DUAL DYNAMICAL SYSTEMS FOR CIRCLE ENDOMORPHISMS

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Abstract. We show that every uniformly asymptotically affine circle endomorphism has a uniformly asymptotically conformal (UAC) extension to the complex plane. Then we use the UAC extension to construct the dual dynamical system, the dual annulus, and the dual circle expanding map.

Introduction

We recapitulate basic properties of uniformly asymptotically affine circle degree $d > 1$ endomorphisms. Then we show how to use the Beurling-Ahlfors extension to realize any uniformly asymptotically affine system as the restriction to the circle of a uniformly asymptotically conformal system. (Most of this exposition is a recapitulation of work of Guizhen Cui.) Then we introduce dual dynamical systems, dual Cantor sets, and show that Thompson's F-group acts naturally as a group of isometries of the degree 2 UAC Teichmüller space. Finally, we show that the dual Cantor set completes projects a natural way to a dual circle with a given symmetric structure depending on the dynamical system. Also, the scaling function is continuous in this circle.

1. Circle endomorphisms

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$ be the unit circle. The map $\pi : \mathbb{R} \to \mathbb{S}^1$ defined by

$$\pi(x) = e^{2\pi i x}$$

realizes $\mathbb{R}$ as the universal covering of $\mathbb{S}^1$ with covering π and covering group $\mathbb{Z}$. π induces an isomorphism from $\mathbb{R}/\mathbb{Z}$ onto $\mathbb{S}^1$.

Let d be the degree of an orientation preserving covering f from $\mathbb{S}^1$ onto itself and assume $1 < d < \infty$. f is an endomorphism of $\mathbb{S}^1$ and it necessarily has one fixed point p. By selecting an orientation preserving Möbius transformation A that preserves the unit disk with $A(p) = 1$, we may shift consideration of the map f to the map $\tilde{f} = A \circ f \circ A^{-1}$. $\tilde{f}$ has the same dynamical properties as f

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and it fixes the point 1. Therefore, without loss of generality, we may assume to begin with that f fixes the point $p = 1$. We denote the homeomorphic lift of f by F. F is uniquely determined by f if we assume it has the following properties:

i) F is a homeomorphism of $\mathbb{R}$,

ii) $\pi \circ F = f \circ \pi$,

iii) $F(0) = 0$.

Note that $F(x+1) = F(x) + d$. In this paper we refer either to f or to its unique corresponding lift F as a circle endomorphism. We denote the $n-$fold composition of f with itself by f^n. Similarly, F^n is the $n-$fold composition of F.

Definition 1. A circle homeomorphism h is called quasisymmetric if there is a constant $M \geq 1$ such that for all real numbers x and t

$$\frac{1}{M} \leq \frac{H(x+y) - H(x)}{H(x) - H(x-y)} \leq M, \quad \forall x \in \mathbb{R}, \ \ \forall t > 0. \tag{1}$$

The expression

$$\rho(x,y) = \frac{H(x+y) - H(x)}{H(x) - H(x-y)}$$

is called the distortion function for H.

Definition 2. A circle homeomorphism h is called symmetric (or asymptotically affine) if it is quasisymmetric and if there exists a function $\epsilon(y) \to 0^+$ as $y \to 0^+$ such that

$$\frac{1}{1+\epsilon(t)} \leq \frac{H(x+y) - H(x)}{H(x) - H(x-y)} \leq 1 + \epsilon(y). \tag{2}$$

A positive function $\epsilon(y)$ defined for positive values of t is called vanishing if $\epsilon(y) \to 0$ as $y \to 0^+$.

We say asymptotically affine in this definition since the ratio in the middle expression in (2) is identically equal to one if and only if H is affine, that is, if $H(x) = ax + b$, for some $a \neq 0$.

Definition 3. A circle endomorphism f of degree d is called uniformly symmetric or uniformly asymptotically affine (UAA) if all of the inverse branches of f^n, $n = 1, 2, \ldots$, are symmetric uniformly. More precisely, f^n is UAA if there is a bounded positive function $\epsilon(y)$ with $\epsilon(t) \to 0^+$ as $y \to 0^+$ such that for all positive integers n and all real numbers x,

$$\frac{1}{1+\epsilon(y)} \leq \frac{F^{-n}(x+y) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x-y)} \leq 1 + \epsilon(y). \tag{3}$$

We say uniformly since the ratio in the middle expression in (3) approaches 1 when y approaches 0 independently of the number n of compositions of F.

2. Beurling-Ahlfors Extensions of symmetric mappings

Consider all possible extensions of H to self-mappings $\tilde{H}$ of the upper half plane $\mathbb{H}$ and let

$$K(H) = \inf\{K(\tilde{H}) : \tilde{H} \text{ extends } H\}.$$

Note that if $K(H) = 1$, then H is affine, that is, $H(x) = ax + b, a \neq 0$. Similarly, H is also affine if $M = M(H) = 1$ in the M-condition (1). Thus, we may take both $M(H)$ and $K(H)$ as measurements of the extent to which H fails to be affine. A well-known result of Beurling and Ahlfors [2] shows that $M(H)$ and $K(H)$ are simultaneously finite and there are estimates for $M(H)$ in terms of $K(H)$ and vice-versa. Moreover, $M(H)$ and $K(H)$ simultaneously approach 1.

The Beurling-Ahlfors extension procedure provides a canonical extension $\tilde{H}$ of any quasisymmetric homeomorphism H such that the Beltrami coefficient μ of $\tilde{H}$ has the following property. There is a vanishing function $\eta(y)$ such that $|\mu(x+iy)| \leq \eta(y)$ if, and only if, there is a vanishing function $\epsilon(y)$ such that $|\rho(x,y)| \leq 1 + \eta(y)$.

In this paper we require a more intricate estimate that involves comparing the distortion functions two quasisymmetric homeomorphisms of the real axis H_0 and H_1 fixing 0, 1 and ∞.

Suppose the distortion functions $\rho_0(x,y)$ and $\rho_1(x,y)$ of H_0 and H_1 satisfy the inequality

$$|\rho_0(x,y) - \rho_1(x,y)| \leq \epsilon(y),$$

where $\epsilon(y)$ approaches 0 as y approaches 0. Suppose furthermore that μ_1 and μ_2 are the Beltrami coefficients of the Beurling Ahlfors extensions $\tilde{H}_0$ and $\tilde{H}_1$. That is,

$$\mu_0(z) = \frac{\tilde{H}_{0\overline{z}}}{\tilde{H}_{1z}} \text{ and } \mu_1(z) = \frac{\tilde{H}_{1\overline{z}}}{\tilde{H}_{1z}}.$$

Then there is a vanishing function $\eta(y)$ depending on $\epsilon(y)$ such that

$$|\mu_0(x+iy) - \mu_1(x+iy)| \leq \eta(y).$$

We take the following formulas as the definition of the Beurling-Ahlfors extension:

$$\tilde{H} = U + iV,$$

where

$$U(x,y) = \frac{1}{2y}\int_{x-y}^{x+y} H(s)ds = \frac{1}{2}\int_{-1}^{1} H(x+ky)dk \tag{4}$$

and

$$V(x,y) = \frac{1}{y}\int_{x}^{x+y} H(s)ds - \frac{1}{y}\int_{x-y}^{x} H(s)ds. \tag{5}$$

In (4) and (5) we have chosen a normalization slightly different from the one given in [1] so that the extension of the identity is the identity and so that this extension is affinely natural in the sense the extension is natural for affine maps, by which we mean that for affine maps A and B,

$$\widetilde{A\circ H\circ B} = A\circ \tilde{H}\circ B$$

and

$$\widetilde{id_{\mathbb{R}}} = \ id_{\mathbb{C}}.$$

Except for minor modifications all of the following calculations are taken from [3] . In addition to the distortion function

$$\rho(x,y) = \frac{H(x+y)-H(x)}{H(x)-H(x-y)},$$

we will need the weighted distortion function

$$\rho(x,y,k) = \frac{H(x+ky)-H(x)}{H(x)-H(x-y)}.$$

Note that

$$\int_0^1 \rho(x,y,k)dk = \frac{1}{H(x)-H(x-y)}\left(\frac{1}{y}\int_x^{x+y} H(s)ds - H(x)\right) \tag{6}$$

and

$$\int_0^1 \rho(x,-y,k)dk = \frac{1}{H(x+y)-H(x)}\left(H(x) - \frac{1}{y}\int_{x-y}^{x} H(s)ds\right). \tag{7}$$

Let

$$\begin{array}{lcl} L &=& H(x)-H(x-y) \\ R &=& H(x+y)-H(x) \\ L' &=& H(x) - \frac{1}{y}\int_{x-y}^{x} H(t)dt, \\ R' &=& \frac{1}{y}\int_x^{x+y} H(t)dt - H(x). \end{array} \tag{8}$$

and let $\rho_+ = \int_0^1 \rho(x,y,k)dk$ and $\rho_- = \int_0^1 \rho(x,-y,k)dk$. Then

$$\begin{array}{lcl} \rho(x,y) &=& R/L \\ \rho_+(x,y) &=& R'/L \\ \rho_-(x,y) &=& L'/R. \end{array} \tag{9}$$

Notice that for symmetric homeomorphisms all three of the quantities ρ, ρ_+ and ρ_- approach $+1$ as y approaches zero. The complex dilatation of $\tilde{H}$ is given by

$$\mu(z) = \frac{K(z)-1}{K(z)+1}$$

where

$$K(z) = \frac{\tilde{H}_z + \tilde{H}_{\overline{z}}}{\tilde{H}_z - \tilde{H}_{\overline{z}}} = \frac{(U+iV)_z + (U+iV)_{\overline{z}}}{(U+iV)_z - (U+iV)_{\overline{z}}} =$$

$$\frac{(U+iV)_x - i(U+iV)_y + (U+iV)_x + i(U+iV)_y}{(U+iV)_x - i(U+iV)_y - (U+iV)_x - i(U+iV)_y} =$$

$$\frac{U_x + iV_x}{V_y - iU_y}.$$

Thus

$$K(z) = \frac{1+ia}{b-ic},$$

where $a = V_x/U_x, b = V_y/U_x$ and $c = U_y/U_x$.

To find estimates for these three ratios we must find expressions for the four partial derivatives of U and V in (4) and (5). In the notation of (8)

$$\begin{aligned} U_x &= \tfrac{1}{2y}(R+L), \\ V_x &= \tfrac{1}{y}\left(R-L\right), \\ V_y &= \tfrac{1}{y}\left(R+L\right) - \tfrac{1}{y}\left(R'+L'\right), \\ U_y &= \tfrac{1}{2y}\left(R-L\right) - \tfrac{1}{2y}\left(R'-L'\right). \end{aligned}$$

Thus

$$\begin{aligned} a(1+\rho) &= 2\tfrac{R-L}{R+L}\cdot\tfrac{R+L}{L}, \\ b(1+\rho) &= 2\tfrac{R+L-R'-L'}{R+L}\cdot\tfrac{R+L}{L} = 2\left(R/L + 1 - R'/L - (R/L)(L'/R)\right), \\ c(1+\rho) &= \tfrac{R-L-R'+L'}{R+L}\cdot\tfrac{R+L}{L} = R/L - 1 - R'/L + (R/L)(L'/R). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} a &= \tfrac{2(\rho-1)}{\rho+1}, \\ b &= \tfrac{2(\rho+1-\rho_+-\rho\rho_-)}{\rho+1}, \\ c &= \tfrac{\rho-1+\rho_+-\rho\rho_-}{\rho+1}. \end{aligned} \tag{10}$$

Theorem 1. [Cui [3]**]** *Let H_1 and H_0 be two quasisymmetric homeomorphisms of $\mathbb{R}$ normalized to fix 0 and 1. Let ρ_1 and ρ_0 be the symmetric distortions of H_1 and H_0, and let μ_1 and μ_0 be the Beltrami coefficients of $\widetilde{H_1}$ and $\widetilde{H_0}$. Then if there is a vanishing function $\eta(y)$ such that $|\rho_1(x,y) - \rho_0(x,y)| < \epsilon(y)$, then there is a vanishing function $\eta(y)$ such that*

$$|\mu_1(z) - \mu_0(z)| < \eta(y).$$

Proof. Since $K(z) = (1+ia)/(b-ic)$, $K(z)+1 = (1+ia+b-ic)/(b-ic)$, we have

$$\mu_1(z) - \mu_0(z) = \frac{K_1(z)-1}{K_1(z)+1} - \frac{K_0(z)-1}{K_0(z)+1} = 2\frac{K_1(z)-K_0(z)}{(K_1(z)+1)(K_0(z)+1)} =$$

$$2\frac{(1+ia_1)(b_0-ic_0) - (1+ia_0)(b_1-ic_1)}{(1+ia_1+b_1-ic_1)(1+ia_0+b_2-ic_0)} =$$

$$2\frac{(a_1-a_0)(ib_1+c_1)+(b_0-b_1)(1+ia_1)+(c_1-c_0)(i-a_1)}{(1+ia_1+b_1-ic_1)(1+ia_0+b_2-ic_0)}. \tag{11}$$

From the first equation in (10), a_1 lies between -2 and 2. Also note that since $\tilde{H}$ is quasiconformal,

$$|K(z)|=\frac{|1+ia|}{|b-ic|}=\left|\frac{\tilde{H}_z+\tilde{H}_{\overline{z}}}{\tilde{H}_z-\tilde{H}_{\overline{z}}}\right|\le\frac{|\tilde{H}_z|+|\tilde{H}_{\overline{z}}|}{|\tilde{H}_z|-|\tilde{H}_{\overline{z}}|}\le K,$$

and, therefore,

$$|ib+c|=|b-ic|\ge 1/K.$$

From the equations (10)

$$1+b=\frac{\rho+1+2(\rho+1-\rho_+-\rho\rho_-)}{\rho+1}$$

and

$$2(a-c)=2\left(\frac{\rho+1-\rho_++\rho\rho_--2}{\rho+1}\right).$$

Therefore,

$$|1+b|+2|a-c|\ge|1+b+2(a-c)|\ge\frac{5(\rho+1)-2\rho_+-2\rho\rho_--2\rho_++2\rho\rho_--4}{\rho+1}=$$

$$\frac{5(\rho+1)-4\rho_+-4}{\rho+1}>\frac{\rho+1}{\rho+1}=1,$$

because $\rho_+<\rho$. Since $|1+b|+2|a-c|\ge 1$, either $|1+b|\ge 1/2$ or $|a-c|\ge 1/4$. Of course, the same inequalities are true for a_1,b_1,c_1 and for a_0,b_0,c_0 as are true for a,b,c. Thus, $(1/4)$ is a lower bound for each factor in the denominator of (11).

On the other hand the coefficients (ib_1+c_1), $(1+ia_1)$ and $(i-a_1)$, in the numerator of (11) are bounded above because of the equations in (10). These equations also show that if $\rho_0\to\rho_1$, then a_0,b_0 and c_0 approach a_1,b_1 and c_1, respectively. Consequently, $|\mu_1(z)-\mu_0(z)|\to 0$ as $y\to 0$ uniformly in x. $\square$

For the previous argument we have needed the following two lemmas.

Lemma 1. *Suppose $\rho_0(x,y)$ and $\rho_1(x,y)$ are the distortion functions for normalized quasisymmetric mappings H_0 and H_1 of the real axis. Suppose also that there is a vanishing function $\eta(y)$ such that $|\rho_0(x,y)-\rho_1(x,y)|\le\eta(y)$. Then there is a vanishing function $\tilde{\eta}(y)$ such that*

$$|\rho_0(x,y,k)-\rho_1(x,y,k)|\le\tilde{\eta}(y)$$

independently of $0<k\le 1$ and of $x\in\mathbb{R}$.

Proof. For $0\le k\le 1$, put kR equal to the interval from $H(x)$ to $H(x+ky)$, and as before, L is the interval from $H(x-y)$ to $H(x)$. Then, by an affine change of coordinates, L and kR transform to the interval $[-1,0]$ and $[0,k]$. So the extremal length $\Lambda(k)$ of the family of curves joining $[-\infty,H(x-y)]$ to

$[H(x), H(x+ky)]$ is equal to the extremal length of the curve family joining $[-\infty, -1]$ to $[0, k.]$ From the Beurling-Ahlfors extension theorem, the Beltrami coefficients μ_0 and μ_1 of the Beurling-Ahlfors extension satisfies

$$|\mu_0(z) - \mu_1(z)| \leq \epsilon(y),$$

where $\epsilon(y)$ is a positive function which approaches zero as $|y|$ approaches 0. Thus,

$$\left|\log \frac{\Lambda(k_1)}{\Lambda(k_0)}\right| \leq \epsilon(y).$$

Let $\Lambda'(k)$ be the derivative with respect to k of $\Lambda(k)$. By relabelling if necessary, we may assume that $k_0 < k_1$ and by the mean value theorem applied to $\log \Lambda(k)$, there is a number k between k_0 and k_1 such that

$$\log \Lambda(k_1) - \log \Lambda(k_0) = (k_1 - k_0)\frac{\Lambda'(k)}{\Lambda(k)}.$$

Thus, we obtain the estimate

$$|k_1 - k_0| \leq \frac{\Lambda(k)}{\Lambda'(k)}\ \epsilon(s). \tag{12}$$

To calculate $\Lambda'(k)$ we make a quasiconformal deformation $f^k(z)$ that keeps $-1, 0$ and ∞ fixed and moves k_0 to k_1. It is given by

$$f^k(z) = \begin{cases} z + \frac{k-k_0}{k_0}\overline{z} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{13}$$

Note that $f^{k_0}(k_0) = k_0, f^k(k_0) = k$, and $f^{k_1}(k_0) = k_1$. The Beltrami coefficient of f^k is $\frac{k-k_0}{k_0}$ if $x > 0$ and 0 if $x < 0$, and the derivative of this Beltrami coefficient with respect to k is $\frac{1}{k_0}$ for $x > 0$ and 0 for $x < 0$. From the formula for the derivative of the extremal length of an annulus with respect to a variable Beltrami coefficient in the direction of increasing k, ([**?**],[**?**],[**?**] [**?**]), we obtain

$$\Lambda'(k) = \frac{1}{k_0}\frac{2}{\pi}\int\int_{x>0}\left|\frac{k(k+1)}{z(z+1)(z-k)}\right| dxdy.$$

Moreover, for some number $c > 0$ and some number $\beta > 0$,

$$\int\int_{x>0}\left|\frac{k(k+1)}{z(z+1)(z-k)}\right| dxdy \geq ck\log\frac{1}{k}$$

for all k with $0 < k < \beta$. To see this put

$$g(k) = \int\int_{x>0,}\left|\frac{k(k+1)}{z(z+1)(z-k)}\right| dxdy,$$

and note that $g(k) = g_1(k) + g_2(k)$ where

$$g_1(k) = \int\int_{x>0,|z|<1}\left|\frac{k(k+1)}{z(z+1)(z-k)}\right| dxdy$$

and

$$g_2(k) = \int\int_{x>0,|z|>1} \left| \frac{k(k+1)}{z(z+1)(z-k)} \right| dxdy.$$

Obviously $0 \leq g_2(k) \leq (constant)k$ and

$$g_1(k) \geq k \int\int_{x>0,|z|<1} \left| \frac{1}{z(z-k)} \right| dxdy. \tag{14}$$

Substituting kz for z in the integrand of (14), we obtain

$$g_1(k) \geq k \int\int_{x>0,|z|<1/k} \left| \frac{1}{z(z-1)} \right| dxdy.$$

The last integral is greater than or equal to a constant times $\log(1/k)$ and, in summary, we conclude that there is a constant c such that for sufficiently small $k > 0$,

$$g(k) \geq ck \log(1/k).$$

Putting these inequalities together with the obvious estimate

$$\Lambda(k) \leq C \log \frac{1}{k},$$

we obtain

$$\frac{\Lambda(k)}{\Lambda'(k)} \leq \frac{cCk_0 \log \frac{1}{k}}{k \log \frac{1}{k}}.$$

Since $k_0 < k$ we conclude that

$$k_1 - k_0 \leq cC\epsilon(y),$$

which implies that

$$\rho_1(x, y, k) - \rho_0(x, y, k) \leq cC\epsilon(y).$$

A similar argument shows the parallel statement for $\rho(x, -y, k)$. □

Lemma 2. *Suppose $\rho_1(x, y)$ and $\rho_2(x, y)$ are the distortion functions for normalized quasisymmetric mappings of the real axis H_1 and H_0. Suppose also that there is a vanishing function $\eta(y)$ such that $|\rho_1(x, y) - \rho_0(x, y)| \leq \eta(y)|$. Then $|\rho_{1+}(x, y) - \rho_{0+}(x, y)| \leq \eta(y)$ and $|\rho_{1-}(x, y) - \rho_{0-}(x, y)| \leq \eta(y)$ independently of $x \in \mathbb{R}$.*

Proof. The hypothesis implies there is a vanishing function $\tilde{\eta}(y)$ such that $|\rho_1(x, y, k) - \rho_0(x, y, k)| \leq \tilde{\eta}(y)$ uniformly for $0 < k \leq 1$. Therefore, $|\rho_{1+}(x, y) - \rho_{0+}(x, y)| \leq \int_0^1 |\rho_1(x, y, k) - \rho_0(x, y, k)| dk \leq \tilde{\eta}(s)$. The same type of argument works for ρ_- because $\rho_-(x, y) = \int_0^1 \rho(x, -y, k) dk$. □

3. The UAA Teichmüller space

The endomorphism $p(z) = z^m$ of $\mathbb{S}^1$ is a degree m circle endomorphism and its lift via the covering mapping π is $P(x) = m\ x$. That is, $P(0) = 0$ and $\pi \circ P = p \circ \pi$. Obviously, p^n is UAA with constant $M = 1$. We will ultimately learn that the restriction to the unit circle of the ratio of Blaschke products,

$$f(z) = \frac{\prod_{j=1}^{k+m} \frac{z-\alpha_j}{1-\overline{\alpha_j} z}}{\prod_{j=1}^{k} \frac{z-\beta_j}{1-\overline{\beta_j} z}},$$

for sufficiently small $|\alpha_j|$ and $|\beta_j|$, is also a degree m UAA circle endomorphism.

Theorem 2. *Given any degree m UAA circle endomorphism f, there is exists a unique quasisymmetric map h such that $h \circ p \circ h^{-1} = f$, where $p(z) = z^m$.*

Proof. We begin by using the dynamics of the iterations of p and f to construct a self map H of $\mathbb{R}$ satisfying

i) $H(0) = 0$,

ii) $H \circ T = T \circ H$ and

iii) $H \circ M = F \circ H$.

From $H(0) = 0$ and $H \circ T^k(0) = T^k \circ H(0)$, we conclude that $H(k) = k$. Note that $F \circ T(0) = T^m \circ F(0)$ and $F(0) = 0$ implies that $F(1) = m$. Also, $F \circ T = T^m \circ F$ implies

$$F^n \circ T = T^{m^n}$$

and so $F^n(1) = m^n$. Since F is an increasing homeomorphism, $F(0) = 0$ and $F^n(1) = m^n$, we may select numbers $a_{j,n}$ between 0 and 1 such that $F^n(a_{j,n}) = j$ for integers j and n with $0 < j < m^n$. Then, by definition, if we put $H(j/m^n) = a_{j,n}$, we obtain

$$H \circ P^n(j/m^n) = H(j) = j \quad \text{and} \quad F^n \circ H(j/m^n) = F^n(a_{j.n}) = j.$$

This defines H on a dense set of the unit interval with the property that for points x in the dense set $H \circ P(x) = F \circ H(x)$. We extend H to a dense subset of the interval $[k-1, k]$ by requiring that $H \circ T^k = T^k \circ H$.

In the definition of a UAA system put $x = j/m^n$ and $t = 1/m^n$. Then we obtain

$$\frac{1}{M} \le \frac{a_{j+1,n} - a_{j,n}}{a_{j,n} - a_{j-1,n}} \le M.$$

This is what is required to construct a quasiconformal homeomorphism $\tilde{H}$ of $\mathbb{C}$ with restriction to the real axis equal to H and with a Beltrami coefficient μ such that $||\mu||_\infty < 1$. □

Definition 4. The Teichmüller space $\mathcal{T}(m)$ consists of all UAA circle endomorphisms of degree $m > 1$ factored by an equivalence relation. Two endomorpisms f_0 and f_1 representing elements of $\mathcal{T}(m)$ are equivalent if, and only if, there is a symmetric homeomorphism h of $\mathbb{S}^1$ such that $h \circ f_0 \circ h^{-1} = f_1$. Since the dynamics of the mappings T, M and F uniquely determine the points $a_{j,n}$ the mapping H is unique.

4. UAC Systems

If f^n is a UAA circle endomorphism, it is possible that f has a reflection invariant extension $\tilde{f}$ defined in a small annulus $r < |z| < 1/r$ such that

$$\tilde{f}(1/\overline{z}) = 1/\overline{\tilde{f}(z)},$$

and such that for every $\epsilon > 0$ there exists a possibly smaller annulus $U = \{z : r' < |z| < 1/r'\}$ such that

$$K_z(\tilde{f}^{-n}) < 1 + \epsilon \tag{15}$$

for all z in U. Here $K_z(g)$ is the dilatation of g at z and inequality (15) is meant to hold for almost every z in U and for all positive integers n. If such an extension exists, then $\tilde{f}^n$ is called a uniformly asymptotically conformal dynamical (UAC) system.

Lemma 3. *If $\tilde{f}^n$ is a UAC system acting a neighborhood of $\mathbb{S}^1$, then $f^n =$ the restriction of $\tilde{f}^n$ to $\mathbb{S}^1$ is a UAA system.*

Lemma 4. *For any UAC system $\tilde{f}^n$ acting on a neighborhood of $\mathbb{S}^1$ with $\tilde{f}(1) = 1$, there is a unique lift $\tilde{F}$ to an infinite strip containing $\mathbb{R}$ and bounded by lines parallel to $\mathbb{R}$ such that*

(1) $\pi \circ \tilde{F} = \tilde{f} \circ \pi$,
(2) $\tilde{F}(0) = 0$
(3) $\tilde{F} \circ T = T^m \circ \tilde{F}$, *and*
(4) $\tilde{F}$ *preserves the real axis, in fact,* $\tilde{F}(\overline{z}) = \overline{\tilde{F}(z)}$.

Lemma 5. *In the notation of the previous lemma, if $\tilde{f}^n$ is UAC then $\tilde{F}^n$ is UAC in the sense that for every $\epsilon > 0$, there is a $\delta > 0$ such that if the absolute value of $y = Im\ z$ is less than δ, then*

$$K_z(\tilde{F}^{-n}) < 1 + \epsilon.$$

Theorem 3. *If f is a UAA system acting on the unit circle, then there exists a UAC system $\tilde{f}$ acting in a neighborhood of the circle such that the restriction of $\tilde{f}$ to the circle is equal to f.*

Proof. Let (F, T) be the lift to the real axis of the system f such that $F(0) = 0$, $F \circ T = T^m \circ F$ and such that $\pi \circ F = f \circ \pi$. By Theorem 2 there is a quasisymmetric homeomorphism H of $\mathbb{R}$ fixing 0 and 1 such that
i) $H \circ M \circ H^{-1} = F$ where $M(x) = m\ x$, and
ii) $H \circ T \circ H^{-1} = T$ where $T(x) = x + 1$.
By Lemma 5 it will suffice to find an extension $\tilde{F}$ of F such that
i) $\tilde{F} \circ T(z) = T^m \circ \tilde{F}(z)$

and

ii) the Beltrami coefficients $\mu_{\tilde{F}^{-n}}$ of $\tilde{F}^{-n}$ satisfy

$$|\tilde{F}^{-n}(x+iy)| \leq \epsilon(y)$$

where $\epsilon(y)$ is independent of n and x.

We define $\tilde{F}$ to be $\tilde{H} \circ M \circ \tilde{H}^{-1}$. Since $\tilde{H}$ extends H, clearly $\tilde{F}$ extends F. Since have $F^{-n} \circ H$ and H have quasisymmetric distortions ρ_1 and ρ_2 satisfying

$$|\rho_1(x,y) - \rho_2(x,y)| \leq \epsilon(y),$$

by Theorem 1 the Beltrami coefficients $\mu_{\widetilde{F^{-n} \circ H}}$ and $\mu_{\tilde{H}}$ satisfy

$$|\mu_{\widetilde{F^{-n} \circ H}}(z) - \mu_{\tilde{H}}(z)| \leq \eta(y),$$

where $\eta(y)$ is independent of $n > 0$. Since

$$\widetilde{F^{-n} \circ} H = H \widetilde{\circ M}^{-n} = \tilde{H} \circ M^{-n},$$

we conclude that

$$|\mu_{\tilde{H}}(m^{-n}z) - \mu(z)| \leq \eta(y).$$

Also, since the Beurling-Ahlfors extension is affinely natural, $\mu(z+1) = \mu(z)$ and $\tilde{H} \circ T \circ \tilde{H}^{-1}(z) = T(z)$. We conclude that $\tilde{F} = \tilde{H} \circ M \circ \tilde{H}^{-1}$ and T form a uniformly asymptotically conformal circle endomorphism of degree m. $\square$

5. Dual UAC systems, scaling functions and solenoid functions

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