrix}L & M\\ M & N\end{pmatrix},$$

then the metric g^t_{ij} on Σ_t is

$$(g_t)_{ij}=\left\{\left[\begin{pmatrix}L & M\\ M & N\end{pmatrix}\sinh t-\begin{pmatrix}\lambda^2 & 0\\ 0 & \lambda^2\end{pmatrix}\cosh t\right]\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right\}^2.$$

In complex coordinates the corresponding hermitian forms are thus

$$h_0 = \frac{1}{2}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}^2$$

and

$$h_t = \frac{1}{2}\begin{pmatrix} \frac{(L+N)}{2\lambda}\sinh t - \lambda\cosh t & \frac{(L-N-2iM)}{2\lambda}\sinh t \\ \frac{(L-N+2iM)}{2\lambda}\sinh t & \frac{(L+N)}{2\lambda}\sinh t - \lambda\cosh t \end{pmatrix}^2.$$

From this we compute the "change of complex structure" matrix to be

$$\begin{aligned} \sqrt{h_0}\,(h_t)^{-1}\sqrt{h_0} &= \begin{pmatrix} \beta_t & \alpha_t \\ \bar{\alpha}_t & \beta_t \end{pmatrix}^2 \\ &= 4\begin{pmatrix} \frac{(L+N)}{2\lambda^2}\sinh t - \cosh t & \frac{(L-N-2iM)}{2\lambda^2}\sinh t \\ \frac{(L-N+2iM)}{2\lambda^2}\sinh t & \frac{(L+N)}{2\lambda^2}\sinh t - \cosh t \end{pmatrix}^2. \end{aligned}$$

It follows that

$$\begin{aligned} \mu_t &= \frac{\bar{\alpha}_t}{\beta_t} = \frac{(L-N+2iM)\tanh t}{(L+N)\tanh t - 2\lambda^2} \\ \mu_{+\infty} &= \frac{(L-N+2iM)}{(L+N)-2\lambda^2} \\ &\text{and} \\ \frac{\mu_{+\infty}}{\mu_t} &= 1 - \frac{2\lambda^2(1-\tanh t)}{((L+N)-2\lambda^2)\tanh t}. \end{aligned}$$

Thus $\frac{\mu_{+\infty}}{\mu_t}$ depends only on t if and only if $\dfrac{2\lambda^2}{(L+N)-2\lambda^2}$ does, *i.e.* if and only if $\dfrac{(L+N)}{2\lambda^2}$ is independent of z and $\bar{z}$ (x and y, respectively). That is, if and only if the trace of $(II_0)^i_j$, the mean curvature, is constant. This establishes the equivalence of (1) and (6).

We have (7) by noting in the above that if

$$c = \frac{(L+N)}{2\lambda^2}$$

then

$$\mu_t = -\frac{(L - N + 2iM)}{2\lambda^2} \cdot \frac{\tanh t}{(1 - c\tanh t)}.$$

Since the Z, Z–part of Π, i.e. $\dot{A}$, is precisely

$$\dot{A} = -\frac{(L - N - 2iM)}{2}$$

and is holomorphic, the proof is concluded. $\square$

COROLLARY 1. *If Σ_0 has constant mean curvature, then Σ_{t_0} ($t_0 \neq 0$) does also if and only if either*

1. $\tanh t_0 = \frac{1}{c}$ *(so that $c \neq 0$) and $H(\Sigma_{t_0}) = -H(\Sigma_0)$, or*
2. *every surface Σ_t has constant mean and constant intrinsic curvatures.*

In the second case, the surfaces Σ_t are all either pieces of metric spheres, horospheres, cylinders equidistant from a geodesic, or surfaces equidistant from a totally geodesic plane.

PROOF: Given that Σ_0 has constant mean curvature c, we have

$$h_t = \frac{1}{2}\lambda^2 \begin{pmatrix} c\sinh t - \cosh t & \dot{A}\lambda^{-2}\sinh t \\ \bar{\dot{A}}\lambda^{-2}\sinh t & c\sinh t - \cosh t \end{pmatrix}^2.$$

We note that

$$\sqrt{h_{t_0}}\,(h_t)^{-1}\sqrt{h_{t_0}} \;=\; \frac{1}{\det(h_{t_0})}\begin{pmatrix} D & C \\ \bar{C} & D \end{pmatrix}^2.$$

where

$$\begin{aligned} C &= \dot{A}\lambda^{-2}(\sinh t_0 \cosh t - \cosh t_0 \sinh t) \\ D &= (c^2 - |\dot{A}\lambda^{-2}|^2)\sinh t_0 \sinh t \\ &\qquad -c(\sinh t_0 \cosh t + \cosh t_0 \sinh t) + \cosh t_0 \cosh t. \end{aligned}$$

The conformal deformation of $N_t^{t_0} = N_t \circ (N_{t_0})^{-1}$, as noted in §3, is thus

$$\mu_t^{t_0} = \frac{\bar{\dot{A}}\lambda^{-2}(\tanh t_0 - \tanh t)}{(c^2 - |\dot{A}\lambda^{-2}|^2)\tanh t_0 \tanh t - c(\tanh t_0 + \tanh t) + 1}.$$

and similarly for $N_{+\infty}^{t_0} = N_{+\infty} \circ (N_{t_0})^{-1}$,

$$\mu_{+\infty}^{t_0} = \frac{\bar{\dot{A}}\lambda^{-2}(\tanh t_0 - 1)}{(c^2 - |\dot{A}\lambda^{-2}|^2)\tanh t_0 - c(\tanh t_0 + 1) + 1}.$$

This yields

$$\begin{aligned} \frac{\mu_{+\infty}^{t_0}}{\mu_t^{t_0}} &= \frac{(c^2 - |\dot{A}\lambda^{-2}|^2)\tanh t_0 \tanh t - c(\tanh t_0 + \tanh t) + 1}{(c^2 - |\dot{A}\lambda^{-2}|^2)\tanh t_0 - c(\tanh t_0 + 1) + 1} \\ &\qquad \times \left(\frac{\tanh t_0 - 1}{\tanh t_0 - \tanh t}\right) \\ &= \tanh t + \frac{(c\tanh t_0 - 1)(\tanh t - 1)}{(c^2 - |\dot{A}\lambda^{-2}|^2)\tanh t_0 - c(\tanh t_0 + 1) + 1} \\ &\qquad \times \left(\frac{\tanh t_0 - 1}{\tanh t_0 - \tanh t}\right) \end{aligned}, \tag{5.3}$$

which depends only on t if and only if either

1. $\tanh t_0 = \frac{1}{c}$, or
2. $|\dot{A}\lambda^{-2}|$ is constant.

Case 1: If $\tanh t_0 = \frac{1}{c}$, then

$$\begin{aligned} \mu_t^{t_0} &= \frac{\bar{\dot{A}}\lambda^{-2}(1 - c\tanh t)}{-|\dot{A}\lambda^{-2}|^2 \tanh t} \\ &= \frac{(1 - c\tanh t)}{-\dot{A}\lambda^{-2}\tanh t}, \\ \mu_{+\infty}^{t_0} &= \frac{\bar{\dot{A}}\lambda^{-2}(1 - c)}{-|\dot{A}\lambda^{-2}|^2} \\ &= \frac{(1 - c)}{-\dot{A}\lambda^{-2}}. \end{aligned}$$

It follows that

$$\frac{\mu_{+\infty}^{t_0}}{\mu_t^{t_0}} = \frac{(1 - c)\tanh t}{(1 - c\tanh t)}.$$

Letting $t = \tau + t_0$ we have

$$\mu_t^{t_0} = \frac{\bar{\dot{A}}\lambda^{-2}(1 - c^2)}{-|\dot{A}\lambda^{-2}|^2}\frac{\tanh\tau}{(c\tanh\tau + 1)} = \frac{(1 - c^2)}{-\dot{A}\lambda^{-2}}\frac{\tanh\tau}{1 + c\tanh\tau}$$

and by equivalence 7 we have $H(\Sigma_{t_0}) = -c$.

Case 2: $|\dot{A}\lambda^{-2}|$ being constant implies that every μ_t is constant, whence, as

$$|\mu_t| = \frac{\tanh t}{1 - c\tanh t} \cdot \frac{|k_1 - k_2|}{2},$$

we deduce that k_1 and k_2 are both constant.

The final conclusion follows from a result of Cartan [5]. He completely classified surfaces in $\mathbf{H}^3$ for which H and K are constant. These are called isoparametric surfaces.

THEOREM. [5] *If Σ_0 is a surface in $\mathbf{H}^3$ for which H and K are constant then one of the following cases must hold:*

1. *Σ_0 is a piece of a metric sphere;*
2. *Σ_0 is a piece of a horosphere;*
3. *Σ_0 is a piece of a surface equidistant from a hyperbolic geodesic; or*
4. *Σ_0 is a piece of a surface equidistant from a totally geodesic plane in $\mathbf{H}^3$.*

This completes the proof of the corollary. □

COROLLARY 2. *Σ_0 is minimal if and only if $\mu_t(p) = f(t) \cdot g(p)$ with $f : [-\infty, +\infty] \to \mathbf{R} \cup \{\infty\}$ odd. In this case $f(t) = \tanh t$.*

PROOF: This is immediate from equivalences 1, 6, and 7 of theorem 1. □

COROLLARY 3. *If $H(\Sigma_0) = c$ and the N_t are not all conformal, we can recover c by knowing only the conformal deformations to $\pm\infty$, i.e. by knowing $\mu_{\pm\infty}$.*

PROOF: From equivalence (7),

$$\mu_{-\infty}(p) = \frac{-1}{1+c} g(p) \qquad \text{and} \qquad \mu_{+\infty}(p) = \frac{1}{1-c} g(p).$$

If $g \not\equiv 0$ then

$$M = \frac{\mu_{-\infty}}{\mu_{+\infty}} = \frac{c-1}{c+1}$$

so that

$$c = \frac{M+1}{1-M}. \qquad \square$$

Corollary 4. *If $\Sigma_0 \hookrightarrow \mathbf{H}^3$ and $H(\Sigma_0) = 1$, let $\dot{A} = ZZ$–part of Π on Σ_0. Then $N_{+\infty}$ is anti-conformal and $N_{-\infty}$ has conformal distortion $-\frac{1}{1+c}\bar{\dot{A}}\lambda^{-2}$.*

Proof: If $H(\Sigma_0) = 1$ then

$$\mu_t = \frac{\tanh t}{1-\tanh t}\bar{\dot{A}}\lambda^{-2}$$

so that $\mu_{+\infty}$ is infinite. Thus $N_{+\infty}$ is anticonformal and $\mu_{-\infty} = \frac{1}{2}\bar{\dot{A}}\lambda^{-2}$ as desired. $\square$

Surfaces $\Sigma_0 \hookrightarrow \mathbf{H}^3$ with constant mean curvature $H(\Sigma_0) = 1$ have been considered by Bryant [4].

Corollary 5. *If Σ_0 has constant mean curvature, then the map $\mu : [-\infty, +\infty] \to B(\Sigma_0)$ actually maps into $B^\infty(\Sigma_0)$, and is continuous if and only if $|k_1 - k_2|$ is bounded, i.e. if and only if the directional curvatures of Σ_0 are bounded.*

Proof:

$$|\mu_t| = \frac{\tanh t}{1 - c\tanh t} \cdot \frac{|k_1 - k_2|}{2},$$

as in general

$$|\mu_t| = \frac{|\frac{k_1-k_2}{2}||\tanh t|}{|1 - \frac{k_1+k_2}{2}\tanh t|}. \qquad \square$$

6 Constant mean curvature surfaces at infinity, the local theory

We assume now that $\Sigma_{+\infty}$ is the embedded image of a surface patch in $\mathbf{S}^2_\infty$. Since a family of parallel surfaces in $\mathbf{H}^3$ is determined by a metric on $\Sigma_{+\infty}$ which induces the same conformal structure on $\Sigma_{+\infty}$ as $\mathbf{S}^2_\infty$ (note 4, §4), we make the following definition.

DEFINITION: $\Sigma_{+\infty}$ will be called a surface patch at infinity of $\mathbf{H}^3$ if it is the conformally embedded image in $\mathbf{S}^2_\infty$ of a metric surface patch. A surface at infinity will refer to the conformally immersed image in $\mathbf{S}^2_\infty$ of a metric surface.

We observed in §4 that if ρ is the support function for Σ_0 from any $\mathbf{x} \in \mathbf{H}^3$ then the hermitian form h^t is given by

$$\begin{aligned} h^t &= \frac{1}{2}e^{2(\rho'+t)} \begin{pmatrix} (\partial\bar{\partial}\rho')e^{-2(\rho'+t)} + 1 & (\partial^2\rho' - (\partial\rho')^2)e^{-2(\rho'+t)} \\ (\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2(\rho'+t)} & (\partial\bar{\partial}\rho')e^{-2(\rho'+t)} + 1 \end{pmatrix}^2 \\ &= \begin{pmatrix} \beta_t & \alpha_t \\ \bar{\alpha}_t & \beta_t \end{pmatrix}^2 \end{aligned}$$

where $\rho' = \rho - \log(1 + |z|^2)$. We will now need the expressions introduced at the end of §4 for ρ', h_t, $(\Pi_t)^i_j$, $\hat{h}_{+\infty}$, and $(\Pi_{+\infty})^i_j$.

First we make several preliminary observations. We have the following proposition.

PROPOSITION 5. *A surface $\Sigma_{+\infty}$ at infinity of $\mathbf{H}^3$ must have $H(\Sigma_{+\infty}) = \pm 1$.*

PROOF: This is immediate from the observation following the theorem of §4. □

Following our definition of $(\Pi_{+\infty})^i_j$ we'll take as our normalization that $\Sigma_{+\infty}$ has mean curvature then this $H(\Sigma_{+\infty}) \equiv -1$.

Secondly, we measure the conformal distortion from $\Sigma_{+\infty}$ to Σ_t by considering $N_t^{+\infty} = N_t \circ (N_{+\infty})^{-1} : \Sigma_{+\infty} \to \Sigma_t$. The conformal deformation $\mu_t^{+\infty}$ for $N_t^{+\infty}$ is thus

$$\mu_t^{+\infty} = \frac{\partial_{\bar{z}} N_t^{+\infty}}{\partial_z N_t^{+\infty}},$$

where z is a conformal parameter on $\Sigma_{+\infty}$. We take z to be a standard coordinate on $\mathbf{S}^2_\infty$, and compute the conformal deformation from $\Sigma_{+\infty}$ to be

$$\mu_t^{+\infty} = \frac{\bar{\alpha}_t}{\beta_t} = \frac{(\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2(\rho'+t)}}{(\partial\bar{\partial}\rho')e^{-2(\rho'+t)} + 1}.$$

Consequently

$$\mu_{-\infty}^{+\infty} = \frac{\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2}{\partial\bar{\partial}\rho'},$$

and

$$\frac{\mu_{-\infty}^{+\infty}}{\mu_t^{+\infty}} = 1 + e^{2t}\frac{1}{e^{-2\rho'}(\partial\bar{\partial}\rho')}.$$

This yields, from the definition of $\hat{K}_{+\infty}$ in §4, the following theorem.

Theorem 2. *The image of $\mu_{(\ )}^{+\infty} : [-\infty, +\infty] \to B(\Sigma_{+\infty})$ lies in a radial line in $B(\Sigma_{+\infty})$ if and only if $\hat{K}_{+\infty}$ is constant.*

Hence when the image $\mu_{[-\infty,+\infty]}^{+\infty}$ lies in a radial line in $B(\Sigma_{+\infty})$ we have

$$\mu_t^{+\infty} = \frac{(\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2(\rho'+t)}}{1 + ce^{-2t}} \tag{6.1}$$

for some $c \in \mathbf{R}$.

Finally, we have the following proposition.

Proposition 6. *If $\mu_{t_0}^{+\infty} \equiv 0$ for some $t_0 \in [-\infty, +\infty)$ then this is so for all $t \in [-\infty, +\infty]$.*

Proof: If $N_{t_0}^{+\infty}$ is conformal then we see from (6.1) that $\bar{\partial}^2\rho - (\bar{\partial}\rho)^2 \equiv 0$. Thus $\mu_t^{+\infty} \equiv 0$ in t. □

Thus when the equivalence of theorem 2 holds we have four possible cases:

1. $\mu_t^{+\infty} \equiv 0$;
2. $\mu_t^{+\infty} = f(t) \cdot \mu_0^{+\infty}$ with $\mu_0^{+\infty} \not\equiv 0$ and $f(t)$ is bounded and never infinite.

3. $\mu_t^{+\infty} = f(t) \cdot \mu_0^{+\infty}$ with $\mu_0^{+\infty} \not\equiv 0$ and $|f(t)| \to \infty$ as $t \to -\infty$; or

4. $\mu_t^{+\infty} = f(t) \cdot \mu_{t_0}^{+\infty}$ with $\mu_{t_0}^{+\infty} \not\equiv 0$, $f(t)$ is infinite for some finite value of t and bounded as $t \to -\infty$.

The image $\mu_{[-\infty,+\infty]}^{+\infty} \subset B(\Sigma_{+\infty}) \cup \infty$ in these four cases are shown in the adjoining figures 1—4.
We treat these separately.

Case 1: This should be compared to case 2 of corollary 1, §5. As there, $\mu_t^{+\infty} \equiv 0$ implies that $N_t^{t_0} = N_t^{+\infty} \circ (N_{t_0}^{+\infty})^{-1}$ is conformal for all t_0, t. Thus all Σ_t are either pieces of spheres, horospheres, cylinders about a geodesic, or surfaces equidistant from a totally geodesic plane.

Case 2: Here $e^{-2\rho'}(\partial\bar{\partial}\rho')$ is a nonzero constant. In fact it is a positive constant $c > 0$, as otherwise $\mu_{t_0}^{+\infty}$ would be infinite for some value of $t_0 \in [-\infty, +\infty)$, contrary to the assumption in this case. Thus we see that ρ' satisfies

$$\partial\bar{\partial}\rho' = ce^{2\rho'},$$

which is precisely the equation for constant negative curvature. As was shown in §4, we may rescale the metric on $\Sigma_{+\infty}$ by a constant and thus assume $\hat{K}_{+\infty} \equiv -1$. Thus, we must have

$$\rho = \log\left(|\psi'| \cdot \frac{1+|z|^2}{1-|\psi|^2}\right)$$

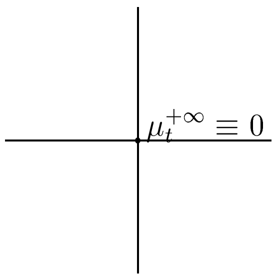

Figure 1:

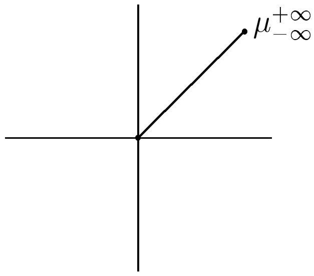

Figure 2:

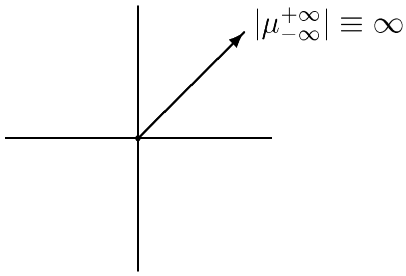

Figure 3:

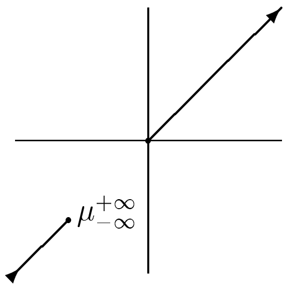

Figure 4:

where $\psi : \Sigma_{+\infty} \to \mathbf{D}$. In words, we have shown that in this case the metric on $\Sigma_{+\infty}$ is the pull–back of the complete constant -1 curvature hyperbolic metric on $\mathbf{D}$ via some locally univalent map $\psi : \Sigma_{+\infty} \to \mathbf{D}$.

Note that

$$\mu_t^{+\infty} = \frac{(\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2(\rho'+t)}}{1+e^{-2t}}.$$

We quickly check that for no $t_0 \in [-\infty, +\infty)$ is Σ_{t_0} a surface of constant mean curvature.

Using the formula for the transformation of Beltrami parameters [1, §1C] we obtain

$$\begin{aligned} \mu_t^{t_0} &= s \cdot \left[\frac{\mu_t^{+\infty} - \mu_{t_0}^{+\infty}}{1 - \mu_t^{+\infty}\overline{\mu_{t_0}^{+\infty}}} \circ N_{+\infty}^{t_0} \right] \\ &= s \cdot \left[\frac{(e^{-2t} - e^{-2t_0})(\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2\rho'}}{(1+e^{-2t})(1+e^{-2t_0}) - e^{-2(t+t_0)}|(\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2)e^{-2\rho'}|^2} \circ N_{+\infty}^{t_0} \right] \end{aligned}$$

where s is a t independent function of modulus 1 on Σ_{t_0}. The ratio $\frac{\mu_{+\infty}^{t_0}}{\mu_t^{t_0}}$ is dependent only on t (see (5.3)) if and only if $\bar{\partial}^2\rho' - (\bar{\partial}\rho')^2 \equiv 0$, reducing to case 1. One may see in the same fashion that $\mu_{[-\infty,+\infty]}^{-\infty}$ does not lie in a radial line of $B(\Sigma_{-\infty}) \cup \infty$.

Case 3: Evidently $\mu_{-\infty}^{+\infty} \equiv \infty$ if and only if $\partial\bar{\partial}\rho' \equiv 0$. By equation (6.1) we see that $\hat{K}_{+\infty} \equiv 0$, or $(\Sigma_{+\infty}, \hat{g}_{+\infty})$ is flat. As $N_{-\infty}^{+\infty}$ is anticonformal and $\Sigma_{+\infty}$ has its normal pointing into the $\mathbf{H}^3$ while $\Sigma_{-\infty}$ has its normal pointing out of $\mathbf{H}^3$, if we change the normal on $\Sigma_{-\infty}$ to point inward we give it the oriented conformal structure induced by $\hat{\mathbf{C}}$. This makes $N_{-\infty}^{+\infty}$ holomorphic. It was already noticed by Bianchi [3] (see §497, equation (64) in particular) that if $f : \Omega \subset \mathbf{C} \to \mathbf{C}$ is holomorphic then the family of geodesics joining $\theta \in \Omega$ to $f(\theta)$ is orthogonally integrable (for a discussion orthogonal integrability in this setting see [9]) in the sense that there is a family of parallel surfaces Σ_t orthogonal to these geodesics and the maps $N_t^{t_0}$ are obtained by unit speed flow along the geodesics. The author thanks Peter Doyle for pointing out this reference.

As in case 2, we may see that for no $t \in (-\infty, +\infty)$ is Σ_t a constant mean curvature surface in $\mathbf{H}^3$. But now $\mu_{[-\infty,+\infty]}^{-\infty}$ does lie in a radial line of

$B(\Sigma_{-\infty}) \cup \infty$ (compare this with case 1 in corollary 1, §5). To see this we note that

$$\mu_t^{+\infty} = (\bar{\partial}^2 \rho' - (\bar{\partial}\rho')^2) e^{-2(\rho'+t)}$$

and compute $\mu_t^{-\infty}$ as in case 2. The Beltrami parameter for the map $N_t^{-\infty} = N_t^{+\infty} \circ (N_{-\infty}^{+\infty})^{-1}$ is

$$\mu_t^{-\infty} = s \cdot \left[\frac{\mu_t^{+\infty} - \mu_{-\infty}^{+\infty}}{1 - \mu_t^{+\infty}\overline{\mu_{-\infty}^{+\infty}}} \circ N_{+\infty}^{-\infty} \right] = s \cdot \left[\frac{1}{\partial^2 \rho' - (\partial\rho')^2} e^{2(\rho'+t)} \circ N_{+\infty}^{-\infty} \right],$$

with s a t independent function of modulus 1 on $\Sigma_{-\infty}$. Thus $\dfrac{\mu_{+\infty}^{-\infty}}{\mu_t^{-\infty}}$ is independent of z and $\Sigma_{-\infty}$ is also a (possibly singular) surface of constant curvature 1 at infinity of $\mathbf{H}^3$. Away from $N_{-\infty}^{+\infty}(\theta)$ where θ is a zero of $\partial^2\rho' - (\partial\rho')^2$, $\Sigma_{-\infty}$ is regular, and the singularities of $\Sigma_{-\infty}$ are precisely at these zeroes.

Note that the above equation shows that if $A_{+\infty}$ ($A_{-\infty}$) denotes the zz–part of the second fundamental form of $\Sigma_{+\infty}$ ($\Sigma_{-\infty}$, respectively), then $A_{+\infty}$ and $A_{-\infty}$ are inverse conjugates in the sense that $\bar{A}_{-\infty} = \dfrac{1}{A_{+\infty}}$.

We now collect the information in this case.

Proposition 7. *If $\Omega \subset \hat{\mathbf{C}}$ is a connected domain and $f : \Omega \to \hat{\mathbf{C}}$ is a holomorphic function then there exists a family of parallel surfaces Σ_t, $t \in [-\infty, +\infty]$, such that the limiting surfaces are precisely $\Sigma_{+\infty} = \Omega$ and $\Sigma_{-\infty} = f(\Omega)$.*

Proposition 8. *If $\Sigma_{+\infty}$ is a surface at infinity of $\mathbf{H}^3$ for which $\hat{K}_{+\infty} = 0$ (i.e. $\partial\bar{\partial}\rho' = 0$) then $\Sigma_{-\infty}$ is also a (possibly singular) surface at infinity satisfying this same condition. The metrics on $\Sigma_{\pm\infty}$ are all flat away from possible singularities of $\Sigma_{-\infty}$, i.e. pulled back from $\mathbf{C}$ by locally univalent functions $\psi_{\pm\infty} : \Sigma_{\pm\infty} \to \mathbf{C}$. The zz–parts of their respective second fundamental forms are inverse conjugates of each other. The singularities of $\Sigma_{-\infty}$ occur at the images under $N_{-\infty}^{+\infty}$ of the zeros of the zz–part of the second fundamental form of $\Sigma_{+\infty}$.*

Case 4: Here $\mu_{t_0}^{+\infty} \equiv \infty$ for some finite t_0, so that by equation (6.1) we must have $c < 0$. Again rescaling, as in §4, we may assume that we have $(\partial\bar{\partial}\rho')e^{-2\rho'} = -1$, or

$$\partial\bar{\partial}\rho' = -e^{-2\rho'}.$$

This is the equation for constant curvature $\hat{K}_{+\infty} \equiv 1$. In this case the metric on $\Sigma_{+\infty}$ is pulled back by some locally univalent map $\psi : \Sigma_{+\infty} \to \hat{\mathbf{C}}$ with a spherical metric.

Among the surfaces Σ_t there is another surface Σ_{t_0} (t_0 is finite) of constant mean curvature. If $c = 1$ then $t_0 = 0$. This may be seen as in cases 2 and 3. Since $N_{+\infty}^{t_0}$ is anticonformal we see that $H(\Sigma_{t_0}) \equiv 1$ as in corollary 4, §4.

We summarize in

THEOREM 3. *If $\Sigma_{+\infty}$ is a surface patch in $\mathbf{S}^2_\infty$ for which $\hat{K}_{+\infty}$ is constant then one of the following must hold:*

1. *The Σ_t are all pieces of spheres, horospheres, cylinders about a geodesic, or surfaces equidistant from a totally geodesic plane.*

2. *The metric on $\Sigma_{+\infty}$ is obtained by pulling back a hyperbolic metric on $\mathbf{D}$ via some locally univalent map $\psi : \Sigma_{+\infty} \to \mathbf{D}$. No other surface in the family Σ_t has constant mean curvature.*

3. *The metric on $\Sigma_{+\infty}$ is obtained by pulling back a flat metric on $\mathbf{C}$ via some locally univalent map $\psi : \Sigma_{+\infty} \to \mathbf{C}$ and the map $N_{-\infty}^{+\infty}$ is anticonformal. The possibly singular surface $\Sigma_{-\infty}$ is also a flat constant mean curvature surface patch in $\mathbf{S}^2_\infty$.*

4. *The metric on $\Sigma_{+\infty}$ is obtained by pulling back a spherical metric on $\hat{\mathbf{C}}$ via some locally univalent map $\psi : \Sigma_{+\infty} \to \hat{\mathbf{C}}$. In the family Σ_t there is a finite t_0 for which the possibly singular Σ_{t_0} has constant mean curvature.*

Compare this with corollary 1, §5.

Now compare the following with theorem 1.

THEOREM 4. *Let $\Sigma_{+\infty} \subset \mathbf{S}^2_\infty$ be a surface patch at infinity of $\mathbf{H}^3$. The following are equivalent:*

1. *$\hat{K}_{+\infty}$ is constant;*

2. *$(\Pi_{+\infty})_{ij}$ is harmonic;*

3. *the zz–part of* $(\Pi_{+\infty})_{ij}$ *(i.e. the quadratic differential* $(\partial^2\rho' - (\partial\rho')^2)$*) is holomorphic on* $\Sigma_{+\infty}$*;*

4. $N_t^{+\infty} : \Sigma_{+\infty} \to \Sigma_t$ *is harmonic for any* $t \in [-\infty, +\infty]$*;*

5. *there exists* $t \in [-\infty, +\infty)$ *such that* $N_t^{+\infty} : \Sigma_{+\infty} \to \Sigma_t$ *is harmonic;*

6. $\mu_t^{+\infty}(p) = f(t) \cdot g(p)$ *where* $f : [-\infty, +\infty] \to \mathbf{R} \cup \{\infty\}$ *is injective and* $g \in B(\Sigma_{+\infty})$*;*

7. $\mu_t^{+\infty}(p) = f(t) \cdot g(p)$ *where* $f(t) = 0,\ \dfrac{1}{1+e^{-2t}},\ e^{-2t}$*, or* $\dfrac{1}{1-e^{-2t}}$*, and* $g \in \overline{Q(\Sigma_{+\infty})} \times e^{-2\rho'}$.

This suggests a definition.

DEFINITION. If a surface $\Sigma_{+\infty}$ at infinity of $\mathbf{H}^3$ satisfies the above equivalences it will be called a *surface of constant mean curvature at infinity of* $\mathbf{H}^3$.

PROOF: That (1) and (6) are equivalent was already seen in theorem 2.

The equivalence of (1) and (2) is a repetition of the proof of lemma 3, §3, noting from proposition 4, §4, that we have the Codazzi–Mainardi equations satisfied in this setting.

We show now that (1) is equivalent to (3). Differentiating $e^{-2\rho'}(\partial\bar{\partial}\rho') = c$ with respect to z gives

$$0 = \partial\left[e^{-2\rho'}(\partial\bar{\partial}\rho')\right] = e^{-2\rho'}\bar{\partial}(\partial^2\rho' - (\partial\rho')^2),$$

so that indeed $\partial^2\rho' - (\partial\rho')^2$ is holomorphic. Reversing the same equality, as well as its conjugate, gives that

$$0 = \partial\left[e^{-2\rho'}(\partial\bar{\partial}\rho')\right] = \bar{\partial}\left[e^{-2\rho'}(\partial\bar{\partial}\rho')\right]$$

if $\partial^2\rho' - (\partial\rho')^2$ is holomorphic, yielding the desired equivalence.

To see now the other equivalences, we note that if $e^{-2\rho'}(\partial\bar{\partial}\rho') = c$ then letting $A = \partial^2\rho' - (\partial\rho')^2$ we have

$$h^t = \frac{1}{2}e^{2(\rho'+t)}\begin{pmatrix} 1+ce^{-2t} & Ae^{-2(\rho'+t)} \\ \bar{A}e^{-2(\rho'+t)} & 1+ce^{-2t} \end{pmatrix}^2,$$

$$(\Pi_{+\infty})_{ij} = \begin{pmatrix} e^{2\rho} & 2A\,d\tau \\ 2\bar{A}\,d\tau & e^{2\rho} \end{pmatrix}$$

and

$$\mu_t^{+\infty} = \frac{e^{-2t}}{1+ce^{-2t}}\bar{A}e^{-2\rho'}.$$

The zz–part of h^t is $e^{-2t}(1+ce^{-2t})A$ which is holomorphic, implying (4) and (5), while (7) is evident from the third expression above.

Clearly (7) implies (1), and (4) implies (5), so the proof will be complete when we show that (5) implies (1). The proof is essentially a repeat of the proof that (5) implies (1) in theorem 1. Assume $N_{t_0}^{+\infty}$ is harmonic. Since

$$h^t \;=\; \frac{1}{2}e^{2(\rho'+t_0)}\begin{pmatrix} (\partial\bar{\partial}\rho')e^{-2(\rho'+t_0)}+1 & Ae^{-2(\rho'+t_0)} \\ \bar{A}e^{-2(\rho'+t_0)} & (\partial\bar{\partial}\rho')e^{-2(\rho'+t_0)}+1 \end{pmatrix}^2,$$

the zz–part is $((\partial\bar{\partial}\rho')e^{-2(\rho'+t_0)}+1)A$. We assume that this is holomorphic, to wit

$$0 = \bar{\partial}\left[((\partial\bar{\partial}\rho')e^{-2(\rho'+t_0)}+1)A\right] = (\partial\bar{A})Ae^{-2t}e^{-2\rho'} + \partial\bar{\partial}\rho'(\bar{\partial}A)e^{-2t}e^{-2\rho'} + \bar{\partial}A.$$

Writing out the real and imaginary parts of this, with $A = u+iv$, the following system of equations in $u_x - v_y$ and $u_y + v_x$ results:

$$\begin{array}{rrl} (\partial\bar{\partial}\rho' e^{-2\rho'}+u+e^{2t})(u_x-v_y) & +v(u_y+v_x) & = 0 \\ v(u_x-v_y) & +(\partial\bar{\partial}\rho' e^{-2\rho'}-u+e^{2t})(u_y+v_x) & = 0 \end{array}.$$

Here, either there is only the trivial solution or there are nontrivial solutions. In the first case, $\bar{\partial}A = \partial K_{+\infty} = 0$ and its conjugate, whence $K_{+\infty}$ is constant. In the second case, $-\frac{e^{2t}}{2}$ is everywhere an eigenvalue of $\Pi_{+\infty}$. As in the proof of theorem 1, one checks that the other eigenvalue is also constant, whence $K_{+\infty}$ is again constant. In either case we have the desired conclusion. □

7 The Ahlfors–Weill examples

Naturally occurring examples of constant mean curvature surfaces at infinity have already been observed [18], although not in these terms. What follows is a brief description of these examples.

Let $A \in Q(\mathbf{D})$ and consider the linear ODE

$$f'' + Af = 0 \qquad \text{on } \mathbf{D}. \tag{7.1}$$

Let f_1, f_2 be independent solutions such that

$$\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \end{pmatrix}_{z=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.2}$$

It is easily seen that the function $F_{\infty,A} : \mathbf{D} \to \hat{\mathbf{C}}$ given by $F_{\infty,A} = \frac{f_1}{f_2}$ is locally univalent and satisfies the Schwarzian differential equation

$$\{F_{\infty,A}, z\} = 2A \qquad (\{f,z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2).$$

Nehari showed [12] that if $\|A\|_\infty = \sup_{z\in\mathbf{D}} |A(z)\cdot(1-|z|^2)^2| \leq 1$ then actually $F_{\infty,A}$ is globally univalent. Ahlfors and Weill expanded on this result to show that if $\|A\|_\infty < 1$ then $F_{\infty,A}$ extends to $\mathbf{S}^1 = \partial\mathbf{D}$ and $F_{\infty,A}(\mathbf{S}^1)$ is a quasicircle. Furthermore, they produced a quasireflection of $F_{\infty,A}$, call it $F_{0,A}$, which is a quasiconformal map $F_{0,A} : \mathbf{D} \cup \mathbf{S}^1 \to \hat{\mathbf{C}}$ agreeing with $F_{\infty,A}$ on $\mathbf{S}^1$ and has image $F_{0,A}(\mathbf{D}) = \hat{\mathbf{C}} \setminus F_{\infty,A}(\mathbf{D} \cup \mathbf{S}^1)$. Letting $1/\mathbf{D} = \{z \in \mathbf{C} : 1/z \in \mathbf{D}\}$, the map

$$G_A(z) = \begin{cases} \overline{F_{\infty,A}(\frac{1}{\bar z})} & z \in 1/\mathbf{D} \\ \overline{F_{0,A}(z)} & z \in \mathbf{D} \cup \mathbf{S}^1 \end{cases}$$

is a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ holomorphic on $1/\mathbf{D}$. This extension $F_{0,A}$ is given explicitly in terms of the f_1, f_2 of (7.2) by

$$\overline{F_{0,A}(z)} = \frac{z\overline{f_1(z)} + (1 - z\bar z)\overline{f_1'(z)}}{z\overline{f_2(z)} + (1 - z\bar z)\overline{f_2'(z)}} \tag{7.3}$$

and satisfies

$$\mu_A(z) = \frac{\partial_{\bar z} G_A}{\partial_z G_A} = \begin{cases} 0 & z \in 1/\mathbf{D} \\ \overline{A(z)}(1 - z\bar z)^2 & z \in \mathbf{D} \end{cases}. \tag{7.4}$$

In [18], the author showed that the phenomena described and analyzed by Ahlfors and Weill are special boundary cases of more general phenomena. To be specific, let $PSU(1,1)$ act as orientation preserving Möbius transformations on $\mathbf{D}$, and $PSL(2,\mathbf{C})$ as isometries on $\mathbf{H}^3$ in our upper half-space model as usual:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (Z,T) = \left(\frac{(aZ+b)\overline{(cZ+d)} + a\bar{c}T^2}{|cZ+d|^2+|c|^2T^2}, \frac{T}{|cZ+d|^2+|c|^2T^2} \right).$$

It was shown in [18] that the ODE (7.1) generates a monodromy map $\mathrm{Mon}_A : PSU(1,1) \to PSL(2,\mathbf{C})$ given by

$$\begin{aligned} &\mathrm{Mon}_A \left[\frac{1}{(1-z\bar{z})^{1/2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \begin{pmatrix} e^{-\mathbf{i}\theta} & \\ & e^{\mathbf{i}\theta} \end{pmatrix} \right] \\ &= \frac{1}{(1-z\bar{z})^{1/2}} \begin{pmatrix} f_1(z) & \bar{z}f_1(z) + (1-z\bar{z})f_1'(z) \\ f_2(z) & \bar{z}f_2(z) + (1-z\bar{z})f_2'(z) \end{pmatrix} \begin{pmatrix} e^{\mathbf{i}\theta} & \\ & e^{-\mathbf{i}\theta} \end{pmatrix}. \end{aligned} \tag{7.5}$$

So by means of Mon_A we can consider the action of $PSU(1,1)$ on $\mathbf{H}^3$ as

$$\gamma : (Z,T) \mapsto \mathrm{Mon}_A(\gamma) \cdot (Z,T).$$

If $\mathrm{stab}(\mathbf{x})$ denotes the stabilizing group of $\mathbf{x}$ as a subgroup of the appropriate ambient group, we observe that the definition of Mon_A yields $\mathrm{Mon}_A(\mathrm{stab}(0)) \subset \mathrm{stab}(0,T)$ for all $T \in \mathbf{R}_+$. Hence for all $T \in \mathbf{R}_+$ we have an induced map $F_{T,A} : \mathbf{D} \to \mathbf{H}^3$ via the factored diagram

$$\begin{array}{ccc} PSU(1,1) & \xrightarrow{\mathrm{Mon}_A} & PSL(2,\mathbf{C}) \\ \downarrow \mathrm{stab}(0) & \longrightarrow & \downarrow \mathrm{stab}(0,T) \\ \mathbf{D} & \xrightarrow{F_{T,A}} & \mathbf{H}^3. \end{array}$$

The $F_{T,A}$ have been given explicitly in terms of the f_1, f_2 of (7.2) as

$$\begin{aligned} F_{T,A}(z) &= \mathrm{Mon}_A \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \cdot (0,T) \\ &= \left(\frac{\mathrm{num}}{\mathrm{denom}}, \frac{(1-z\bar{z})T}{\mathrm{denom}} \right), \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} \mathrm{num} &= (\bar{z}f_1(z) + (1-z\bar{z})f_1'(z))\overline{(\bar{z}f_2(z) + (1-z\bar{z})f_2'(z))} \\ &\quad + f_1(z)\overline{f_2(z)}T^2, \\ \mathrm{denom} &= |\bar{z}f_2(z) + (1-z\bar{z})f_2'(z)|^2 + |f_2(z)|^2T^2. \end{aligned}$$

We may also extend this action continuously to $T = 0$ and $T = \infty$ to get the $F_{0,A}$ and $F_{\infty,A}$ of Ahlfors and Weill, from the beginning of this section.

It was shown in [18] that the maps $F_{T,A}$ quasiconformally interpolate between the $F_{\infty,A}$ and the $F_{0,A}$ in that the Beltrami differential of $F_{T,A}$ is

$$\mu_{T,A} = -\frac{\overline{A(z)} \cdot (1 - z\bar{z})^2}{(T^2 + 1)}.$$

Thus if $t = \log T$, the images $\Sigma_t = F_{T,A}(\mathbf{D})$ form a family of surfaces such that the maps $M_{t,A} = F_{T,A} \circ F^{-1}_{\infty,A} : \Sigma_{+\infty} \to \Sigma_t$ have Beltrami differentials

$$\nu_{t,A} = \frac{\partial_{\bar{z}} M_{t,A}}{\partial_z M_{t,A}} = \frac{1}{(e^{2t} + 1)} \left(-\overline{A(z)}(1 - z\bar{z})^2 \right) \tag{7.7}$$

which are a radial family. To deduce that $\Sigma_{+\infty}$ is a surface of constant mean curvature $+1$ we still need to show that the Σ_t are parallel. But this follows from the fact that

$$\|\partial_T F_{T,A}(z)\|_{\mathbf{H}^3} = \frac{1}{T}$$

which is independent of $z \in \mathbf{D}$. Hence these $\Sigma_{+\infty}$ are examples of the precise phenomena desired.

For the record, we note the directional, mean and intrinsic curvatures for $F_{T,A}(\mathbf{D})$. Letting $m = |A(z) \cdot (1 - |z|^2)^2|$ we have

$$\begin{aligned} k_1 &= \frac{m^2 + (T^4 - 1) + 2T^2 m}{m^2 - (T^2 + 1)^2}, \\ k_2 &= \frac{m^2 + (T^4 - 1) - 2T^2 m}{m^2 - (T^2 + 1)^2}, \\ H &= \frac{k_1 + k_2}{2} = \frac{m^2 + (T^4 - 1)}{m^2 - (T^2 + 1)^2} \\ K &= k_1 k_2 - 1 = \frac{4T^2}{m^2 - (T^2 + 1)^2}. \end{aligned} \tag{7.8}$$

As we observed in theorem 3, $\Sigma_{+\infty}$ is endowed with a metric of constant negative curvature. We observe this here by letting $T = e^t$ and computing $\hat{K}_{+\infty}$ as in §4. We obtain

$$\hat{K}_{+\infty} = \lim_{t \to +\infty} \frac{e^{2t}}{4} \frac{4e^{2t}}{m^2 - (e^{2t} + 1)^2} = -1.$$

Two notes are in order here —

Note 6. The $\Sigma_{+\infty}$ are locally univalent images of $\mathbf{D}$. If we think of $\Sigma_{+\infty}$ not as the image of $\mathbf{D}$ but rather as an immersion of $\mathbf{D}$, we can push the hyperbolic metric on $\mathbf{D}$ (thought of as $\mathbf{H}^2$) forward onto $\Sigma_{+\infty}$ locally. Thus one point in $\mathbf{S}^2_\infty$ may have many sheets of $\Sigma_{+\infty}$ containing it, and correspondingly many different metrics, one for each sheet. Here the map $F_{0,A}$ is thus an isometry from $\mathbf{H}^2$ to $\Sigma_{+\infty}$, so that $\Sigma_{+\infty}$ is complete. We will discuss this further in the next two sections.

Note 7. There is a nice geometric interpretation of the Σ_t as envelopes of horospheres which can be deduced from [7] and [18]. We sketch this.

These examples are obtained locally by considering a surface patch $\Omega \subset \mathbf{S}^2_\infty$ as part of the image of $\mathbf{D}$ under a locally univalent conformal map $f : \mathbf{D} \to \mathbf{S}^2_\infty$. In this case we take ρ to be

$$\rho = \log\left(\frac{1}{|f'|} \cdot \frac{1+|f(z)|^2}{1-|z|^2}\right) = \log\left(\frac{ds_{\mathbf{H}^2}(p)}{ds_{\mathbf{S}^2}(f(p))}\right).$$

This means that for any $p \in \mathbf{D}$, precompose by a Möbius transformation so that we may assume $p = 0$, and then normalize f so that $f(z) = z + O(z^3)$ by Möbius transformations. Then $\nu_{0\in\mathbf{S}^2_\infty}$ is the horosphere going through $(0,1) \in \mathbf{C} \times \mathbf{R}_+$ (as following note 5 of §4).

In these cases one checks that $\hat{g}^\infty_{ij}$ is, up to a constant, the Poincaré metric on Ω obtained by pushing forward that on $\mathbf{D} = \mathbf{H}^2$. This is possible as f is locally univalent.

In closing this section it is important to note that in neither the geometric envelopes of horospheres treatment [7] nor the more computational treatment [18] was it necessary to have $\Sigma_{+\infty}$ embedded in $\mathbf{S}^2_\infty$. At no time in the computations of [18] was $\|A\|_\infty < 1$ used, and the discussion in [7] is strictly local. Hence we observe that the entire discussion of this section holds for *any* $A \in Q(\mathbf{D})$.

8 Ahlfors–Weill: the flat case

For any $A \in Q(\mathbf{C})$, we can construct complete constant mean curvature surfaces at infinity of $\mathbf{H}^3$ in a fashion completely analogous to the construction of the preceding section. Again let f_1, f_2 be independent solutions of

$$f'' + Af = 0 \qquad \text{on } \mathbf{C}.$$

such that

$$\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2' \end{pmatrix}_{z=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote by $\mathrm{Isom}(1, \mathbf{C})$ the group of isometries of $\mathbf{C}$. Here the monodromy map $\mathrm{Mon}_A : \mathrm{Isom}(1, \mathbf{C}) \to PSL(2, \mathbf{C})$ given by

$$\text{(8.1)} \qquad \begin{aligned} &\mathrm{Mon}_A \left[\begin{pmatrix} e^{\mathbf{i}\theta} & \\ & e^{-\mathbf{i}\theta} \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} f_1(z) & f_1'(z) \\ f_2(z) & f_2'(z) \end{pmatrix} \begin{pmatrix} e^{\mathbf{i}\theta} & \\ & e^{-\mathbf{i}\theta} \end{pmatrix}. \end{aligned}$$

Again by means of Mon_A we consider the action of $\mathrm{Isom}(1, \mathbf{C})$ on $\mathbf{H}^3$ as

$$\gamma : (Z, T) \mapsto \mathrm{Mon}_A(\gamma) \cdot (Z, T).$$

Since $\mathrm{Mon}_A(\mathrm{stab}(0)) \subset \mathrm{stab}(0, T)$ for all $T \in \mathbf{R}_+$, we have an induced map $F_{T,A} : \mathbf{C} \to \mathbf{H}^3$ via the factored diagram

$$\begin{array}{ccc} \mathrm{Isom}(1, \mathbf{C}) & \overset{\mathrm{Mon}_A}{\longrightarrow} & PSL(2, \mathbf{C}) \\ \downarrow \mathrm{stab}(0) & \longrightarrow & \downarrow \mathrm{stab}(0, T) \\ \mathbf{C} & \overset{F_{T,A}}{\longrightarrow} & \mathbf{H}^3. \end{array}$$

We have the following explicit form for the $F_{T,A}$ in terms of f_1 and f_2:

$$\text{(8.2)} \qquad \begin{aligned} F_{T,A}(z) &= \mathrm{Mon}_A \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \cdot (0, T) \\ &= \left(\frac{f_1'(z)\overline{f_2'(z)} + f_1(z)\overline{f_2(z)}T^2}{|f_2'(z)|^2 + |f_2(z)|^2 T^2}, \frac{T}{|f_2'(z)|^2 + |f_2(z)|^2 T^2} \right), \end{aligned}$$

and again extend this action continuously to $T = 0$ and $T = \infty$ to get

$$\text{(8.3)} \qquad F_{\infty,A}(z) = \frac{f_1(z)}{f_2(z)} \qquad \text{and} \qquad F_{0,A}(z) = \frac{f_1'(z)}{f_2'(z)}.$$

Since $F_{0,A}$ is conformal, $\overline{F_{0,A}}$ is anticonformal with conformal deformation $\mu_{0,A} \equiv \infty$. As $F'_{0,A} = \dfrac{-A}{(f_2')^2}$, we see that the singularities of $F_{0,A}$ come from precisely the zeroes of A.

If we compute the deformations of the $F_{T,A}$ we have

$$\mu_{T,A} = -\bar{A}T^{-2}.$$

Again, for completeness, the curvatures of the Σ_t are included. We have

$$\begin{aligned} k_1 &= \frac{|A| + T^2}{|A| - T^2} \\ k_2 &= \frac{|A| - T^2}{|A| + T^2} \\ H &= \frac{|A|^2 + T^4}{|A|^2 - T^4} \\ K &= 0. \end{aligned} \tag{8.4}$$

Clearly we also have $\hat{K}_{+\infty} \equiv 0$. This can be checked as was done in the hyperbolic case at the end of §7.

The relationship between entire functions and $Q(\mathbf{C})$, i.e. families of parallel surfaces generated by a complete flat constant mean curvature 1 surface at infinity of $\mathbf{S}^2_\infty$, is easily understood. We have the following evident theorem.

THEOREM 5. *The vector space* $Q(\mathbf{C})$ *is naturally isomorphic to* (*entire functions*)/ (*affine transformations of* $\mathbf{C}$), *where the affine transformations act on the space of entire functions as* $(az+b)\cdot f(z) = a^2 f(az+b)$.

One consequence is that, for $a \in \mathbf{R}$, the entire function $A_1(z) = a^2 A(az)$ reparametrizes the family of parallel surfaces generated by the complete constant mean curvature surface at infinity arising from A. This is seen through formula (8.2) as follows:

If f_1, f_2 are our normalized solutions to $f'' + Af = 0$, and g_1, g_2 are those for $g'' + A_1 g = 0$, then $g_i(z) = a^{-1/2} f_i(az)$. Thus

$$
\begin{aligned}
F_{T,A_1}(z) &= \mathrm{Mon}_{A_1}\begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \cdot (0,T) \\
&= \left(\frac{g_1'(z)\overline{g_2'(z)} + g_1(z)\overline{g_2(z)}T^2}{|g_2'(z)|^2 + |g_2(z)|^2T^2}, \frac{T}{|g_2'(z)|^2 + |g_2(z)|^2T^2} \right) \\
&= \left(\frac{a^{-1}f_1'(az)\overline{f_2'(az)} + af_1(az)\overline{f_2(az)}T^2}{a^{-1}|f_2'(az)|^2 + a|f_2(az)|^2T^2}, \right. \\
&\qquad \left. \frac{T}{a^{-1}|f_2'(az)|^2 + a|f_2(az)|^2T^2} \right) \qquad (8.5) \\
&= \left(\frac{f_1'(az)\overline{f_2'(az)} + f_1(az)\overline{f_2(az)}T^2}{|f_2'(az)|^2 + |f_2(az)|^2(aT)^2}, \right. \\
&\qquad \left. \frac{aT}{|f_2'(az)|^2 + |f_2(az)|^2(aT)^2} \right) \\
&= F_{aT,A}(az).
\end{aligned}
$$

We note the following for general interest. The proof was suggested by A. Hinkkanen.

PROPOSITION 9. *There are locally univalent conformal maps $F : \mathbf{C} \to \mathbf{S}^2_\infty$ for which the image is the whole sphere and each $\theta \in \mathbf{S}^2_\infty$ has infinitely many preimages.*

PROOF: Let A be a polynomial of odd degree n on $\mathbf{C}$, and consider the ODE

$$f'' + Af = 0.$$

If f is a nonzero solution to this ODE it has fractional order $\frac{n+2}{2}$, and so has infinitely many zeroes [16]. It follows that if f_1 and f_2 are independent solutions then $F = \frac{f_1}{f_2}$ is locally univalent and has infinitely many zeroes and poles. To see that F assumes the value $c \in \mathbf{C} \setminus \{0\}$ infinitely often, consider that $F - c = \frac{f_1 - cf_2}{f_2}$ and $f_1 - cf_2$ vanishes infinitely often. $\square$

9 Existence and uniqueness of complete constant mean curvature surfaces at infinity

Suppose that Σ_0 has its metric given conformally by $ds^2 = \lambda^2|dZ|^2$, that

$$(II_0)_{ij} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

and that $H(\Sigma_0) = \dfrac{L+N}{2\lambda^2} \equiv c \in (-1,1)$. Then $A = \dfrac{L-N-2iM}{2}$ is holomorphic. The intrinsic curvature of Σ_0 is given by

$$\begin{aligned} K &= k_0^1 k_0^2 - 1 \\ &= \frac{LN - M^2}{\lambda^4} - 1 \\ &= c^2 - A\bar{A}\lambda^{-4} - 1 \\ &\leq -(1-c^2) \quad < \quad 0, \end{aligned}$$

whence the universal cover of Σ_0 is conformally $\mathbf{D}$, with coordinate z. In this case, $(\Pi_0)_{ij}$ pulls back to give the quadratic differential $(\tilde{\Pi}_0)_{ij} = Adz^2 + 2c\lambda^2|\frac{dZ}{dz}|^2 dzd\bar{z} + \bar{A}d\bar{z}^2$ on $\mathbf{D}$. The author considers this of particular interest in the case where Σ_0 is a complete surface, in which case the following issues arise naturally:

QUESTION: To what extent does the holomorphic quadratic differential A, the mean curvature c, and the completeness of the metric $\lambda^2|dZ|^2$ determine the surface Σ_0? Or more precisely, given a holomorphic quadratic differential A on $\mathbf{D}$, and a constant $c \in (-1,1)$, does there exist a complete surface $\Sigma_0 \hookrightarrow \mathbf{H}^3$ with $H(\Sigma_0) = c$ and $\tilde{\Pi}_{zz} = A$? Given (A,c) for which this problem is solvable, to what extent is a solution unique?

EXAMPLE. If $A \equiv 0$ and $c \in (-1,1)$ then one readily sees that any surface Σ_0 inducing the trivial quadratic differential on $\mathbf{D}$ satisfies $k_1 = k_2 = c$ globally, i.e. Σ_0 is a piece of a surface parallel to a totally geodesic plane. The completeness of Σ_0 implies that it is precisely a surface parallel to a totally geodesic plane. Hence for $A \equiv 0$ we have both existence and uniqueness up to ambient isometry.

Karen Uhlenbeck [17] addressed this issue. She showed that if A is a sufficiently small quadratic differential on $\mathbf{D}$ then there exists a complete Σ_0

when $c = 0$ inducing A. In [2], Anderson proved that, for example, if $\mathcal{C}$ is a quasicircle in $\mathbf{S}^2_\infty$ then there is a stable complete minimal embedding of $\mathbf{D}$ as $\Sigma_0 \hookrightarrow \mathbf{H}^3$ with $\partial_\infty \Sigma_0 = \mathcal{C}$. By work of Epstein [8], one may see that when $\mathcal{C}$ is sufficiently 'close' to a circle (we won't discuss what this means) then the quadratic differentials arising in Uhlenbeck's work parametrize the minimal surfaces produced by Anderson.

Bennett Palmer [13] asked the same question for space-like surfaces immersed in symmetric constant curvature k three dimensional Lorentzian manifolds L_k. He established existence and uniqueness given that the holomorphic quadratic differential A on $\mathbf{D}$ is automorphic with respect to a co-compact Fuchsian group. In fact, Palmer also considered when the other conformal simply connected Riemann surfaces ($\mathbf{C}$, $\hat{\mathbf{C}}$) immerse as space-like constant mean curvature surfaces in L_k.

To date, little progress has been made toward the resolution of the above question. We do note, however, that the case of constant mean curvature $c \in (-1, 1)$ is equivalent to the case $c = 0$. This follows immediately from Uhlenbeck's work.

COROLLARY 6. *If* $\|A\rho^{-2}\|_\infty < (1-c^2)^{-\frac{1}{2}}$ *then there is an immersed complete surface* $\Sigma_0 \hookrightarrow \mathbf{H}^3$ *of constant mean curvature* c *so that* $\tilde{\Pi}_{zz}(\Sigma_0) = A$.

PROOF: To see this, note that the Gauss equation for λ in the case of $c \in (-1, 1)$ is

$$\Delta \log \lambda = 2A\bar{A}\lambda^{-2} + 2(1 - c^2)\lambda^2,$$

and letting $\lambda = \frac{1}{2}e^{\frac{1}{2}\phi}$ yields

$$\Delta\phi = 16A\bar{A}e^{-\phi} + (1 - c^2)e^{\phi}.$$

If we now let $\psi = \phi + \log(1 - c^2)$ we get

$$\Delta\psi = 16A\bar{A}(1 - c^2)e^{-\psi} + e^{\psi},$$

i.e. the Gauss equation for the metric of a minimal surface with $\tilde{\Pi}_{zz} = \sqrt{1 - c^2}A$. $\square$

It is worth noting that as $c \to 1$ this is solvable for more quadratic differentials, and when $c = \pm 1$ we get exactly the Liouville equation for constant Gauss curvature -1.

We now examine a variant of the question above, and give a parametrization of complete constant mean curvature surfaces at infinity of $\mathbf{H}^3$. As was observed in §6, these must necessarily have mean curvature ± 1. We begin with two lemmas.

LEMMA 5. *If $\Sigma_{+\infty}$ is a surface patch at infinity of $\mathbf{H}^3$, and if $N_{t_0}^{+\infty} : \Sigma_{+\infty} \to \Sigma_{t_0}$ is conformal for some t_0, then H and K are constant on Σ_{t_0}, or equivalently the directional curvatures k_1 and k_2 on Σ_{t_0} are constant.*

PROOF: If $N_{t_0}^{+\infty} : \Sigma_{+\infty} \to \Sigma_{t_0}$ is conformal, then at least locally so is $N_{+\infty}^{t_0}$. Thus by theorem 2, $\mu_{+\infty}^{t_0} \equiv 0$ which implies that $\mu_t^{t_0} \equiv 0$ for all t and we are in the second case of corollary 1. This also follows from proposition 6 and the expression for $(\Pi_t)_j^i$ in §4. $\square$

LEMMA 6. *Under the assumptions of lemma 5, the $N_t^{+\infty}$ are conformal for all $t \in [-\infty, +\infty]$ with at most one exception.*

PROOF: This is immediate. The exceptional possibility is that the $N_t^{+\infty}$ may focus on a point (when the Σ_t are all pieces of spheres or horospheres – in this latter case the focal point is at infinity) or on a line (when the Σ_t are all pieces of geodesic cylinders). $\square$

The following theorem is an immediate consequence of the above two lemmas and the theorem of Cartan mentioned in §5.

THEOREM 6. *If $\Sigma_{+\infty}$ is complete and the $N_t^{+\infty}$ are conformal the for all t we have one of the following cases:*

1. *$\Sigma_{+\infty} = \mathbf{S}^2_\infty$ and the Σ_t are concentric spheres;*

2. *$\Sigma_{+\infty} = \mathbf{S}^2_\infty \setminus \{\theta\}$ for some $\theta \in \mathbf{S}^2_\infty$, endowed with the affine structure of $\mathbf{C}$, and the Σ_t are horospheres centered at θ;*

3. *$\Sigma_{+\infty} = \mathbf{S}^2_\infty \setminus \{\theta_1, \theta_2\}$ endowed with the complete flat structure of $\mathbf{C}^* \cong \mathbf{C}/(z \mapsto z+1)$, and the Σ_t are the equidistant surfaces from the geodesic connecting θ_1 and θ_2; or*

4. *$\Sigma_{+\infty}$ is a disk with the hyperbolic metric bounded by a geometric circle in $\mathbf{S}^2_\infty$, and the Σ_t are the surfaces in $\mathbf{H}^3$ equidistant from the totally geodesic plane spanning $\mathbf{C}$.*

And finally we have

THEOREM 7. *Given a holomorphic quadratic differential A on either* $\mathbf{C}$ *or* $\mathbf{D}$*, there exists a unique complete surface $\Sigma_{+\infty}$ of constant mean curvature ± 1 at infinity of* $\mathbf{H}^3$ *with A as the holomorphic part of its second fundamental form. In the first case, $\Sigma_{+\infty}$ is flat and in the second it is hyperbolic. Thus the surfaces of constant mean curvature at $\partial_\infty \mathbf{H}^3$ are parametrized up to isometry by holomorphic quadratic differentials on either* $\mathbf{C}$ *or* $\mathbf{D}$*, with the exception of the sphere $\mathbf{S}^2_\infty$ itself.*

PROOF: The existence of such structures has been established in §7 and 8. This amounted, in effect, to solving the Schwarzian differential equation $\{f, z\} = A$ on $\mathbf{C}$ or $\mathbf{D}$. We are left with the problem of showing that the family of equidistant surfaces in $\mathbf{H}^3$ arising therefrom is unique.

This, though, is the content of the theorem of §4 and theorem 3.

Thus our support function arises from one of the two Ahlfors–Weill constructions.

It follows that except in the cases covered by theorem 6 the complete surfaces at infinity are precisely those arising from solving the Schwarzian differential equation, and the corresponding deformation of surfaces in $\mathbf{H}^3$ is given by extension of the Ahlfors–Weill construction. In the exceptional cases we see that we have the sphere itself, the quadratic differential $A = 0$ on $\mathbf{C}$ (case 2 of theorem 6), the quadratic differential $A = \mathit{const} \neq 0$ on $\mathbf{C}$ (case 3 of theorem 6, see also the comments after theorem 5 as to why all constants give the same foliation here), and finally the quadratic differential $A = 0$ on $\mathbf{D}$ (case 4 of theorem 6). This finishes the proof. $\square$

This can be reformulated as

THEOREM 8. *The complete constant mean curvature surfaces at infinity of* $\mathbf{H}^3$ *are in a natural 1–1 correspondence with quadratic differentials on either* $\mathbf{S}^2 = \hat{\mathbf{C}}$*,* $\mathbf{C}$*, or* $\mathbf{D}$*. The map from quadratic differentials to surfaces (up to ambient isometry) is given by solving the Schwarzian differential equation, and the inverse is via the map from surfaces to the zz–part of their second fundamental forms.*

References

[1] Ahlfors, L. V., *Lectures on Quasiconformal Mappings.* Wadsworth & Brooks/Cole, Monterey, CA (1987).

[2] Anderson, M., Complete minimal hypersurfaces in hyperbolic n-manifolds, *Comm. Math. Helv.* **58** (1983), 264–290.

[3] Bianchi, L., *Lezioni di Geometria Differenziale, vol. II, pt. II.* Zanichellin, Bologna (1930).

[4] Bryant, R. L., Surfaces of mean curvature one in hyperbolic space, *Asterisque* **154-5** (1987), 321–347.

[5] Cartan, E., Familles de surfaces isoparametriques dans les espaces à courbure constante, *Annali di Math.* **17** (1938), 177–191.

[6] Eells, J. & Lemaire, L., A report on harmonic maps, *Bull. London Math. Soc.* **10** (1978), 1–68.

[7] Epstein, C. L., Envelopes of horospheres and Weingarten surfaces in hyperbolic 3-space. *preprint,* (1984).

[8] —, The hyperbolic Gauss map and quasiconformal reflections, *J. reine angew. Math.* **372** (1986), 96–135.

[9] —, Orthogonally integrable line fields in $\mathbf{H}^3$, *Comm. Pure & App. Math.* **23** (1985), 599–608.

[10] Gauss, C. F., *Disquisitiones Generales Circa Superficies Curvas.* Gottingen (1827). Translated as *General Investigation of Curved Surfaces.* Raven Press, Hewlett, New York (1965).

[11] Hawley, N., *unpublished notes.*

[12] Nehari, Z., Schwarzian derivatives and schlicht functions, *Bull. AMS* **55** (1949), 545–551.

[13] Palmer, B., Spacelike constant mean curvature surfaces in pseudo-Riemannian spaceforms. *Ann. Global Anal. & Geom.* **8** (1990), 217–226.

[14] Porteous, I. R., The normal singularities of a submanifold, *J. Diff. Geom.* **5** (1971), 543–564.

[15] Spivak, M., *A Comprehensive Introduction to Differential Geometry, II.* Publish or Perish, Boston (1970).

[16] Titchmarsh, E. C., *The Theory of Functions.* Oxford U. Press, Oxford (1939).

[17] Uhlenbeck, K., Closed minimal surfaces in hyperbolic 3–space, *Ann. Math. Studies* **103** (1985), 147–168.

[18] Velling, J. A., A geometric interpretation of the Ahlfors–Weill mappings and an induced foliation of $\mathbf{H}^3$, *Holomorphic Functions and Moduli, I.* MSRI Pubs. **10** (1988), 193–202.