CHARACTERIZATION OF THE ASYMPTOTIC TEICHMÜLLER SPACE OF THE OPEN UNIT DISK THROUGH SHEARS

JINHUA FAN* AND JUN HU**

ABSTRACT. We give a parametrization of the asymptotic Teichmüller space $AT(\mathbb{D})$ of the open unit disk through a space of equivalent classes of the shear functions induced by quasisymmetric homeomorphisms on the Farey tesselation of the disk. Then using the parametrization, we define a metric on $AT(\mathbb{D})$. Two other related metrics are also introduced on $AT(\mathbb{D})$ by using cross-ratio distortions or quadrilateral dilatations under the boundary maps on degenerating sequences of quadruples or quadrilaterals. We show that the topologies induced by the three metrics are equivalent to the one induced by the Teichmüller metric on $AT(\mathbb{D})$. In addition, we revisit and give a complete proof to Šarić's characterization of quasisymmetric homeomorphisms in terms of shear functions.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane and centered at the origin, and let $\mathbb{S}^1$ be the boundary of $\mathbb{D}$. Denote by $T(\mathbb{D})$ the universal Teichmüller space and $AT(\mathbb{D})$ the asymptotic Teichmüller space of $\mathbb{D}$. Recently, two characterizations of the Teichmüller topology on $T(\mathbb{D})$ are given in [18] and [21]: one uses a uniform weak* topology defined on the Thurston parametrization of $T(\mathbb{D})$ ([23] or see elaborations and further investigations in [7], [10], and [20]) - the space of all Thurston-bounded measured geodesic laminations $\mathcal{L}$ on $\mathbb{D}$; the other is comprised of a parametrization of $T(\mathbb{D})$ by shear functions s induced by quasisymmetric homeomorphisms of $\mathbb{S}^1$ on the Farey tesselation of $\mathbb{D}$ and a metric defined on the parametrization. In the following, by a topological characterization of $AT(\mathbb{D})$ we mean a parametrization of $AT(\mathbb{D})$ and a topology or metric on the parametrization equivalent to or with the induced topology equivalent to the Teichmüller topology on $AT(\mathbb{D})$. Very recently, a topological characterization of $AT(\mathbb{D})$ is given in [6] by using an asymptotically uniform weak* topology on a parametrization of $AT(\mathbb{D})$ by equivalent classes of laminations $\mathcal{L}$ in the Thurston parametrization of $T(\mathbb{D})$. In this paper, we give a topological characterization of $AT(\mathbb{D})$ by introducing a metric on a parametrization of $AT(\mathbb{D})$ by equivalent classes of the shear functions s representing $T(\mathbb{D})$.

There are different models for $T(\mathbb{D})$ and $AT(\mathbb{D})$. In this paper, the main model we use for $T(\mathbb{D})$ is the collection $QS(\mathbb{S}^1)$ of all quasisymmetric homeomorphisms of $\mathbb{S}^1$ fixing three points $1, -1$ and i; correspondingly, the one we use for $AT(\mathbb{D})$ is the quotient space $S(\mathbb{S}^1)\backslash QS(\mathbb{S}^1)$, where $S(\mathbb{S}^1)$ is the collection of all symmetric

homeomorphisms in $QS(\mathbb{S}^1)$. Before we give the statements of our theorems, we first recall some background.

Consider $\mathbb{D}$ as the hyperbolic plane. The Farey tesselation $\mathcal{F}$ of $\mathbb{D}$ is a locally finite idea triangulation of $\mathbb{D}$ that is invariant under the group of the isometries of $\mathbb{D}$ generated by the reflections in the hyperbolic metric with respect to the geodesics of $\mathcal{F}$ (see Section 4 for the details). The endpoints of the geodesics in $\mathcal{F}$ are called the tips of $\mathcal{F}$ and we denote the collection of all tips by $\mathcal{P}$. Each homeomorphism h of $\mathbb{S}^1$ induces a real-valued function $s_h : \mathcal{F} \to \mathbb{R}$ as follows, which is called the *shear function* or *coordinate* of h. Given each edge $e \in \mathcal{F}$, let (Δ, Δ_1) be the pair of the two adjacent triangles of $\mathcal{F}$ sharing a common boundary at e. Then $s_h(e)$ is defined to be the shear of the image pair $(h(\Delta), h(\Delta_1))$, that is the signed hyperbolic distance between the orthogonal projections of the third vertices of $h(\Delta)$ and $h(\Delta_1)$ to their common edge (see Section 4 for its interpretation through the cross-ratio distortion under h on a quadruple of four points on $\mathbb{S}^1$).

Let $\{e_n^p\}_{n \in \mathbb{Z}}$ be a fan in $\mathcal{F}$ with tip p (see Section 4 for the definition), and let $s_n^p = s_h(e_n^p)$. Define

(1.1)
$$s(p; m, k) = e^{s_m^p} \frac{1 + e^{s_{m+1}^p} + e^{(s_{m+1}^p + s_{m+2}^p)} + \cdots + e^{(s_{m+1}^p + s_{m+2}^p + \cdots + s_{m+k}^p)}}{1 + e^{-s_{m-1}^p} + e^{-(s_{m-1}^p + s_{m-2}^p)} + \cdots + e^{-(s_{m-1}^p + s_{m-2}^p + \cdots + s_{m-k}^p)}}$$

for any $m, k \in \mathbb{Z}$. This quantity can be interpreted as the cross-ratio distortion of h on a quadruple of four points on $\mathbb{S}^1$ (see Section 4). Then define

$$M_s(p) = \sup_{m,k \in \mathbb{Z}} |\log s(p; m, k)| \text{ and } M_s = \sup_{p \in \mathcal{P}} M_s(p).$$

We call M_s the *shear norm* of h. We say that a circle homeomorphism h of $\mathbb{S}^1$ has a *finite shear norm* if M_s is finite.

It is stated in Theorem A of [21] that a real-valued function $s : \mathcal{F} \to \mathbb{R}$ is equal to the shear function s_h of a quasisymmetric homeomorphism h of $\mathbb{S}^1$ if and only if s has a finite shear norm. Let $\mathcal{X}$ be the collection of all real-valued functions $s : \mathcal{F} \to \mathbb{R}$ with finite shear norms. Then $\mathcal{X}$ gives a parametrization of $T(\mathbb{D})$. Given two elements $s_1, s_2 \in \mathcal{X}$, define

$$M_{s_1,s_2}(p) = \sup_{m,k \in \mathbb{Z}} |\log \frac{s_1(p; m, k)}{s_2(p; m, k)}| \text{ and } d_S(s_1, s_2) = \sup_{p \in \mathcal{P}} M_{s_1,s_2}(p).$$

Then d_S defines a metric on $\mathcal{X}$, called the *shear metric*. In Theorem B of [21], Šarić showed that the parametrization $\mathcal{X}$ of $T(\mathbb{D})$ equipped with the metric d_S is topologically equivalent to $T(\mathbb{D})$ under the Teichmüller metric d_T. More explicitly, he proved that a sequence of points converges to a point in $T(\mathbb{D})$ under d_T if and only if the sequence of the corresponding shear functions converges to the shear function of the limiting point under d_S.

In this paper, we give a parametrization of the asymptotic Teichmüller space $AT(\mathbb{D})$ via a space of equivalent classes of shear functions, define a metric metric on the parametrization, and show that the parametrization with the metric is a topological characterization of $AT(\mathbb{D})$ under the Teichmüller metric d_{AT}.

We first introduce a pseudo metric d_{AS} on $\mathcal{X}$ (defined at the beginning of Section 5). This pseudo metric introduces an equivalent relation on $\mathcal{X}$ when two elements s and s' of $\mathcal{X}$ are defined to be equivalent if $d_{AS}(s, s') = 0$. We denote by $[s]$ the equivalent class of s and by $\mathcal{AX}$ the quotient space under this equivalent relation. Then d_{AS} induces a metric on $\mathcal{AX}$, which is continued to be denoted by d_{AS} and

called the *asymptotic shear metric* on $\mathcal{AX}$. In the following two theorems, we show that $\mathcal{AX}$ is a parametrization of $AT(\mathbb{D})$ and $(\mathcal{AX}, d_{AS})$ is topologically equivalent to $(AT(\mathbb{D}), d_{AT})$, where d_{AT} is the Teichmüller metric on $AT(\mathbb{D})$.

Theorem 1. *Let $h, h' \in T(\mathbb{D})$ and let s and s' be the shear functions induced by h and h' respectively. Then $[h] = [h']$ if and only if $d_{AS}(s, s') = 0$, where $[h]$ denote a point in the asymptotic Teichmüller space $AT(\mathbb{D})$.*

Theorem 2. *Let $[h] \in AT(\mathbb{D})$ and let $\{[h_n]\}_{n=1}^{\infty}$ be a sequence of points in $AT(\mathbb{D})$. Assume that s_n and s are the shear functions induced by h_n and h respectively. Then $d_{AT}([h_n], [h]) \to 0$ if and only if $d_{AS}([s_n], [s]) \to 0$.*

The Douady-Earle extensions [2] of circle homeomorphisms of $\mathbb{S}^1$ have played an important role in the study of Teichmüller spaces and asymptotic Teichmüller spaces. For some recent applications of these extensions, we refer to [5], [17], [18], [21] and [6]. Further investigation on regularities or generalizations of the Douady-Earle extensions to large classes of circle maps have been developed in [11]-[15] and [16].

The conformal naturality and one local regularity near the origin of the Douady-Earle extensions together play a crucial role in the proof of Theorem A in [21]. That local regularity is summarized as Lemma 2.2 in [21], which states as follows. Let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the extended real line, which represents the boundary of the hyperbolic plane when it takes the upper half plane $\mathbb{H}$ as a model. Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of orientation-preserving homeomorphisms h_n of $\widehat{\mathbb{R}}$ fixing three points 0, 1 and ∞, and let μ_n be the Beltrami coefficient of the Douady-Earle extension $ex(h_n)$ of h_n for each n. If there exists $c_0 \geq 1$ such that

$$-c_0 \leq h_n(-1) \leq -\frac{1}{c_0}$$

for all n, then there exists a neighborhood U of the imaginary unit $i \in \mathbb{H}$ and a constant $0 < c < 1$ such that

$$||\mu_n|_U||_\infty \leq c < 1$$

for all n.

Theorem A of [21] gives a necessary and sufficient condition for $s : \mathcal{F} \to \mathbb{R}$ to be induced by a quasisymmetric (resp. symmetric) homeomorphism of $\mathbb{S}^1$. The necessity of the condition can be easily obtained, see Remark 1 in Section 4 (resp. Remark 2 in Section 5). The main part of the proof of Theorem A is to show that $M_s < \infty$ is sufficient for s to be induced by a quasisymmetric homeomorphism of $\mathbb{S}^1$. This part is proved there by applying the above local regularity of the Douady-Earle extensions, that is Lemma 2.2 in [21]. Counter-examples have been found for that lemma. Using an algorithm called the MAY operator [1], the first counter-example was tested numerically and then proved in [13]. This means that there is a gap in [21] in the proof of the sufficiency of the condition for s to be induced by a quasisymmetric homeomorphism. In Section 3 of this paper, we first provide a counter-example to Lemma 2.2 in [21]. It takes less effort to check this example satisfying the assumption but failing the conclusion of Lemma 2.2 in [21] than to verify the counter-example given in [13]. Then we develop a new lemma for being used to bridge the gap in [21] in the proof of Theorem A. In Section 4, we apply the new lemma to bridge the gap by improving the sufficient condition for s to be induced by a quasisymmetric homeomorphism as follows.

Theorem 3. *Let h be a homeomorphism of $\widehat{\mathbb{R}}$ and let s be the shear function induced by h. Then*

$$K(ex(h)) < C(M_s);$$

that is, the maximal dilatation of the Douady-Earle extension of h is bounded from above by a positive constant only depending on the shear norm of h.

In the course of developing and proving our Theorems 1 and 2, two related metrics on $AT(\mathbb{D})$ come into place naturally: one is defined by using cross-ratio distortions under the boundary maps on degenerating sequences of quadruples and the other is defined by using quadrilateral dilatations under the boundary maps on degenerating sequences of quadrilaterals. Tracing back to similar metrics introduced on the universal Teichmüller space $T(\mathbb{D})$, in the remaining part of this introduction section we first recall the two metrics on $T(\mathbb{D})$ defined by using cross-ratio distortions and quadrilateral dilatations respectively; then we define the corresponding metrics on $AT(\mathbb{D})$.

For each $h \in T(\mathbb{D})$, let

$$K_e(h) = \inf_{f|_{\mathbb{S}^1}=h} K(f),$$

where the infimum is taken over all quasiconformal extensions of h to $\mathbb{D}$. Then the Teichmüller metric d_T is defined as

$$d_T(h_1, h_2) = \frac{1}{2} \log K_e(h_2 \circ h_1^{-1})$$

for any two points $h_1, h_2 \in T(\mathbb{D})$.

Given a quadruple $Q = \{a, b, c, c\}$ of four points a, b, c and d on $\mathbb{S}^1$ arranged in counter-clockwise order, let $M(Q)$ denote the conformal modulus of the (topological) quadrilateral $\mathbb{D}(a, b, c, d)$. Each orientation-preserving homeomorphism h of $\mathbb{S}^1$ maps $\mathbb{D}(a, b, c, d)$ to another quadrilateral $\mathbb{D}(h(a), h(b), h(c), h(d))$. Correspondingly, we let $M(h(Q))$ denote the conformal modulus of $\mathbb{D}(h(a), h(b), h(c), h(d))$. Given $h \in T(\mathbb{D})$, the maximal quadrilateral dilatation of h is defined as

$$K_m(h) = \sup_Q \{\frac{M(h(Q))}{M(Q)}\},$$

where the supremum is taken over all quadruples Q. Given any two points $h_1, h_2 \in T(\mathbb{D})$, we define

$$d_M(h_1, h_2) = \frac{1}{2} \log K_m(h_2 \circ h_1^{-1}).$$

By definition, it is clear that

$$d_M(h_1, h_2) = \frac{1}{2} \sup_Q |\log \frac{M(h_2(Q))}{M(h_1(Q))}|.$$

It is well known that

$$\frac{1}{K(f)} \leq \frac{M(h(Q))}{M(Q)} \leq K(f)$$

for any quasiconformal extension f of h and any quadruple Q. It follows that for any $h \in T(\mathbb{D})$,

$$K_m(h) \leq K_e(h),$$

and for any two points in $[h_1], [h_2] \in T(\mathbb{D})$,

$$d_M(h_1, h_2) \leq d_T(h_1, h_2).$$

Given a homeomorphism h of $\mathbb{S}^1$ and a quadruple $Q = \{a, b, c, d\}$ of four points on $\mathbb{S}^1$ arranged in the counter-clockwise order, let $h(Q)$ be the image quadruple $\{h(a), h(b), h(c), h(d)\}$ and let $cr(Q)$ denote the following cross ratio of Q:

$$(1.2) \qquad cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Given any two points $h_1, h_2 \in T(\mathbb{D})$, we define

$$d_C(h_1, h_2) = \sup_{cr(Q)=1} |\log \frac{cr(h_2(Q))}{cr(h_1(Q))}|.$$

One has known that d_M and d_C are metrics on $T(\mathbb{D})$ and the three metrics d_T and d_M and d_C on $T(\mathbb{D})$ are topologically equivalent. Since we cannot find references explicitly giving these two statements and their proofs are not long, we state them as the following folklore theorems and provide their proofs in Section 2.

Theorem 4 (Folklore Theorem 1). *Both d_M and d_C define metrics on $T(\mathbb{D})$.*

Theorem 5 (Folklore Theorem 2). *The three metrics d_T, d_M and d_C on $T(\mathbb{D})$ are topologically equivalent.*

An orientation-preserving homeomorphism h of $\mathbb{S}^1$ is said to be *symmetric* if

$$(1 + \delta(x, t))^{-1} \le \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \le 1 + \delta(x, t),$$

where $\delta(x, t) \to 0$ uniformly in $x \in \mathbb{S}^1$ as $t \to 0$. By letting $T_0(\mathbb{D})$ be the subspace of $T(\mathbb{D})$ whose elements are symmetric homeomorphisms, the asymptotic Teichmüller space can be expressed as the quotient space

$$AT(\mathbb{D}) = T(\mathbb{D})/T_0(\mathbb{D}).$$

Given any $h \in T(\mathbb{D})$, we denote by $[h]$ the corresponding point in $AT(\mathbb{D})$. This space was studied by Gardiner and Sullivan in [9] and the asymptotic Teichmüller spaces of Riemann surfaces were studied in [3] [4], [5], [8] and etc.

Parallel to d_T, d_M and d_C on $T(\mathbb{D})$, three metrics can be defined on $AT(\mathbb{D})$.

Let f be a quasiconformal conformal mapping on $\mathbb{D}$. The *boundary dilatation* $H(f)$ of f is defined as

$$H(f) = \inf_E K(f|_{\mathbb{D} \setminus E}),$$

where the infimum is taken over all compact subsets E of $\mathbb{D}$. Given any point $[h] \in AT(\mathbb{D})$, the boundary dilatation of $[h]$ is defined as

$$K_{ae}([h]) = \inf_{f|_{\mathbb{S}^1} \in [h]} H(f).$$

The Teichmüller distance between two points $[h_1], [h_2] \in AT(\mathbb{D})$ is defined as

$$d_{AT}([h_1], [h_2]) = \frac{1}{2} \log K_{ae}([h_2 \circ (h_1)^{-1}]).$$

Given a topological quadrilateral $Q = \mathbb{D}(a, b, c, d)$ or quadruple $Q = \{a, b, c, d\}$, we define the *minimal scale* of Q as

$$s(Q) = \min\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

Then a sequence $\{Q_n = \mathbb{D}(a_n, b_n, c_n, d_n)\}_{n=1}^{\infty}$ of topological quadrilaterals is said to be *degenerating* if $s(Q_n) \to 0$ as $n \to \infty$. A sequence $\{Q_n = \{a_n, b_n, c_n, d_n\}\}_{n=1}^{\infty}$ of quadruples is said to be *degenerating* if $cr(Q_n) = 1$ for each n and $s(Q_n) \to 0$ as

$n \to \infty$. For any $[h] \in AT(\mathbb{D})$, we define the asymptotically maximal quadrilateral dilatation of $[h]$ as

$$K_{am}([h]) = \sup_{\{Q_n\}} \limsup_{n \to \infty} \{\frac{M(h(Q_n))}{M(Q_n)}\},$$

where the supremum is taken over all degenerating sequences of topological quadrilaterals $\{Q_n\}_{n=1}^{\infty}$. Given any two points $[h_1], [h_2] \in AT(\mathbb{D})$, we define

$$d_{AM}([h_1], [h_2]) = \frac{1}{2} \log K_{am}([h_2 \circ h_1^{-1}]).$$

Clearly,

$$d_{AM}([h_1], [h_2]) = \frac{1}{2} \sup_{\{Q_n\}} \limsup_{n \to \infty} |\log \frac{M(h_2(Q_n))}{M(h_1(Q_n))}|,$$

where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of topological quadrilaterals.

From the proof of Theorem 1 of [25], it follows that for any $[h] \in AT(\mathbb{D})$,

$$K_{ae}([h]) \geq K_{am}([h]).$$

Thus, for any two points $[h_1], [h_2] \in AT(\mathbb{D})$,

$$d_{AM}([h_1], [h_2]) \leq d_{AT}([h_1], [h_2]).$$

Given any two points $[h_1], [h_2] \in AT(\mathbb{D})$, we define

$$d_{AC}([h_1], [h_2]) = \sup_{\{Q_n\}} \limsup_{n \to \infty} |\log \frac{cr(h_2(Q_n))}{cr(h_1(Q_n))}|,$$

where the supremum is taken over all degenerating sequences $\{Q_n\}_{n=1}^{\infty}$ of quadruples.

In Section 2, we also show the following two theorems.

Theorem 6. *Both d_{AM} and d_{AC} define metrics on $AT(\mathbb{D})$.*

Theorem 7. *The three metrics d_{AT}, d_{AM} and d_{AC} on $AT(\mathbb{D})$ are topologically equivalent.*

The paper is arranged as follows. We first give proofs of Theorems 4-7 in the second section. In the third section, we provide a counter-example to Lemma 2.2 in [21] and introduce a new lemma. Then in the fourth section we apply the new lemma to bridge the gap in the proof of Theorem A in [21] by showing Theorem 3 directly. The proofs of Theorems 1 and 2 are given in the fifth section.

2. Topological equivalence of three metrics on $T(\mathbb{D})$ and $AT(\mathbb{D})$

In this section, we show Theorems 4-7. Let $\rho_{(0,-1)}$ be the Poincaré density on $\mathbb{C}\backslash\{0, -1\}$ and let $d_{\rho_{(0,-1)}}(\alpha, \beta)$ be the distance under $\rho_{(0,-1)}$ between two points $\alpha, \beta \in \mathbb{C}\backslash\{0, -1\}$. The following lemma was proved in [24].

Lemma 1. *For any quadruples Q_1 and Q_2 on $\mathbb{S}^1$,*

$$(2.1) \qquad d_{\rho_{(0,-1)}}(cr(Q_1), cr(Q_2)) = |\log \frac{M(Q_1)}{M(Q_2)}|.$$

Proof of Theorem 4. It is easy to check that d_Q and d_C are symmetric and satisfy the triangle inequality. To prove d_M and d_C to be metrics, we only need to check that $d_M(h_1, h_2) = 0$ or $d_C(h_1, h_2) = 0$ implies $h_1 = h_2$.

We first show that $d_M(h_1, h_2) = 0$ implies $d_C(h_1, h_2) = 0$. If on the contrary, $d_C(h_1, h_2) > 0$, then there exist $m > 0$ and a quadruple $Q = (a, b, c, d) \in \mathbb{S}^1$ with $cr(Q) = 1$ such that

$$\frac{cr(h_1(Q))}{cr(h_2(Q))} > 1 + m.$$

Then Lemma 1 implies

$$|\log \frac{M(h_1(Q))}{M(h_2(Q))}| > 0,$$

which is a contradiction to $d_M(h_1, h_2) = 0$.

Now we show that $d_C(h_1, h_2) = 0$ implies $h_1 = h_2$. Let $A(z) = -\frac{z-i}{z+i}$, which is a Möbius transformation from $\mathbb{H}$ to $\mathbb{D}$, and let $\tilde{h}_1 = A^{-1} \circ h_1 \circ A$ and $\tilde{h}_2 = A^{-1} \circ h_2 \circ A$. Then $d_C(h_1, h_2) = 0$ means $cr(\tilde{h}_1(Q)) = cr(\tilde{h}_2(Q))$ for any quadruple $Q = \{a, b, c, d\} \subset \widehat{\mathbb{R}}$ with $cr(Q) = 1$. Since $\tilde{h}_1$ and $\tilde{h}_2$ fix $0, 1$ and ∞, by applying the condition $cr(\tilde{h}_1(Q)) = cr(\tilde{h}_2(Q))$ repeatedly on appropriate quadruples $Q \subset \mathbb{Z}$, we first obtain that $\tilde{h}_1(x) = \tilde{h}_2(x)$ for all $x \in \mathbb{Z}$. Then by using quadruples $Q \subset \frac{\mathbb{Z}}{2}$, we conclude that $\tilde{h}_1(x) = \tilde{h}_2(x)$ for all $x \in \frac{\mathbb{Z}}{2}$. Inductively, $\tilde{h}_1(x) = \tilde{h}_2(x)$ for all $x \in \frac{\mathbb{Z}}{2^n}$ for each $n \in \mathbb{N}$. Since $\cup_{n=1}^{\infty} \frac{\mathbb{Z}}{2^n}$ is dense in $\widehat{\mathbb{R}}$, it follows that $\tilde{h}_1(x) = \tilde{h}_2(x)$ for all $x \in \widehat{\mathbb{R}}$; that is, $h_1 = h_2$.   □

Proof of Theorem 5. Let $h \in T(\mathbb{D})$ and let $\{h_n\}_{n=1}^{\infty}$ be a sequence of points in $T(\mathbb{D})$.

At first, since $d_M(h_n, h) \leq d_T(h_n, h)$, it follows that $d_T(h_n, h) \to 0$ implies $d_M(h_n, h) \to 0$ as $n \to \infty$.

Secondly, we show that $d_M(h_n, h) \to 0$ implies $d_C(h_n, h) \to 0$ as $n \to \infty$. If on the contrary this is not true, then there exists a subsequence of $\{h_n\}_{n=1}^{\infty}$, denoted again by $\{h_n\}_{n=1}^{\infty}$ for simplicity, such that $d_C(h_n, h) > \varepsilon_0$ for all n and some $\varepsilon_0 > 0$. Thus there is a sequence $\{Q_n = \{a_n, b_n, c_n, d_n\}\}_{n=1}^{\infty} \subset \mathbb{S}^1$ of quadruples with $cr(Q_n) = 1$, such that for all n,

$$\frac{cr(h_n(Q_n))}{cr(h(Q_n))} > e^{\varepsilon_0}.$$

Since h is a given quasisymmetric homeomorphism and $cr(Q_n) = 1$, $\{cr(h(Q_n))\}_{n=1}^{\infty}$ stays in a compact subset of the positive half real axis. Thus there exists $\varepsilon_0' > 0$ such that for all n,

$$d_{\rho(0, -1)}(cr(h_n(Q_n)), cr(h(Q_n))) > \varepsilon_0'.$$

Using Lemma 1, we conclude that for all n,

$$|\log \frac{M(h_n(Q_n))}{M(h(Q_n))}| > \varepsilon_0'.$$

This is a contradiction to the assumption that $d_Q(h_n, h) \to 0$ as $n \to \infty$.

Finally, we show that $d_C(h_n, h) \to 0$ implies $d_T(h_n, h) \to 0$ as $n \to \infty$. If on the contrary this is not true, then by passing to a subsequence we may assume that $d_T(h_n, h) > \varepsilon_1$ for all n and some $\varepsilon_1 > 0$.

Let $ex(h_n)$ be the Douady-Earle extension of h_n. Since $d_C(h_n, h) \to 0$ as $n \to \infty$, there exists $C > 0$ such that for all n and any quadruple Q with $cr(Q) = 1$,

$$|\log cr(h_n(Q))| < C.$$

Using the main theorem in [11], there exists $K_0 > 1$ such that for all n,

$$K(ex(h_n)) \leq K_0.$$

Thus $\{h_n\}$ is a bounded sequence in $T(\mathbb{D})$. Passing to subsequences, we may assume that $\{h_n\}_{n=1}^{\infty}$ converges to a quasisymmetric homeomorphism h' uniformly on $\mathbb{S}^1$ and also converges to a point h'' in $T(\mathbb{D})$ under the metric d_T. Note that h' and h'' fix the same three points as each h_n does. Using the conclusions in the previous two steps, we obtain $d_C(h_n, h'') \to 0$ as $n \to \infty$. Then for each quadruple Q with $cr(Q) = 1$,

$$cr(h''(Q)) = \lim_{n \to \infty} cr(h_n(Q)) = cr(h'(Q)).$$

Therefore, $h' = h''$. Since $d_c(h_n, h) \to 0$ as $n \to \infty$, we concluded that $h = h' = h''$. Thus, $d_T(h_n, h) \to 0$ as $n \to \infty$. This is a contradiction to the assumption that $d_T(h_n, h) > \varepsilon_1$ for all n.

We complete the proof. $\square$

Proof of Theorem 6. Using definitions, it is easy to check that d_{AM} and d_{AC} are symmetric and satisfy the triangle inequality. To prove d_{AM} and d_{AC} to be metrics, it remains to show that $d_{AM}([h_1], [h_2]) = 0$ or $d_{AC}([h_1], [h_2]) = 0$ implies $[h_1] = [h_2]$.

Using the same argument to show that $d_M(h_1, h_2) = 0$ implies $d_C(h_1, h_2) = 0$ in the proof of Theorem 4, one can see that $d_{AM}([h_1], [h_2]) = 0$ implies $d_{AC}([h_1], [h_2]) = 0$.

In the following, we apply some properties of Douady-Earle extensions to show that $d_{AC}([h_1], [h_2]) = 0$ implies $[h_1] = [h_2]$. Suppose on the contrary that $[h_1] \neq [h_2]$, which means $h_1 \circ (h_2)^{-1}$ is not symmetric. Let $ex(h_1)$ and $ex(h_2)$ be the Douady-Earle extensions of h_1 and h_2 respectively. Using a result of [5], we know that $ex(h_1) \circ (ex(h_2))^{-1}$ is not asymptotic conformal on $\mathbb{D}$, which means that there exist a constant $\varepsilon_0 > 0$ and a sequence $\{D_n\}_{n=1}^{\infty}$ of hyperbolic disks in $\mathbb{D}$ of diameter 1 with the Euclidean distance from D_n to $\mathbb{S}^1$ approaching 0 as $n \to \infty$ such that

$$(2.2) \qquad \|\mu_{ex(h_1)}|_{D_n} - \mu_{ex(h_2)}|_{D_n}\|_{L^\infty} \geq \varepsilon_0$$

for all n. Let D_0 be the hyperbolic disk on $\mathbb{D}$ of diameter 1 and centered in 0, and assume that $\gamma_n \in M\ddot{o}b(\mathbb{D})$ and $\gamma_n(D_0) = D_n$, where $M\ddot{o}b(\mathbb{D})$ is the group of all Möbius transformation preserving $\mathbb{D}$. Let A_{1n} and $A_{2n} \in M\ddot{o}b(\mathbb{D})$ such that $A_{1n} \circ h_1 \circ \gamma_n$ and $A_{2n} \circ h_2 \circ \gamma_n$ fix $1, -1, i$ for all n. Given any quadruple $Q \in \mathbb{S}^1$ with $cr(Q) = 1$, from the fact that D_n converges to the boundary $\mathbb{S}^1$, one can see that $s(\gamma_n(Q)) \to 0$ as $n \to \infty$. Using the assumption that $d_{AC}([h_1], [h_2]) = 0$, we obtain

$$(2.3) \qquad \lim_{n \to \infty} \frac{cr(A_{1n} \circ h_1 \circ \gamma_n(Q))}{cr(A_{2n} \circ h_2 \circ \gamma_n(Q))} = 1.$$

Let $ex(A_{1n} \circ h_1 \circ \gamma_n)$ and $ex(A_{2n} \circ h_2 \circ \gamma_n)$ be the Douady-Earle extensions of $A_{1n} \circ h_1 \circ \gamma_n$ and $A_{2n} \circ h_2 \circ \gamma_n$ respectively. Since these quasiconformal mappings fix three common points and have constant maximal dilatations, passing to subsequences,

we may assume that they converge to two quasisymmetric homeomorphisms $\widetilde{h}_1$ and $\widetilde{h}_2$ uniformly on $\mathbb{S}^1$. Using (2.3) and the convergences, we obtain

$$cr(\widetilde{h}_1(Q)) = cr(\widetilde{h}_2(Q))$$

for any quadruple $Q \in \mathbb{S}^1$ with $cr(Q) = 1$. Using the fact $\widetilde{h}_1$ equals to $\widetilde{h}_2$ at three fixed point and Theorem 4, we conclude that $\widetilde{h}_1 = \widetilde{h}_2$. Using a convergence property of Douady-Earle extensions, $Belt(ex(A_{1n} \circ h_1 \circ \gamma_n))$ and $Belt(ex(A_{2n} \circ h_2 \circ \gamma_n))$ converge to $Belt(ex(\widetilde{h}_1))$ and $Belt(ex(\widetilde{h}_2))$ uniformly on D_0 respectively; that is,

$$(2.4) \qquad ||\mu_{ex(A_{1n} \circ h_1 \circ \gamma_n)}|_{D_0} - \mu_{ex(A_{2n} \circ h_2 \circ \gamma_n)}|_{D_0}||_{L^\infty} \to 0.$$

On the other hand, by (2.2) and the conformal naturality of Douady-Earle extensions,

$$||\mu_{ex(A_{1n} \circ h_1 \circ \gamma_n)}|_{D_0} - \mu_{ex(A_{2n} \circ h_2 \circ \gamma_n)}|_{D_0}||_{L^\infty}$$
$$= \quad ||\mu_{ex(h_1)}|_{D_n} - \mu_{ex(h_2)}|_{D_n}||_{L^\infty} \geq \varepsilon_0.$$

This is a contradiction to (2.4). Thus, $h_1 \circ (h_2)^{-1}$ is symmetric and hence $[h_1] = [h_2]$. $\qquad\square$

Lemma 2. [12] *Let h be an orientation-preserving homeomorphism of $\mathbb{S}^1$ and $ex(h)$ be the Douady-Earle extension of h to the closed unit disk $\mathbb{D}$. Let $p \in \mathbb{S}^1$ and I_p be an open arc on $\mathbb{S}^1$ containing p and symmetric with respect to p. Assume that*

$$\|h|_{I_p}\|_{cr} = \sup_{cr(Q)=1} |\log cr(h(Q))| < \infty,$$

where the supremum is over all quadruple $Q \in I$. Then there exists an open hyperbolic half plane U_p with p at the middle of its boundary on $\mathbb{S}^1$ such that

$$\log K(ex(h)|U_p) \leq C_1 \|h|_{I_p}\|_{cr} + C_2$$

for two universal positive constants C_1 and C_2, where $K(ex(h)|U_p)$ is the maximal dilatation of $ex(h)$ on U_p.

Proof of Theorem 7. Let $[h] \in AT(\mathbb{D})$ and $\{[h_n]\}_{n=1}^\infty$ be a sequence of points in $AT(\mathbb{D})$. We prove the equivalence of the topologies induced by the three metrics d_{AT}, d_{AM} and d_{AC} in the same order as we show the equivalence of the topologies induced by d_T, d_M and d_C in Theorem 5.

At first, as pointed out in the introduction, $d_{AM}([h_n], [h]) \leq d_{AT}([h_n], [h])$. Thus, $d_{AT}([h_n], [h]) \to 0$ implies $d_{AM}([h_n], [h]) \to 0$ as $n \to \infty$.

Secondly, we show $d_{AM}([h_n], [h]) \to 0$ implies $d_{AC}([h_n], [h]) \to 0$ as $n \to \infty$. If on the contrary this is not true, then there exists a subsequence of $\{[h_n]\}_{n=1}^\infty$, denoted again by $\{[h_n]\}_{n=1}^\infty$ for simplicity, such that $d_{AC}([h_n], [h]) > \varepsilon_0$ for all n and some $\varepsilon_0 > 0$. Then, for each n, there is a degenerating sequence $\{Q_{n_j} = \{a_{n_j}, b_{n_j}, c_{n_j}, d_{n_j}\}\}_{j=1}^\infty$ of quadruples and $\varepsilon_0 > 0$ such that, for each n,

$$\lim_{j \to \infty} \frac{cr(h_n(Q_{n_j}))}{cr(h(Q_{n_j}))} > e^{\varepsilon_0}.$$

Thus, we can choose a degenerating sequence $\{Q_n = \{a_n, b_n, c_n, d_n\}\}_{n=1}^\infty$ of quadruples such that for each n,

$$\frac{cr(h_n(Q_n))}{cr(h(Q_n))} > e^{\varepsilon_0}.$$

Using the same argument in the second step in the proof of Theorem 5, we obtain

$$d_{\rho_{(0,-1)}}(cr(h_n(Q_n)), cr(h(Q_n))) > \varepsilon'_0$$

for all n and some $\varepsilon'_0 > 0$. By Lemma 1, for all n,

$$|\log \frac{M(h_n(Q_n))}{M(h(Q_n))}| > \varepsilon'_0,$$

which is a contradiction to the assumption that $d_{AQ}([h_n], [h]) \to 0$ as $n \to \infty$.

Finally, we show that $d_{AC}([h_n], [h]) \to 0$ implies $d_{AT}([h_n], [h]) \to 0$. Suppose on the contrary that $d_{AT}([h_n], [h]) \not\to 0$. Passing to a subsequence, we may assume that, for all n,

$$(2.5) \qquad\qquad d_{AT}([h_n], [h]) > \varepsilon > 0.$$

Let $ex(h_n)$ and $ex(h)$ be the Douady-Earle extensions of h_n and h respectively. Using (2.5), we can choose a sequence of hyperbolic disks $\{D_n\}_{n=1}^{\infty} \subset \mathbb{D}$ of diameter 1 with the Euclidean distance from D_n to $\mathbb{S}^1$ approaching 0 as $n \to \infty$ such that, for all n,

$$(2.6) \qquad\qquad \|\mu_{ex(h_n)}|_{D_n} - \mu_{ex(h)}|_{D_n}\|_{L^{\infty}} \geq \epsilon.$$

Let D_0 be the hyperbolic disk on $\mathbb{D}$ of diameter 1 and centered in 0, and assume that $\gamma_n \in M\ddot{o}b(\mathbb{D})$ and $\gamma_n(D_0) = D_n$. Let A_{1n} and $A_{2n} \in M\ddot{o}b(\mathbb{D})$ such that $A_{1n} \circ h_n \circ \gamma_n$ and $A_{2n} \circ h \circ \gamma_n$ fix $1, -1, i$ for all n. Now given a quadruple Q with $cr(Q) = 1$, one can check that $s(\gamma(Q)) \to 0$ as $n \to \infty$. Using the assumption that $d_{AC}([h_n], [h]) \to 0$ as $n \to \infty$, we obtain

$$(2.7) \qquad\qquad \lim_{n \to \infty} |\frac{cr(A_{1n} \circ h_n \circ \gamma_n(Q))}{cr(A_{2n} \circ h \circ \gamma_n(Q))}| = 1.$$

Since $d_{AC}([h_n], [h]) \to 0$ as $n \to \infty$, $\{d_{AC}([h_n], [Id])\}_{n=1}^{\infty}$ is bounded. Using the definition of d_{AC} and Lemma 2, we know that $\{H(ex(h_n))\}_{n=1}^{\infty}$ is bounded. For each n, we can choose $0 < r_n < 1$ such that $\frac{1+\|\mu_n\|_{\infty}}{1-\|\mu_n\|_{\infty}} \leq H(ex(h_n)) + 1$, where μ_n is defined as

$$\mu_n(z) = \begin{cases} \mu_{ex(h_n)}(z), & r_n \leq |z| < 1, \\ 0, & |z| < r_n. \end{cases}$$

Let f_n be the normalized (i.e., fixing three points 1, -1 and i) quasiconformal homeomorphism of $\mathbb{D}$ with the Beltrami coefficient μ_n, and $\widetilde{h}_n = f_n|_{\mathbb{S}^1}$. Then $\widetilde{h}_n \in [h]$ for each n and $\{K(ex(\widetilde{h}_n))\}_{n=1}^{\infty}$ is bounded. For simplicity, we denote $\{[\widetilde{h}_n]\}_{n=1}^{\infty}$ by $\{[h_n]\}_{n=1}^{\infty}$. In other words, by replacing representatives of $[h_n]'s$, we may assume that $\{K(ex(h_n))\}_{n=1}^{\infty}$ is bounded.

Since all $A_{1n} \circ h_n \circ \gamma_n$ and $A_{2n} \circ h \circ \gamma_n$ are normalized to fix three points, passing to subsequences we may assume that $A_{1n} \circ h_n \circ \gamma_n$ and $A_{2n} \circ h \circ \gamma_n$ converge uniformly to quasisymmetric maps h^* and $\widehat{h}^*$ respectively on $\mathbb{S}^1$. Using (2.7) and the convergences, we obtain

$$cr(h^*(Q)) = cr(\widehat{h}^*(Q))$$

for any quadruple Q with $cr(Q) = 1$. Thus, $h^* = \widehat{h}^*$. By a convergence property of Douady-Earle extensions, we obtain

$$(2.8) \qquad\qquad \|\mu_{ex(A_{1n} \circ h_n \circ \gamma_n)}|_{D_0} - \mu_{ex(A_{2n} \circ h \circ \gamma_n)}|_{D_0}\|_{L^{\infty}} \to 0.$$

On the other hand, (2.6) and the conformal naturality of Douady-Earle extensions imply that for each n,

$$||\mu_{ex(A_{1n}\circ h_n\circ\gamma_n)}|_{D_0} - \mu_{ex(A_{2n}\circ h\circ\gamma_n)}|_{D_0}||_{L^\infty}$$
$$= ||\mu_{ex(h_n)}|_{D_n} - \mu_{ex(h)}|_{D_n}||_{L^\infty} \geq \epsilon$$

This is a contradiction to (2.8). Thus, $d_{AC}([h_n],[h]) \to 0$ implies $d_{AT}([h_n],[h]) \to 0$ as $n \to \infty$.

$\square$

3. A COUNTER-EXAMPLE AND A NEW LEMMA

In this section, we construct a sequence $\{h_n\}_{n=2}^\infty$ of orientation-preserving homeomorphisms of $\mathbb{S}^1$ such that for each $n \geq 2$, h_n fixes four points ± 1 and $\pm i$ and the Douady-Earle extension $ex(h_n)$ of h_n fixes the origin, but the maximal dilatation $K(ex(h_n))(0)$ of $ex(h_n)$ at the origin goes to ∞ as $n \to \infty$. Let A be the Möbius transformation from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$ mapping $-1, -i$ and 1 to $-1, 0$ and 1 respectively. Then the sequence $\{A \circ f_n \circ A^{-1}\}_{n=2}^\infty$ is a counter-example to Lemma 2.2 in [21] (the statement of this lemma is recalled in the introduction).

Given each $n \geq 2$, we first define the map h_n on the circular arc of $\mathbb{S}^1$ in the first quadrant; that is, h_n maps the circular arc from 1 to $e^{\frac{\pi}{2}[1-\frac{1}{n}]i}$ linearly onto the arc from 1 to $e^{\frac{\pi}{2n}i}$ and maps the arc from $e^{\frac{\pi}{2}[1-\frac{1}{n}]i}$ to i linearly onto the arc from $e^{\frac{\pi}{2n}i}$ to i. Clearly, both 1 and i are fixed under h_n and $e^{\frac{\pi}{2}[1-\frac{1}{n}]i}$ is mapped to $e^{\frac{\pi}{2n}i}$. Through conjugation under taking conjugacies of complex numbers, h_n is extended to be defined on the arc of $\mathbb{S}^1$ in the fourth quadrant. Finally, through conjugation under taking mirror images with respect to the imaginary axis, h_n is extended to be defined on $\mathbb{S}^1$. Note that h_2 is equal to the identity map on $\mathbb{S}^1$. Since each $h_n, n \geq 2$, is symmetric with respect to the origin, using definition we see that the Douady-Earle extension $ex(h_n)$ fixes the origin. The main work is to show that

$$(3.1) \qquad |Belt(ex(h_n))(0)| \to 1 \ \text{ as } \ n \to \infty.$$

Before starting to show this convergence, let us first recall some formulas to express the Beltrami coefficient of the Douady-Earle extension at the origin.

Let h be an orientation-preserving homeomorphism of $\mathbb{S}^1$. Assume that the Douady-Earle extension $ex(h)$ is normalized at the origin; that is, $ex(h)(0) = 0$. Then the Beltrami coefficient of $ex(h)$ at the origin can be explicitly expressed as follows [2]. Given any point $z \in \mathbb{D}$, $ex(h)(z)$ is equal to the unique point $w \in \mathbb{D}$ such that

$$F(z,w) = 0,$$

where

$$(3.2) \qquad F(z,w) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{h(\xi) - w}{1 - \bar{w}h(\xi)} \cdot \frac{1 - |z|^2}{|z - \xi|^2} |d\xi|.$$

If we let

$$(3.3) \qquad c_1 = \frac{\partial F}{\partial z}(0,0) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \bar{\xi}h(\xi)|d\xi|, \ \ c_{-1} = \frac{\partial F}{\partial \bar{z}}(0,0) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \xi h(\xi)|d\xi|$$

and

$$(3.4) \qquad d_1 = \frac{\partial F}{\partial w}(0,0) = -1, \ \ d_{-1} = \frac{\partial F}{\partial \bar{w}}(0,0) = \frac{1}{2\pi}\int_{\mathbb{S}^1} h(\xi)^2 |d\xi|,$$

then

$$(3.5) \qquad \frac{\partial w}{\partial \bar{z}}(0) = -\frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial \bar{z}}(0,0) - \frac{\partial F}{\partial \bar{w}}(0,0)\overline{\frac{\partial F}{\partial z}(0,0)}}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \bar{w}}(0,0)|^2} = \frac{c_{-1} + d_{-1}\bar{c}_1}{1 - |d_{-1}|^2}.$$

and

$$(3.6) \qquad \frac{\partial w}{\partial z}(0) = -\frac{\overline{\frac{\partial F}{\partial w}(0,0)}\frac{\partial F}{\partial z}(0,0) - \frac{\partial F}{\partial \bar{w}}(0,0)\overline{\frac{\partial F}{\partial \bar{z}}(0,0)}}{|\frac{\partial F}{\partial w}(0,0)|^2 - |\frac{\partial F}{\partial \bar{w}}(0,0)|^2} = \frac{c_1 + d_{-1}\bar{c}_{-1}}{1 - |d_{-1}|^2}.$$

Therefore the Beltrami coefficient $Belt(ex(h))$ at 0 is equal to

$$(3.7) \qquad\qquad Belt(ex(h))(0) = \frac{c_{-1} + d_{-1}\bar{c}_1}{c_1 + d_{-1}\bar{c}_{-1}}.$$

If the boundary homeomorphism h satisfies $h(\bar{\xi}) = \overline{h(\xi)}$ and $h(-\bar{\xi}) = -\overline{h(\xi)}$, then one can rewrite

$$(3.8) \qquad c_1 = \frac{2}{\pi}Re\int_0^{\frac{\pi}{2}} \bar{\xi}h(\xi)|d\xi|, \ \ c_{-1} = \frac{2}{\pi}Re\int_0^{\frac{\pi}{2}} \xi h(\xi)|d\xi|$$

and

$$(3.9) \qquad\qquad d_{-1} = \frac{2}{\pi}Re\int_0^{\frac{\pi}{2}} h^2(\xi)|d\xi|.$$

Now let $h = h_n$. In the following, we show that both c_1 and c_{-1} converge to $\frac{2}{\pi}$ and d_{-1} converges to 1 as $n \to \infty$. It follows that $Belt(ex(h_n))(0)$ converges to 1 as $n \to \infty$.

We first show that c_1 converges to $\frac{2}{\pi}$ as $n \to \infty$.

Let $\theta_n = \frac{\pi}{2n}$, where $n \geq 2$. Then for each $\xi = e^{i\theta}$ with $0 \leq \theta \leq \frac{\pi}{2} - \theta_n$, $h(\xi) = 1 + O(\frac{1}{n})$. Clearly,

$$\int_0^{\frac{\pi}{2}} \bar{\xi}h(\xi)|d\xi| = \int_0^{\frac{\pi}{2}-\theta_n} \bar{\xi}h(\xi)|d\xi| + \int_{\frac{\pi}{2}-\theta_n}^{\frac{\pi}{2}} \bar{\xi}h(\xi)|d\xi|.$$

Then

$$\int_0^{\frac{\pi}{2}-\theta_n} \bar{\xi}h(\xi)|d\xi| = \int_0^{\frac{\pi}{2}-\theta_n} \bar{\xi}(1 + O(\frac{1}{n}))|d\xi| = \int_0^{\frac{\pi}{2}-\theta_n} \bar{\xi}|d\xi| + O(\frac{1}{n})$$

$$= \int_0^{\frac{\pi}{2}} \bar{\xi}|d\xi| + O(\frac{1}{n}) = 1 - i + O(\frac{1}{n})$$

and

$$|\int_{\frac{\pi}{2}-\theta_n}^{\frac{\pi}{2}} \bar{\xi}h(\xi)|d\xi|| \leq \int_{\frac{\pi}{2}-\theta_n}^{\frac{\pi}{2}} |\bar{\xi}h(\xi)||d\xi| = \theta_n = O(\frac{1}{n}).$$

Thus

$$\int_0^{\frac{\pi}{2}} \bar{\xi}h(\xi)|d\xi| = 1 - i + O(\frac{1}{n})$$

and then

$$c_1 = \frac{2}{\pi} + O(\frac{1}{n}).$$

It follows that c_1 converges to $\frac{2}{\pi}$ as $n \to \infty$.

Similarly, one can show that c_{-1} converges to $\frac{2}{\pi}$ as $n \to \infty$.

It remains to show that d_{-1} converges to 1 as $n \to \infty$. Since for each $\xi = e^{i\theta}$ with $0 \leq \theta \leq \frac{\pi}{2} - \theta_n$, $h(\xi) = 1 + O(\frac{1}{n})$ implies $h^2(\xi) = 1 + O(\frac{1}{n})$. Then

$$
\begin{aligned}
d_{-1} &= \frac{2}{\pi} Re(\int_0^{\frac{\pi}{2}-\theta_n} h^2(\xi)|d\xi| + \int_{\frac{\pi}{2}-\theta_n}^{\frac{\pi}{2}} h^2(\xi)|d\xi|) \\
&= \frac{2}{\pi} Re(\int_0^{\frac{\pi}{2}-\theta_n} (1 + O(\frac{1}{n}))|d\xi| + O(\frac{1}{n})) = \frac{2}{\pi} Re(\frac{\pi}{2} + O(\frac{1}{n})) = 1 + O(\frac{1}{n}).
\end{aligned}
$$

Thus d_{-1} converges to 1 as $n \to \infty$. We complete the proof of (3.1).

In the second half of this section, we introduce a new lemma which is used in the next section to prove Theorem 3. This bridges the gap in the proof of Theorem A in [21].

Lemma 3. *Let $M > 0$ and h be an orientation-preserving homeomorphism of $\mathbb{S}^1$. Assume that $p_1 = \frac{3+4i}{5}$, $p_2 = -\bar{p}$, $p_3 = -p$ and $p_4 = \bar{p}$. If $|\log cr(h(Q))| < M$ holds for any of the following five quadruples*

$$Q_0 = \{-1, -i, 1, i\}, Q_1 = \{-1, 1, p_1, i\}, Q_2 = \{-1, 1, i, p_2\},$$

$$Q_4 = \{-1, p_3, -i, 1\} \text{ and } Q_4 = \{-1, -i, p_4, 1\},$$

then there exist a universal small neighborhood U of the origin and a constant $C > 0$, only depending on U and M, such that

$$K(ex(h)|_U) < C.$$

Proof. Using the conformal naturality of Douady-Earle extensions and invariance of the cross-ratio distortion norm of h under postcomposition by Möbius transformations, we can assume that the Douady-Earle extension $ex(h)$ fixes the origin. Using Lemma 3.6 in [17], the conclusion of this lemma follows from the following claim.

Claim. *There exist $0 < s_0 < \pi$ (universal, one may take $s_0 = \frac{2\pi}{3}$) and $\epsilon > 0$ (only depending on M) such that for any circular arc Γ of length $> s_0$ and $< \pi$, the length of $h(\gamma)$ is greater than ϵ.*

We divide the proof of this claim into the following four steps.

Step 1. In this step, we show there exists a constant $\epsilon_1 > 0$ only depending on M such that for each half circle Γ bounded by two of the four points $\{\pm 1, \pm i\}$, the length $|h(\Gamma)|$ of $h(\Gamma)$ is greater than ϵ_1. For example, we show in detail how the two conditions $ex(h)(0) = 0$ and $|\log cr(h(Q_1))| < M$ imply that the length $|h(\widehat{-i, 1, i})|$ is greater than ϵ_1.

Let $A = -1$, $B = -i$, $C = 1$ and $D = i$, and let $A' = h(A)$, $B' = h(B)$, $C' = h(C)$, $D' = h(D)$ and $p'_k = h(p_k)$, where $k = 1, 2, 3, 4$.

Using the normalization condition $ex(h)(0) = 0$, it is shown in Step 1 of the proof of Lemma 1 in [11] that

$$|A'\widehat{B'C'}D'| \geq \frac{\pi}{3},$$

where $A'\widehat{B'C'}D'$ represents the image under h of the three quarter of the circle from -1 to i and through $-i$ and 1 in the counterclockwise direction. Furthermore,

it is shown in Case 1 in Step 2 of the proof of Lemma 1 in [11] that

$$|\widehat{B'C'D'p_2'}| > \frac{\pi}{3},$$

where $\widehat{B'C'D'p_2'}$ represents the image under h of the circular arc from $-i$ to p_2 and through 1 and i in the counterclockwise direction.

Now we apply the condition $|\log cr(h(Q_1))| < M$ to show that there exists $\epsilon_1 > 0$ such that

$$|\widehat{B'C'D'}| > \epsilon_1.$$

Suppose that $l = |\widehat{B'C'D'}|$ is very small. Then $l_1 = |\widehat{C'p_1'D'}|$ is also very small since $C'p_1'D' \subset B'C'D'$. It also follows that

$$|\widehat{D'p_2'}| = |\widehat{B'C'D'p_1'}| - |\widehat{B'C'D'}| > \frac{\pi}{3} - l$$

and

$$|\widehat{A'B'C'}| = |\widehat{A'B'C'D'}| - |\widehat{C'p_1'D'}| > \frac{2\pi}{3} - l_1 > \frac{2\pi}{3} - l.$$

Then

$$|\log cr(h(Q_1))| = |\log \frac{2\sin\frac{|\widehat{C'p_1'D'}|}{2} \times 2\sin\frac{|\widehat{p_2'A'}|}{2}}{2\sin\frac{|\widehat{A'B'C'}|}{2} \times 2\sin\frac{|\widehat{D'p_2'}|}{2}}|.$$

Clearly,

$$\frac{2\sin\frac{|\widehat{C'p_1'D'}|}{2} \times 2\sin\frac{|\widehat{p_2'A'}|}{2}}{2\sin\frac{|\widehat{A'B'C'}|}{2} \times 2\sin\frac{|\widehat{D'p_2'}|}{2}} < \frac{2\sin\frac{l}{2} \times 2}{2\sin(\frac{\pi}{3} - \frac{l}{2}) \times 2\sin(\frac{\pi}{6} - \frac{l}{2})}.$$

Thus this quotient approaches 0 as $l \to 0$, which implies that $|\log cr(h(Q_1))|$ goes to ∞ as $l \to 0$. This is a contradiction to $|\log cr(h(Q_1))| < M$. Therefore, there exists $\epsilon_1 > 0$ such that $|\widehat{B'C'D'}| > \epsilon_1$.

Similarly, using $ex(h)(0) = 0$ and $|\log cr(h(Q_3))| < M$, we show $|\widehat{C'D'A'}| > \epsilon_1$; using $ex(h)(0) = 0$ and $|\log cr(h(Q_4))| < M$, we show $|\widehat{D'A'B'}| > \epsilon_1$; using $ex(h)(0) = 0$ and $|\log cr(h(Q_1))| < M$, we show $|\widehat{A'B'C'}| > \epsilon_1$.

Step 2. Using the estimates on the lengths of the images of the four half circles obtained in Step 1 and the condition that $|\log cr(h(Q_0))| < M$, the strategy to show the existence of ϵ_1 also imply that there exists $\epsilon_2 > 0$ such that the length of the image under h of each of the four quarters $\widehat{Cp_1D}$, $\widehat{Dp_2A}$, $\widehat{Ap_3B}$ and $\widehat{Bp_4C}$ of the circle is greater than ϵ_2.

Step 3. Using the estimates on the lengths of the images of the four half circles obtained in Step 1, the estimates on the lengths of the images of the four quarters obtained in Step 2, and the condition that $|\log cr(h(Q_k))| < M$ for $k = 1, 2, 3, 4$, the strategy to show the existence of ϵ_1 also imply that there exists $\epsilon_3 > 0$ such that the length of the image under h of each of the eight circular arc bounded by two adjacent points among $\{-1, p_3, -i, p_4, 1, p_1, i, p_2\}$ is greater than ϵ_3.

Step 4. Let $s_0 = \frac{2\pi}{3}$ and $\epsilon = \epsilon_3$. Then for any circular arc Γ with length $s_0 < |\Gamma| < \pi$, Γ contains at least one circular arc between two adjacent points among $\{-1, p_3, -i, p_4, 1, p_1, i, p_2\}$ since two longest adjacent arcs among those eight arcs comprise an arc of length $2\arcsin\frac{4}{5} < \frac{2\pi}{3}$. Therefore, the length $|h(\Gamma)| > \epsilon_3 = \epsilon$.

$\square$

4. The Farey tesselation, shear maps, and proof of Theorem 3

We first introduce some background on the Farey tesselation and shear maps. Then we prove Theorem 3.

In the following, we use the upper half plane $\mathbb{H}$ as a model for the hyperbolic plane and recall the Farey tesselation and the shear map, which were introduced by Penner [19] and furthered studied by Saric [21] [22].

Let $\triangle_0$ be the idea geodesic triangle on $\mathbb{H}$ with vertices $0, 1$ and ∞ and let Γ be the group generated by the hyperbolic reflections to the sides of $\triangle_0$. The Farey tesselation $\mathcal{F}$ is the collection of the geodesics on the Γ-orbits of the edges of $\triangle_0$. The set of the endpoints of the geodesics in $\mathcal{F}$ is equal to $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.

Let $e \in \mathcal{F}$ and let $a, c \in \widehat{\mathbb{R}}$ be the endpoints of e. Assume that $(\triangle_1, \triangle_2)$ is a pair of two triangles on $\mathbb{H}$ with disjoint interiors and sharing a common boundary e. Besides the vertices a, c, we denote the third vertices of $\triangle_1$ and $\triangle_2$ by b and d respectively. Given a homeomorphism $h : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$, the shear $s_h(e)$ mentioned in the introduction can be precisely defined by

$$s_h(e) = \log cr(\tilde{h}(a, b, c, d)),$$

where $cr(\tilde{h}(a, b, c, d))$ is the cross ratio defined by (1.2) in the introduction. Then $h : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ induces a map $s_h : \mathcal{F} \to \mathbb{R}$, which is called the shear map or function or coordinate of $\tilde{h}$. Conversely, given any map $s : \mathcal{F} \to \mathbb{R}$, there is a unique injective map h_s from the vertices $\widehat{\mathbb{Q}}$ of the Farey tesselation $\mathcal{F}$ into $\widehat{\mathbb{R}}$ such that h_s fixes $0, 1$ and ∞ and the shear map of h_s is equal to s. We call h_s the characteristic map of s.

A fan $\mathcal{F}_p$ of geodesics in $\mathcal{F}$ with tip $p \in \widehat{\mathbb{Q}}$ consists of all edges of $\mathcal{F}$ that have a common endpoint at p. Each fan $\mathcal{F}_p$ has a natural order. Take a horocycle C tangent to $\widehat{\mathbb{R}}$ at p and choose an orientation on C such that the corresponding horoball is to the left of C. Let e, e' be two geodesics in $\mathcal{F}_p$. Define $e < e'$ if the point $e \cap C$ comes before the point $e' \cap C$. This natural order on $\mathcal{F}_p$ gives a one-to-one correspondence between $\mathcal{F}_p$ and the set $\mathbb{Z}$ of integers, and any two such correspondences differ by a translation in $\mathbb{Z}$.

Given a shear map s and a fan $\mathcal{F}_p = \{e_n^p\}_{n \in \mathbb{Z}}$, the quantity $s(p; m, k)$ is defined by (1.1) in the introduction.

Remark 1. Let $s : \mathcal{F} \to \mathbb{R}$ be a map from $\mathcal{F}$ to $\mathbb{R}$ and let h_s be the characteristic map of s. Assume that $e_j^p, j = m - k, m, m + k$, are three geodesics in a fan $\mathcal{F}_p$. Let a_j^p be the other endpoint of e_j^p besides p, where $j = m - k, m, m + k$. Using the definition of cross ratio given by (1.2) and through pre-composition by a Möbius transformation to arrange p at ∞, one can easily see that for any $m, k \in \mathbb{Z}$,

$$cr(\{p, a_{m-k}^p, a_m^p, a_{m+k}^p\}) = 1.$$

Similarly, through post-composition by a Möbius transformation to arrange $h_s(p)$ at ∞, we see

$$s(p; m, k) = cr(h_s(\{p, a_{m-k}^p, a_m^p, a_{m+k}^p\})).$$

Using the definitions of d_S and d_C in the introduction, we know that for any $h_1, h_2 \in T(\mathbb{D})$,

$$d_S(s_1, s_2) \leq d_C(h_1, h_2),$$

where s_1 and s_2 are the shear functions induced by h_1 and h_2.

Substituting h_2 by id in the previous inequality, one can see that the shear norm M_{s_1} of h_1 is finite is a necessary condition for s_1 to be induced by a quasisymmetric map. This verifies the necessity of the condition in Theorem A of [21] for a shear function to be induced by a quasisymmetric homeomorphism.

Using the previous inequality and Theorem 5, we can conclude that as $n \to \infty$,

$$d_T(h_n, h) \to 0 \ \text{implies} \ d_S(s_n, s) \to 0.$$

It is shown in Theorem B of [21] that as $n \to \infty$,

$$d_T(h_n, h) \to 0 \ \text{if and only if} \ d_S(s_n, s) \to 0.$$

Let $A = -\frac{z-i}{z+i}$ be the Möbius transformation form $\mathbb{H}$ to $\mathbb{D}$ that maps $0, 1$ and ∞ to $1, i$ and -1 respectively. In the following, given a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ of $\mathbb{S}^1$, we denote by $\widetilde{h} = A^{-1} \circ h \circ A$, which is a homeomorphism of $\widehat{\mathbb{R}}$ fixing $0, 1$ and ∞. Then Lemma 3 converts to the following corollary.

Corollary 1. *Let $M > 0$ and $\widetilde{h}$ be a homeomorphism of $\widehat{\mathbb{R}}$ with $0, 1$ and ∞ fixed. If $|\log cr(\widetilde{h}(Q))| < M$ holds for any of the following five quadruples*

$$Q_0 = \{\infty, -1, 0, 1\}, Q_1 = \{\infty, 0, \frac{1}{2}, 1\}, Q_2 = \{\infty, 0, 1, 2\},$$

$$Q_3 = \{\infty, -2, -1, 0\} \ \text{and} \ Q_4 = \{\infty, -1, -\frac{1}{2}, 0\},$$

then there exist a constant $C > 0$ and a neighborhood U of i, only depending on M, such that

$$K(ex(\widetilde{h})|_U) < C.$$

Proof of Theorem 3. Given a homeomorphism $\tilde{h}_n$ of $\widehat{\mathbb{R}}$, we denote by s_n the shear map of $\tilde{h}_n$. Suppose that Theorem 3 fails. Then there exists a sequence $\{\widetilde{h}_n\}_{n=1}^{\infty}$ of homeomorphisms of $\widehat{\mathbb{R}}$ such that $M_{s_n} < M$ for all n and $K(ex(\widetilde{h}_n)) \to \infty$ as $n \to \infty$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points on $\mathbb{H}$ such that $|\mu_{ex(\widetilde{h}_n)}(z_n)| \to 1$ as $n \to \infty$. Denote by $\triangle_n$ the ideal triangle in $\mathcal{F}$ such that $z_n \in \overline{\triangle}_n$. Let $A_n \in PSL(2, \mathbb{Z})$ such that $A_n(\triangle_n) = \triangle_0$, and let $B_n \in PSL(2, \mathbb{R})$ such that $\widehat{h}_n = B_n \circ \widetilde{h}_n \circ (A_n)^{-1}$ fixes $0, 1$ and ∞.

Let $z'_n = A_n(z_n)$. Consider the following two cases of $\{z'_n\}_{n=1}^{\infty}$. One is that a subsequence of $\{z'_n\}_{n=1}^{\infty}$ stays in a compact subset of $\overline{\triangle}_0$, the other is that $\{z_n\}_{n=1}^{\infty}$ has a subsequence converging to an endpoint of $\triangle_0$ as $n \to \infty$. In the second case, without loss of generality we may assume that z_n converges to ∞ as $n \to \infty$.

In the first case, we briefly denote the subsequence by $\{z'_n\}_{n=1}^{\infty}$ again. Since $M_{s_n} < M$ for all n, the sequence of the shear maps $s_n \circ A_n^{-1}$ corresponding to the homeomorphisms $\widehat{h}_n = B_n \circ \widetilde{h}_n \circ (A_n)^{-1}$ has a convergent subsequence in the sense that for any $e \in \mathcal{F}$, the sequence of real number $s_{n_k} \circ A_{n_k}^{-1}(e)$ converges as $k \to \infty$. It is clear that the limiting shear map $s_\infty : \mathcal{F} \to \mathbb{R}$ satisfies $M_{s_\infty} < M$. Since all maps $\widehat{h}_{n_k}$ are normalized to fix three points, it is shown on pages 2418-9 in [21] that the convergent subsequence $\{\widehat{h}_{n_k}\}_{k=1}^{\infty}$ converges pointwisely to a homeomorphism $\widehat{h}_\infty$ inducing the shear map s_∞. Then $|\mu_{ex(\widehat{h}_{n_k})}|$ converges to $|\mu_{ex(\widehat{h}_\infty)}|$ uniformly on any compact subset of $\mathbb{H}$. Since $ex(\widehat{h}_\infty)$ is a real analytic diffeomorphism, it follows that for any compact Ω of $\mathbb{H}$, there exists $0 < a < 1$ such that $|\mu_{ex(\widehat{h}_\infty)}| < a$

on Ω. On the other hand, by the conformal naturality of Douady-Earle extensions,

$$|\mu_{ex(\widehat{h}_{n_k})}(z'_{n_k})| = |\mu_{ex(\widetilde{h}_{n_k})}(z_{n_k})| \to 1 \ as \ k \to \infty.$$

This is a contradiction.

Now we show that there will be a contradiction coming up in the second case too. Let λ_n be the greatest even integer less than or equal to $Im(z'_n)$ and let λ'_n be the real number such that $\widehat{h}^*_n(x) = \lambda'_n \widehat{h}_n(\lambda_n x)$ fixes $0, 1$ and ∞. We show that $\widehat{h}^*_n$ satisfies the conditions in Corollary 1 for all n. For example, we verify that if $Q = \{\infty, -1, 0, 1\}$, then for all n,

$$|\log cr(\widehat{h}^*_n(Q))| \le M.$$

Verifications of the above inequality for the other four cases of Q in Corollary 1 are similar. Therefore, we skip them here.

Let e^∞_k be the edge in $\mathcal{F}$ that connects the integer k to ∞. Using the fan $\{e^\infty_k\}_{k \in \mathbb{Z}}$, we obtain

$$
\begin{aligned}
|\log cr(\widehat{h}^*_n(\{\infty, -1, 0, 1\}))| &= |\log cr(\widehat{h}_n(\{\infty, -\lambda_n, 0, \lambda_n\}))| \\
&= |\log cr(\widetilde{h}_n \circ A_n^{-1}(\{\infty, -\lambda_n, 0, \lambda_n\}))| \le M_{s_n} \le M.
\end{aligned}
$$

Let $w_n = (1/\lambda_n)z'_n$. Then w_n converges to the point i as $n \to \infty$. Therefore,

$$|\mu_{ex(\widehat{h}^*_n)}(w_n)| = |\mu_{ex(\widehat{h}_n)}(z'_n)| = |\mu_{ex(\widetilde{h}_n)}(z_n)| \to 1$$

as $n \to \infty$. This is a contradiction to Corollary 1.

We complete the proof. $\qquad \square$

5. A METRIC CHARACTERIZATION OF $AT(\mathbb{D})$ THROUGH EQUIVALENT CLASSES OF SHEAR MAPS

For each edge e in $\mathcal{F}$, we define the Farey generation $g(e)$ of e as follows. Each boundary edge of $\triangle_0$ has Farey generation 0. If an edge $e \in \mathcal{F}$ is obtained by n reflections of an edge of $\triangle_0$ (where n is the smallest such number), then the Farey generation $g(e)$ of e is defined to be n.

Given any two points $h_1, h_2 \in T(\mathbb{D})$, let s_1 and s_2 be the shear maps induced by h_1 and h_2 respectively. For each fan $\mathcal{F}_p = \{e^p_n\}_{n \in \mathbb{Z}}$, we let

$$AM_{s_1, s_2}(p) = \limsup_{\min\{g(e^p_{m+k}), g(e^p_{m-k})\} \to \infty} |\log \frac{s_1(p; m, k)}{s_2(p; m, k)}|,$$

and

$$d_{AS}(s_1, s_2) = \sup_{p \in \mathcal{P}} AM_{s_1, s_2}(p),$$

where $\mathcal{P}$ is the collection of all endpoints of the geodesics in $\mathcal{F}$.

Recall that $\mathcal{X}$ is the collection of all real-valued functions $s : \mathcal{F} \to \mathbb{R}$ with finite shear norms, and $\mathcal{X}$ gives a parametrization of $T(\mathbb{D})$. Then it is clear that d_{AS} defines a pseudo metric on $\mathcal{X}$ and hence introduces an equivalent relation on $\mathcal{X}$ by defining two elements s and s' of $\mathcal{X}$ to be equivalent if $d_{AS}(s, s') = 0$. We denote by $[s]$ the equivalent class of s and by $\mathcal{AX}$ the quotient space under this equivalent relation. The pseudo metric d_{AS} on $\mathcal{X}$ induces a a metric on $\mathcal{AX}$, which is called the *asymptotic shear metric* on $\mathcal{AX}$ and still denoted by d_{AS}. In this section, we prove our Theorems 1 and 2, that is, we show that $\mathcal{AX}$ is a parametrization of $AT(\mathbb{D})$ and $(\mathcal{AX}, d_{AS})$ is topologically equivalent to $(AT(\mathbb{D}), d_{AT})$, where d_{AT} is the Teichmüller metric on $AT(\mathbb{D})$.

Remark 2. Let e_j^p be an edge in $\mathcal{F}$ with tip p and let a_j^p be the endpoint of e_j^p not equal to p. Note first that $\min\{g(e_{m+k}^p), g(e_{m-k}^p)\} \to \infty$ is equivalent to say that the minimal scale of the corresponding quadruple $\{p, a_{m-k}^p, a_m^p, a_{m+k}^p\} \to 0$. From the definitions of d_{AS} and d_{AC}, it follows that for any two points $[h_1]$ and $[h_2]$ in $AT(\mathbb{D})$,

$$d_{AS}([s_1], [s_2]) \leq d_{AC}([h_1], [h_2]).$$

From Theorem 7, we know that $d_{AT}([h_n], [h]) \to 0$ implies $d_{AS}([s_n], [s]) \to 0$ as $n \to \infty$. Our Theorem 2 states that $d_{AT}([h_n], [h]) \to 0$ if and only if $d_{AS}([s_n], [s]) \to 0$ as $n \to \infty$.

Remark 3. If h_2 is the identity map, then $s_2 \equiv 0$ and $s_2(p; m, k) \equiv 1$. Thus

$$d_{AS}(s_1, 0) = \sup_{p \in \mathcal{P}} \ \limsup_{\min\{g(e_{m+k}^p), g(e_{m-k}^p)\} \to \infty} |\log s_1(p; m, k)|.$$

It has already been proved in Theorem A of [21] that h_1 is symmetric if and only if $d_{AS}(s_1, 0) = 0$. This means the equivalent class containing the identity map is the collection $S(\mathbb{S}^1)$ of all symmetric homeomorphisms in $QS(\mathbb{S}^1)$. This is a special case of our Theorem 1.

Proof of Theorem 1. Given two points h and h' in $T(\mathbb{D})$, let s and s' be the shear maps of $\widetilde{h} = A^{-1} \circ h \circ A$ and $\widetilde{h}' = A^{-1} \circ h' \circ A$ respectively. If $[h] = [h']$, then it follows from Remark 2 and Theorem 7 that $d_{AS}(s, s') = 0$.

In the following, we apply some ideas in [21] to show that $d_{AS}(s, s') = 0$ implies $[h] = [h']$. Suppose on the contrary this is not true. Then there exist two points h and h' in $T(\mathbb{D})$ such that $d_{AS}(s, s') = 0$ but $[h] \neq [h']$. Using the same notation in the previous section, we let $A = -\frac{z-i}{z+i}$ and $\widetilde{h} = A^{-1} \circ h \circ A$ and $\widetilde{h}' = A^{-1} \circ h' \circ A$. Denote by $ex(\widetilde{h})$ and $ex(\widetilde{h}')$ the Douady-Earle extensions of $\widetilde{h}$ and $\widetilde{h}'$ respectively. Since $[h] \neq [h']$, there exist a constant $c > 0$ and a sequence of points $z_n \in \mathbb{H}$ leaving every compact subset of $\mathbb{H}$ such that

$$(5.1) \qquad\qquad |\mu_{ex(\widetilde{h})}(z_n) - \mu_{ex(\widetilde{h}')}(z_n)| \geq c.$$

Denote by $\triangle_n$ the ideal triangle in $\mathcal{F}$ such that $z_n \in \overline{\triangle}_n$ and by A_n the element in $PSL(2, \mathbb{Z})$ such that $A_n(\triangle_n) = \triangle_0$. Let B_n and B_n' be the elements in $PSL(2, \mathbb{R})$ such that $\widetilde{h}_n = B_n \circ \widetilde{h} \circ (A_n)^{-1}$ and $\widetilde{h}_n' = B_n' \circ \widetilde{h}' \circ (A_n)^{-1}$ fixing $0, 1$ and ∞.

Let $z_n' = A_n(z_n)$. It suffices to consider the following two cases: one is that $\{z_n'\}_{n=1}^\infty$ has a subsequence staying in a compact set of $\overline{\triangle}_0$ and the other is that z_n has a subsequence converging to one of the endpoint of $\triangle_0$ as $n \to \infty$. In the second case, without loss of generality we assume that z_n converges to ∞ as $n \to \infty$.

In the first case, for brevity in notation we continue to denote by $\{z_n'\}_{n=1}^\infty$ the subsequence staying in a compact set of $\overline{\triangle}_0$. Given an edge $e \in \mathcal{F}$, we claim that the Farey generation $g(A_n^{-1}(e))$ goes to ∞ as $n \to \infty$. This is clear if e is an edge e_0 of $\triangle_0$ (since $A_n^{-1}(e_0)$ is an edge of $\triangle_n$). For any other edge $e \in \mathcal{F}$, we note that for any $n \in \mathbb{N}$,

$$g(A_n^{-1}(e)) - g(A_n^{-1}(e_0)) = g(e) - g(e_0),$$

which implies that $g(A_n^{-1}(e)) \to \infty$ as $n \to \infty$. Now using the assumption $d_{AS}(s, s') = 0$, we know for each $e \in \mathcal{F}$, $s \circ A_n^{-1}(e)$ and $s' \circ A_n^{-1}(e)$ converge to the same limiting value. Therefore, $\{\widetilde{h}_n\}_{n=1}^\infty$ and $\{\widetilde{h}_n'\}_{n=1}^\infty$ converge to the same limiting homeomorphism pointwisely. Since $\{K(ex(\widetilde{h}_n))\}_{n=1}^\infty$ and $\{K(ex(\widetilde{h}_n'))\}_{n=1}^\infty$

are bounded and all $\tilde{h}_n$ and $\tilde{h}'_n$ are normalized to fix three points, passing to subsequences we may assume that $\{\tilde{h}_n\}_{n=1}^\infty$ and $\{\tilde{h}'_n\}_{n=1}^\infty$ converge uniformly to the same quasisymmetric homeomorphism of $\widehat{\mathbb{R}}$. Thus,

$$|\mu_{ex(\tilde{h})}(z_n) - \mu_{ex(\tilde{h}')}(z_n)| = |\mu_{ex(\tilde{h}'_n)}(z'_n) - \mu_{ex(\tilde{h}_n)}(z'_n)| \to 0$$

as $n \to \infty$. This is a contradiction to (5.1).

In the second case, let $\lambda_{n,1}$ be the greatest integer less than or equal to $Im(z'_n)$. Then $\lambda_{n,1} \to \infty$ as $n \to \infty$. Let $\beta_{n,1}$ and $\beta'_{n,1}$ be the two real numbers such that $\widehat{h}_{n,1}(x) = \beta_{n,1}\tilde{h}_n(\lambda_{n,1}x)$ and $\widehat{h}'_{n,1}(x) = \beta'_{n,1}\tilde{h}'_n(\lambda_{n,1}x)$ fix $0, 1$ and ∞.

Let e_k^∞ be the edge in $\mathcal{F}$ connecting an integer k to ∞. At first we claim that there exists at most one $k_0 \in \mathbb{Z}$ such that the Farey generation $g(A_n^{-1}(e_{\lambda_{n,1}k_0}^\infty))$ is bounded as $n \to \infty$. In the fan $\{e_k^\infty\}_{k \in \mathbb{Z}}$, there are $|\lambda_{n,1}k_0 - \lambda_{n,1}k|$ edges between $e_{\lambda_{n,1}k_0}^\infty$ and $e_{\lambda_{n,1}k}^\infty$. Given any two integers $k \neq k_0$, since $|\lambda_{n,1}k_0 - \lambda_{n,1}k| \to \infty$ as $n \to \infty$, it follows that $g(A_n^{-1}(e_{\lambda_{n,1}k}^\infty))$ is unbounded if $g(A_n^{-1}(e_{\lambda_{n,1}k_0}^\infty))$ is bounded. Thus, the claim follows.

We now show that for all $k \in \mathbb{Z}$,

(5.2) $$\lim_{n \to \infty} \widehat{h}_{n,1}(k) = \lim_{n \to \infty} \widehat{h}'_{n,1}(k).$$

We divide the proof into three cases according to the value of k_0: $k_0 = 0$, $k_0 < 0$ or $k_0 > 0$.

If $k_0 = 0$, then the Farey generations $g(A_n^{-1}(e_{-\lambda_{n,1}}^\infty))$ and $g(A_n^{-1}(e_{\lambda_{n,1}}^\infty))$ go to ∞ as $n \to \infty$. Since $d_{AS}(s, s') = 0$, it follows from Remarks 1 and 2 that

$$\frac{s \circ A_n^{-1}(\infty, 0, \lambda_{n,1})}{s' \circ A_n^{-1}(\infty, 0, \lambda_{n,1})} = \frac{cr(\tilde{h}'_n(-\lambda_{n,1}, 0, \lambda_{n,1}, \infty))}{cr(\tilde{h}_n(-\lambda_{n,1}, 0, \lambda_{n,1}, \infty))}$$
$$= \frac{cr(\widehat{h}'_{n,1}(-1, 0, 1, \infty))}{cr(\widehat{h}_{n,1}(-1, 0, 1, \infty))} = \frac{\widehat{h}'_{n,1}(-1)}{\widehat{h}_{n,1}(-1)} \to 1 \text{ as } n \to \infty.$$

Thus,

$$\lim_{n \to \infty} \widehat{h}_{n,1}(-1) = \lim_{n \to \infty} \widehat{h}'_{n,1}(-1).$$

Using the symmetric triple $(-1, 1, 3)$, the assumption $\widehat{h}_{n,1}(1) = \widehat{h}'_{n,1}(1)$ and the equality of the above two limits, we first obtain

$$\lim_{n \to \infty} \widehat{h}_{n,1}(3) = \lim_{n \to \infty} \widehat{h}'_{n,1}(3).$$

Then we use $(1, 2, 3)$ and then $(2, 3, 4)$ and so on to obtain that

$$\lim_{n \to \infty} \widehat{h}_{n,1}(k) = \lim_{n \to \infty} \widehat{h}'_{n,1}(k)$$

for each $k \in \mathbb{N}$. Thirdly, for each $k \in \mathbb{N}$, we use the triple $(-k, 0, k)$ to obtain that

$$\lim_{n \to \infty} \widehat{h}_{n,1}(-k) = \lim_{n \to \infty} \widehat{h}'_{n,1}(-k).$$

Using the normalization condition, it is clear that

$$\lim_{n \to \infty} \widehat{h}_{n,1}(0) = \lim_{n \to \infty} \widehat{h}'_{n,1}(0).$$

Therefore, (5.2) holds if $k_0 = 0$.

If $k_0 < 0$, we first use the triple $(0, 1, 2)$ to obtain

$$\lim_{n \to \infty} \widehat{h}_{n,1}(2) = \lim_{n \to \infty} \widehat{h}'_{n,1}(2).$$

Using the same argument as in the case when $k_0 = 0$, we obtain for each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \widehat{h}_{n,1}(k) = \lim_{n\to\infty} \widehat{h}'_{n,1}(k).$$

Thirdly, for each $k \in \mathbb{Z}\backslash k_0$, we use $(-k, 0, k)$ to obtain that

$$\lim_{n\to\infty} \widehat{h}_{n,1}(k) = \lim_{n\to\infty} \widehat{h}'_{n,1}(k).$$

Finally, we use $(k_0 - 1, k_0, k_0 + 1)$ to obtain that

$$\lim_{n\to\infty} \widehat{h}_{n,1}(k_0) = \lim_{n\to\infty} \widehat{h}'_{n,1}(k_0).$$

Therefore, we show (5.2) if $k_0 < 0$.

The proof of (5.2) for $k_0 > 0$ is very similar to what we have done in the case $k_0 < 0$. We skip it.

For each $r \in \mathbb{N}$, let $\lambda_{n,r}$ be the greatest integral multiple of 2^{r-1} less than or equal to $Im(z'_n)$. Let $\beta_{n,r}$ and $\beta'_{n,r}$ be the real numbers such that $\widehat{h}_{n,r}(x) = \beta_{n,r}\widetilde{h}_n(\lambda_{n,r}x)$ and $\widehat{h}'_{n,r}(x) = \beta'_{n,r}\widetilde{h}_n(\lambda_{n,r}x)$ fix $0, 1$ and ∞. Using the same argument in the case when $r = 1$, it is easy to see that there exists at most one integer $k_{r-1} \in \mathbb{Z}$ such that the Farey generation $g(A_n^{-1}(e^\infty_{\lambda_{n,r}k_{r-1}}))$ is bounded as $n \to \infty$. Now we claim that for each $r \in \mathbb{N}$ and each $k \in \mathbb{Z}$,

$$(5.3) \qquad \lim_{n\to\infty} \widehat{h}_{n,r}\left(\frac{k}{2^{r-1}}\right) = \lim_{n\to\infty} \widehat{h}'_{n,r}\left(\frac{k}{2^{r-1}}\right).$$

This claim can be proved by an induction on r. We have just shown the case when $r = 1$. Assume that the claim is true for $r = m$, we only need to show the claim is true for $r = m + 1$ and k is odd. Given each $k \in \mathbb{Z}$, we can choose an odd integer $k' \in \mathbb{Z}$ such that both $\frac{k-k'}{2^{m-1}}$ and $\frac{k+k'}{2^{m-1}}$ are not equal to k_{r-1}. Since the limits of $\widehat{h}_{n,r}$ and $\widehat{h}'_{n,r}$ are the same at the points $\frac{k-k'}{2^{m-1}}$ and $\frac{k-k'}{2^{m-1}}$, using the symmetric triple $\left(\frac{k-k'}{2^{m-1}}, \frac{k}{2^m}, \frac{k+k'}{2^{m-1}}\right)$ we obtain the claim (5.3) for $r = m + 1$ and an odd integer k.

For each $r \in \mathbb{N}$, since $\{K(ex(\widehat{h}_{n,r}))\}_{n=1}^\infty$ and $\{K(ex(\widehat{h}'_{n,r}))\}_{n=1}^\infty$ are bounded and all the maps in these two sequences are normalized to fix three points, passing to subsequences we may assume that $\{\widehat{h}_{n,r}\}_{n=1}^\infty$ and $\{\widehat{h}'_{n,r}\}_{n=1}^\infty$ converge uniformly to two quasisymmetric homeomorphisms of $\widehat{\mathbb{R}}$, which agree on $\{\frac{k}{2^{r-1}} : k \in \mathbb{Z}.\}$ because of the previous claim (5.3). Then for each $r \in \mathbb{N}$, we choose a sufficiently big n_r such that for each $k \in \mathbb{Z}$,

$$\left|\widehat{h}_{n_r,r}\left(\frac{k}{2^{r-1}}\right) - \widehat{h}'_{n_r,r}\left(\frac{k}{2^{r-1}}\right)\right| \leq 1/r.$$

Since $Im(z'_{n_r}) \to \infty$ as $n \to \infty$, we can also choose n_r large enough such that $1 \leq Im(z'_{n_r})/\lambda_{n_r,r} < 2$. Then $\{\frac{z'_{n_r}}{\lambda_{n_r,r}}\}_{r=1}^\infty$ stays in a compact subset of $\mathbb{H}$. Because $\{\frac{k}{2^r} : k \in \mathbb{Z}, r \in \mathbb{N}\}$ is dense in $\widehat{\mathbb{R}}$, the two sequences $\{\widehat{h}_{n_r,r}\}_{r=1}^\infty$ and $\{\widehat{h}'_{n_r,r}\}_{r=1}^\infty$ converge pointwisely to the same quasisymmetric homeomorphism of $\widehat{\mathbb{R}}$. Using a property of Douady-Earle extensions, we conclude that

$$(5.4) \qquad \left|\mu_{ex(\widehat{h}_{n_r,r})}\left(\frac{z'_{n_r}}{\lambda_{n_r,r}}\right) - \mu_{ex(\widehat{h}'_{n_r,r})}\left(\frac{z'_{n_r}}{\lambda_{n_r,r}}\right)\right| \to 0 \text{ as } r \to \infty.$$

On the other hand, using (5.1) we obtain for each $r \in \mathbb{N}$,

$$
\begin{aligned}
|\mu_{ex(\widehat{h}_{n_r,r})}(\tfrac{z'_{n_r}}{\lambda_{n_r,r}}) - \mu_{ex(\widehat{h}'_{n_r,r})}(\tfrac{z'_{n_r}}{\lambda_{n_r,r}})| &= |\mu_{ex(\widetilde{h}_{n_r,r})}(z'_{n_r,r}) - \mu_{ex(\widetilde{h}'_{n_r,r})}(z'_{n_r,r})| \\
&= |\mu_{ex(\widetilde{h})}(z_{n_r,r}) - \mu_{ex(\widetilde{h}')}(z_{n_r,r})| \geq c > 0.
\end{aligned}
$$

This is a contradiction to (5.4). $\qquad\square$

Proof of Theorem 2. By Theorem 7 and Remark 2, we see that $d_{AT}([h_n],[h]) \to 0$ implies $d_{AS}([s_n],[s]) \to 0$ as $n \to \infty$.

It remains to show that $d_{AS}([s_n],[s]) \to 0$ implies $d_{AT}([h_n],[h]) \to 0$ as $n \to \infty$. Suppose that on the contrary this implication is not true. Then there exist $[h] \in AT(\mathbb{D})$ and a sequence $\{[h_n]\}_{n=1}^{\infty}$ in $AT(\mathbb{D})$ such that $d_{AS}([s_n],[s])$ goes to 0 but $d_{AT}([h_n],[h])$ dose not go to 0 as $n \to \infty$. Passing to a subsequence, we may assume that

$$d_{AT}([h_n],[h]) \geq c'$$

for some $c' > 0$ and all n. Then there exist a constant $c > 0$ and a sequence of points $z_n \in \mathbb{H}$ leaving every compact subset of $\mathbb{H}$ such that

(5.5) $$|\mu_{ex(\widetilde{h})}(z_n) - \mu_{ex(\widetilde{h}_n)}(z_n)| \geq c.$$

Using a very similar strategy in the proof of the previous theorem, a contradiction can be derived as follows.

Denote by $\triangle_n$ the ideal triangle in $\mathcal{F}$ such that $z_n \in \overline{\triangle}_n$ and by A_n the element in $PSL(2,\mathbb{Z})$ such that $A_n(\triangle_n) = \triangle_0$. Let B_n and B'_n be the elements in $PSL(2,\mathbb{R})$ such that $\widehat{h}_n = B_n \circ \widetilde{h} \circ (A_n)^{-1}$ and $\widehat{h}'_n = B'_n \circ \widetilde{h}_n \circ (A_n)^{-1}$ fix $0, 1$ and ∞. Let $z'_n = A_n(z_n)$. Consider the following two cases: one is that the sequence $\{z'_n\}_{n=1}^{\infty}$ has a subsequence staying in a compact set of $\overline{\triangle}_0$ and the other is that without loss of generality, we may assume z_n goes to ∞ as $n \to \infty$.

In the first case, we denote the subsequence by $\{z'_n\}_{n=1}^{\infty}$ again. For each $e \in \mathcal{F}$, $g(A_n^{-1}(e))$ goes to ∞ as $n \to \infty$. Using the assumption that $d_{AS}([s_n],[s])$ approaches 0 as $n \to \infty$, we see that $s \circ A_n^{-1}(e)$ and $s_n \circ A_n^{-1}(e)$ converge to the same limit for each $e \in \mathcal{F}$. Using the normalization condition, we conclude that $\{\widehat{h}_n\}_{n=1}^{\infty}$ and $\{\widehat{h}'_n\}_{n=1}^{\infty}$ converge pointwisely to the same quasisymmetric homeomorphism of $\widehat{\mathbb{R}}$. Using a property of Douady-Earle extensions,

(5.6) $$|\mu_{ex(\widehat{h}'_n)}(z'_n) - \mu_{ex(\widehat{h}_n)}(z'_n)| \to 0 \ \text{ as } \ n \to \infty.$$

On the other hand, (5.5) implies

$$|\mu_{ex(\widehat{h}'_n)}(z'_n) - \mu_{ex(\widehat{h}_n)}(z'_n)| = |\mu_{ex(\widetilde{h})}(z_n) - \mu_{ex(\widetilde{h}_n)}(z_n)| \geq c > 0.$$

This is a contradiction to (5.6).

In the second case, for each $r \in \mathbb{N}$, let $\lambda_{n,r}$ be the greatest integral multiple of 2^{r-1} less than or equal to $Im(z'_n)$. Let $\beta_{n,r}$ and $\beta'_{n,r}$ be the real numbers such that $\widehat{h}_{n,r}(x) = \beta_{n,r}\widetilde{h}_n(\lambda_{n,r}x)$ and $\widehat{h}'_{n,r}(x) = \beta'_{n,r}\widetilde{h}_n(\lambda_{n,r}x)$ fix $0, 1$ and ∞. Using the assumption that $d_{AS}([s_n],[s]) \to 0$ as $n \to \infty$ and following the exact same details used to derive a contradiction for the second case in the proof of Theorem 1, we can conclude that for each $r \in \mathbb{N}$, $\{\widehat{h}_{n,r}\}_{n=1}^{\infty}$ and $\{\widehat{h}'_{n,r}\}_{n=1}^{\infty}$ converge pointwisely to two quasisymmetric homeomorphisms of $\widehat{\mathbb{R}}$ that agree on the set $\{\frac{k}{2^{r-1}}, k \in \mathbb{Z}\}$. Then the part of the last paragraph in the proof of Theorem 1 after the first sentence also forms the rest of the proof for Theorem 2. $\qquad\square$

References

1. W. Abikoff, *Conformal barycenters and the Douady-Earle extension - A discrete dynamical approach*, Jour. d'Analyse Math. 86 (2002) 221-234.
2. A. Douady and C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. 157 (1986) 23-48.
3. C. J. Earle, F. P. Gardiner, N. Lakic, *Asymptotic Teichmüller space. I. The complex structure*, In the tradition of Ahlfors and Bers, 17-38, Contemp. Math., 256, Amer. Math. Soc., Providence, RI, 2000.
4. C. J. Earle, F. P. Gardiner, N. Lakic, *Asymptotic Teichmüller space. II. The metric structure*, In the tradition of Ahlfors and Bers III, 187-219, Contemp. Math. 355, Amer. Math. Soc., Providence, RI, 2004.
5. C. J. Earle, V. Markovic, D. Šarić, *Barycentric extension and the Bers embedding for asymptotic Teichmüller space*, In Complex manifolds and hyperbolic geometry (ed. C. J. Earle, W. J. Harvey and S. Recillas-Pishmish), 87–105, Contemp. Math. 311, Amer. Math. Soc., Providence, RI, 2002.
6. J. Fan and J. Hu, *Topological characterization of an asymptotic Teichmüller space through measured geodesic laminations*, Preprint, Graduate Center of CUNY, 2013.
7. F. P. Gardiner, J. Hu and N. Lakic, *Earthquake curves*, In Complex Manifolds and Hyperbolic Geometry (ed. C. J. Earle, W. J. Harvey and S. Recillas-Pishmish), 141–195, Contemp. Math. 311, Amer. Math. Soc., Providence, RI, 2002.
8. F. P. Gardiner, N. Lakic, *Quasiconformal* Teichmüller *spaces*, Math Surveys and Monogr. 76, Amer. Math. Soc., (2000).
9. F. P. Gardiner, D. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. 114 (1992) 683–736.
10. J. Hu, *Earthquake measure and cross-ratio distortion*, In the tradition of Ahlfors and Bers, III, 285–308, Contemp. Math., 355, Amer. Math. Soc., Providence, RI, 2004.
11. J. Hu and O. Muzician, *Cross-ratio distortion and Douady-Earle extension: I. A new upper bound on quasiconformality*, Jour. of London Math. Soc. 86(2) (2012) 387-406.
12. J. Hu and O. Muzician, *Cross-ratio distortion and Douady-Earle extension: II. Quasiconformality and asymptotic conformality are local*, Jour. d'Analyse Math. 117(1) (2012) 249–271.
13. J. Hu and O. Muzician, *Cross-ratio distortion and Douady-Earle extension: III. How the maximal dilatation of the extension near the origin is controlled by the distortion of the boundary map on finitely many points*, in preparation.
14. J. Hu and O. Muzician, *Conformally natural extensions of continuous circle maps: I. The case when the pushforward measure has no atom*, Contemp. Math. 575 (2012) 171-198.
15. J. Hu and O. Muzician, *Conformally natural extensions of continuous circle maps: II. The general case*, preprint 2013.
16. J. Hu and S. Pal, *Boundary differentiability of Doudy-Earle extensions of diffeomorphisms of* $\mathbb{S}^n$, preprint of 2013, to appear in Pure and Applied Mathematics Quarterly.
17. V. Markovic, *Quasisymmetric groups*, J. Amer. Math. Soc. 19 (2006) 673–715.
18. H. Miyachi and D. Šarić, *Uniform Weak* Topology and earthquake in the hyperbolic plane*, Proc. London Math. Soc. 105 (2012) 1123-1148.
19. R.C Penner, *Universal constructions in Teichmüller theory*, Adv. in Math. 98 (1993) 143–215.
20. D. Šarić, *Real and complex earthquakes*, Trans. Amer. Math. Soc. 358 (2006) 233-249.
21. D. Šarić, *Circle homeomorphisms and shears*, Geom. Topol. 14 (2010) 2405–2430.
22. D. Šarić, *Zygmund vector fields, Hilbert transformation and Fourier coeddicients in shear coodinates*, to appear in Amer. J. Math.
23. W. Thurston, *Earthquakes in two-dimensional hyperbolic geometry*, In Low-dimensional topology and Kleinian groups, LMS. Lecture Note Ser. 112, Cambridge Univ. Press, Cambridge (1986) 91–112.
24. C. Wu, *The cross-ratio distortion of integrably asymptotic affine homeomorphism of unit circle*, Sci. China Math. 55 (2012) 625–632.
25. S. Wu, *Moduli of quadrilaterals and extremal quasiconformal extensions of quasisymmetric functions*, Comment. Math. Helv. 72 (1997) 593–604.

DEPARTMENT OF APPLIED MATHEMATICS, NANJING UNIVERSITY OF SCIENCE AND TECHNOLOGY, NANJING 210094, PRC

E-mail address: `jinhuafan@hotmail.com`

DEPARTMENT OF MATHEMATICS, BROOKLYN COLLEGE OF CUNY, BROOKLYN, NY 11210, AND, PH.D. PROGRAM IN MATHEMATICS, GRADUATE CENTER OF CUNY, 365 FIFTH AVENUE, NEW YORK, NY 10016

E-mail address: `junhu@brooklyn.cuny.edu or JHu1@gc.cuny.edu`