TOPOLOGICAL CHARACTERIZATION OF AN ASYMPTOTIC TEICHMÜLLER SPACE THROUGH MEASURED GEODESIC LAMINATIONS

JINHUA FAN* AND JUN HU**

ABSTRACT. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and centered at the origin, and let $\mathcal{ML}_b(\mathbb{D})$ be the collection of Thurston bounded measured geodesic laminations on $\mathbb{D}$. We introduce an equivalent relation on $\mathcal{ML}_b(\mathbb{D})$ such that the earthquake measure map induces a bijection between the asymptotic Teichmüller space $AT(\mathbb{D})$ and the quotient space $\mathcal{AML}_b(\mathbb{D})$ of $\mathcal{ML}_b(\mathbb{D})$ under the equivalent relation. Furthermore, we introduce a topology on $\mathcal{AML}_b(\mathbb{D})$ under which the bijection is a homeomorphism between $AT(\mathbb{D})$ and $\mathcal{AML}_b(\mathbb{D})$ with respect to the Teichmüller metric on $AT(\mathbb{D})$. Corresponding results are also developed for a bijection and then a homeomorphism between the tangent space $\mathcal{AZ}(\mathbb{S}^1)$ of $AT(\mathbb{D})$ at a base point and $\mathcal{AML}_b(\mathbb{D})$ with respect to the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ and the defined topology on $\mathcal{AML}_b(\mathbb{D})$.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane and centered at the origin, and let $T(\mathbb{D})$ be the universal Teichmüller space and $\mathcal{ML}_b(\mathbb{D})$ the collection of Thurston bounded measured geodesic laminations on $\mathbb{D}$. By Thurston's earthquake theory [21], for each quasisymmetric homeomorphism h of the unit circle $\mathbb{S}^1$, there is a unique Thurston bounded measured geodesic lamination λ induced by any earthquake representation E^λ of h. This correspondence introduces a bijection between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$, which is called *the earthquake measure map* and denoted by

$$\mathcal{EM} : T(\mathbb{D}) \to \mathcal{ML}_b(\mathbb{D}) : [h] \mapsto \lambda. \tag{1.1}$$

For background on earthquake representations of quasisymmetric maps, Thurston bounded measured geodesic laminations, and their relationships, and for further developments in Thurston's earthquake theory, we refer to [6], [9], [12], [15], [17], [19] and [20].

Furthermore, for any closed hyperbolic Riemann surface S, Kerckhoff [15] showed that the earthquake measure map $\mathcal{EM}$ (1.1) induces a homeomorphism between the Teichmüller space $T(S)$ of S and the space $\mathcal{ML}_b(S)$ of measured geodesic laminations on S with respect to the Teichmüller metric on $T(S)$ and a weak* topology on $\mathcal{ML}_b(S)$.

Lately, by introducing a uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$ (see Definition 3 in Section 2.5), Miyachi and Šarić [16] proved that the earthquake measure map $\mathcal{EM}$ is a homeomorphism between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ with respect to the Teichmüller metric on $T(\mathbb{D})$ and the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$. They also pointed out that since the invariance of a quasisymmetric map under a Fuchsian group implies the invariance of the corresponding measured lamination under the group, the same result holds for the restriction of $\mathcal{EM}$ between the Teichmüller space $T(S)$ of any geometrically infinite Riemann surface S and its representation $\mathcal{ML}_b(S)$ by Thurston bounded measured geodesic laminations on S.

Let $\mathcal{Z}(\mathbb{S}^1)$ be the quotient space of Zygmund bounded continuous tangent vector fields on $\mathbb{S}^1$ modulo quadratic polynomials, which is the tangent space of the universal Teichmüller $T(\mathbb{D})$ space at a base point. Gardiner proved in [5] that given any element $V \in \mathcal{Z}(\mathbb{S}^1)$, there is a unique element $\lambda \in \mathcal{ML}_b(\mathbb{D})$ such that any infinitesimal earthquake map $\dot{E}^\lambda$ (see Section 7.1) satisfying $V = \dot{E}^\lambda|_{\mathbb{S}^1}$ modulo a quadratic polynomial. Therefore, it introduces a bijection between $\mathcal{Z}(\mathbb{S}^1)$ and $\mathcal{ML}_b(\mathbb{D})$, which is called *the infinitesimal earthquake measure map* and denoted by

$$\mathcal{E\dot{M}} : \mathcal{Z}(\mathbb{S}^1) \to \mathcal{ML}_b(\mathbb{D}) : V \mapsto \lambda. \tag{1.2}$$

It is also proved in [16] that $\mathcal{E\dot{M}}$ is a homeomorphism with respect to the cross-ratio norm topology on $\mathcal{Z}(\mathbb{S}^1)$ (see Section 7.1) and the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$.

Let $T_0(\mathbb{D})$ be the subspace of $T(\mathbb{D})$ whose elements are represented by asymptotically conformal homeomorphisms of $\mathbb{D}$. The quotient space $T_0(\mathbb{D})\backslash T(\mathbb{D})$ is called the asymptotic Teichmüller space of quasiconformal homeomorphisms of $\mathbb{D}$, denoted by $AT(\mathbb{D})$. This space was studied by Gardiner and Sullivan in [8] and the asymptotic Teichmüller spaces of Riemann surfaces are studied in [2], [3], [4], [7] and etc.

The main work of this paper consists of the followings.

(1) Introduce an equivalent relation $\sim$ on $\mathcal{ML}_b(\mathbb{D})$ such that the earthquake measure map $\mathcal{EM}$ (1.1) induces a bijection $\widehat{\mathcal{EM}}$ between the asymptotic Teichmüller space $AT(\mathbb{D})$ and the quotient space $\mathcal{ML}_b(\mathbb{D})/\sim$.

(2) Prove that the bijection $\widehat{\mathcal{EM}}$ in (1) is a homeomorphism between $AT(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})/\sim$ with respect to the asymptotic Teichmüller metric on $AT(\mathbb{D})$ and a newly defined asymptotically uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})/\sim$.

(3) Show that the infinitesimal earthquake measure map $\mathcal{E\dot{M}}$ (1.2) induces a bijection and then a homeomorphism $\widehat{\mathcal{E\dot{M}}}$ between the tangent space $\mathcal{AZ}(\mathbb{S}^1)$ (see Section 7.3) of $AT(\mathbb{D})$ at a base point and $\mathcal{ML}_b(\mathbb{D})/\sim$ with respect to the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ and the asymptotic uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})/\sim$.

Given a Thurston bounded measured geodesic lamination λ, we denote by E^λ an earthquake map inducing λ (see Section 2.3 for background) and by $[\lambda]$ the equivalent class of λ (see Definition 4 in Section 3).

Theorem 1. *Given two points $[[h]]$ and $[[h']]$ in $AT(\mathbb{D})$, assume that $h = E^\lambda|_{\mathbb{S}^1}$ and $h' = E^{\lambda'}|_{\mathbb{S}^1}$. Then $[[h]] = [[h']]$ if and only if $[\lambda] = [\lambda']$.*

Now we let $\mathcal{AML}_b(\mathbb{D})$ be the quotient space of $\mathcal{ML}_b(\mathbb{D})$ under the equivalent relation. Theorem 1 implies that the earthquake measure map $\mathcal{EM}$ between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ induces a bijection between $AT(\mathbb{D})$ and $\mathcal{AML}_b(\mathbb{D})$, which is called

the induced earthquake measure map and denoted by

$$\widehat{\mathcal{EM}} : AT(\mathbb{D}) \to \mathcal{AML}_b(\mathbb{D}) : [[h]] \mapsto [\lambda], \qquad (1.3)$$

where $h = E^\lambda|_{\mathbb{S}^1}$.

Two metrics are commonly introduced on the asymptotic Teichmüller space $AT(\mathbb{D})$. As a quotient space, $AT(\mathbb{D})$ inherits a quotient metric from the Teichmüller metric on $T(\mathbb{D})$. Another metric is defined by using boundary dilatations. From [3] and [7], it is known that these two metrics are equal to each other. For this reason, they are simply called the asymptotic Teichmüller metric on $AT(\mathbb{D})$ and the topology induced by this metric is called the asymptotic Teichmüller topology on $AT(\mathbb{D})$.

Following the previous pattern in thinking, $\mathcal{AML}_b(\mathbb{D})$ first inherits a quotient topology from the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$ (see Definition 5 in Section 4). With respect to the quotient topologies on $AT(\mathbb{D})$ and $\mathcal{AML}_b(\mathbb{D})$, one can easily see that Theorem 1 and the homeomorphic property of $\mathcal{EM}$ between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ [16] imply the following corollary.

Corollary 1. *The induced earthquake measure map $\widehat{\mathcal{EM}}$ (1.3) is a homeomorphism with respect to the quotient topology on $AT(\mathbb{D})$ from the Teichmüller topology on $T(\mathbb{D})$ and the quotient topology on $\mathcal{AML}_b(\mathbb{D})$ from the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$.*

Corresponding to the other boundary-dilatation definition of the asymptotical Teichmüller topology on $AT(\mathbb{D})$, we introduce another topology, namely the asymptotic uniform weak* topology, on $\mathcal{AML}_b(\mathbb{D})$ (see Definition 8 in Section 6) and show the following theorem.

Theorem 2. *The induced earthquake measure map $\widehat{\mathcal{EM}}$ (1.3) is a homeomorphism with respect to the asymptotically Teichmüller topology on $AT(\mathbb{D})$ and the asymptotic uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.*

As a consequence of Theorem 2 and Corollary 1, we obtain the following.

Corollary 2. *The quotient topology on $\mathcal{AML}_b(\mathbb{D}))$ from the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D}))$ is equivalent to the asymptotically uniform weak* topology.*

Let $\mathcal{AZ}(\mathbb{S}^1)$ be the tangent space of the asymptotic Teichmüller space $AT(\mathbb{D})$ at a base point (see Section 7.3).

Theorem 3. *Given two points $[V]$ and $[V']$ in $\mathcal{AZ}(\mathbb{S}^1)$, assume that $V = \dot{E}^\lambda|_{\mathbb{S}^1}$ and $V' = \dot{E}^{\lambda'}|_{\mathbb{S}^1}$. Then $[V] = [V']$ if and only if $[\lambda] = [\lambda']$.*

It follows that the infinitesimal earthquake measure map $\mathcal{E\dot{M}}$ (1.2) induces a bijection between $\mathcal{AZ}(\mathbb{S}^1)$ and $\mathcal{AML}_b(\mathbb{D})$, which is called *the induced infinitesimal earthquake measure map* and denoted by

$$\widehat{\mathcal{E\dot{M}}} : \mathcal{AZ}(\mathbb{S}^1) \to \mathcal{AML}_b(\mathbb{D}) : [V] \mapsto [\lambda], \qquad (1.4)$$

where $V = \dot{E}^\lambda|_{\mathbb{S}^1}$.

Under the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ (see Section 7.6), we prove the following theorem.

Theorem 4. *The induced infinitesimal earthquake measure map $\widehat{\mathcal{E\dot{M}}}$ (1.4) is a homeomorphism with respect to the asymptotic cross-ratio norm topology on $\mathcal{AZ}(\mathbb{S}^1)$ and the asymptotic uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.*

In the course of developing a proof of Theorem 2, we introduce, and then study their relationship, the asymptotic Thurston norm $||\lambda||_{\widehat{Th}}$ (see Definition 7 in Section 5) of a Thurston bounded measured geodesic lamination λ and the strong asymptotic cross-ratio distortion norm $||h||_{\widehat{cr}}$ (see Definition 6 in Section 5) of a quasisymmetric circle homeomorphism h. We obtain the following theorem.

Theorem 5. *Let h be a quasisymmetric homeomorphism of $\mathbb{S}^1$ and λ_h the measured geodesic lamination induced by an earthquake representation of h. There exists a universal constant $C > 0$ such that*

$$\|h\|_{\widehat{cr}} \leq C\|\lambda_h\|_{\widehat{Th}}. \tag{1.5}$$

In Section 6, the previous theorem is used to prove the continuity of the inverse of the induced earthquake measure map $\widehat{\mathcal{EM}}$ in Theorem 2.

To prove Theorem 4, we apply the following relationship between the asymptotic cross-ratio norm $||V||_{\widehat{cr}}$ (see Definition 11 in Section 7.5) of a Zygmund bounded continuous tangent vector field V on $\mathbb{S}^1$ and the asymptotic Thurston norm $||\lambda_V||_{\widehat{Th}}$ of the measured geodesic lamination λ_V induced by any infinitesimal earthquake representation of V (see Section 7.1).

Theorem 6. *There exists a universal constant $C > 0$ such that*

$$\|\lambda_V\|_{\widehat{Th}} \leq C\|V\|_{\widehat{cr}} \tag{1.6}$$

for each $V \in \mathcal{Z}(\mathbb{S}^1)$.

Remark 1. The Teichmüller space $T(S)$ of any hyperbolic Riemann surface S is embedded into the universal Teichmüller $T(\mathbb{D})$ and $\mathcal{ML}_b(S)$ is embedded into $\mathcal{ML}_b(\mathbb{D})$. Therefore, one can see that the homeomorphic property of the earthquake measure map $\widehat{\mathcal{EM}}$ between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$ continues to hold on the restriction of $\widehat{\mathcal{EM}}$ between $T(S)$ and $\mathcal{ML}_b(S)$. Unfortunately, the asymptotic Teichmüller space $AT(S)$ of a Riemann surface S of infinite type can no longer be embedded as a subspace of the asymptotic Teichmüller space $AT(\mathbb{D})$. Therefore, one can not claim immediately that after the work of this paper, there is a similar topological characterization of $AT(S)$ in terms of a quotient space of $\mathcal{ML}_b(S)$. To obtain such a result for a Riemann surface S of infinite type, different strategies and more techniques need to be developed.

The paper is arranged as follows. Some background and definitions are given in Section 2. Then we prove Theorem 1 in Section 3, Corollary 1 in Section 4, Theorem 5 in Section 5, and finally Theorem 2 in Section 6. In the seventh and last section, we show Theorems 3, 4 and 6.

2. Preliminaries

2.1. **Teichmüller space and asymptotic Teichmüller space.** Let $\mathbb{D}$ be the open unit disk in the complex plane and centered at the origin, $\mathbb{S}^1 = \partial\mathbb{D}$, and let QS be the set of all quasisymmetric homeomorphisms of $\mathbb{S}^1$. The universal Teichmüller space $T(\mathbb{D})$ is the quotient space $T(\mathbb{D}) = M\ddot{o}b(\mathbb{D})\backslash QS$, where $M\ddot{o}b(\mathbb{D})$ is the group of all Möbius transformation preserving $\mathbb{D}$ and it acts on QS through

post-compositions. Given any $h \in QS$, we denote by $[h]$ the corresponding point in $T(\mathbb{D})$. The Teichmüller metric on $T(\mathbb{D})$ is defined as

$$d_T([h_1], [h_2]) = \frac{1}{2} \log \inf_{f|_{\mathbb{S}^1} = h_2 \circ h_1^{-1}} K(f),$$

where f is a quasiconformal homeomorphism of $\mathbb{D}$ and $K(f)$ is the maximal dilatation of f.

Let S_0 be the collection of all symmetric homeomorphisms of $\mathbb{S}^1$. Clearly, $M\ddot{o}b(\mathbb{D}) \subset S_0$. The quotient space $S_0 \backslash QS$ is called the asymptotic Teichmüller space on $\mathbb{D}$, denoted by $AT(\mathbb{D})$. By letting $T_0(\mathbb{D})$ be the subspace of $T(\mathbb{D})$ whose elements are represented by asymptotic conformal mappings on $\mathbb{D}$, the asymptotic Teichmüller space can also be expressed as

$$AT(\mathbb{D}) = T_0(\mathbb{D}) \backslash T(\mathbb{D}).$$

Given any $h \in QS$, we denote by $[[h]]$ the corresponding point in $AT(\mathbb{D})$. As a quotient space of $T(\mathbb{D})$, the quotient metric on $AT(\mathbb{D})$ is defined as

$$\widehat{d}_{AT}([[h_1]], [[h_2]]) = \inf d_T([\tilde{h}_1], [\tilde{h}_2]), \qquad (2.1)$$

where the infimum is taken over all $\tilde{h}_1 \in [[h_1]]$ and $\tilde{h}_2 \in [[h_2]]$. Using boundary dilatation, one can define another metric on $AT(\mathbb{D})$ by

$$d_{AT}([[h_1]], [[h_2]]) = \frac{1}{2} \log \inf_{f|_{\mathbb{S}^1} = h_2 \circ h_1^{-1}} \inf_E K(f|_{\mathbb{D} \backslash E}), \qquad (2.2)$$

where f is a quasiconformal homeomorphism of $\mathbb{D}$ and the first infimum from the left is taken over all compact subsets E of $\mathbb{D}$. It is known in [3] and [7] that

$$d_{AT} = \widehat{d}_{AT},$$

which are called the *asymptotic Teichmüller metric* on $AT(\mathbb{D})$.

2.2. Measured laminations on $\mathbb{D}$.
A complete oriented geodesic g on $\mathbb{D}$ is uniquely determined by an ordered pair of two distinct endpoints, the initial and the terminal points of g. Thus the space of all oriented geodesics on $\mathbb{D}$ is naturally identified with $\mathbb{S}^1 \times \mathbb{S}^1 \backslash diag$, where $diag$ is the diagonal set of the product space $\mathbb{S}^1 \times \mathbb{S}^1$. Let $\mathcal{G}$ be the set of all un-oriented complete hyperbolic geodesic on $\mathbb{D}$, then $\mathcal{G} = (\mathbb{S}^1 \times \mathbb{S}^1 \backslash diag)/ \sim$, where the equivalence is defined by $(a, b) \sim (b, a)$. We denote by $\lceil a, b \rceil$ be the equivalence class of $(a, b) \in \mathbb{S}^1 \times \mathbb{S}^1 \backslash diag$. Note that the topology on $\mathcal{G}$ is the induced topology from $\mathbb{S}^1 \times \mathbb{S}^1 \backslash diag$.

A geodesic lamination $\mathcal{L}$ is a collection of disjoint un-oriented complete geodesics which foliates a closed subset of $\mathbb{D}$. Equivalently, a geodesic lamination $\mathcal{L}$ can be identified as a closed subset of $\mathcal{G}$ such that any two geodesics presented by two different elements in $\mathcal{L}$ don't intersect in $\mathbb{D}$ (they may share one common endpoint). Each complete geodesic in $\mathcal{L}$ is called a leaf of $\mathcal{L}$. A stratum of $\mathcal{L}$ is either a geodesic of $\mathcal{L}$ or a component of the complement of $\mathcal{L}$ in $\mathbb{D}$.

By a measured geodesic lamination $(\mathcal{L}, \lambda)$ we mean a nonnegative, locally finite, Borel measure on the space $\mathcal{G}$ with support equal to $\mathcal{L}$. We often briefly say that λ is a measured lamination with support $|\lambda|$. Each measured lamination induces a transverse measure along the support $|\lambda|$. Given any hyperbolic geodesic segment I of length ≤ 1, the measure $\lambda(I)$ is equal to $\lambda(I \cap |\lambda|)$.

2.3. **Earthquakes and earthquake measures.** Earthquake maps in the hyperbolic plane $\mathbb{D}$ (and on any hyperbolic Riemann surface) were introduced by Thurston [21]. Let $\mathcal{L}$ be a geodesic lamination on $\mathbb{D}$. An earthquake E along a geodesic lamination $\mathcal{L}$ is an injective and surjective map $E : \mathbb{D} \to \mathbb{D}$ satisfying

(1) the restriction of E on each stratum A of $\mathcal{L}$ is the restriction of a Möbius transformation, which maps $\mathbb{D}$ onto $\mathbb{D}$, on A, and,

(2) for any two strata A and B, the comparison isometry

$$cmp(A, B) = (E|_A)^{-1} \circ E|_B : \mathbb{D} \to \mathbb{D}$$

is a hyperbolic translation whose axis weakly separates A and B, and which translates B to the left as viewed from A.

An earthquake E on $\mathbb{D}$ continuously extends to a homeomorphism of the boundary $\mathbb{S}^1$ ([21]), we denote by $E|_{\mathbb{S}^1}$ the restriction of the extension to $\mathbb{S}^1$. The converse statement is so-called Thurston's theorem [21], which says that for any orientation-preserving homeomorphism h of $\mathbb{S}^1$, there is an earthquake map E^λ such that $h = E^\lambda|_{\mathbb{S}^1}$. We call E^λ an earthquake representation of h.

Each earthquake E along a lamination $\mathcal{L}$ induced a transverse measure to $\mathcal{L}$, which is called the earthquake measure λ induced by E. An earthquake measure corresponds to a measured geodesic lamination. Therefore, each earthquake map $(E, \mathcal{L})$ induced a measure geodesic lamination λ with $|\lambda| = \mathcal{L}$. It is also a fact that given a orientation-preserving homeomorphism h of $\mathbb{S}^1$, although the earthquake representation of h is not necessarily unique, the induced earthquake measure or measured lamination λ_h is unique. More precisely, two homeomorphisms h_1 and h_2 determine the same measured lamination λ if and only if $h_2 = \gamma \circ h_1$ for some $\gamma \in M\ddot{o}b(\mathbb{D})$. Therefore, Thurston's earthquake representation induces an injective map from the space of the right cosets of $M\ddot{o}b(\mathbb{D})$ in the group of orientation-preserving homeomorphisms of $\mathbb{S}^1$ into the space of measured laminations on $\mathbb{D}$ by associating each coset with the corresponding measured lamination.

For any measured lamination λ and $\gamma \in M\ddot{o}b(\mathbb{D})$, we denote by $\gamma^*\lambda$ a measured lamination, called the pull-back of λ by γ, which is supported on $\gamma^{-1}(|\lambda|)$ and with the transverse measure evaluated by $\lambda \circ \gamma$. For an orientation preserving homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ and the earthquake map $E^\lambda|_{\mathbb{S}^1} = h$, we have that $h \circ \gamma = E^{\gamma^*\lambda}|_{\mathbb{S}^1}$.

2.4. **Earthquake measure map.** A measured lamination λ is Thurston bounded if the Thurston's norm

$$\|\lambda\|_{Th} = \sup_I \lambda(I)$$

is finite, where I runs over all geodesic arcs in $\mathbb{D}$ with unit length. Let $\mathcal{ML}_b(\mathbb{D})$ be the set of bounded measured laminations on $\mathbb{D}$. The following theorem of Thurston is well known, for which we refer to [6], [9], and [17] for different proofs.

Theorem A. Let h be an orientation preserving homeomorphism of $\mathbb{S}^1$ and let E^λ be an earthquake on $\mathbb{D}$ such that $h = E^\lambda|_{\mathbb{S}^1}$. Then the earthquake measure λ is Thurston bounded if and only if h is quasisymmetric.

Because of Thurston's earthquake presentation of orientation-preserving homeomorphisms of $\mathbb{S}^1$ and Theorem A, a bijection between $T(\mathbb{D})$ and $\mathcal{ML}_b(\mathbb{D})$, called

the earthquake measure map in [16], is defined by

$$\mathcal{EM} : T(\mathbb{D}) \ni [h] \to \lambda \in \mathcal{ML}_b(\mathbb{D}),$$

where $h = E^\lambda|_{\mathbb{S}^1}$.

2.5. **Weak* topology and Uniform Weak* topology on $\mathcal{ML}_b(\mathbb{D})$.** A box of geodesics B in $\mathcal{G}$ is the quotient under the equivalence $\sim$ of the product $[a, b] \times [c, d]$ of two disjoint closed arcs in $\mathbb{S}^1$, where $[a, b]$ (resp. $[c, d]$) is the arc in $\mathbb{S}^1$ from a (resp. c) to b (resp. d) in counterclockwise order on $\mathbb{S}^1$. We will write somewhat incorrectly $B = [a, b] \times [c, d]$ instead of $B = ([a, b] \times [c, d]) / \sim$. In the following of this paper, we call $B^* = [-i, 1] \times [i, -1]$ the standard box.

Definition 1. *Let $B = ([a, b] \times [c, d]) / \sim$, the Liouville measure L is a non-trivial, Möbius group invariant Borel measure on $\mathcal{G}$ defined by*

$$L(B) = |\log cr_L(B)| = |\log |cr_L(\{a, b, c, d\})|| = |\log|\frac{(a-c)(b-d)}{(a-d)(b-c)}||,$$

where $cr_L(B) = cr_L(\{a, b, c, d\}) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$.

Remark 2. The cross ratio $cr_L(B) = cr_L(\{a, b, c, d\})$ of a box B or a quadruple $\{a, b, c, d\}$ is used in [16] and [18]. In [6] and [9], a different cross ratio of a box B or a quadruple $\{a, b, c, d\}$ is used, that is

$$cr(B) = cr(\{a, b, c, d\}) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Since we need to quote results from the papers by Saric and Miyachi and the ones by Gardiner, Hu and Lakic, we will use both of them in the paper. Fortunately,

$$cr_L(B) = cr_L(\{a, b, c, d\}) = 1 + cr(\{a, b, c, d\}) = 1 + cr(B).$$

Definition 2. *A sequence $\{\lambda_n\}_{n=1}^\infty$ of Borel measures on $\mathcal{G}$ converges in the weak* topology to a Borel measure λ if*

$$\lim_{n\to\infty} \int_{\mathcal{G}} f d\lambda_n = \int_{\mathcal{G}} f d\lambda$$

for any continuous function f on $\mathcal{G}$ with compact support.

For any box $B \in \mathcal{G}$ with $L(B) = \log 2$, in the rest of this paper, we will always use γ_B to stand for the element of $M\ddot{o}b(\mathbb{D})$ such that $\gamma_B(B^*) = B$. Meanwhile, for any measured geodesic lamination λ on $\mathbb{D}$, $(\gamma_B)^*\lambda$ stands for the pullback of λ by γ_B.

Definition 3. *A sequence $\{\lambda_n\}_{n=1}^\infty \subset \mathcal{ML}_b(\mathbb{D})$ converges to $\lambda \in \mathcal{ML}_b(\mathbb{D})$ in the uniform weak* topology if for any continuous function f on $\mathcal{G}$ with compact support $supp(f) \subset B^*$,*

$$\lim_{n\to\infty} \{\sup_B \int_{B^*} f d[(\gamma_B)^*\lambda_n - (\gamma_B)^*\lambda]\} = 0,$$

where the supremum is taken over all boxes B with $L(B) = \log 2$.

Theorem B. [20] *Let λ and λ_n, $n = 1, 2, 3, \cdots$, be uniformly bounded earthquake measures (i.e., $\|\lambda\|_{Th}, \|\lambda_n\|_{Th} \leq M < \infty$ for all n) in $\mathcal{ML}_b(\mathbb{D})$. If λ_n converges to λ in the weak* topology, then $E^{\lambda_n}|_{\mathbb{S}^1}$ converges to $E^\lambda|_{\mathbb{S}^1}$ pointwisely on $\mathbb{S}^1$ (i.e., for each $x \in \mathbb{S}^1$, $E^{\lambda_n}|_{\mathbb{S}^1}(x) \to E^{\lambda_n}|_{\mathbb{S}^1}(x)$ as $n \to \infty$) when the earthquakes $E^\lambda|_{\mathbb{S}^1}$ and $E^{\lambda_n}|_{\mathbb{S}^1}$, $n = 1, 2, 3, \cdots$, are properly normalized.*

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In this section, we prove Theorem 1.

For any box $B = [a, b] \times [c, d]$, we define the *minimal scale* $s(B)$ of B as

$$s(B) = \min\{|a - b|, |b - c|, |c - d|, |d - a|\}.$$

A sequence $\{B_n\}_{n=1}^{\infty} \subset \mathcal{G}$ of boxes is said to be *degenerating* if $L(B_n) = \log 2$ for all n and

$$\lim_{n \to \infty} s(B_n) = 0.$$

Definition 4. *Given two bounded measured laminations λ and λ' in $\mathcal{ML}_b(\mathbb{D})$, we say that λ is equivalent to λ' if*

$$\sup_{\{B_i\}} \limsup_{i \to \infty} \int_{B^*} f d((\gamma_{B_i})^* \lambda - (\gamma_{B_i})^* \lambda') = 0 \qquad (3.1)$$

for any continuous function f on $\mathcal{G}$ with compact support $supp(f) \subset B^*$, where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes. Denote by $[\lambda]$ the equivalence class of λ.

Proof of the necessity part of Theorem 1. Assume that $[[h]] = [[h']]$ and λ and λ' are the measured laminations determined by h and h' respectively. We need to show $[\lambda] = [\lambda']$. Suppose that this is not true. Then the condition (3.1) does not hold. Thus there exist a degenerating sequence $\{B_i\}_{i=1}^{\infty}$ of boxes and a continuous function f on $\mathcal{G}$ with compact support $supp(f) \subset B^*$ such that

$$\int_{B^*} f d((\gamma_{B_i})^* \lambda - (\gamma_{B_i})^* \lambda') > \delta \qquad (3.2)$$

for some positive constant δ and all positive integers i. Let

$$\lambda_i = (\gamma_{B_i})^* \lambda \text{ and } \lambda'_i = (\gamma_{B_i})^* \lambda'. \qquad (3.3)$$

Then $\{\|\lambda_i\|_{Th}\}_{i=1}^{\infty}$ and $\{\|\lambda'_i\|_{Th}\}_{i=1}^{\infty}$ are bounded because $\|\lambda_i\|_{Th} = \|\lambda\|_{Th}$ and $\|\lambda'_i\|_{Th} = \|\lambda'\|_{Th}$. It follows that there exist two subsequences $\{\lambda_{i_j}\}_{j=1}^{\infty} \subset \{\lambda_i\}_{i=1}^{\infty}$ and $\{\lambda'_{i_j}\}_{j=1}^{\infty} \subset \{\lambda'_i\}_{i=1}^{\infty}$ such that λ_{i_j} weakly converges to $\widehat{\lambda}$ and λ'_{i_j} weakly converges to $\widehat{\lambda'}$. Following (3.2), $\widehat{\lambda} \neq \widehat{\lambda'}$. For simplicity of notation, we rename the subsequences to be $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\lambda'_i\}_{i=1}^{\infty}$.

Let $A_i, A'_i \in M\ddot{o}b(\mathbb{D})$ such that $h_i = A_i \circ E^{\lambda_i}|_{\mathbb{S}^1}$ and $h'_i = A'_i \circ E^{\lambda'_i}|_{\mathbb{S}^1}$ are normalized to fix $1, i$ and -1. Assume also that $E^{\widehat{\lambda}}$ and $E^{\widehat{\lambda'}}$ are normalized to fix $1, i$ and -1. Then by Theorem B,

$$h_i = A_i \circ E^{\lambda_i}|_{\mathbb{S}^1} \to E^{\widehat{\lambda}}|_{\mathbb{S}^1} = \widehat{h} \text{ and } h'_i = A'_i \circ E^{\lambda'_i}|_{\mathbb{S}^1} \to E^{\widehat{\lambda'}}|_{\mathbb{S}^1} = \widehat{h'} \qquad (3.4)$$

pointwisely on $\mathbb{S}^1$. Since $\widehat{\lambda} \neq \widehat{\lambda'}$, it follows that

$$\widehat{h} \circ (\widehat{h'})^{-1} \notin M\ddot{o}b(\mathbb{D}). \qquad (3.5)$$

Let $ex(\cdot)$ be the Douady-Earle extension operator. Then it follows from (3.5) that

$$ex(\widehat{h}) \circ (ex(\widehat{h'}))^{-1} \notin M\ddot{o}b(\mathbb{D}). \qquad (3.6)$$

Since $[[h]] = [[h']]$, using Theorem 4 in [4], $ex(h) \circ (ex(h'))^{-1}$ is asymptotically conformal. Thus given any $\epsilon > 0$, there exists a compact subset K of $\mathbb{D}$ such that

$$\|Belt(ex(h))|_{\mathbb{D}-K}) - Belt(ex(h'))|_{\mathbb{D}-K})\|_{\infty} \leq \epsilon. \qquad (3.7)$$

Since h_i (resp. h_i') differs from h (resp. h') only by precomposition and post-composition by Möbius transformations, the conformal naturality of Douady-Earle extensions implies that $ex(h_i)$ and $ex(h_i')$ are quasiconformal mappings with maximal dilatations as the same as the extensions of h and h' respectively. Because $ex(h_i)$ and $ex(h_i')$ are normalized to fix three points 1, i and -1 on $\mathbb{S}^1$, passing to subsequences we may further assume that $ex(h_i)$ and $ex(h_i')$ converges to quasiconformal homeomorphisms of $\mathbb{D}$ uniformly on the closure $\overline{\mathbb{D}}$. Since we have already known that h_i and h_i' converges to $\widehat{h}$ and $\widehat{h}'$ pointwisely on $\mathbb{S}^1$, it follows that they converge to the limit functions uniformly on $\mathbb{S}^1$. Using the properties of Douady-Earle extensions in [1], we obtain

$$Belt(ex(h_i)) \to Belt(ex(\widehat{h})) \text{ and } Belt(ex(h_i')) \to Belt(ex(\widehat{h}')) \qquad (3.8)$$

uniformly on any compact subset of $\mathbb{D}$. Again by the conformal naturality of Douady-Earle extensions,

$$ex(h_i) = A_i \circ ex(h) \circ \gamma_{B_i}, \ ex(h_i') = A_i' \circ ex(h') \circ \gamma_{B_i}. \qquad (3.9)$$

By taking Beltrami coefficients of the left and of the right side of (3.9), we get that

$$Belt(ex(h_i)) = Belt(ex(h)) \circ \gamma_{B_i} \frac{\overline{\partial \gamma_{B_i}/\partial z}}{\partial \gamma_{B_i}/\partial z},$$

$$Belt(ex(h_i')) = Belt(ex(h')) \circ \gamma_{B_i} \frac{\overline{\partial \gamma_{B_i}/\partial z}}{\partial \gamma_{B_i}/\partial z}. \qquad (3.10)$$

Then

$$Belt(ex(h_i)) - Belt(ex(h_i'))$$

$$= [Belt(ex(h)) \circ \gamma_{B_i} - Belt(ex(h')) \circ \gamma_{B_i}] \frac{\overline{\partial \gamma_{B_i}/\partial z}}{\partial \gamma_{B_i}/\partial z}. \qquad (3.11)$$

Recall that γ_{B_i} is the Möbius transformation mapping the standard box B^* to B_i and $s(B_i)$ converges to 0 as $i \to \infty$. Then γ_{B_i} maps the origin to the intersection point O_i of the diagonals of B_i and O_i approaches to $\mathbb{S}^1$ as $i \to \infty$. It follows that for any compact K of $\mathbb{D}$, $\gamma_{B_i}(K)$ converges to $\mathbb{S}^1$ as $i \to \infty$. Thus $Belt(ex(h_i)) - Belt(ex(h_i'))$ converges to 0 as $i \to \infty$ uniformly on any compact subset of $\mathbb{D}$. Combining this property with (3.8), we obtain

$$Belt(ex(\widehat{h})) - Belt(ex(\widehat{h}')) = 0.$$

Hence

$$ex(\widehat{h}) \circ (ex(\widehat{h}'))^{-1} \in M\ddot{o}b(\mathbb{D}). \qquad (3.12)$$

This is a contradiction to (3.6). Therefore, $[\lambda] = [\lambda']$.   □

The following two lemmas are developed to prove the sufficiency of Theorem 1.

Lemma 1. *Suppose that* $\{\gamma_{B_i}\}_{i=1}^{\infty} \subset M\ddot{o}b(\mathbb{D})$ *and* $s(\gamma_{B_i}(B^*)) \to 0$ *as* $i \to \infty$. *Then given any box* $B = [a, b] \times [c, d]$, $s(\gamma_{B_i}(B)) \to 0$ *as* $i \to \infty$.

Proof. Suppose on the contrary that there exists a box $B = [a, b] \times [c, d]$ such that

$$s(\gamma_{B_i}(B)) \nrightarrow 0$$

as $i \to \infty$. Then there is a subsequence $\{\gamma_{B_{i_j}}\}_{j=1}^{\infty}$ of $\{\gamma_{B_i}\}_{i=1}^{\infty}$ such that $\gamma_{B_{i_j}}(B) \to [a', b'] \times [c', d']$ as $j \to \infty$, where a', b', c' and d' are four distinct points on $\mathbb{S}^1$. For simplicity of notation, we rename the subsequence $\{\gamma_{B_{i_j}}\}_{j=1}^{\infty}$ to be $\{\gamma_{B_i}\}_{i=1}^{\infty}$. Let $\gamma_{B_i}(B^*) = [a_i, b_i] \times [c_i, d_i]$ and $\gamma_{B_i}(B) = [a_i', b_i'] \times [c_i', d_i']$. Since $s(\gamma_{B_i}(B^*)) \to 0$

as $i \to \infty$, passing to a subsequence we may assume that two of the four point $\{a_i, b_i, c_i, d_i\}$ converges to a point x on $\mathbb{S}^1$. Since $L(B_i) = \log 2$ for each i, it follows that there are at least three of $\{a_i, b_i, c_i, d_i\}$ converging to x. Passing to a subsequence one more time we may assume that a_i, b_i and c_i converge to x as $i \to \infty$. The fourth point d_i may converge to another point y on $\mathbb{S}^1$. There are two pints of $\{a', b', c', d'\}$ different from x and y, namely, a' and b'. Then the hyperbolic distance δ_i between the geodesic connecting a'_i and b'_i and the geodesic connecting a_i and b_i converges to ∞ as $i \to \infty$ since a'_i and b'_i converge to two different points a' and b' and a_i and b_i converge to the same point x. On the other hand, since the hyperbolic distance is preserved under Möbius transformation, it follows that δ_i is equal to the hyperbolic distance between the geodesic connecting a and b and the geodesic connecting $-i$ and 1, which is a constant. This is a contradiction. Therefore, the conclusion of the lemma follows. $\qquad\square$

Lemma 2. *Let h_1 and h_2 be two quasisymmetric homeomorphisms of $\mathbb{S}^1$. Then $h_1 \circ (h_2)^{-1}$ is symmetric provided that*

$$\sup_{\{B_n\}} \limsup_{n \to \infty} |L(h_1(B_n)) - L(h_2(B_n))| = 0, \qquad (3.13)$$

where the supremum is taken over all degenerating sequences $\{B_n\}_{n=1}^{\infty}$ of boxes.

Proof. Suppose on the contrary that $h_1 \circ (h_2)^{-1}$ is not symmetric. Let $ex(h_1)$ and $ex(h_2)$ be the Douady-Earle extensions of h_1 and h_2 respectively. Then $ex(h_1) \circ (ex(h_2))^{-1}$ is not asymptotic conformal on $\mathbb{D}$, which means that there exist a constant $\epsilon > 0$ and a sequence $\{D_n\}_{n=1}^{\infty}$ of hyperbolic disks in $\mathbb{D}$ of diameter 1 with the Euclidean distance from D_n to $\mathbb{S}^1$ approaching 0 as $n \to \infty$ such that

$$\|Belt(ex(h_1)|_{D_n}) - Belt(ex(h_2)|_{D_n})\|_{L^{\infty}} \geq \epsilon \qquad (3.14)$$

for all n. Let D_0 be the hyperbolic disk on $\mathbb{D}$ of diameter 1 and centered in 0, and assume that $\gamma_n \in M\ddot{o}b(\mathbb{D})$ and $\gamma_n(D_0) = D_n$. Let A_{1n} and $A_{2n} \in M\ddot{o}b(\mathbb{D})$ such that $A_{1n} \circ h_1 \circ \gamma_n$ and $A_{2n} \circ h_2 \circ \gamma_n$ fix $1, -1, i$ for all n. Using the assumption (3.13) and applying Lemma 1 to γ_n, given any box given box B with $L(B) = \log 2$, we obtain

$$\lim_{n \to \infty} |L(A_{1n} \circ h_1 \circ \gamma_n(B)) - L(A_{2n} \circ h_2 \circ \gamma_n(B))| = 0. \qquad (3.15)$$

Let $ex(A_{1n} \circ h_1 \circ \gamma_n)$ and $ex(A_{2n} \circ h_2 \circ \gamma_n)$ be the Douady-Earle extensions of $A_{1n} \circ h_1 \circ \gamma_n$ and $A_{2n} \circ h_2 \circ \gamma_n$ respectively. Since these quasiconformal mappings fix three common points and have constant maximal dilatations, passing to subsequences we may assume that $A_{1n} \circ h_1 \circ \gamma_n$ and $A_{2n} \circ h_2 \circ \gamma_n$ converge uniformly to quasisymmetrics $\widehat{h}_1$ and $\widehat{h}_2$ respectively. Using (3.15) and the convergence, we obtain

$$L(\widehat{h}_1(B)) = L(\widehat{h}_2(B))$$

for any box B with $L(B) = \log 2$. By the normalized condition at three points, we conclude that $\widehat{h}_1 = \widehat{h}_2$. By the convergence properties of Douady-Earle extensions, $Belt(ex(A_{1n} \circ h_1 \circ \gamma_n))$ and $Belt(ex(A_{2n} \circ h_2 \circ \gamma_n))$ converge to $Belt(ex(\widehat{h}_1))$ and $Belt(ex(\widehat{h}_2))$ uniformly on D_0; that is,

$$||Belt(ex(A_{1n} \circ h_1 \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2n} \circ h_2 \circ \gamma_n)|_{D_0})||_{L^{\infty}} \to 0 \qquad (3.16)$$

as $n \to \infty$. On the other hand, by the conformal naturality of Douady-Earle extensions and (3.14),

$$
\begin{aligned}
&||Belt(ex(A_{1n} \circ h_1 \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2n} \circ h_2 \circ \gamma_n)|_{D_0})||_{L^\infty} \\
=\ &||Belt(ex(h_1)|_{D_n}) - Belt(ex(h_2)|_{D_n})||_{L^\infty} \geq \epsilon > 0.
\end{aligned}
$$

This is a contradiction to (3.16), so $h_1 \circ (h_2)^{-1}$ is symmetric. □

Proof of the sufficiency part of Theorem 1. We prove $[[h]] = [[h']]$ if $[\lambda] = [\lambda']$. Suppose on the contrary that $[[h]] \neq [[h']]$, which means $h' \circ h^{-1}$ is not symmetric. Then the condition (3.13) in Lemma 2 does not hold. Thus there exists a degenerating sequence $\{B_i\}_{i=1}^\infty$ of boxes such that

$$|L(h(B_i)) - L(h'(B_i))| \geq \delta \tag{3.17}$$

for some positive constant δ and all positive integers i.

Let $\lambda_i = (\gamma_{B_i})^*\lambda$ and $\lambda_i' = (\gamma_{B_i})^*\lambda'$. We show that $\lambda_i - \lambda_i'$ converges to 0 in the weak* topology as $i \to \infty$. It suffices to show that for any box B with $L(B) = \log 2$ and any continuous function f on $\mathcal{G}$ with a compact support $supp(f) \subset B$,

$$\lim_{i \to \infty} \int_B f d(\lambda_i - \lambda_i') = 0. \tag{3.18}$$

Let $\gamma \in M\ddot{o}b(\mathbb{D})$ such that $B = \gamma(B^*)$. Then

$$
\begin{aligned}
\int_B f d(\lambda_i - \lambda_i') &= \int_{B^*} f \circ \gamma d(\gamma^*(\lambda_i) - \gamma^*(\lambda_i')) \\
&= \int_{B^*} f \circ \gamma d((\gamma_{B_i} \circ \gamma)^*\lambda - (\gamma_{B_i} \circ \gamma)^*\lambda').
\end{aligned}
\tag{3.19}
$$

By Lemma 1, we know $s(\gamma_{B_i} \circ \gamma(B^*)) \to 0$ as $i \to \infty$. Then the definition of $[\lambda] = [\lambda']$ implies that the last integral in the previous expression (3.19) converges to 0 as $i \to \infty$, which means (3.18) holds.

Since $\{\|\lambda_i\|_{Th}\}_{i=1}^\infty$ and $\{\|\lambda_i'\|_{Th}\}_{i=1}^\infty$ are uniformly bounded, passing to subsequences we may assume that they converge in the weak* topology. Then the two weak* limits are equal to each other. Now by applying Theorem B, there exist $\{A_i\}_{i=1}^\infty$ and $\{A_i'\}_{i=1}^\infty$ in $M\ddot{o}b(\mathbb{D})$ such that the two sequences $\{E^{\lambda_i}|_{S^1} = A_i \circ h \circ \gamma_{B_i}\}_{i=1}^\infty$ and $\{E^{\lambda_i'}|_{S^1} = A_i' \circ h' \circ \gamma_{B_i}\}_{i=1}^\infty$ converge to the same quasisymmetric map pointwisely on $\mathbb{S}^1$. Thus

$$\lim_{i \to \infty} |L(A_i \circ h \circ \gamma_{B_i}(B^*)) - L(A_i' \circ h' \circ \gamma_{B_i}(B^*))| = 0. \tag{3.20}$$

On the other hand,

$$|L(A_i \circ h \circ \gamma_{B_i}(B^*)) - L(A_i' \circ h' \circ \gamma_{B_i}(B^*))| = |L(h(B_i)) - L(h'(B_i))|.$$

It follows that (3.20) is a contradiction to (3.17). Thus our assumption $[[h]] \neq [[h']]$ is false. Therefore, $[[h]] = [[h']]$ if $[\lambda] = [\lambda']$. □

4. QUOTIENT UNIFORM WEAK* TOPOLOGY

In this section, we first give the definition of the quotient uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$; then we prove Corollary 1.

Definition 5. *A sequence $\{[\lambda_n]\}_{n=1}^{\infty} \subset \mathcal{AML}_b(\mathbb{D})$ converges to $[\lambda] \in \mathcal{AML}_b(\mathbb{D})$ in the quotient uniform weak* topology if for any continuous function f on $\mathcal{G}$ with compact support $supp(f) \subset B^*$,*

$$\inf_{\widehat{\lambda}_n \in [\lambda_n], \widehat{\lambda} \in [\lambda]} \sup_B | \int_{B^*} f d((\gamma_B)^* \widehat{\lambda}_n - (\gamma_B)^* \widehat{\lambda})| \to 0 \text{ as } n \to \infty,$$

where the supremum is taken over all boxes $B \in \mathcal{G}$ with $L(B) = \log 2$.

Proof of Corollary 1. We first show that the induced earthquake measure map $\widehat{\mathcal{EM}}$ is continuous. Let $[\lambda_n] = \widehat{\mathcal{EM}}([[h_n]])$ and $[\lambda] = \widehat{\mathcal{EM}}([[h]])$. If $d_{AT}([[h_n]], [[h]]) \to 0$ as $n \to \infty$, then there exist $h'_n \in [[h_n]]$ and $h' \in [[h]]$ such that $d_T([h'_n], [h']) \to 0$ as $n \to \infty$. Let $\lambda'_n = \mathcal{EM}([h'_n])$ and $\lambda' = \mathcal{EM}([h'])$. Theorem 1 implies that $\lambda' \in [\lambda]$ and $\lambda'_n \in [\lambda_n]$. Since $\mathcal{EM}$ is continuous (Theorem 1 in [16]), it follows that $\lambda'_n \to \lambda'$ in the uniform weak* topology on $\mathcal{ML}_b(\mathbb{D})$. Using Definition 5, we conclude that $[\lambda_n] \to [\lambda]$ in the quotient uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.

Now we show that $\widehat{\mathcal{EM}}^{-1}$ is continuous. Suppose $[\lambda_n] = \widehat{\mathcal{EM}}([[h_n]]) \to [\lambda] = \widehat{\mathcal{EM}}([[h]])$ in the quotient uniform weak* topology as $n \to \infty$. By definition, there exist $\lambda'_n \in [\lambda_n]$ and $\lambda' \in [\lambda]$ such that $\lambda'_n \to \lambda'$ in the uniform weak* topology as $n \to \infty$. Let $[h'_n] = \mathcal{EM}^{-1}(\lambda'_n)$ and $[h'] = \mathcal{EM}^{-1}(\lambda')$. Theorem 1 implies that $h'_n \in [[h_n]]$ and $h' \in [[h]]$. Since $\mathcal{EM}^{-1}$ is continuous (Theorem 1 in [16]), it follows that $d_T([h'_n], [h']) \to 0$ as $n \to \infty$. Thus $d_{AT}([[h_n]], [[h]]) \to 0$ as $n \to \infty$. $\square$

5. Asymptotic Thurston's norm and strong asymptotic cross-ratio distortion norm

Let Q be a quadruple consisting of four points a, b, c, d on the unit circle arranged in the counter-clockwise direction, denoted by $Q = \{a, b, c, d\}$. We use $cr(Q)$ to denote the following cross ratio:

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

For a quasisymmetric homeomorphism h of $\mathbb{S}^1$, $cr(h(Q))$ denotes

$$cr(h(Q)) = \frac{(h(b) - h(a))(h(d) - h(c))}{(h(c) - h(b))(h(d) - h(a))}.$$

In [6] and [9], the cross-ratio distortion norm $||h||_{cr}$ is defined to be

$$||h||_{cr} = \sup_{cr(Q)=1} |\log cr(h(Q))|,$$

where the supremum is taken over all quadruples Q with $cr(Q) = 1$.

Let λ_h be the measured lamination induced by an earthquake representation of h. It is shown in [6] that there is a universal positive constant C such that

$$||\lambda_h||_{Th} \leq C||h||_{cr}$$

for any quasisymmetric homeomorphism h of $\mathbb{S}^1$. Then the inverse is proved in [9]; that is

$$||h||_{cr} \leq C||\lambda_h||_{Th} \tag{5.1}$$

for a universal positive constant C. Therefore, the cross-ratio distortion norm and the Thurston norm are bi-Lipschitz. In this section, we also briefly denote λ_h by λ.

Definition 6. *The strong asymptotic cross-ratio distortion norm of a quasisymmetric homeomorphism h of $\mathbb{S}^1$ is defined as*

$$\|h\|_{\widehat{cr}} = \sup_{\{Q_i\}} \limsup_{i \to \infty} |\log cr(h(Q_i))|,$$

where the supremum is taken over all sequences $\{Q_i\}_{i=1}^{\infty}$ of quadruples such that $cr(Q_i) = 1$ for all i and $S_{max}(Q_i) \to 0$ as $i \to \infty$, and where $S_{max}(Q)$ is the maximum scale of Q; that is,

$$S_{max}(Q) = \max\{|a - b|, |b - c|, |c - d|, |d - a|\}.$$

Remark 3. Note first that in the previous definition of $\|h\|_{\widehat{cr}}$, we require the maximal scale $S_{max}(Q)$ of Q to approach 0. Note secondly that $\|h\|_{\widehat{cr}} \leq M$ if and only if for any arbitrary small positive ϵ, there exists $\delta > 0$ such that for any quadruple Q with $S_{max}(Q) < \delta$,

$$|\log cr(h(Q))| < M + \epsilon.$$

The proof for the "if" part is trivial; the "only if" part can be easily shown by proof by contradiction.

Definition 7. *The asymptotic Thurston norm $\|\lambda\|_{\widehat{Th}}$ of a bounded measured geodesic lamination λ is defined as*

$$\|\lambda\|_{\widehat{Th}} = \sup_{\{I_n\}} \limsup_{n \to \infty} \lambda(I_n),$$

where the supremum is taken over all sequences $\{I_n\}_{n=1}^{\infty}$ of closed geodesic segments in $\mathbb{D}$ of hyperbolic length 1 such that the Euclidean distance from I_n to $\mathbb{S}^1$ goes to 0 as $n \to \infty$.

Similarly, one can see that $\|\lambda\|_{\widehat{Th}} \leq M$ if and only if for any arbitrary small positive ϵ, there exists $\delta > 0$ such that for any geodesic segment I of hyperbolic length 1, if the Euclidean distance from I to $\mathbb{S}^1$ is less than δ, then $\lambda(I) < M + \epsilon$.

In this section, we prove Theorem 5. The strategy of the proof is as the same as the one used in [9] to prove the inequality $\|h\|_{cr} \leq C\|\lambda_h\|_{Th}$, but all considerations have to be arranged near the boundary of the open unit disk. For the completeness of the paper, we sketch the proof. In order to do so, we recall a few results showed in [6] and [9]. The following two lemmas, which are enumerated as Corollaries 1 and 2 in [6] with proofs, present monotone changes of the cross ratio of the image of a quadruple under such changed earthquake maps that leaves are moved in certain patterns without changing their measures.

Lemma 3 ([6]). *Let $Q = \{a, b, c, d\}$ be a quadruple on the real line with $-\infty \leq a < b < c < d$, $c \leq s \leq d$ and $d < t$. Suppose that $A_{(s,t)}$ is the hyperbolic Möbius transformation with repelling fixed point at s and attracting fixed point at t and derivative at the repelling fixed point equal to $\lambda > 1$. Suppose $f_{(s,t)} : \mathbb{R} \to \mathbb{R}$ is defined to be equal to $A_{(s,t)}$ on the interval $[s, t]$ and equal to the identity on the complement of $[s, t]$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ considered as a function of two variables $s \in [c, d]$ and $t \in (d, +\infty)$ decreases in s for each fixed t and increases in t for each fixed s.*

Lemma 4 ([6]). *With the same notation as in the previous lemma, suppose $b \leq s \leq c$ and $d \leq t$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ is increasing in s for each fixed t and also increasing in t for each fixed s.*

By using Lemma 3 and 4, the proof of the inequality (5.1) given in [9] is reduced to derive similar inequalities in three cases, which are summarized into Propositions 3, 4 and 5 there. In order to sketch a proof of our Theorem 5, we recall these three propositions.

Let h denote an orientation-preserving circle homeomorphism, $(E, \mathcal{L})$ an earthquake representation of h, and λ the induced earthquake measure by $(E, \mathcal{L})$. There are three universal positive constants C_0, C_1 and C_2, independent of f, λ and Q, such that the following three propositions hold.

Proposition 1 ([9]). *If $cr(Q) = 1$ and a, b, c belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of h, then*

$$0 \leq \ln cr(h(Q)) \leq C_1 C_2 ||\lambda||_{Th}.$$

Proposition 2 ([9]). *If $cr(Q) = 1$ and a, c belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of h, then*

$$0 \leq \ln cr(h(Q)) \leq 2C_1 C_2 ||\lambda||_{Th}.$$

Proposition 3 ([9]). *If $cr(Q) = 1$ and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ which separates a, b from c, d, then*

$$|\ln(cr(Q))| \leq (C_0 + 2C_1 C_2)||\lambda||_{Th}.$$

For a quadruple $Q = \{a, b, c, d\}$ of four points a, b, c, d on $\mathbb{S}^1$ arranged in the counter-clockwise direction, we notice that $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ between a and c is perpendicular to the one $\overline{bd}$ between b and d. Denote by e the intersection point between $\overline{ac}$ and $\overline{bd}$, and by $\overline{ea}$ (resp. $\overline{eb}$, $\overline{ec}$, $\overline{ed}$) the geodesic ray from e to a (resp. b, c, d).

Given two points x and y on the unit circle, we use $[x, y]$ (resp. (x, y), $[x, y)$, $(x, y]$) to denote the closed (resp. open, half open and half closed) arc on $\mathbb{S}^1$ from x to y in the counter-clockwise direction. Careful examinations of the proofs of the previous propositions in [9] enable us to state them in more elaborated ways as follows.

Corollary 3. *If $cr(Q) = 1$ and a, b, c belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of h, then*

$$0 \leq \ln cr(h(Q)) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ed}} \lambda|_{\overline{ed}}(I),$$

where $\lambda|_{\overline{ed}}$ is the restriction of λ on the collection of leaves of $\mathcal{L}$ connecting points in $[c, d)$ to points in $(d, a]$.

Corollary 4. *If $cr(Q) = 1$ and a, c belong to the same stratum of the earthquake representation $(E, \mathcal{L})$ of h, then*

$$0 \leq \ln cr(h(Q)) \leq 2C_1 C_2 \sup_{l(I)=1, I \subset \overline{bd}} \lambda|_{\overline{bd}}(I),$$

where $\lambda|_{\overline{bd}}$ is the restriction of λ on the collection of the leaves of $\mathcal{L}$ connecting points in $[a, c)$ to points in $[c, a)$.

Corollary 5. *If $cr(Q) = 1$ and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ which separates a, b from c, d, then*

$$-(C_0 + 2C_1 C_2) \max\{ \sup_{l(I)=1, I \subset \beta} \lambda|_{L_I}(I), \sup_{l(I)=1, I \subset \overline{ea}} \lambda|_{L_{II}}(I), \sup_{l(I)=1, I \subset \overline{ec}} \lambda|_{L_{IV}}(I)\} \leq$$
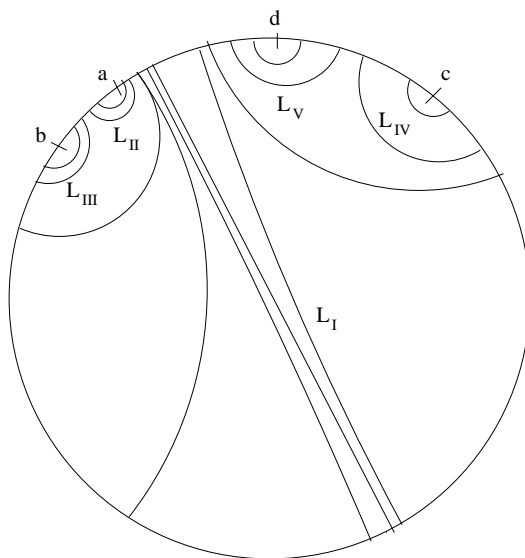
FIGURE 1. Five subcollections of the leaves of $\mathcal{L}$ described in Corollary 5.

$$\ln(cr(Q)) \leq$$

$$(C_0 + 2C_1C_2) \max\{ \sup_{l(I)=1, I \subset \beta} \lambda|_{L_I}(I), \sup_{l(I)=1, I \subset \overline{eb}} \lambda|_{L_{III}}(I), \sup_{l(I)=1, I \subset \overline{ed}} \lambda|_{L_V}(I)\},$$

where β is the common perpendicular segment between the geodesics $\overline{ab}$ and $\overline{cd}$, L_I is the collection of the geodesic lines in $\mathcal{L}$ that connect points of the arc $[d, a]$ to points of the arc $[b, c]$, L_{II} is the collection of the lines in $\mathcal{L}$ that connect points of the arc (d, a) to points of the arc (a, b), L_{III} is the collection of the lines in $\mathcal{L}$ that connect points of the arc (a, b) to points of the arc (b, c), L_{IV} is the collection of the lines in $\mathcal{L}$ that connect points of the arc (b, c) to points of the arc (c, d), and finally L_V is the collection of the lines in $\mathcal{L}$ that connect points of the arc (c, d) to points of the arc (d, a).

Now we prove our Theorem 5.

Proof. Let $M = ||\lambda||_{\widehat{Th}}$. Given any $\epsilon > 0$, there exists $0 < r < 1$ such that for any geodesic segment I contained in the annulus $W_r = \{z : r < |z| < 1\}$ with hyperbolic length 1, $\lambda(I) \leq M + \epsilon$. Then there exists $\delta > 0$ such that for any quadruple $Q = \{a, b, c, d\}$ with $S_{max}(Q) < \delta$, the geodesic connecting any two points in Q is contained in the annulus W_r and both the common perpendicular geodesic segment between the geodesics $\overline{ab}$ and $\overline{cd}$ and the one between $\overline{bc}$ and $\overline{da}$ are also contained in W_r.

Let $C = C_0 + 2C_1C_2$. We show that for any quadruple Q with $cr(Q) = 1$ and $S_{max}(Q) < \delta$,

$$|\ln cr(h(Q))| \leq C(M + \epsilon). \tag{5.1}$$

We divide the proof into three cases.

Case 1: The quadruple Q has three points belonging to the same stratum. There are four subcases: either a, b, c or b, c, d or c, d, a or d, a, b belong to the same stratum.

Corollary 3 implies in these four subcases respectively that either

$$0 \leq \ln cr(h(Q)) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ed}} \lambda|_{\overline{ed}}(I)$$

or

$$0 \leq \ln cr(h(\{b,c,d,a\})) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ea}} \lambda|_{\overline{ea}}(I),$$

$$0 \leq \ln cr(h(\{c,d,a,b\})) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{eb}} \lambda|_{\overline{eb}}(I),$$

$$0 \leq \ln cr(h(\{d,a,b,c\})) \leq C_1 C_2 \sup_{l(I)=1, I \subset \overline{ec}} \lambda|_{\overline{ec}}(I).$$

Since the values of the previous four supremos are less than or equal to $M + \epsilon$ and since

$$cr(h(\{b,c,d,a\})) = \frac{1}{cr(h(Q))}, \ cr(h(\{c,d,a,b\})) = cr(h(Q)) \text{ and}$$

$$cr(h(\{d,a,b,c\})) = \frac{1}{cr(h(Q))},$$

it follows that

$$|\ln cr(h(Q))| \leq C_1 C_2 (M + \epsilon) < C(M + \epsilon).$$

Case 2: The quadruple Q has two opposite points belonging to the same stratum. Then either a and c or b and d belong to the same stratum. By Corollary 4 and a similar fashion in reasoning as in Case 1, we obtain

$$|\ln cr(h(Q))| \leq 2C_1 C_2 (M + \epsilon) < C(M + \epsilon).$$

Case 3: The quadruple Q has no opposite points belonging to the same stratum. Then either there exists a geodesic line in $\mathcal{L}$ that separates a and b from c and d or there exists a geodesic line in $\mathcal{L}$ that separates b and c from d and a. By Corollary 5 and a similar fashion in reasoning as in Case 1, we obtain

$$|\ln cr(h(Q))| \leq (C_0 + 2C_1 C_2)(M + \epsilon) = (M + \epsilon).$$

Therefore no matter which case happens, we obtain that, for any sequence $\{Q_i\}_{i=1}^{\infty}$ of quadruples with $cr(Q_i) = 1$ for all i and $S_{max}(Q_i) \to 0$ as $i \to \infty$,

$$\limsup_{i \to \infty} |\ln cr(h(Q_i))| \leq C(M + \epsilon).$$

Thus

$$||h||_{\widehat{cr}} \leq C(M + \epsilon).$$

Since ϵ is an arbitrarily small positive, it follows that

$$||h||_{\widehat{cr}} \leq CM = C||\lambda||_{\widehat{Th}}.$$

$\square$

6. HOMEOMORPHIC PROPERTY OF THE INDUCED EARTHQUAKE MEASURE MAP

In this section, after giving the definition of asymptotically uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$ and some lemmas, we prove Theorem 2.

Definition 8. *A sequence $\{[\lambda_n]\}_{n=1}^\infty \subset \mathcal{AML}_b(\mathbb{D})$ converges to $[\lambda] \in \mathcal{AML}_b(\mathbb{D})$ in the asymptotically uniform weak* topology if for any continuous function f on $\mathcal{G}$ with compact support $supp(f) \subset B^*$,*

$$\sup_{\{B_i\}} \limsup_{i \to \infty} \left| \int_{B^*} f d((\gamma_{B_i})^*(\lambda_n) - (\gamma_{B_i})^*(\lambda)) \right| \to 0 \ as \ n \to \infty, \qquad (6.1)$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^\infty$ of boxes.

Remark 4. Theorem 1 implies that for each n, the value defined by the supremum of the lim sup's in (6.1) is independent of the choices of representatives for $[\lambda_n]$ and $[\lambda]$.

Theorem C. [14] *Let h be an orientation-preserving homeomorphism of $\mathbb{S}^1$ and $ex(h)$ be the Douady-Earle extension of h to the closed unit disk $\overline{\mathbb{D}}$. Let $p \in \mathbb{S}^1$ and I_p be an open arc on $\mathbb{S}^1$ containing p and symmetric with respect to p. If $\|h|_{I_p}\|_{cr} < \infty$, then there exists an open hyperbolic half plane U_p with p at the middle of its boundary on $\mathbb{S}^1$ such that $\log K(ex(h)|U_p) \le C_1\|h|_{I_p}\|_{cr} + C_2$ for two universal positive constants C_1 and C_2, where $K(ex(h)|U_p)$ is the maximal dilatation of $ex(h)$ on U_p.*

Remark 5. It is shown in [13] that there exists a universal constant $C > 0$ such that

$$\log K(ex(h)) \le C\|h\|_{cr}$$

for any orientation-preserving homeomorphism h of $\mathbb{S}^1$. As a corollary to this result or the previous Theorem C, $ex(h)$ is quasiconformal if $\|h\|_{cr}$ is finite.

Lemma 5. *There is a universal constant $C_0 > 0$ such that for any measured lamination $\lambda \in \mathcal{ML}_b(\mathbb{D})$,*

$$\frac{1}{C_0}\|\lambda\|_{\widehat{Th}} \le \sup_{\{B_i\}} \limsup_{i \to \infty} \lambda(B_i) \le \|\lambda\|_{\widehat{Th}}, \qquad (6.2)$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^\infty$ of boxes.

Proof. This lemma is the asymptotic version of Lemma 2.1 in [16] and the proof is similar. Let $B = [a, b] \times [c, d]$ be a box in $\mathcal{G}$. The measure $\lambda(B)$ is obtained as follows. Without loss of generality, we assume that a, b, c and d lie on $\mathbb{S}^1$ in the counterclockwise order. Let I be the common perpendicular geodesic segment between the hyperbolic geodesics $\overline{ad}$ and $\overline{bc}$. Then any geodesic contained in B intersects I. By definition, $\lambda(B)$ is equal to $\lambda(I \cap |\lambda|_B)$, where $|\lambda|_B$ denote the collection of all the geodesics of $|\lambda|$ contained in B. If $L(B) = 2$, then I is contained in a geodesic segment I_0 of hyperbolic length 1. Furthermore, $s(B)$ goes to 0 if and only if Euclidean distance between I_0 and $\mathbb{S}^1$ goes to 0. Thus, by definition,

$$\sup_{\{B_i\}} \limsup_{i \to \infty} \lambda(B_i) \le \|\lambda\|_{\widehat{Th}}.$$

Now we show the other half of the double inequality. We recall a fact used in the proof of Lemma 2.1 in [16]; that is, there is a universal constant $L_0 > 0$ such that for any geodesic segment I of hyperbolic length 1 transversely intersecting a

leaf l of λ, if J is the geodesic segment of length L_0 orthogonally intersecting l at $I \cap l$ with the intersecting point at the midpoint of J, then any leaf of λ intersecting I must intersect J. A more general version of this fact can be stated as follows. For any positive number c, there exists a positive number $L_0(c)$ such that for any geodesic segment I of hyperbolic length c transversely intersecting a leaf l of λ, if J is the geodesic segment of length $L_0(c)$ and orthogonally intersecting l at $I \cap l$ with the intersecting at the midpoint of J, then any leaf of λ intersecting I must intersect J; furthermore, $L_0(c)$ approaches 0 as c goes to 0.

Let γ be a Möbius transformation from $\mathbb{H}$ onto $\mathbb{D}$ such that $\gamma^{-1}(J) = [1, e^{L_0(c)}]i$ and $\gamma^{-1}(l) = \{z : |z| = e^{L_0(c)/2}\} \cap \mathbb{H}$. Consider the box

$$B_0 = [-e^{3L_0(c)/2}, -e^{-L_0(c)/2}] \times [e^{-L_0(c)/2}, e^{3L_0(c)/2}].$$

Through an elementary calculation, we can see that if a leave of the pullback $\gamma^*(\lambda)$ of λ under γ intersects $\gamma^{-1}(J)$, then it is contained in the box B_0. Let $B(I) = \gamma(B_0)$. Then any leave of λ intersecting J is contained in $B(I)$. It follows that for any geodesic segment I of hyperbolic length c,

$$\lambda(I) \leq \lambda(J) \leq \lambda(B(I)). \tag{6.3}$$

One can also see that the Euclidean distance from I to $\partial\mathbb{D}$ goes to 0 if and only if the Euclidean distance from J to $\partial\mathbb{D}$ goes to 0. Furthermore, from the constructions of J and $B(I)$, it is also true that the Euclidean distance from J to $\partial\mathbb{D}$ goes to 0 if and only if $s(B(I))$ goes to 0. Now by setting c at a value c_0 such that

$$L(B_0) = 2\log\cosh(L_0(c_0)/2) = \log 2.$$

Following (6.3), we conclude that for generating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes,

$$\|\lambda\|_{\widehat{Th}} \leq ([\frac{1}{c_0}] + 1) \sup_{\{B_i\}} \limsup_{i\to\infty} \lambda(B_i),$$

where $[\frac{1}{c_0}]$ stands for the integral part of $\frac{1}{c_0}$. We complete the proof. $\qquad\square$

Lemma 6. *If a sequence $\{[\lambda_n]\}_{n=1}^{\infty} \subset \mathcal{AML}_b(\mathbb{D})$ converges to $[\lambda] \in \mathcal{AML}_b(\mathbb{D})$ in the asymptotically uniform weak* topology, then there is a sequence $\{[\lambda_n']\}_{n=1}^{\infty} \subset \mathcal{AML}_b(\mathbb{D})$ such that $[\lambda_n'] = [\lambda_n]$ for all n and $\{\|\lambda_n'\|_{Th}\}_{n=1}^{\infty}$ is bounded.*

Proof. We show first that if $\{[\lambda_n]\}_{n=1}^{\infty} \subset \mathcal{AML}_b(\mathbb{D})$ converges to $[\lambda] \in \mathcal{AML}_b(\mathbb{D})$ in the asymptotically uniform weak* topology, then $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded.

Let us follow the same notation and reasoning given in the last two paragraphs of the proof of the previous lemma. Now we set c at a value c_0' such that

$$L(B_0) = 2\log\cosh(L_0(c_0')/2) = \frac{1}{3}\log 2.$$

Then for any measured lamination λ and any geodesic segment I of hyperbolic length c_0', there is a box $B(I)$ of Liouville measure $\frac{1}{3}\log 2$ such that every leave of λ intersecting I is contained in $B(I)$.

Now suppose that $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is not bounded. Then there exists a sequence $\{I_n\}_{n=1}^{\infty}$ of geodesic segments of hyperbolic length c_0' approaching to $\partial\mathbb{D}$ such that $\{\lambda_n(I_n)\}_{n=1}^{\infty}$ is not bounded. Passing to a subsequence, we may assume that

$$\lambda_n(I_n) \to \infty \text{ as } n \to \infty.$$

For each n, $L(B(I_n)) = \frac{1}{3}\log 2$. Then $B(I_n)$ sits in the middle of a larger box of Liouville measure $\log 2$, denote it by B_n.

Let γ_{B_n} be the Möbius transformation mapping the standard box B^* to B_n. Let B_m^* be the box sitting in the middle of B^* with Liouville measure $\frac{1}{3}\log 2$. Now let f be a continuous real function between 0 and 1 with $supp(f) \subset B^*$ and taking value 1 on B_m^*. Then

$$\int_{B^*} f d((\gamma_{B_n})^* \lambda_n) = \int_{B_n} f \circ (\gamma_{B_n})^{-1} d\lambda_n \geq \lambda_n(B(I_n)) \geq \lambda_n(I_n) \to \infty$$

as $n \to \infty$. On the other hand,

$$\int_{B^*} f d((\gamma_{B_n})^* \lambda) \leq \lambda(B_n).$$

Using Lemma 2.1 of [16], we know $\lambda(B_n) \leq ||\lambda||_{Th}$; or using the previous lemma, we know

$$\limsup_{n \to \infty} \lambda(B_n) \leq ||\lambda||_{\widehat{Th}}.$$

Therefore, $\lambda(B_n)$ has to be bounded. The above two estimates make a contradiction to the assumption that $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. Therefore, $\{||\lambda_n||_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded.

Let $h_n = E^{\lambda_n}|_{\mathbb{S}^1}$ and $ex(h_n)$ be the Douady-Earle extension. Since $\{||\lambda_n||_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded, Theorem 5 and Theorem C together imply that the boundary dilatation $H(ex(h_n)) = \inf_E K(ex(h_n)|_{\mathbb{D}\setminus E})$ is bounded by a positive constant independent of n, where the infimum is taken over all compact subsets E of $\mathbb{D}$. Then there exists $M' > 0$ such that $d_{AT}([[h_n]], [[0]]) < M'$ for all n. Thus for each n, there exists $h_n' \in [[h_n]]$ such that $d_T([h_n'], [0]) < M'$. Let $\lambda_n' = \mathcal{EM}([h_n'])$, then $\lambda_n' \in [\lambda_n]$. It follows that $[\lambda_n'] = [\lambda_n]$ for each n and $\{||\lambda_n'||_{Th}\}_{n=1}^{\infty}$ is a bounded sequence. We complete the proof.   □

Similar to the proof of Lemma 2, one can show the following lemma.

Lemma 7. *Let $[[h]] \in AT(\mathbb{D})$ and $\{[[h_n]]\}_{n=1}^{\infty}$ be a sequence of points in $AT(\mathbb{D})$. Then $[[h_n]]$ converges to $[[h]]$ in the asymptotic Teichmüller topology on $AT(\mathbb{D})$ provided that*

$$\sup_{\{B_i\}} \limsup_{i \to \infty} |L(h_n(B_i)) - L(h(B_i))| \to 0 \quad as \; n \to \infty, \tag{6.4}$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes.

Proof. Suppose that $[[h_n]]$ does not converge to $[[h]]$ in the asymptotic Teichmüller topology on $AT(\mathbb{D})$. Let $ex(h_n)$ and $ex(h)$ be the Douady-Earle extensions of h_n and h respectively, where $n \in \mathbb{N}$. From the definition given by (2.2), it follows that the boundary dilatation of $ex(h_n) \circ (ex(h))^{-1}$ does not converge to 0 as $n \to \infty$. Passing to a subsequence, we may assume that the boundary dilatation of $ex(h_n) \circ (ex(h))^{-1}$ is greater than a positive number ϵ for all n. Thus there exists a sequence of hyperbolic disks $\{D_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ of diameter 1 with the Euclidean distance from D_n to $\mathbb{S}^1$ approaching 0 as $n \to \infty$ such that

$$||Belt(ex(h_n)|_{D_n}) - Belt(ex(h)|_{D_n})||_{L^\infty} \geq \epsilon, \tag{6.5}$$

for all n. Let D_0 be the hyperbolic disk on $\mathbb{D}$ of diameter 1 and centered in 0, and assume that $\gamma_n \in M\ddot{o}b(\mathbb{D})$ and $\gamma_n(D_0) = D_n$. Let A_{1n} and $A_{2n} \in M\ddot{o}b(\mathbb{D})$ such that $A_{1n} \circ h_n \circ \gamma_n$ and $A_{2n} \circ h \circ \gamma_n$ fix $1, -1, i$ for all n. Using the assumption (6.4) and applying Lemma 1 to γ_n, for any box B with $L(B) = \log 2$, we obtain

$$\lim_{n \to \infty} |L(A_{1n} \circ h_n \circ \gamma_n(B)) - L(A_{2n} \circ h \circ \gamma_n(B))| = 0. \tag{6.6}$$

Using Remark 2, one can show that the condition (6.4) implies that there exists a constant $M > 1$ such that for each n,

$$\frac{1}{M} \leq \sup_{\{B_i\}} \limsup_{i \to \infty} cr(h_n(B_i)) \leq M,$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes. Using the definition of $\|h\|_{\widehat{cr}}$, we obtain for each n,

$$\|h_n\|_{\widehat{cr}} \leq \sup_{\{B_i\}} \limsup_{i \to \infty} |\log cr(h_n(B_i))| \leq \log M,$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes. Theorem C implies that the sequence $\{H(ex(h_n))\}_{n=1}^{\infty}$ of the boundary dilatation of $ex(h_n)$ is bounded, which means that $H(ex(h_n) < M'$ for each n and some constant $M' > 1$. For each n, there exists $0 < r_n < 1$ such that the maximal dilatation $K(ex(h_n))(z) < M'$ for any z with $r_n < |z| < 1$. Let μ_n be defined as

$$\mu_n(z) = \begin{cases} Belt(ex(h_n))(z) & \text{if } r_n \leq |z| < 1, \\ 0 & \text{if } |z| < r_n. \end{cases}$$

For each n, let f_n be the normalized (i.e., 1, -1 and i are fixed) quasiconformal homeomorphism of $\mathbb{D}$ with the Beltrami coefficient μ_n, and let $\widetilde{h}_n = f_n|_{\mathbb{S}^1}$. Clearly, $\widetilde{h}_n \in [[h_n]]$ and $K(f_n) < M'$ for each n. Using one of the main theorems in [1], $\{K(ex(\widetilde{h}_n))\}_{n=1}^{\infty}$ is bounded. Therefore, we conclude that one can replace the representatives of $[[h_n]]$'s such that $\{K(ex(h_n))\}_{n=1}^{\infty}$ is bounded. For simplicity of notation, we continue to denote these representatives by h_n's.

Let $ex(A_{1n} \circ h_n \circ \gamma_n)$ and $ex(A_{2n} \circ h \circ \gamma_n)$ be the Douady-Earle extensions of $A_{1n} \circ h_n \circ \gamma_n$ and $A_{2n} \circ h \circ \gamma_n$ respectively. These quasiconformal mappings fix three common points. Using the conformal naturality of Douady-Earle extensions, their maximal dilatations are uniformly bounded. Passing to subsequences, we may assume that $A_{1n} \circ h_n \circ \gamma_n$ and $A_{2n} \circ h \circ \gamma_n$ converge uniformly to quasisymmetric homeomorphisms $\widehat{\widetilde{h}}$ and $\widehat{h}$ on $\mathbb{S}^1$ respectively. Using (6.4) and the convergence, we obtain

$$L(\widehat{\widetilde{h}}(B)) = L(\widehat{h}(B))$$

for any box B with $L(B) = \log 2$. Thus, $\widehat{\widetilde{h}} = \widehat{h}$. Again using the convergence properties of Douady-Earle extensions, $Belt(ex(A_{1n} \circ h_n \circ \gamma_n))$ and $Belt(ex(A_{2n} \circ h \circ \gamma_n))$ converge to $Belt(ex(\widehat{\widetilde{h}}))$ and $Belt(ex(\widehat{h}))$ uniformly on D_0; that is,

$$\|Belt(ex(A_{1n} \circ h_n \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2n} \circ h \circ \gamma_n)|_{D_0})\|_{L^{\infty}} \to 0 \qquad (6.7)$$

as $n \to \infty$. On the other hand, by the conformal naturality of Douady-Earle extensions,

$$\begin{aligned} &\|Belt(ex(A_{1n} \circ h_n \circ \gamma_n)|_{D_0}) - Belt(ex(A_{2n} \circ h \circ \gamma_n)|_{D_0})\|_{L^{\infty}} \\ = \ &\|Belt(ex(h_n)|_{D_n}) - Belt(ex(h)|_{D_n})\|_{L^{\infty}} \geq \epsilon. \end{aligned}$$

This is a contradiction to (6.7). Therefore, the conclusion of the lemma has to hold. □

Now we prove Theorem 2.

Proof of Theorem 2. We first show that the induced earthquake measure map $\widehat{\mathcal{EM}}$ is continuous. Assume that $d_{AT}([[h_n]], [[h]]) \to 0$ as $n \to \infty$. Let $[\lambda_n] = \widehat{\mathcal{EM}}([h_n])$ and $[\lambda] = \widehat{\mathcal{EM}}([h])$. We need to prove $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. Since $d_{AT}([[h_n]], [[h]]) \to 0$, there exist $h'_n \in [[h_n]]$ and $h' \in [[h]]$ such that $d_T([h'_n], [h']) \to 0$ as $n \to \infty$. Let $\lambda'_n = \mathcal{EM}([h'_n])$ and $\lambda' = \mathcal{EM}([h'])$. By Theorem 1, $\lambda'_n \in [\lambda_n]$ and $\lambda' = [\lambda]$. Using the continuity of $\mathcal{EM}$, we know that λ'_n converges to λ' in the uniform weak* topology. Then for any continuous function f on $\mathcal{G}$ with compact support $\text{supp}(f) \subset B^*$ and any box B with the Liouville measure $L(B) = \log 2$,

$$\lim_{n \to \infty} [\sup_B | \int_{B^*} f d((\gamma_B)^* \lambda'_n - (\gamma_B)^* \lambda')|] = 0.$$

Clearly, for any degenerating sequences $\{B_i\}_{i=1}^\infty$ of boxes,

$$\sup_{\{B_i\}} \limsup_{i \to \infty} | \int_{B^*} f d((\gamma_i)^*(\lambda_n) - (\gamma_{B_i})^*(\lambda))| \le \sup_B | \int_{B^*} f d((\gamma_B)^* \lambda'_n - (\gamma_B)^* \lambda')|.$$

Therefore

$$\sup_{\{B_i\}} \limsup_{i \to \infty} | \int_{B^*} f d((\gamma_{B_i})^*(\lambda_n) - (\gamma_{B_i})^*(\lambda))| \to 0 \ (n \to \infty).$$

By Definition 8, $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. Therefore $\widehat{\mathcal{EM}}$ is continuous.

Next we show that the inverse $\widehat{\mathcal{EM}}^{-1}$ is continuous. Suppose not, then there exists a sequence $\{[\lambda_n]\}_{n=1}^\infty$ of points in $\mathcal{AML}_b(\mathbb{D})$ such that $[\lambda_n]$ converges to a point $[\lambda]$ of $\mathcal{AML}_b(\mathbb{D})$ in the asymptotic uniform weak* topology but $[[h_n]] = \widehat{\mathcal{EM}}^{-1}([\lambda_n])$ does not converge to $[[h]] = \widehat{\mathcal{EM}}^{-1}([\lambda])$ in the asymptotic Teichmüller topology on $\mathcal{AT}(\mathbb{D})$.

Using Lemma 6 to replace representatives if necessary, we may assume that $\{\|\lambda_n\|_{Th}\}_{n=1}^\infty$ is bounded.

Since $[[h_n]]$ does not converge to $[[h]]$ in the asymptotic Teichmüller topology, applying Lemma 7 and passing to a subsequence, we may assume that there exist $\epsilon > 0$ and a degenerating sequences $\{B_n\}_{n=1}^\infty$ of boxes such that

$$\lim_{n \to \infty} |L(h(B_n)) - L(h_n(B_n))| \ge \epsilon. \tag{6.8}$$

Now we show that if $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology, then $\{(\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda\}_{n=1}^\infty$ converges to 0 in the weak* topology. It suffices to show that for any box B with $L(B) = \log 2$ and any continuous function f on $\mathcal{G}$ with a compact support $supp(f) \subset B$,

$$\lim_{n \to \infty} \int_B f d((\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda) = 0.$$

Let $\gamma \in M\ddot{o}b(\mathbb{D})$ such that $B = \gamma(B^*)$. Then

$$
\begin{aligned}
\int_B f d((\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda) &= \int_{B'} f \circ \gamma d(\gamma^*((\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda)) \\
&= \int_{B^*} f \circ \gamma d(\gamma^*((\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda)) \\
&= \int_{B^*} f \circ \gamma d((\gamma_{B_n} \circ \gamma)^* \lambda_n - (\gamma_{B_n} \circ \gamma)^* \lambda)).
\end{aligned}
$$

By Lemma 1, we know $s(\gamma_{B_n} \circ \gamma(B^*)) \to 0$ as $n \to \infty$. Then the definition of $[\lambda_n]$ converging to $[\lambda]$ in the asymptotically uniform weak* implies that the last integral in the previous expression converges to 0 as $n \to \infty$. Thus, $(\gamma_{B_n})^*\lambda_n - (\gamma_{B_n})^*\lambda$ converges to 0 in the weak* topology as $n \to \infty$.

Since $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^\infty$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^\infty$ are uniformly Thurston bounded, it follows that $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^\infty$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^\infty$ contain a pair of converging subsequences in the weak* topology. Then the weak* limits of the converging subsequences are the same. For simplicity of notation, we continue to denote such subsequences by $\{(\gamma_{B_n})^*\lambda_n\}_{n=1}^\infty$ and $\{(\gamma_{B_n})^*\lambda\}_{n=1}^\infty$. Furthermore, using Theorem B, we know there exist $\{A_n\}_{n=1}^\infty$ and $\{C_n\}_{n=1}^\infty$ in $M\ddot{o}b(\mathbb{D})$ such that the two sequences $\{E^{(\gamma_{B_n})^*\lambda_n}|_{\mathbb{S}^1} = A_n \circ h_n \circ \gamma_{B_n}\}_{n=1}^\infty$ and $\{E^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1} = C_n \circ h \circ \gamma_{B_n}\}_{n=1}^\infty$ converge to the same quasisymmetric map pointwisely on $\mathbb{S}^1$. Thus

$$\lim_{n\to\infty} |L(h(B_n)) - L(h_n(B_n))|$$

$$= \lim_{n\to\infty} |L(A_n \circ h \circ \gamma_{B_n}(B^*)) - L(C_n \circ h_n \circ \gamma_{B_n}(B^*))| = 0,$$

which is a contradiction to (6.8). Therefore $[[h_n]]$ converges to $[[h]]$ in the asymptotic Teichmüller topology. Thus, $\widehat{\mathcal{EM}}^{-1}$ is continuous. $\square$

7. Induced infinitesimal earthquake measure map and asymptotic cross-ratio norm topology

In this section, we consider the infinitesimal version of the induced earthquake measure map. As pointed out in Section 2.1, the universal Teichmüller space $T(\mathbb{D})$ is the quotient space $M\ddot{o}b(\mathbb{D})\backslash QS$, where QS is the collection of quasisymmetric homeomorphisms of $\mathbb{S}^1$. The tangent space of $T(\mathbb{D})$ at a base point is characterized by the space $\mathcal{Z}(\mathbb{S}^1)$ (or $\mathcal{Z}(\mathbb{D})$) of Zygmund bounded continuous tangent vector fields on $\mathbb{S}^1$ (resp. $\mathbb{D}$). By developing infinitesimal versions of Beurling-Ahlfors extensions, Gardiner and Sullivan showed in [8] that the tangent space of $T_0(\mathbb{D})$ at a base point is characterized by a subspace $\mathcal{Z}_0(\mathbb{S}^1)$ (or $\mathcal{Z}_0(\mathbb{D})$) of $\mathcal{Z}(\mathbb{S}^1)$ (resp. $\mathcal{Z}(\mathbb{D})$). It follows that the tangent space of $AT(\mathbb{D})$ is the quotient space $\mathcal{Z}(\mathbb{S}^1)/\mathcal{Z}_0(\mathbb{S}^1)$ (resp. $\mathcal{Z}(\mathbb{D})/\mathcal{Z}_0(\mathbb{D})$), denoted by $\mathcal{AZ}(\mathbb{S}^1)$ (resp. $\mathcal{AZ}(\mathbb{D})$). The work of this section is to introduce a topology on $\mathcal{AZ}(\mathbb{S}^1)$ under which the infinitesimal earthquake measure map is a homeomorphism between $\mathcal{AZ}(\mathbb{S}^1)$ and $\mathcal{AML}_b(\mathbb{D})$ with respect to the asymptotic uniform weak* topology on $\mathcal{AML}_b(\mathbb{D})$.

7.1. Zygmund space and infinitesimal earthquake measure map.

A continuous tangent vector field V on $\mathbb{S}^1$ can be viewed as a continuous function from $\mathbb{S}^1$ to the complex plane $\mathbb{C}$. It is said to be Zygmund bounded if

$$|V(e^{2\pi i(x+t)}) + V(e^{2\pi i(x-t)}) - 2V(e^{2\pi ix})| \leq M|t|, \tag{7.1}$$

for all $x \in [0, 1)$, $0 < t < \frac{1}{2}$ and some $M > 0$.

Let Q be a quadruple consisting of four points a, b, c, d on the unit circle arranged in the counter-clockwise direction, denoted by $Q = \{a, b, c, d\}$. It is defined in [7] that

$$V[Q] = \frac{V(b) - V(a)}{b - a} + \frac{V(d) - V(c)}{d - c} - \frac{V(c) - V(b)}{c - b} - \frac{V(d) - V(a)}{d - a}$$

and

$$\|V\|_{cr} = \sup_Q |V[Q]|,$$

where the supremum is taken over all quadruples Q with $cr(Q) = 1$.

One can show that $\|V\|_{cr} = 0$ if and only if V is a quadratic polynomial. Furthermore, it is true that V is Zygmund bounded if and only if $\|V\|_{cr}$ is finite. We let $\mathcal{Z}(\mathbb{S}^1)$ be the space of Zygmund bounded tangent vector fields on $\mathbb{S}^1$ modulo quadratic polynomials.

Let $\lambda \in \mathcal{ML}_b(\mathbb{D})$. For each $t \geq 0$, let h_t be a quasisymmetric homeomorphism of $\mathbb{S}^1$ defined by an earthquake map on $\mathbb{D}$ inducing $t\lambda$. Suppose that h_t fixes three common points for all $t \geq 0$, which is called an earthquake curve determined by $t\lambda, t \geq 0$. It is shown in [6] that $h_t(z)$ is differentiable on t at each point $z \in \mathbb{S}^1$ and furthermore

$$\frac{d}{dt}h_t(z)|_{t=0} = \int_{\mathcal{G}} E_{ab}(z)d\lambda(a,b) \text{ modulo a quadratic polynomial,}$$

where

$$E_{ab}(z) = \begin{cases} 0 & \text{for } z \text{ outside of } [a,b], \\ \frac{(z-a)(z-b)}{a-b} & \text{for } z \in [a,b]. \end{cases}$$

Here we have an agreement that for each geodesic line connecting a, b in $\mathcal{L}$, $[a,b]$ denotes the short arc on $\mathbb{S}^1$ between a and b and in the counter-clockwise direction.

We denote by

$$V_\lambda = \dot{E}^\lambda|_{\mathbb{S}^1} = \int_{\mathcal{G}} E_{ab}(z)d\lambda(a,b).$$

Then the integral introduces an injective map $\dot{E}$ from $\mathcal{ML}_b(\mathbb{D})$ into $\mathcal{Z}(\mathbb{S}^1)$ [5], and $\dot{E}^\lambda$ is called an infinitesimal earthquake map determined by λ. Conversely, Gardiner [5] showed that for any $V \in \mathcal{Z}(\mathbb{S}^1)$, there exists a $\lambda_V \in \mathcal{ML}_b(\mathbb{D})$ such that

$$V(z) = \int_{\mathcal{G}} E_{ab}(z)d\lambda_V(a,b) \text{ modulo a quadratic polynomial.}$$

Furthermore, if two V's differ by a quadratic polynomial, then the corresponding λ's are the same. Therefore, $\dot{E}$ is a bijection between $\mathcal{ML}_b$ and $\mathcal{Z}(\mathbb{S}^1)$. The inverse of $\dot{E}$ is often called the infinitesimal earthquake measure map, and it is denoted as

$$\mathcal{E}\dot{\mathcal{M}} : \mathcal{Z}(\mathbb{S}^1) \to \mathcal{ML}_b(\mathbb{D}) : V \mapsto \lambda_V.$$

7.2. **Pointwise Convergence of infinitesimal earthquake maps.** The infinitesimal version of Theorem B is also showed in [16], which can be improved as the following proposition.

Proposition 4. *Let $\lambda \in \mathcal{ML}_b(\mathbb{D})$ and let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathcal{ML}_b(\mathbb{D})$ with uniformly bounded Thurston norms. Then λ_n converges to λ in the weak* topology if and only if $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ converges to $\dot{E}^\lambda|_{\mathbb{S}^1}$ pointwisely on $\mathbb{S}^1$ when all $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$ are properly normalized.*

Proof. It is proved in [16] that if $\{\lambda_n\}_{n=1}^\infty$ converges to λ in the weak* topology, then $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ converges to $\dot{E}^\lambda|_{\mathbb{S}^1}$ pointwisely on $\mathbb{S}^1$ when all $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$ are properly normalized. We only need to prove the other direction.

We normalize $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$, by subtracting quadratic polynomials, such that $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^\lambda|_{\mathbb{S}^1}$ vanish at three common points of $\mathbb{S}^1$. Assume on the contrary that λ_n does not converge to λ in the weak* topology. By the same argument in [19], there exists a subsequence $\{\lambda_{n_j}\}_{j=1}^\infty$ of $\{\lambda_n\}_{n=1}^\infty$ weakly converging to $\kappa \in \mathcal{ML}_b(\mathbb{D})$. For simplicity, we rename $\{\lambda_{n_j}\}_{j=1}^\infty$ to be $\{\lambda_n\}_{n=1}^\infty$. The assumption that λ_n does not converge to λ in the weak* topology implies $\kappa \neq \lambda$. Normalizing $\dot{E}^\kappa|_{\mathbb{S}^1}$ as the

same as $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$, it follows that $\dot{E}^{\lambda_n}|_{\mathbb{S}^1} \to \dot{E}^{\kappa}|_{\mathbb{S}^1}$ pointwise as $n \to \infty$. By the assmption that $\dot{E}^{\lambda_n}|_{\mathbb{S}^1}$ pointwise converges to $\dot{E}^{\lambda}|_{\mathbb{S}^1}$, we know that $\dot{E}^{\kappa}|_{\mathbb{S}^1} = \dot{E}^{\lambda}|_{\mathbb{S}^1}$. Thus $\kappa = \lambda$, which is a contradiction. $\qquad\square$

7.3. **Asymptotic cross-ratio vanishing equivalence on $\mathcal{Z}(\mathbb{S}^1)$.** The tangent space of $AT(\mathbb{D})$ at a base point is the quotient space $\mathcal{Z}(\mathbb{S}^1)/\mathcal{Z}_0(\mathbb{S}^1)$, where an element V of $\mathcal{Z}(\mathbb{S}^1)$ belongs to $\mathcal{Z}_0(\mathbb{S}^1)$ provided that the constant M in (7.1) converges to 0 independent of x as $|t| \to 0$. In this subsection, we first introduce an alternative definition of the elements in $\mathcal{Z}_0(\mathbb{S}^1)$ by using $V[Q]$.

In the previous sections, we have used two cross ratios. One is used to define the Liouville measure of a box $B = [a,b] \times [c,d]$ and the other is used to define the cross-ratio distortion norm. They are used in different situations based on different purposes and the quantitative results are different. In the following, one can see that their infinitesimal versions only differ up to multiplication by 2.

Definition 9. *Let $B = [a,b] \times [c,d]$ be a box of geodesics, where a,b,c,d lie on $\mathbb{S}^1$ in the counter-clockwise direction. For any $V \in \mathcal{Z}(\mathbb{S}^1)$, we set*

$$V_L[B] = \frac{V(a) - V(c)}{a - c} + \frac{V(b) - V(d)}{b - d} - \frac{V(a) - V(d)}{a - d} - \frac{V(b) - V(c)}{b - c},$$

and the cross-ratio norm $\|V\|_{cr_L}$ of V is defined by

$$\|V\|_{cr_L} = \sup_B |V_L[B]|,$$

where the supremum is taken over all B with Liouville measure $L(B) = \log 2$.

Proposition 5. *For any $Q = \{a,b,c,d\}$ with $cr_L(Q) = 2$,*

$$V[Q] = 2V_L[Q].$$

Proof. Let $V \in \mathcal{Z}(\mathbb{S}^1)$. Given any $Q = \{a,b,c,d\}$ with $cr_L(Q) = 2$, we consider that a, b, c and d are temporarily fixed. Let $f_t(z) = z + tV$. Then $f_t(Q)$ is a quadruple of four distinct points when $|t|$ is sufficiently small.

Clearly, $cr_L(Q) = 2$ if and only if $cr(Q) = 1$. By the definitions of $V[Q]$ and $V_L[Q]$, we obtain

$$
\begin{aligned}
V_L[Q] &= \frac{d}{dt} \ln cr_L(f_t(Q))|_{t=0} = \frac{d}{dt} \ln(1 + cr(f_t(Q)))|_{t=0} \\
&= \frac{\frac{d}{dt} cr(f_t(Q))}{1 + cr(f_t(Q))}|_{t=0} = \frac{1}{2} \frac{d}{dt} cr(f_t(Q))|_{t=0} \\
&= \frac{1}{2} \frac{d}{dt} e^{\ln cr(f_t(Q))}|_{t=0} = \frac{1}{2} e^{\ln cr(f_t(Q))} \frac{d}{dt} \ln cr(f_t(Q))|_{t=0} \\
&= \frac{1}{2} V[Q].
\end{aligned}
$$

Therefore, $V[Q] = 2V_L[Q]$. $\qquad\square$

Now we introduce an alternative characterization of the elements in $\mathcal{Z}_0(\mathbb{D})$.

Lemma 8 ([11])**.** *If two tangent vector fields V and $\tilde{V}$ satisfy $\tilde{V}(x) = \frac{V(\gamma(x))}{\gamma'(x)}$ for an element $\gamma \in \text{Möb}(\mathbb{D})$, then for any quadruple Q of four points a,b,c,d on the unit circle in the counter-clockwise order,*

$$\tilde{V}[Q] = V[\gamma(Q)] \quad (\text{or } V[Q] = \tilde{V}[\gamma^{-1}(Q)]).$$

Similar to the definition of a degenerating sequence of boxes in $\mathcal{G}$, we define a degenerating sequence of quadruples to be a sequence $\{Q_n\}_{n=1}^{\infty}$ of quadruples Q_n such that $cr(Q_n) = 1$ for all n and $s(Q_n) \to 0$ as $n \to \infty$, where $s(Q)$ is the minimum scale of $Q = \{a, b, c, d\}$; that is,

$$s(Q) = \min\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

Proposition 6. *A continuous tangent vector field V on $\mathbb{S}^1$ belongs to $\mathcal{Z}_0(\mathbb{S}^1)$ if and only if*

$$\sup_{\{B_n\}} \limsup_{n\to\infty} V_L[B_n] = 0 \ \text{ or } \ \sup_{\{Q_n\}} \limsup_{n\to\infty} V[Q_n] = 0, \tag{7.2}$$

where the supremum is taken over all degenerating sequences $\{B_i\}_{i=1}^{\infty}$ of boxes or all degenerating sequences $\{Q_i\}_{i=1}^{\infty}$ of quadruples.

Proof. Let V be a continuous tangent vector field on $\mathbb{S}^1$ and $p \in \mathbb{S}^1$, and let γ_p be an orientation-preserving Möbius transformation from $\mathbb{H}$ onto $\mathbb{D}$ mapping i to the origin, 0 to p and ∞ to $-p$. Assume that $V_p(z) = V(\gamma_p(z))/\gamma_p'(z)$. We note that $V \in \mathcal{Z}_0(\mathbb{S}^1)$ if and only if $V_p \in \mathcal{Z}_0(\mathbb{R})$ for any $p \in \mathbb{S}^1$, where $\mathcal{Z}_0(\mathbb{R})$ denotes the space of all continuous functions V_p defined on $\mathbb{R}$ satisfying

$$\frac{V_p(x+t) + V_p(x-t) - 2V_p(x)}{t} = \delta(x, t)$$

for any points x and t on $\mathbb{R}$, and $\delta(x, t)$ converges to 0 uniformly on x as $t \to 0$.

In the following, we first show that if V satisfies the condition (7.2) then $V \in \mathcal{Z}_0(\mathbb{S}^1)$. By the above note, it suffices to show that $V_p \in \mathcal{Z}_0(\mathbb{R})$ for any $p \in \mathbb{S}^1$.

Let p be a point on $\mathbb{S}^1$. Using a lemma in [11] (see Lemma 10 in Subsection 7.5), we see that $V_p[Q] = V[\gamma_p(Q)]$. Given any quadruple $Q = \{x-t, x, x+t, \infty\}$,

$$\frac{V_p(x+t) + V_p(x-t) - 2V_p(x)}{t} = V_p[Q] = V[\gamma_p(Q)].$$

Applying the condition (7.2) to V on the quadruples $\gamma_p(Q)$, we can show that $V_p \in \mathcal{Z}_0(\mathbb{R})$.

Conversely, assuming that $V \in \mathcal{Z}_0(\mathbb{S}^1)$, we want to show it satisfies condition (7.2). Suppose not, it follows that there exist $\epsilon > 0$ and a degenerating sequence $\{Q_n\}_{n=1}^{\infty}$ of quadruples such that

$$|V[Q_n]| > \epsilon$$

for each n.

Passing to a subsequence, we may assume that a_n, b_n, c_n and d_n converge to a, b, c and d on $\mathbb{S}^1$ respectively. Using the conditions that $cr(Q_n) = 1$ for each n and $s(Q_n) \to 0$ as $n \to \infty$, we conclude that the set $\{a, b, c, d\}$ contains at most two distinct points, namely a and d. Now let p be a point on $\mathbb{S}^1$ such that $-p$ is different from a and d. Then $V_p(z) = V(\gamma_p(z))/\gamma_p'(z) \in \mathcal{Z}_0(\mathbb{R})$. Now we apply the infinitesimal Beurling-Ahlfors extension of V_p to the upper half plane $\mathbb{H}$ introduced by Gardiner and Sullivan in [8]. It is shown there that if $V_p \in \mathcal{Z}_0(\mathbb{R})$, then $\mu = \bar{\partial}V_p$ is a Beltrami coefficient vanishing when approaching the boundary $\mathbb{R}$. Note that if n is big enough, Q_n is outside a neighborhood of $-p$ on $\mathbb{S}^1$, and hence $\gamma_p^{-1}(Q_n)$ is contained in a compact subset K of $\mathbb{R}$. Denote by $Q_n' = \gamma_p^{-1}(Q_n) = \{a_n', b_n', c_n', d_n'\}$. Now applying the measurable Riemann mapping theorem to $t\mu$, the approximation

$$f^{t\mu}(z) = z + tV_p(z) + o(|t|)$$

is uniform on every compact subset of $\mathbb{R}$, where $|t| < 1$ and $f^{t\mu}$ is normalized to fix 0, 1 and ∞ for all t. Furthermore,

$$
\begin{aligned}
V_p(z) &= -\frac{z(z-1)}{\pi} \int \int \frac{\mu(\xi)d\zeta d\eta}{\xi(\xi-1)(\xi-z)} \\
&= \frac{1}{\pi} \int \int \mu(\xi)(\frac{z}{\xi-1} - \frac{z-1}{\xi} - \frac{1}{\xi-z})d\zeta d\eta,
\end{aligned}
$$

where $\xi = \zeta + i\eta$. Then

$$
\begin{aligned}
V[Q_n] &= V_p[Q_n'] \\
&= -\frac{1}{\pi} \int \int \frac{(c_n'-a_n')(d_n'-b_n')}{(\xi-a_n')(\xi-b_n')(\xi-c_n')(\xi-d_n')}\mu(\xi)d\zeta d\eta.
\end{aligned}
$$

By choosing $-p$ at a proper position on $\mathbb{S}^1$ and using the corresponding γ_p, we may assume that $a_n' < b_n' < c_n' < d_n'$ for each sufficiently large n. Passing to a subsequence and without loss of generality, we assume further that $s(Q_n') = c_n' - b_n'$ for all sufficiently large n. For each sufficiently large n, substituting ξ by $\xi = (c_n' - b_n')w + b_n'$ we obtain

$$
V_p[Q_n'] = -\frac{1}{\pi} \int \int \frac{\frac{c_n'-a_n'}{c_n'-b_n'}\frac{d_n'-b_n'}{c_n'-b_n'}}{(w-a_n'')w(w-1)(w-d_n'')}\mu((c_n'-b_n')w+b_n')dudv,
$$

where $w = u + iv$ and $a_n'' = -\frac{b_n'-a_n'}{c_n'-b_n'}$ and $d_n'' = \frac{d_n'-b_n'}{c_n'-b_n'}$.

Since $c_n' - b_n'$ approaches 0 as $n \to \infty$ and μ vanishes near the real line $\mathbb{R}$, it follows that $\mu((c_n'-b_n')w+b_n')$ converges to 0 pointwisely at almost every w as $n \to \infty$. According to the relation between the maximal and minimal scales of Q_n', we consider the following two cases. In one case, the ratios of the maximal scales over the minimal ones of Q_n''s are bounded, and in the other, the ratios converge to ∞ as $n \to \infty$ (by passing to a subsequence if necessary). Using the condition $cr(Q_n') = 1$ for each n, we obtain in the first case, a_n'''s, $\frac{c_n'-a_n'}{c_n'-b_n'}$'s and $\frac{d_n'-b_n'}{c_n'-b_n'}$'s are all bounded; in the other case, either a_n'' converges to -1, $\frac{c_n'-a_n'}{c_n'-b_n'}$ converges to 2 and $\frac{d_n'-b_n'}{c_n'-b_n'}$ converges to ∞ as $n \to \infty$ or a_n'' converges to $-\infty$, $\frac{c_n'-a_n'}{c_n'-b_n'}$ converges to ∞ and $\frac{d_n'-b_n'}{c_n'-b_n'}$ converges to 2 as $n \to \infty$. Applying Lebesgue's dominating convergence theorem, we conclude that in either case, $V[Q_n] = V_p[Q_n']$ converges to 0 as $n \to \infty$. This is a contradiction to the assumption that $V[Q_n] > \epsilon > 0$ for each n. $\qquad\square$

Definition 10. *Given two elements $V, V' \in \mathcal{Z}(\mathbb{S}^1)$, we say that V is equivalent to V', denoted by $V \sim V'$, if*

$$
\sup_{\{B_n\}} \limsup_{n\to\infty} |V_L[B_n] - V_L'[B_n]| = 0 \ \text{ or } \ \sup_{\{Q_n\}} \limsup_{n\to\infty} |V[Q_n] - V'[Q_n]| = 0
$$

where the supremum is taken over all degenerating sequences $\{B_n\}_{n=1}^\infty$ of boxes or all degenerating sequences $\{Q_n\}_{n=1}^\infty$ of quadruples.

For each $V \in \mathcal{Z}(\mathbb{S}^1)$, we denote by $[V]$ the equivalent class of V in $\mathcal{Z}(\mathbb{S}^1)$. Define

$$
\mathcal{AZ}(\mathbb{S}^1) = \mathcal{Z}(\mathbb{S}^1)/\sim .
$$

Using Proposition 6, the following corollary is obvious.

Corollary 6. $\mathcal{AZ}(\mathbb{S}^1) = \mathcal{Z}(\mathbb{S}^1)/\mathcal{Z}_0(\mathbb{S}^1)$.

7.4. **Proof of Theorem 3.**

Proof. We first show that if $[\lambda] = [\lambda']$, then $[V] = [V']$. Suppose that $[V] \neq [V']$. Then there exist $\epsilon > 0$ and a degenerating sequence of $\{B_n\}_{n=1}^\infty$ boxes such that, for all n

$$|\dot{E}^\lambda|_{\mathbb{S}^1}[B_n] - \dot{E}^{\lambda'}|_{\mathbb{S}^1}[B_n]| \geq \epsilon > 0.$$

Then, for all n,

$$|\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B^*] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B^*]| = |V[B_n] - V'[B_n]| \geq \epsilon > 0. \qquad (7.3)$$

Since $\|(\gamma_{B_n})^*\lambda\|_{Th} = \|\lambda\|_{Th}$ and $\|(\gamma_{B_n})^*\lambda'\|_{Th} = \|\lambda'\|_{Th}$, $\{(\gamma_{B_n})^*\lambda\}_{n=1}^\infty$ and $\{(\gamma_{B_n})^*\lambda'\}_{n=1}^\infty$ are uniformly Thurston bounded. Therefore there exist convergent subsequences of $\{(\gamma_{B_n})^*\lambda\}_{n=1}^\infty$ and $\{(\gamma_{B_n})^*\lambda'\}_{n=1}^\infty$ in the weak* topology. For simplicity, we denote them by the same notation. In the proof of the sufficiency of Theorem 1, we have shown that the condition $[\lambda] = [\lambda']$ implies that the limit of $(\gamma_{B_n})^*\lambda$ equals to the limit of $(\gamma_{B_n})^*\lambda'$. We normalize $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$, by adding quadratic polynomials, such that $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$ equal to 0 at three fixed points on $\mathbb{S}^1$. By Proposition 4, the (pointwise) limits of the two sequences $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$ are the same. This is a contradiction to (7.3).

Now we show that if $[V] = [V']$, then $[\lambda] = [\lambda']$. Assume on the contrary that $[\lambda] \neq [\lambda']$, then there exist a degenerating subsequence $\{B_n\}_{n=1}^\infty$ of boxes, and a continuous function f on $\mathcal{G}$ with compact support contained in B^* such that

$$\lim_{n \to \infty} \int_{B^*} f d((\gamma_{B_n})^*\lambda - (\gamma_{B_n})^*\lambda') \geq \epsilon > 0. \qquad (7.4)$$

Since they are uniformly Thurston bounded, there exist convergent subsequences of $\{(\gamma_{B_n})^*\lambda\}_{n=1}^\infty$ and $\{(\gamma_{B_n})^*\lambda'\}_{n=1}^\infty$ in the weak* topology, which we denote by the same notation for simplicity. Using the assumption (7.4), we know that the weak* limit of $(\gamma_{B_n})^*\lambda$ does not equals to the limit of $(\gamma_{B_n})^*\lambda'$. By Proposition 4, the pointwisely limits of the two sequences $\dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}$ are not the same even they vanish at three common points on $\mathbb{S}^1$. Thus there exists a box $B_0 \in \mathcal{G}$ with $L(B_0) = \log 2$ such that

$$\lim_{n \to \infty} |\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B_0]| \geq \delta > 0. \qquad (7.5)$$

On the other hand, using the assumption that $V \sim V'$ and $\lim_{n \to \infty} s(B_n) = 0$, we obtain

$$\lim_{n \to \infty} |\dot{E}^{(\gamma_{B_n})^*\lambda'}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^*\lambda}|_{\mathbb{S}^1}[B_0]|$$

$$= \lim_{n \to \infty} |\dot{E}^{\lambda'}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)] - \dot{E}^\lambda|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)]| \leq \limsup_{s(B) \to 0} |V'[B] - V[B]| = 0.$$

This is a contradiction to (7.5). It follows that $[\lambda] = [\lambda']$. $\qquad \square$

7.5. **Asymptotic Thurston's norm and asymptotic cross-ratio norm.** It is shown in [10] and [11] with two different methods that the cross-ratio norm $||V||_{cr}$ of a vector field $V \in \mathcal{Z}(\mathbb{S}^1)$ and the Thurston's norm $\|\lambda_V\|_{Th}$ of λ_V are bi-Lipschitz equivalent. Now we define the following.

Definition 11. *Given a vector field $V \in \mathcal{Z}(\mathbb{S}^1)$, the asymptotic cross-ratio norm $\|V\|_{\widehat{cr}}$ of $V \in \mathcal{Z}(\mathbb{S}^1)$ is defined to be*

$$\|V\|_{\widehat{cr}} = \sup_{\{Q_n\}} \limsup_{n \to \infty} |V[Q_n]|,$$

where the supreme is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.

In this subsection, we show Theorem 6, which is viewed as an asymptotic version of the result

$$\|\lambda_V\|_{Th} \leq C\|V\|_{cr}$$

for a universal positive constant C in [11]. In the next subsection, we apply this theorem to prove the continuity of the inverse of the induced infinitesimal earthquake measure map $\widehat{\mathcal{E}\mathcal{M}}$.

The strategy of the proof of Theorem 6 is similar to the one used to prove the inequality $\|\lambda_V\|_{Th} \leq C\|V\|_{Th}$ in [11], but extra effort has to be made in order to have the scales $s(Q)$ of selected quadruples Q approach to 0 as disks D of hyperbolic diameter $\leq \frac{1}{2}$ approach the boundary $\mathbb{S}^1$ of $\mathbb{D}$. We first recall three technical lemmas developed in [11].

Let λ denote a Thurston bounded measured lamination and $V = V_\lambda$. Let B be an orientation-preserving Möbius transformation from the upper half plane $\mathbb{H}$ or the unit open disk $\mathbb{D}$ onto $\mathbb{D}$, and $\tilde{\lambda} = (B^*\lambda)$ be the pullback of λ by B (or the pushforward of λ by B^{-1}). And define

$$\tilde{V}(x) = V_{\tilde{\lambda}}(x) = \dot{E}^{\tilde{\lambda}}(x) = \int \int E_{ab}(x) d\tilde{\lambda}(a, b).$$

Lemma 9 ([11]). *The vector fields V and $\tilde{V}$ satisfy*

$$\tilde{V}(x) = \frac{V(B(x))}{B'(x)};$$

that is, $\tilde{V}$ is the pullback of V by B.

Lemma 10 ([11]). *Let B be a Möbius transformation from $\mathbb{D}$ or $\mathbb{H}$ onto $\mathbb{D}$ or $\mathbb{H}$. Assume that two continuous tangent vector fields $\tilde{V}$ and V on $\mathbb{S}^1$ or $\mathbb{R}$ satisfy the condition in the previous lemma. Then for any quadruple Q of four points,*

$$\tilde{V}[Q] = V[B(Q)].$$

Lemma 11 ([11]). *Assume $\rho > 0$, $-\infty \leq a < b < c < d$, and $c \leq s \leq d \leq t$. Let $V(x) = \rho E_{s,t}(x)$ and $Q = \{a, b, c, d\}$. Consider $V[Q]$ as a function of s and t. Then $V[Q] \geq 0$ and $V[Q]$ is an increasing function on t for each fixed s and a decreasing function on s for each fixed t.*

Lemma 12 ([11]). *Assume $\rho > 0$, $-\infty \leq a < b < c < d \leq \infty$, and $b \leq s \leq c$ and $t \geq d$. Let $V(x) = \rho E_{s,t}(x)$ and $Q = \{a, b, c, d\}$. Consider $V[Q]$ as a function of s and t. Then $V[Q] \geq 0$ and $V[Q]$ is increasing on s for each fixed t and also increasing on t for each fixed s.*

Now we prove Theorem 6.

Proof. Let r_0 be a constant between 0 and 1, which will be selected later. Let D denote a closed disk in $\mathbb{D}$ of hyperbolic diameter $\leq r_0$. It suffices to show that there exists a universal positive constant C such that the measure of the leaves of the lamination $\mathcal{L}$ intersecting D is less than or equal to $C\|V\|_{\widehat{cr}}$.

Suppose that D is near the boundary. Let l_1 and l_2 denote the lines in the lamination $\mathcal{L}$ of λ which bound all the lines in $\mathcal{L}$ intersecting D. We label the endpoints of l_1 and l_2 by a, s, c and t in the counter-clockwise order such that a and t are the endpoints of one leave and s and c are the endpoints of the other, and
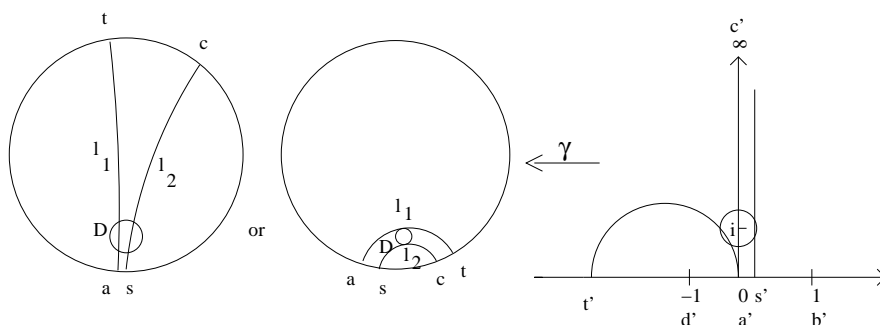
FIGURE 2. Illustration of the quadruples Q and Q' in the proof of Theorem 6.

furthermore the length of the arc between a and s is less than or equal to the length of the arc between c and t. In case that l_1 and l_2 share one or both endpoints, $a = s$ and/or $c = t$.

Let p be a point on the intersection of D with the geodesic connecting a and c. Now we let B be an orientation-preserving Möbius transformation from $\mathbb{H}$ onto $\mathbb{D}$ such that $B^{-1}(p) = i$, $B^{-1}(a) = 0$ and $B^{-1}(c) = \infty$. It also follows that $t' = B^{-1}(t) < 0$, $s' = B^{-1}(s) \geq 0$, and $B(D)$ is a disk containing of i and with hyperbolic diameter $\leq r_0$. Now it is easy to see that if r_0 is small enough, then $t' < -2$ and $0 \leq s' < 1$. Furthermore, we can see that as soon as r_0 is small enough, the properties $t' < -2$ and $0 \leq s' < 1$ hold universally in the sense that for any disk D of diameter r_0 in the hyperbolic metric and any geodesic lamination $\mathcal{L}$, the two variables t' and s' resulting from the previous process satisfy those two inequalities. We thereby choose r_0 to be such a positive constant.

Let $Q' = \{0, 1, \infty, -1\}$, which is a quadruple on $\mathbb{R} \cup \{\infty\}$ with $cr(Q') = -1$. Let $Q = B(Q')$. By using a similar idea to prove Lemma 1, one can show that $s(Q)$ goes to 0 if D approaches the boundary $\mathbb{S}^1$.

Now assume that $\tilde{\lambda}$ and $\tilde{V}$ are as the same as introduced in this subsection. Clearly, $cr(Q) = cr(Q') = -1$. By Lemma 8,

$$\tilde{V}[Q'] = V[Q].$$

Let $\tilde{\mathcal{L}} = B(\mathcal{L})$, where $\mathcal{L}$ is the lamination for λ. In order to estimate $\tilde{V}[Q']$, we divide the lines in the lamination $\tilde{\mathcal{L}}$ that affect the value of $V[Q']$ into three groups. Let $\mathcal{L}_m$ denote the collection of the lines in $\mathcal{L}$ intersecting $B(D)$, $\mathcal{L}_b$ denote the collection of the lines in $\tilde{\mathcal{L}} \setminus \mathcal{L}_m$ connecting points in $(s', 1)$ to points in $(1, \infty)$, and $\mathcal{L}_d$ the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $(t', -1)$ to points in $(-1, 0)$. Denote by $\lambda_k = \tilde{\lambda}|_{\mathcal{L}_k}$ and $V_i = \dot{E}^{\lambda_k}$ for $k = m, b, d$. By the linearity of the operator $\dot{E}$, we obtain

$$\tilde{V}[Q'] = V_m[Q'] + V_b[Q'] + V_d[Q'].$$

For a habit of getting used to the symbols in the course of applying Lemmas 11 and 12, we denote Q' by $\{a', b', c', d'\}$. By Lemma 11,

$$V_d[Q'] = V_d[\{a', b', c', d'\}] \geq 0 \quad \text{and} \quad V_b[\{c', d', a', b'\}] \geq 0.$$

Then

$$V_b[Q'] = V_b[\{a', b', c', d'\}] = -V_b[\{b', c', d', a'\}] = V_b[\{c', d', a', b'\}] \geq 0.$$

Therefore

$$\tilde{V}[Q'] \geq V_m[Q'].$$

In the next, we use Lemma 12 to obtain an explicit lower bound for $V_m[Q']$, which enables us to complete the proof. By Lemma 12, if we move the weights of the geodesic lines in the lamination $\mathcal{L}_m$ to the geodesic line connecting t' to s', then the value of $V_m[\{b', c', d', a'\}]$ is possibly increased, and hence the value of $V_m[Q'] = -V_m[\{b', c', d', a'\}]$ is possibly decreased. Therefore

$$V_m[Q'] \geq (\rho E_{t',s'})[Q'],$$

where $\rho = \tilde{\lambda}(\mathcal{L}_m)$. It is easy to check

$$E_{t',s'}[Q'] = \frac{2E_{t',s'}(d') - E_{t',s'}(a')}{a' - d'}$$

$$= \frac{2 + 2t' + 2s' + t's'}{t' - s'} = \frac{2[-(1+t')] + s'[-(2+t')]}{s' - t'}.$$

Since $0 \leq s' < 1$ and $t' < -2$, $2[-(1+t')] \geq 2$, $s'[-(2+t')] > 0$, and $s' - t' \leq 1 - t'$. It follows that

$$\frac{2[-(1+t')] + s'[-(2+t')]}{s' - t'} \geq \frac{2[-(1+t')]}{1 - t'} = 2\frac{t'+1}{t'-1}.$$

Clearly, $2\frac{t'+1}{t'-1}$ attains its minimal value $\frac{2}{3}$ on the interval $(-\infty, -2]$. Thus

$$V_m[Q'] \geq \frac{2}{3}\rho.$$

In summary,

$$V[Q] = \tilde{V}[Q'] \geq V_m[Q'] \geq \frac{2}{3}\rho,$$

where $\rho = \tilde{\lambda}(\mathcal{L}_m)$, which is equal to the λ measure of the lines of $\mathcal{L}$ intersecting D, D is a closed disk in $\mathbb{D}$ of hyperbolic diameter $\leq r_0$ and $0 < r_0 < 1$, and $cr(Q) = cr(Q') = -1$.

Now let $Q_1' = \{-1, 0, 1, \infty\}$ and $Q_1 = B(Q_1')$. Then $cr(Q_1') = cr(Q_1) = 1$, $s(Q_1) = s(Q)$, and furthermore

$$V[Q_1] = -V[Q] \leq -\frac{2}{3}\rho.$$

Thus

$$|V[Q_1]| \geq \frac{2}{3}\rho.$$

It follows that there exists a universal positive constant C such that

$$||\lambda||_{\widehat{Th}} \leq C||V||_{\widehat{cr}}.$$

$\square$

7.6. **Proof of Theorem 4.** For any $V \in \mathcal{Z}(\mathbb{S}^1)$, define

$$\|V\|_{\widehat{cr_L}} = \sup_{\{B_n\}} \limsup_{n \to \infty} |V[B_n]|,$$

where the supremum is taken over all degenerating sequences $\{B_n\}_{n=1}^{\infty}$ of boxes.

By Proposition 5, the following corollary is obvious.

Corollary 7. *For any $V \in \mathcal{Z}(\mathbb{S}^1)$, $\|V\|_{\widehat{cr}} = 2\|V\|_{\widehat{cr_L}}$.*

Now we prove Theorem 4.

Proof. We first show that $\widehat{\mathcal{E}\dot{\mathcal{M}}}^{-1}$ is continuous. Assume that $[\lambda_n] \to [\lambda]$ in the asymptotically uniform weak* topology, we prove $[V_n] = \widehat{\mathcal{E}\dot{\mathcal{M}}}^{-1}([\lambda_n])$ goes to $[V] = \widehat{\mathcal{E}\dot{\mathcal{M}}}^{-1}([\lambda])$ in the $\|\cdot\|_{\widehat{cr_L}}$ norm. By Corollary 2, since $[\lambda_n] \to [\lambda]$ in the asymptotically uniform weak* topology, there exist $\lambda_n' \in [\lambda_n]$ goes to $\lambda' \in [\lambda]$ in the uniform weak* topology. Let $V_n' = \mathcal{E}\dot{\mathcal{M}}^{-1}(\lambda_n')$ and $V' = \mathcal{E}\dot{\mathcal{M}}^{-1}(\lambda')$, by the homeomorphic property of $\mathcal{E}\dot{\mathcal{M}}$ ([16]), V_n' goes to V' in the $\|\cdot\|_{cr_L}$ norm. Since $\|V\|_{cr_L} \geq \|V\|_{\widehat{cr_L}}$ for any $V \in \mathcal{Z}(\mathbb{S}^1)$, it follows that $[V_n'] = [V_n]$ goes to $[V'] = [V]$ in the $\|\cdot\|_{\widehat{cr_L}}$ norm.

It remains to show that $\widehat{\mathcal{E}\dot{\mathcal{M}}}$ is continuous. Assume that $[V_n] \to [V]$ in the asymptotically uniform weak* topology. Let $[\lambda_n] = \widehat{\mathcal{E}\dot{\mathcal{M}}}([V_n])$ and $[\lambda] = \widehat{\mathcal{E}\dot{\mathcal{M}}}([V])$. We prove $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology.

Since $[V_n] \to [V]$ in the asymptotic cross-ratio norm $\|\cdot\|_{\widehat{cr_L}}$, $\{\|V_n\|_{\widehat{cr_L}}\}_{n=1}^{\infty}$ is bounded. By Corollary 7, $\|V_n\|_{\widehat{cr}} = 2\|V_n\|_{\widehat{cr_L}}$. Hence $\{\|V_n\|_{\widehat{cr}}\}_{n=1}^{\infty}$ is bounded. By Theorem 6, $\{\|\lambda_n\|_{\widehat{Th}}\}_{n=1}^{\infty}$ is bounded. Using Lemma 6 to replace the representatives, we may assume that $\{\lambda_n\}_{n=1}^{\infty}$ is uniformly Thurston bounded.

Suppose on the contrary that $[\lambda_n]$ does not converges to $[\lambda]$ in the asymptotically uniform weak* topology. Then there exist a degenerating sequence $\{B_n\}_{n=1}^{\infty}$ of boxes and a continuous function f on $\mathcal{G}$ with compact support contained in B^* such that

$$\lim_{n \to \infty} \int_{B^*} f d((\gamma_{B_n})^* \lambda_n - (\gamma_{B_n})^* \lambda) > m > 0. \tag{7.6}$$

Since $\|(\gamma_{B_n})^* \lambda_n\|_{Th} = \|\lambda_n\|_{Th}$, $\|(\gamma_{B_n})^* \lambda\|_{Th} = \|\lambda\|_{Th}$ and $\{\|\lambda_n\|_{Th}\}_{n=1}^{\infty}$ is bounded, it follows that $(\gamma_{B_n})^* \lambda_n$ and $(\gamma_{B_n})^* \lambda$ are uniformly Thurston bounded. Therefore there exist convergent subsequences of $\{(\gamma_{B_n})^* \lambda_n\}_{n=1}^{\infty}$ and $\{(\gamma_{B_n})^* \lambda\}_{n=1}^{\infty}$ in the weak* topology, which we denote by the same notation for simplicity. By the assumption (7.6), we see that the weak* limit of $(\gamma_{B_n})^* \lambda_n$ does not equal to the limit of $(\gamma_{B_n})^* \lambda$. By Proposition 4, the pointwise limits of the two sequences $\dot{E}^{(\gamma_{B_n})^* \lambda_n}|_{\mathbb{S}^1}$ and $\dot{E}^{(\gamma_{B_n})^* \lambda}|_{\mathbb{S}^1}$ are not the same even they vanish at the three common points on $\mathbb{S}^1$. Thus there exists a box $B_0 \in \mathcal{G}$ with $L(B_0) = \log 2$ such that

$$\lim_{n \to \infty} |\dot{E}^{(\gamma_{B_n})^* \lambda_n}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^* \lambda}|_{\mathbb{S}^1}[B_0]| \geq \tilde{m} > 0.$$

By Lemma 1, $s(\gamma_{B_n}(B_0)) \to 0$ as $n \to \infty$. Using the assumption that $[V_n] \to [V]$ in the asymptotically uniform weak* topology, we obtain

$$\lim_{n \to \infty} |\dot{E}^{(\gamma_{B_n})^* \lambda_n}|_{\mathbb{S}^1}[B_0] - \dot{E}^{(\gamma_{B_n})^* \lambda}|_{\mathbb{S}^1}[B_0]|$$

$$= \lim_{n \to \infty} |\dot{E}^{\lambda_n}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)] - \dot{E}^{\lambda}|_{\mathbb{S}^1}[(\gamma_{B_n})(B_0)]| \leq \limsup_{s(B) \to 0} |V_n[B] - V[B]| = 0.$$

This is a contradiction to (7.6). It follows that $[\lambda_n]$ converges to $[\lambda]$ in the asymptotically uniform weak* topology. $\qquad\square$

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Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, PRC

E-mail address: `jinhuafan@hotmail.com`

Department of Mathematics, Brooklyn College of CUNY, Brooklyn, NY 11210, and, Ph.D. Program in Mathematics, Graduate Center of CUNY, 365 Fifth Avenue, New York, NY 10016

E-mail address: `junhu@brooklyn.cuny.edu or JHu1@gc.cuny.edu`