Finite Earthquakes and the Associahedron

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Abstract

By using finite earthquakes, we show there exists a homeomorphism Φ between the real analytic Teichmüller space T_n of a set of $n+3$ variable labelled points cyclically arranged on the unit circle and the interior of the n-dimensional associahedron K_{n+2}. We also show how to obtain the faces of a compactification $\overline{T}_n$ of T_n by letting certain finite earthquake parameters approach ∞. The homeomorphism Φ naturally extends to these faces so that they themselves are realized as products of lower dimensional Teichmüller spaces. Without using Teichmüller's theorem, we show that T_n is isomorphic to an n-dimensional open ball. Furthermore, the relationships among the faces of the associahedron K_{n+2} provide a combinatorial view of how pieces of T_n are sewn together to form the interior of K_{n+2}.

1 Introduction

The associahedron K_{n+2} is a complex of dimension n with a combinatorial structure determined by the different associations of $n+2$ letters. It consists of a large number of cubes of dimension n sewn together in a certain way. The exact number is the number of all of the different ways of associating a multiplication on $n+2$ ordered factors. Each cube is labelled by the maximal associations of the $n+2$ letters and the faces of the cubes are labelled by the partial associations. The cubes are sewn together along faces according to their labels. The associahedrons on three, four and five letters, K_3, K_4, and K_5, can be easily constructed as follows.

In the case $n = 1$ we have three factors a, b and c with two associations $a(bc)$ and $(ab)c$. K_3 is a union of two closed line segments labelled $a(bc)$ and $(ab)c$ which are joined together at a central point which is labelled abc.. Clearly, K_3 can be embedded in $\mathbb{R}$. It is a one-dimensional complex.

In the case $n = 2$ we have 4 factors a, b, c and d with five associations,

$$((ab)c)d, (ab)(cd), a((bc)d), (a(bc))d, a(b(cd)).$$

The maximal and partial associations of $abcd$ are arranged in a partially ordered set in Figure 1. We view the diagram as a picture of the ordering and regard the association $abcd$ with no parentheses as its smallest element. Any association that can be obtained from another by adding matching pairs of parentheses is considered larger.

K_4 is a pentagon; its interior is labelled $abcd$, its five sides are labelled with the five partial associations of four letters, and its five vertices are labelled with the five maximal associations. Each vertex is adjacent to a side if the label on

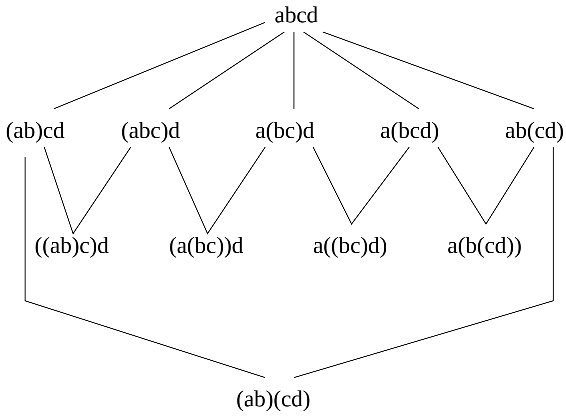

Figure 1: Partial ordering for the partial associations for K_4.

the vertex can be obtained by adding one pair of parentheses to the label on the adjoining side. We see that each edge of K_4 is a one-dimensional associahedron, the edges form a simple closed curve in $\mathbb{R}^2$ and K_4 is represented by the 2-dimensional complex shown in Figure 2.

For K_5 we have 5 factors a, b, c, d and e with 14 maximal associations:

$$\begin{array}{llll} ((ab)(cd))e, & a((bc)(de)), & & \text{two stars} \\ a((b(cd))e), & ((a(bc))d)e, & (ab)(c(de)), & \text{three left accordions} \\ a(b((cd)e)), & (a((bc)d))e, & a(b((cd)e)), & \text{three right accordions} \\ (((ab)c)d)e, & a(((bc)d)e), & (ab)((cd)e), & \multirow{2}{*}{\}\ \text{six fans}} \\ (a(bc))(de), & a(b(c(de))), & (a(b(cd)))e. & \end{array}$$

The explanation for why these associations are called stars, accordions and fans is given in the fourth paragraph of section 2 and in the caption to Figure 4. K_5 is represented by the three dimensional convex polyhedron shown in Figure 3. Its vertices are labelled by the fourteen maximal associations on five letters a, b, c, d and e listed above.

The number of faces of different dimension of K_5 breaks down as follows:

i) one three-dimensional cell,
ii) six pentagonal faces and three rectangular faces,
iii) twenty-one edges, and
iv) fourteen vertices.

In general, K_{n+2} is a simple convex polytope of dimension n (see [3]), whose faces are lower dimensional associahedra or products of lower dimensional associahedra. In the last section of this paper we will review the algorithm for constructing K_{n+2}, $n \geq 2$ that appears in Devadoss [3]. It is obtained by truncating an n-dimensional simplex by codimension one hyperplanes.

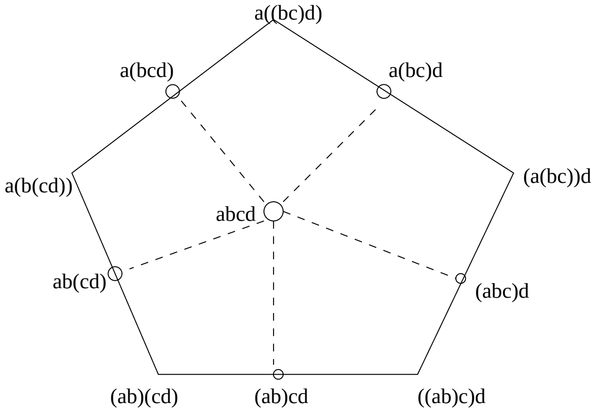

Figure 2: The associahedron K_4.

Now we consider the real analytic Teichmüller space T_n of dimension n. T_n is a certain set of equivalence classes of $n+3$ variable points cyclically arranged on the unit circle. Two such labelled sets $S = \{p_1, \ldots, p_{n+3}\}$ and $S' = \{p'_1, \ldots, p'_{n+3}\}$ are considered equivalent if there exists an orientation preserving Möbius transformation A preserving the circle such that $A(p_j) = p'_j$ for $1 \leq j \leq n+3$. Another way to view T_n is as equivalence classes of orientation preserving homeomorphisms from the circle into the circle restricted to a fixed subset S of the circle consisting of $n+3$ points. Two such maps h_0 and h_1 are equivalent if there is a Möbius transformation A such that $A \circ h_0(p) = h_1(p)$ for every p in S.

In this paper we show how the finite earthquake theorem ([8]) provides a homeomorphism Φ from T_n onto the interior of the associahedron K_{n+2}. By letting certain of the earthquake parameters approach ∞, a compactification $\overline{T}_n$ of T_n is realized in such a way that Φ extends to an isomorphism between $\overline{T}_n$ and K_{n+2}. Thus, we obtain a natural cell structure for T_n without using quadratic differentials or Teichmüller's theorem. The relationship between the faces of K_{n+2} provides a combinatorial arrangement that shows how the pieces of T_n and $\overline{T}_n$ are sewn together to form the interior of K_{n+2} and K_{n+2}.

There are interesting relationships between this cell structure and other important theorems from the Teichmüller theory of a finite set of variable points on a circle. In particular, Teichmüller's theorem, [15], [2], [6], the theorem on the existence of Jenkins-Strebel differentials with given heights, [14], [12], [9], and the infinitesimal earthquake theorem, [5]. These are topics that should be investigated further.

The theory of the associahedron was first developed by Stasheff [13] as a topic in topology and is also related to problems in homological algebra [11] and category theory [18]. In this paper we approach the subject from a different viewpoint, namely, the viewpoint of Teichmüller geometry.

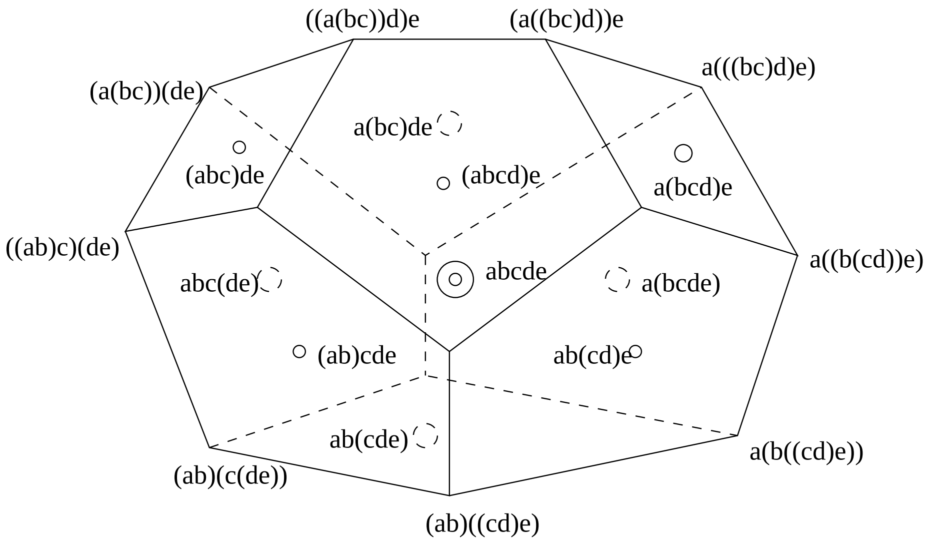

Figure 3: The associahedron K_5.

We are grateful to Reza Chamanara, Bill Harvey, Dennis Sullivan and Noson Yanofsky for their help and encouragement.

2 The finite earthquake theorem.

In this section we revisit a finite version of the well-known earthquake theorem of Thurston in [16], which tells us that any orientation-preserving homeomorphism of the unit circle is realized as the extension to the boundary of an earthquake map on the interior unit disk. An earthquake map on the unit disk is a piecewise Möbius transformation on domains separated by non-intersecting hyperbolic geodesics such that the comparisons of the Möbius transformations on different domains are hyperbolic, have their axes separating the domains, and shift in the same direction. The union of these non-intersecting lines is called the lamination for the earthquake and the amount of shifting is quantified by a transverse measure. Provided the amount of shifting is controlled, the restriction of the earthquake to the boundary of the unit circle is a homeomorphism. In [16] Thurston gave a construction that shows that any homeomorphism of the circle is realized in this manner by an earthquake and that its measure and lamination are uniquely determined. Moreover, up to post composition by a Möbius transformation, the homeomorphism can be reconstructed from its lamination and measure.

In [8] Gardiner and Lakic gave a version of this theorem that applies to the case that the homeomorphism is replaced by a cyclic order preserving map

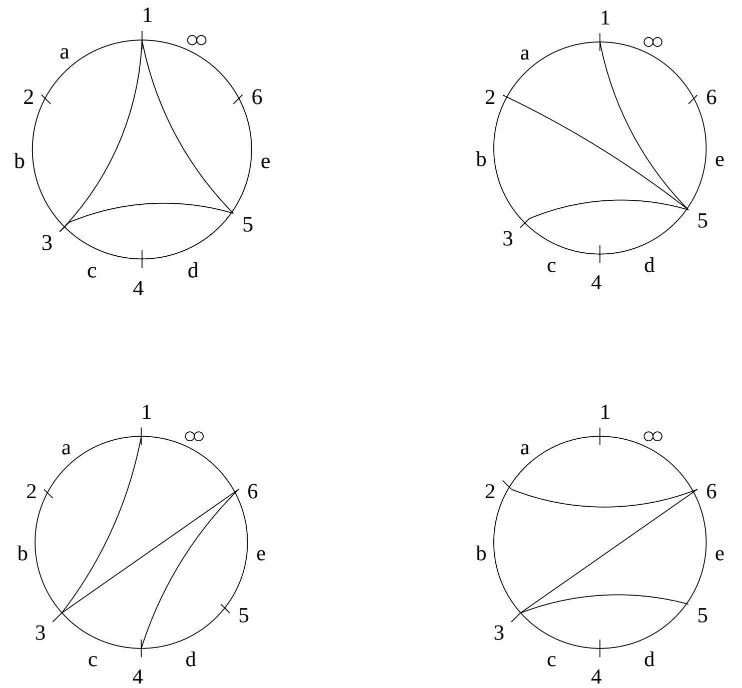

Figure 4: *Different topological types of allowable laminations on six points. The upper left figure is a star corresponding to the association* $((ab)(cd))e$. *The upper right is a fan corresponding to* $(a(b(cd)))e$. *The lower left is a left accordion corresponding to* $(ab)(c(de))$. *The lower right is a right accordion corresponding to* $a(b((cd)e))$. *By rotating these figures one sees that there are two stars, six fans, three left accordions, and three right accordions, making a total of fourteen allowable laminations corresponding to the fourteen possible associations.*

h defined only on a finite subset S of the circle and mapping S bijectively to another subset $h(S)$. In their formulation a lamination $\mathcal{L}$ is *allowable* if it consists of a finite number of non-intersecting geodesics joining points of S *and if it has the property that no line of* $\mathcal{L}$ *joins points adjacent in* S. Using their finite earthquake theorem and a limiting process, they deduce Thurston's theorem for arbitrary homeomorphisms, ([7] and [8]).

In this paper, consideration of only the allowable finite laminations is essential for two reasons:

i) The different topological types of allowable laminations for a finite set S with $n+3$ elements correspond in an obvious way to the different possible associations for the multiplication of $n+2$ letters.

ii) Without this property a finite earthquake is not uniquely determined up to post composition with a Möbius transformation by its lamination and measure.

To see how allowable finite laminations correspond to associations consider the pictures shown in Figure 4. Six points are marked on the unit circle and the intervals between them are labelled in counter-clockwise order with the symbols a, b, c, d, e, and ∞. Each line of the lamination divides the symbols into two sets. Because the finite lamination is allowable, each of these sets contains at least two symbols. To obtain the association corresponding to the lamination, we insert a pair of matching parentheses that contains the set which does not include the symbol ∞. We insert such a pair of matching parentheses for every line of the lamination. In general, this procedure yields a one-to-one correspondence between the maximal allowable finite laminations on subsets of $n+3$ points and the full associations of $n+2$ letters.

A non-negative atomic measure σ associated to an allowable finite lamination $\mathcal{L}$ is a collection of non-negative numbers λ_j associated to lines l_j in $\mathcal{L}$. We write $\sigma(\{l_j\}) = \lambda_j$. Now we explain how the measure σ together with its lamination $\mathcal{L}$ induces a left earthquake that yields a one-to-one order-preserving map h from S onto another set $S' = h(S)$ of $n+3$ points in the unit circle. Note that $\mathcal{L}$ consists of at most $m \leq n$ lines. These lines cut the disk into $m+1$ pieces, which we call the strata of the lamination. We choose any line l_{j_0} of $\mathcal{L}$, stand on one side of that line and look across to the hyperbolic half-plane on the other side. We shift every point of that half plane to the left by an isometry with translation axis equal to the line l_{j_0} and translation length equal to λ_{j_0}. All the lines of the lamination lying in that half plane together with their endpoints on the circle are also shifted to the left. Having done this, denote by s_{j_0} the stratum adjacent to l_{j_0} in that shifted half plane. Now we move to each other geodesic boundary line l_{k_0} of s_{j_0} and repeat the same process on the other side of l_{k_0} as we have done along the line l_{j_0}. We repeat this process indefinitely until we have used every line of $\mathcal{L}$ lying in this half-plane. Then we turn around 180^0 and do the same thing in the half plane lying on the other side of l_{j_0}. In the end, we obtain a map E_{σ,j_0} which is discontinuous along the lines l_{j_0} but which extends to a homeomorphism of the boundary of the unit disc $\partial\mathbb{D}$. E_{σ,j_0} maps the finite set S one-to-one and onto a set S' and it depends on σ and the choice of the line l_{j_0} in $\mathcal{L}$. However, up to post composition by an orientation preserving isometry, it is independent of this choice. That is, if we had started at a different line l_{j_1} of $\mathcal{L}$ and followed the same procedure, then there would be an isometry $A_{0,1}$ of $\mathbb{D}$, such that

$$A_{0,1} \circ E_{\sigma,j_0} = E_{\sigma,j_1}.$$

Since up to post-composition by Möbius transformations the maps $E_{\sigma,j}$ do not depend on j, we denote the coset class $PSL(2,\mathbb{R}) \circ E_{\sigma,j}$ by E_σ.

Given this notation we can now state the finite earthquake theorem given by Gardiner and Lakic in [8].

Theorem 1. (The finite earthquake theorem). *Suppose*

$$S = \{p_1, \ldots, p_{n+3}\} \text{ and } S' = \{p'_1, \ldots, p'_{n+3}\}$$

are two finite subsets of the same cardinality, both arranged in counterclockwise cyclic order on the circle and assume $h(p_j) = p'_j$*. Then there exists a unique allowable lamination* $\mathcal{L}$ *for* S *and a unique measure* σ *supported on* $\mathcal{L}$ *such that, up to post composition by an orientation preserving isometry of* $\mathbb{D}$*, the left earthquake map* E_σ *maps* p_j *to* p'_j*. The measure* σ *and its corresponding lamination are uniquely determined by the locations of the points of* S *and* S'*.*

The key to the proof of this theorem is the following lemma.

Lemma 1. ([8]) *Let* h *be the map from* S *onto* S' *with* $h(p_i) = p'_i$*, where* $1 \leq i \leq n+3$*. There exists an orientation preserving Möbius transformation* A *preserving the unit disk such that the post composition* $A \circ h$ *of* h *by* A *fixes at least three points in* S *and moves all other points counter-clockwise.*

In [8] Gardiner and Lakic show inductively how to find the isometry A and the three fixed points of $A \circ h$. They also give a recurrence process that finds the lamination and the weights that are assigned to its lines. On his website, *http://comet.lehman.cuny.edu/lakic,* Lakic uses this recurrence to give a Maple program which finds the lamination and the measure.

For the convenience of the reader, here we give a proof that constructs the isometry A in the form

$$A = (\textit{hyperbolic translation}) \circ (\textit{parabolic translation}) \circ (\textit{rotation}).$$

Proof. The proof is comprised of the following steps:

1. First, select a rotation A_1 around the Euclidean center of the unit disk such that $A_1 \circ h$ has at least one fixed point p_j in S and put $h_1 = A_1 \circ h$.

2. Second, find a parabolic translation A_2 fixing p_j and preserving the unit circle for which $A_2 \circ h_1$ has at least two fixed points p_k and p_j and for which all other points p_l are either fixed or moved counter-clockwise by $h_2 = A_2 \circ h_1$.

To see that this is possible, change coordinates so that the hyperbolic plane is realized by the upper half plane $\mathbb{H}$ and the point p_j is placed at ∞. Then all of the points p_k and $p'_k = h(p_k)$ are located on the real axis. Put b equal to the minimum value of $p'_k - p_k$ over all values of k not equal to j and let this minimum be realized by k. Then by putting $A_2(x) = z + b$, $h_2 = A_2 \circ h_1$ fixes p_k and p_j and for every other point p_l, $p_l \leq h_2(p_l) = p'_l$.

3. Finally, observe that there exists a hyperbolic translation A_3 with translation axis joining p_k to p_j and such that $A_3 \circ h_2$ has at least three fixed points p_j, p_k and p_l and all other points in S are either fixed or moved counter-clockwise by h_3.

To see that this is possible, change coordinates so that $p_j = \infty$ and $p_k = 0$. Some of the the pairs of points p_l and $p'_l = h_2(p_l)$ may be on the negative real axis and some may be on the positive real axis, but for every such point $p_l \leq p'_l$. If there are any such pairs on the negative real axis, for such pairs p_l and p'_l, let λ be the largest possible value of p'_l/p_l among those values of l for which p_l is negative. Then let $A_3(z) = \lambda z$ and $h_3 = A_3 \circ h_2$ will fix p_j, p_k and p_l

and, for every other point p_m in S either on the positive or negative real axis, $p_m \leq h_3(p_m)$. If there are no such pairs on the negative real axis, all such pairs must lie on the positive real axis and there must be at least one such pair. Put λ equal to the smallest possible value of p'_l/p_l. This time put $A_3(z) = (1/\lambda)z$ and, otherwise, follow the same procedure.

The desired isometry A is equal to $A_3 \circ A_2 \circ A_1$. □

To prove the finite earthquake we start with the lemma which guarantees there is an orientation preserving, non-Euclidean isometry A, such that $A \circ h$ fixes a finite subset of S consisting of three or more points and moves all other points of S to the left. Since there are three such points in S, at least two of them must be non-adjacent. Let l the hyperbolic line that joins these two non-adjacent points and $\mathbb{H}_l$ the half-plane lying on one side of l that contains a third point of S. Since all points of S that lie on the boundary of $\mathbb{H}_l$ are moved to the left, there is one such point which is moved the least to the left. If B_l is a hyperbolic transformation B_l with axis l that moves p_l to p'_l, then $B_l^{-1} \circ A \circ h$ still moves to the left all points of S that are on the boundary of the half-plane $\mathbb{H}_l$. Then we put weight $\sigma(\{l\})$ equal to the translation length of B_l, and repeat the same procedure for the restriction of $B_l^{-1} \circ A \circ h$ on the two half-planes on both sides of l.

To prove the uniqueness assume there are two finite earthquake representations $(E_\sigma, \mathcal{L})$ and $(E'_{\sigma'}, \mathcal{L}')$. We first show no line l of $\mathcal{L}$ can cross any line l' in $\mathcal{L}'$. If l in $\mathcal{L}$ crosses l' in $\mathcal{L}'$, let a and c be the endpoints of l and b and d be the endpoints of l', where a, b, c, d are in counter-clockwise order. Let B be the isometry that maps a, b, c to $h(a), h(b), h(c)$ respectively. Then $B^{-1} \circ h$ fixes a, b, c and moves d counter-clockwise. Now let B' be the hyperbolic isometry with translation axis connecting a to c and moving $B^{-1} \circ h(d)$ back to d. Then $B' \circ B^{-1} \circ h$ fixes a, c, d and moves b clockwise. Since l' belongs to $\mathcal{L}'$ and $B \circ (B')^{-1}$ is the isometry uniquely determined by the values of h at a, c and d, $B \circ (B')^{-1} \circ h$ must move b counter-clockwise. This is a contradiction. Thus no line in $\mathcal{L}$ can cross any line in $\mathcal{L}'$.

If any weight for E' on a line of the lamination exceeds the weight for E on that line, then by holding the two endpoints of that line fixed, one can see that points of S on one side of that line will be moved further to the left by E' than by E. Therefore, the laminations and the weights are unique.

3 The combinatorial structure of T_n

In this section we use finite earthquakes to study the combinatorial structure of T_n. We first introduce an embedding of the associahedron K_{n+2} into a much higher dimensional vector space. Then we show T_n is homeomorphic to the interior of the embedded image of K_{n+2}.

Let P_{n+3} be a convex polygon in the Euclidean plane with $n+3$ sides. Label the sides of the polygon in the counterclockwise direction with the symbols,

$$a_1, a_2, \dots, a_{n+2}, \infty.$$

Consider the set Λ_n of all diagonals, that is, straight line segments joining non-adjacent vertices of P_{n+3}. Notice that Λ_n has N elements where

$$N = \begin{pmatrix} n+3 \\ 2 \end{pmatrix} - (n+3). \tag{1}$$

Let $I = [0, \infty)$, the positive real axis together with 0, and let $\overline{I} = [0, \infty]$, the compactification of I.

Definition. We define A_{n+2} (resp. $\overline{A}_{n+2}$) to be the closed subset of the product space I^N (resp. $\overline{I}^n$) consisting of vectors $v = (v_1, \ldots, v_N)$ such that each $v_j \in I$ (resp. $\overline{I}^N$) and such that any two non-zero entries v_i and v_j of v correspond to diagonals d_i and d_j of P_{n+3} that do not intersect.

Corollary of the definition. *A_{n+2} and $\overline{A}_{n+2}$ are contractible Hausdorff spaces and $\overline{A}_{n+2}$ is compact.*

Proof. $\overline{A}_{n+2}$ is compact because it is a closed subset of the compact product $\overline{I}^N$. A_{n+2} and $\overline{A}_{n+2}$ are contractible because any contraction of I or of $\overline{I}$ can be extended to a contraction of A_{n+2} or $\overline{A}_{n+2}$ since the condition that v be a vector in A_{n+2} or in $\overline{A}_{n+2}$ is preserved throughout the contraction. □

There is a partition of $\overline{A}_{n+2}$ into cells of dimensions $k \leq n$. By a finite *lamination* L of P_{n+3}, we shall mean a set of n or fewer, non-intersecting diagonals. Then we let $F(L)$ be the set of all vectors v in $\overline{A}_{n+2}$ for which the entries v_j corresponding to diagonals d_j in L are equal to ∞ and the other entries are not equal to infinity. Combinatorial formulas for the numbers of the laminations L consisting of fixed numbers of the lines in Λ_n are well-known, [17]. For example the number of L's of maximal size n is equal to the Catalan number

$$\frac{1}{n+2} \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix}. \tag{2}$$

For different laminations L of P_{n+3}, the sets $F(L)$ comprise the interiors of the faces of the associahedron of dimension n.

To introduce Teichmüller coordinates to A_{n+2}, we choose P_{n+3} to be the regular polygon circumscribed by the unit circle. Thus the set S of the vertices of P_{n+3} consists of $n+3$ equally spaced points on the unit circle. We also replace the diagonals by hyperbolic lines joining pairs of non-adjacent points of S. The Teichmüller space $T(S)$ consists of deformations of S to variable sets of $n+3$ cyclically ordered points also lying on the unit circle, factored by $PSL(2, \mathbb{R})$. By the finite earthquake theorem the points of $T(S)$ are parameterized by the vectors v in A_{n+2}. In fact, this parametrization introduces a continuous isomorphism between $T(S)$ (with the Teichmüller metric) and A_{n+2} (with the topology inherited from the product topology of I^N.)

The next theorem is justification for these statements. Before proceeding to it, we recall the definition of the Teichmüller's metric and the Teichmüller topology. The Teichmüller distance between two ordered $(n+3)$-tuples

$p_1, \ldots, p_{n+3}$ and $p'_1, \ldots, p'_{n+3}$ is $\log K(h)$ where $K(h)$ is the smallest possible dilatation of a quasiconformal map h preserving the unit disk such that $h(p_j) = p'_j, 1 \le j \le n+3$.

Lemma 2. *Let $p = (p_1, \ldots, p_{n+3})$ be an ordered set of $n+3$ points on the unit circle. Let U_ϵ be the set of variable points $p' = (p'_1, \ldots, p'_{n+3})$ with that property that there is a Möbius transformation A preserving the unit disk for which*

$$|p_j - A(p'_j)| < \epsilon \text{ for } 1 \le j \le n+3.$$

Then $U_\epsilon(p)$, $\epsilon > 0$ forms a neighborhood basis in the the Teichmüller topology at the point represented by p in T_n.

Proof. First assume p' is close to p in the Teichmüller topology, that is, for a given $\alpha > 0$, there is a quasiconformal map h preserving the unit disk such that $h(p_j) = p'_j$ and such that $K(h) < 1 + \alpha$. Then every extremal length problem for topological quadrilaterals in the unit disk with sides on the unit circle and vertices taken from the points of S must be $K-$quasi-preserved by h. Since these extremal lengths are strictly monotone functions of cross-ratios, the same is true about the cross ratios. That is, there exists a $\beta > 0$ depending on α and the spacing between the points of S, such that

$$|\log \frac{cr(a', b', c', d')}{cr(a, b, c, d)}| < \beta. \tag{3}$$

Here, a, b, c, d and a', b', c', d' are arbitrary quadruples taken from the sets S and S' with $h(a) = a', h(b) = b', h(c) = c', h(d) = d'$ and

$$cr(a, b, c, d) = \frac{b-a}{c-b} \cdot \frac{d-c}{d-a}.$$

We shall denote the quantity inside the absolute value on the left side of (3) by $cr_d^h(Q)$ and call it the cross ratio distortion of h on the quadruple $Q = \{a, b, c, d\}$. For an explanation of the functional relationship between cross ratio distortion and extremal length see [1] or [10].

To complete the proof we must show that if p is in U_ϵ for arbitrarily small ϵ then there is a qquasiconformal map h preserving the unit disk and carrying p_j to p'_j such that $\log K(h) = \delta$ for arbitrarily small δ. Since $K(h) = K(A \circ h \circ B)$ for any Möbius transformations A and B, we may assume the points p_j and p'_j lie on the real axis. We may also assume $p_1 = p'_1 = \infty, p_2 = p'_2, p_3 = p'_3$ and $|p_j - p'_j| < \epsilon$ for $4 \le j \le n+3$. Now choose δ equal to the minimum distance $|p_{j+2} - p_{j+3}|$ for $1 \le j \le n$. For each such p_j construct a square R_j centered at p_j with side length equal to δ and with top and bottom parallel to the x-axis. Notice that the rectangles R_j are disjoint for different values of j. Also, the two diagonals of R_j intersect at p_j and cut R_j into four triangles. If $|p'_j - p_j| < \delta/2$ then p'_j lies in the interior of R_j and the piecewise affine map h that is the identity on the exterior of the union of the rectangles R_j and moves p_j to $p'_j = p_j + \epsilon$ is a quasiconformal. Moreover, it is easy to see that $K(h) \le 1 + C|\epsilon|$. □

Theorem 2. *The earthquake parametrization is a homeomorphism from T_n onto A_{n+2}.*

Proof. Let $S = \{p_1, \ldots, p_{n+3}\}$ be a fixed set of $n+3$ cyclically ordered, equally spaced points on the unit circle. A point in T_n is represented by another set of $n+3$ cyclically ordered points $S' = \{p'_1, \ldots, p'_{n+3}\}$. Two such sets S' and S'' represent the same point if and only if there is a Möbius transformation preserving the unit disc such that $A(p_j) = p'_j$ for $1 \leq j \leq n+3$.

The finite left earthquake parametrization yields a map $\Phi : A_{n+2} \to T_n$. By definition a vector $v \in A_{n+2}$ has a finite number of positive entries $v_1, \ldots, v_m$ with $m \leq n$ while all other entries are equal to zero. Each nonzero entry corresponds to a hyperbolic line l_j to which we assign the weight v_j. The totality of these lines comprise a finite lamination $\mathcal{L}$, and also by definition no two of these lines intersect and none of them join adjacent points of S. We have seen in the Introduction how the data consisting of these lines together with their weights produce a homeomorphism f of the unit circle, well-defined up to post composition by a Möbius transformation. We define $\Phi(v)$ to be the map f which takes S to $S' = f(S)$. By the finite earthquake theorem, Φ is a well-defined bijection from A_{n+2} onto T_n.

We claim that Φ is a homeomorphism. To show that Φ is continuous suppose that v is a vector in A_{n+2} and v' is another vector close to v. That means that every entry v'_j of v' is close to the corresponding entry of v_j of v in the sense that $|v'_j - v_j| < \epsilon$. Reorder the entries of v so that $v_1, \ldots, v_m$ are positive and the rest of its entries are equal to zero. Note that by hypothesis the lines l_j corresponding to v_j for $1 \leq j \leq m$ form the lines of a finite lamination $\mathcal{L}$ and if $0 < \epsilon < \min_{1 \leq j \leq m} v_j$, then v'_j is also positive and, therefore, the lines of the lamination $\mathcal{L}'$ for $\Phi(v')$ include all of the lines of $\mathcal{L}$. There may be other lines of $\mathcal{L}'$, but since $|v'_k - v_k| < \epsilon$ for all k, the values of v'_k for $k > m$ must lie between 0 and ϵ. If we start the construction of the earthquake for v and for v' on the same stratum of $\mathcal{L}$, it is now clear that the two constructed maps h and h' will be nearly equal on S, and this shows that Φ is continuous.

We must show that Φ^{-1} is continuous, that is, that the earthquake parametrization maps T_n to A_{n+2} continuously. Let $[f]$ and $[g]$ be two points in T_n, and $(\sigma_f, \mathcal{L}_f)$ and $(\sigma_g, \mathcal{L}_g)$ be the corresponding earthquake measures and laminations for f and g. By Lemma 2 if $[f]$ and $[g]$ are close in the Teichmüller metric then the cross-ratio distortions of f and g on any quadruple of points contained in S are close. So we assume that the difference of cross-ratio distortions of f and g on every quadruple in S is less than ϵ.

Claim 1: If a geodesic l_f in $\mathcal{L}_f$ crosses a geodesic l_g in $\mathcal{L}_g$, then both the weights λ_f of l_f in σ_f and λ_g of l_g in σ_g are less than ϵ.

Denote the endpoints of l_f by a and c and the endpoints of l_g by b and d and label these endpoints so that the quadruple $Q = \{a, b, c, d\}$ is arranged counterclockwise on the unit circle. Let $cr_d^f(Q)$ and $cr_d^g(Q)$ be the cross-ratio distortions on the quadruple Q under f and g respectively. Under the earthquake map for f, the weights on all other geodesics in $\mathcal{L}_f$ may move b or d further to the left before the movement corresponding to λ_f. Therefore the cross-ratio distortion $cr_d^f(Q)$ is greater than or equal to the cross-ratio distortion of Q under the earthquake map E_l^f with only one geodesic l_l and weight λ_f. But that the cross-ratio distortion of Q under E_l^f is equal to λ_f. Therefore $cr_d^f(Q) \geq \lambda_f$. Similarly, $cr_d^g(\{b,c,d,a\}) \geq \lambda_g$ and so $cr_d^g(Q) = -cr_d^g(\{b,c,d,a\}) \leq -\lambda_g$. Therefore,

$$\epsilon > |cr_d^f(Q) - cr_d^g(Q)| \geq \lambda_f + \lambda_g.$$

Hence $\lambda_f < \epsilon$ and $\mathcal{L}_g < \epsilon$.

Claim 2: If a geodesic l_f is in $\mathcal{L}_f$ but not in $\mathcal{L}_g$ and $\mathcal{L}_g$ has no geodesic crossing l_f, then $\lambda_f < \epsilon$; vice versa, it is also true.

Denote the endpoints of l_f by a and c. Then in S, there must be a point b on one side of l_f and another point d on the other side of l_f such that the four points a, b, c, d lie in the Euclidean closure of a single stratum of $\mathcal{L}_g$. Now let Q be the quadruple consisting of a, b, c, d labelled counterclockwise on the unit circle (exchange the labels of b and d if necessary.) Then $cr_d^f(Q) \geq \lambda_f$ and $cr_d^g(Q) = 0$. Therefore

$$\epsilon > |cr_d^f(Q) - cr_d^g(Q)| \geq \lambda_f.$$

Claim 3: If a geodesic l is contained in both $\mathcal{L}_f$ and $\mathcal{L}_g$, then $|\lambda_f - \lambda_g| < \epsilon$.

The proof of this claim is similar to claim 2 except that now $cr_d^g(Q) = \lambda_g$. Thus

$$\epsilon > |cr_d^f(Q) - cr_d^g(Q)| \geq |\lambda_f - \lambda_g|.$$

Claims 1, 2 and 3 together imply that σ_f and σ_g have distance at most $2n\epsilon$ in the metric on A_{n+2}. Thus the earthquake parametrization maps T_n to A_{n+2} continuously. □

4 The structures of the faces of the associahedron

In this section we study the Teichmüller structures of the interiors of the faces of the asociahedron. More precisely, we show:

Theorem 3. *For every lamination L of P_{n+3}, the interior of the face $F(L)$ is isomorphic to a Teichmüller space or a geometric product of Teichmüller spaces. In particular, $F(\emptyset)$ is isomorphic to T_{n+3} and for any maximal lamination L consisting of n lines in Λ_n, $F(L)$ is a single point.*

Proof. Let U_j be the closure of j-th component of the complement of L in $\overline{\mathbb{D}}$. Each U_j contains a subset S_j of 3 or more points of S on its boundary, where $1 \leq j \leq m$. Let $P(j)$ be the Euclidean convex hull of S_j. Let $A(S, L, \lambda)$ be the set of the elements of A_{n+2} that the coordinates of the diagonals in L are equal to the entries of a fixed vector $\lambda = (\lambda_1, \dots, \lambda_k)$ consisting of non-zero weights. Now let $A(S_j)$ be the set of the elements in A_{n+2} that the coordinates of only those diagonals of $P(j)$ are allowed to be non-zero. Then $A(S, L, \lambda)$ is isomorphic to the direction sum of $A(S_j)$'s, that is,

$$A(S, L, \lambda) \cong A(S_1) \oplus A(S_2) \oplus \cdots \oplus A(S_m).$$

By the finite earthquake theorem and changing hyperbolic geodesics to the Euclidean segments connecting the same endpoints, the points of each $T(S_j)$ is parameterized by those vectors v in $A(S_j)$. Therefore, $A(S, L, \lambda)$ is isomorphic to the product space of $T(S_j)$'s, that is,

$$A(S, L, \lambda) \cong T(S_1) \times T(S_2) \times \cdots \times T(S_m).$$

Note that when L is empty, $A(S, L, \lambda) \cong T(S)$ and when L is maximal, that is, when it contains n lines, then $T(L, \lambda)$ reduces to a single point. By letting all of the weights $\lambda_1, \dots, \lambda_k$ simultaneously approach ∞ we obtain

$$F(L) \cong T(S_1) \times T(S_2) \times \cdots \times T(S_m).$$

□

5 The realization of $\overline{A}_{n+2}$ as a polytope

We choose an increasing homeomorphism from $\overline{I} = [0, \infty]$ onto $J = [0, 1]$. Then there is an induced homeomorphism H from $\overline{I}^N$ onto J^N and we let $\overline{B}_{n+2} = H(\overline{A}_{n+2})$. We will show that there is a homeomorphism from $\overline{B}_{n+2}$ onto a closed unit ball in $\mathbb{R}^n$. More precisely, we obtain

Theorem 4. *There is a a piecewise affine homeomorphism h from $\overline{B}_{n+2}$ onto an n-dimensional simple convex polytope K_{n+2} (commonly called the associahedron of dimension n). The homeomorphism h carries $H \circ F(\emptyset)$ onto the interior of K_{n+2} and for any lamination L of P_{n+3}, it carries $H \circ F(L)$ onto the interior of a face of K_{n+2} of codimension k, where k is the number of the diagonals in L.*

Proof. To proceed with the prooof we must first summarize the algorithm presented by Devadoss in [3] and [4] used to construct K_{n+2}. K_{n+2} is obtained by truncating an n-dimensional simplex by codimension-1 half-planes. We will see the above theorem follows as a natural consequence by cutting K_{n+2} into a union of cubes and labelling the faces of K_{n+2} with the finite subsets of non-intersecting diagonals of P_{n+3}. These subsets correspond to the full and partial associations of $n + 2$ factors written in order.

Let there be given a set of congruent regular $(n+3)$-gons with sides labelled with the symbols $\{x_1, \dots, x_{n+2}, \infty\}$ in cyclic counterclockwise order and each with different diagonal marked in. Also let $\mathcal{G}_n$ be the set of these polygons. We

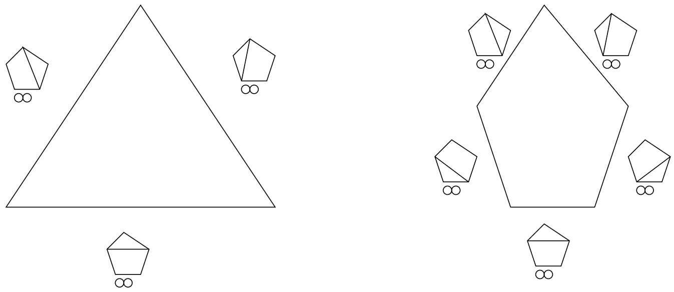

Figure 5: The construction of C_4 through truncation.

will say that two elements G_1 and G_2 in $\mathcal{G}_n$ satisfy the *nonintersecting condition* if the superposition of G_1 onto G_2 by a congruence respecting the labeling of the sides does not have intersecting diagonals.

The diagonal of an element G of $\mathcal{G}_n$ divides the $(n+3)$-gon into two parts. The part that does not contain the side marked with ∞ is called the *free part* of G. Let $\mathcal{G}_n^i$ be the collection of the elements in $\mathcal{G}_n$ that have i sides on their free parts. It is easy to see that the order of $\mathcal{G}_n^i$ is $n+3-i$, where $1 < i < n+2$. In particular, the order of $\mathcal{G}_n^2$ is $n+1$, which is the number of sides (codimension one faces) of an n-dimensional simplex W_n. Arbitrarily label each face of W_n by an element of $\mathcal{G}_n^2$.

Notice that the label on some adjacent faces of W_n do not satisfy the nonintersecting condition. This is an obstruction to the simplex satisfying the assciahedron condition. In order to overcome this obstruction, it is natural to truncate the vertices, bring in more faces and introduce new labels. In fact, this is how it is carried out in [3], and it is organized in an inductive way as follows.

Step I: Truncating two vertices.

Check all vertices (codimension n faces) of W_n and find the two vertices that satisfy the following condition. For each of these two there exists an element G in $\mathcal{G}_n^{n+3-2}$ such that G satisfies the nonintersecting condition with the labels of all faces adjacent to that vertex. Now truncate those two vertices off W_n by two codimension 1 half-planes and label the two new faces (two simplices of dimenson $n-1$) by the corresponding two elements in $\mathcal{G}_n^{n+3-2.}$ Then the labels of the new faces and their adjacent ones satisfy the nonintersecting condition. We continue to call the resulting simple convex polytope W_n. Figure 4 shows how K_4 is constructed.

Step II: Truncating some edges if $n \geq 3$.

Check the edges (codimension $n-1$ faces) of W_n and find those edges that satisfy the following condition. For each of them there exists an element G in $\mathcal{G}_n^{n+3-3}$ such that G satisfies the nonintersecting condition with the labels of all faces adjacent to that edge. Now truncate those edges W_n by codimension 1 half-planes and label the new faces by the corresponding elements of $\mathcal{G}^{n+3-3}$. Then

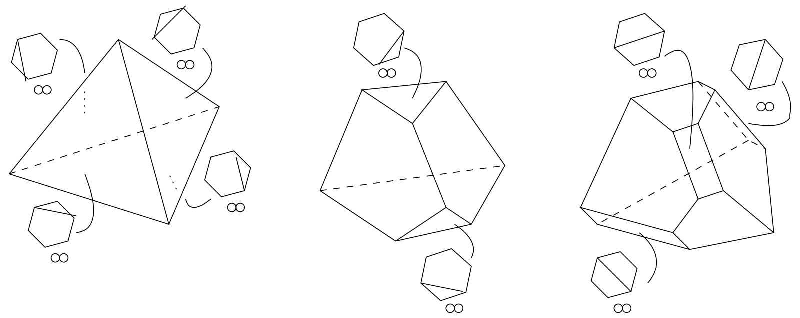

Figure 6: The construction of C_5 through truncation.

the labels of the new faces and their adjacent faces satisfy the nonintersecting condition. We continue to call the resulting simple convex polytope W_n. Figure 5 shows how K_5 is constructed.

Step III: If $n \geq 4$, inductively find the codimension i faces that the labels of their adjacent codimension 1 faces do not satisfy the nonintersecting condition, truncate them by codimension 1 half-planes, and finally label the new codimension 1 faces by the corresponding elements in $\mathcal{G}_n^{n+3-i}$ for $i = 4, 5, \cdots, n$. The resulting simple convex polytope is K_{n+2}.

In the course of truncating W_n to construct K_{n+2}, the truncations of the faces of dimension i add $i+2$ new faces to the polytope and each is labelled by an element in $\mathcal{G}_n^{i+2}$, where $i = 0, 1, \cdots, n-2$. Therefore, K_{n+2} has the number of its codimension 1 faces equal to $\sum_{i=0}^{n-2} |\mathcal{G}_n^{n+1-i}|$, which matches with the number of codimension 1 faces of K_{n+2}.

Now we are ready to begin the proof of Theorem 4. Identify each diagonal of Λ_n with the regular $(n+3)$-gon with that diagonal, that is an element of $\mathcal{G}_n$. Then each maximal lamination L corresponds to the regular $(n+3)$-gon with n non-intersecting diagonals, which labels a vertex of the associahedron K_{n+2} and is equal to the union of the labels of its adjacent codimension-1 faces. Now we view all vertices of K_{n+2} as the centers of the faces dimension 0 and for each 1-dimensional face take any interior point as its center. Then we inductively select the centers for the faces of K_{n+2} of higher and higher dimension. For each 2-dimensional face of K_{n+2}, we take as its center an interior point in the convex hull of the centers of the edges of that face; for each 3-dimensional face of K_{n+2}, we take as its center an interior point of the convex hull of the centers of all 2-dimensional faces of those 3-dimensional faces. We continue inductively until we obtain a center for K_{n+2} itself. In fact, we also use the regular $(n+3)$-gon with nonintersecting diagonals to label these centers as we have labelled the vertices and codimension-1 faces. See Figure 2 for the labels of the centers of K_4 by the associations on four factors and Figure 3 for the labels of the centers of some faces of K_5.

Now given the label V of a vertex of K_{n+2}, denote by 2^V the collection of the labels of the faces of K_{n+2} whose diagonals form a sub-collection of the diagonals of the label of V, and denote by L the corresponding maximal lamination for V. Let Δ be the convex hull of the set consisting of the centers of the faces with labels in 2^V and let Γ be the image under H of the closure of the cube generated by the vectors with non-negative coordinates for the diagonals in L and zero coordinates for the diagonals not in L. Then there exists an affine homeomorphism from Γ onto Δ, which maps the vertices of Γ to the vertices of Δ respectively according to their labels and have linear extensions among the corresponding simplices. All these affine homeomorphisms are pasted together to provide a piecewise affine homeomorphism between $\overline{B}_{n+2}$ and K_{n+2}. Furthermore, this piecewise affine homeomorphism maps the faces of $\overline{B}_{n+2}$ onto the respective faces of K_{n+2}, according to their labels. This completes the proof of the theorem. □

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