Contemporary Mathematics

Extremal Length and Uniformization

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Introduction

The classical theorem that identifies Riemann surfaces of negative Euler characteristic with surfaces carrying a hyperbolic structure is called uniformization. For surfaces that have complex structure it gives simultaneously a common parametrization by a parameter varying in a simply connected domain. For topological reasons the covering domain is connected and simply connected and, this being so, the uniformization theorem says that it must be conformal to either the Riemann sphere, the complex plane or the unit disc.

Since this theorem was first proved more than a century ago by Koebe [**16**], there have been many accounts, nearly all of them relying on some form of potential theory and the classification of open surfaces into those that have and those that do not have Green's functions. The exposition given here differs in that it avoids potential theory replacing it by the notion of extremal length. The proof involves finding moduli of curve families and the existence of conformal mappings from abstract doubly connected Riemann surfaces to planar annuli, that is, regions in the plane of the form $\{z : 1/r < |z| < r\}$.

Our proof starts out the same way as it does in most classical expositions. It is assumed there is given a simply connected Riemann surface X with a local parameter z vanishing at a point $p_0 \in X$ and mapping a neighborhood of p_0 onto the unit disc in the z-plane. Then one forms the disc $D_\epsilon(p_0) = \{p \in X : |z(p)| \leq \epsilon\}$ and tries to uniformize the annular surface $X_\epsilon = X - D_\epsilon(p_0)$. In our approach we consider moduli $M(A_n)$ of annuli A_n contained in X_{ϵ_n} with one boundary component equal to the boundary of $D_{\epsilon_n}(p_0)$, where ϵ_n is a sequence of positive numbers that decrease to 0. Then we let $M(A_n)$ be the extremal length of the family curves contained in A_n and homotopic to the boundary of A_n. We first establish that for such annuli with non-zero modulus there exists conformal maps z_n that map A_n onto a ring domains in the z-plane lying between $|z| = 1$ and $|z| = R_n$ where $M(A_n) = 2\pi/\log R_n$.

There are two cases:

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Case 1. There exists a sequence of annuli A_n such that $M(A_n)$ decreases to 0 as $\epsilon_n \to 0$. In this case we show X is conformal either to the complex plane or to the Riemann sphere.

Case 2. For any such sequence of annuli A_n the moduli $M(A_n)$ are bounded below by a positive number independently of n. In this case we show X is conformal to the unit disc.

In both cases we find uniformizing annuli or uniformizing punctured discs for X_{ϵ_n}. We repeat the same constructions for the surfaces X_{ϵ_n} and obtain conformal maps from X_{ϵ_n} onto annuli in the complex plane. Then we take the limit as ϵ_n approaches 0 and the crux of the argument is to show that these limits exist. The underlying entity that controls the limiting process is a quadratic differential, holomorphic on $X - \{p_0\}$ and with a double pole at p_0 whose regular trajectories are closed and fill $X - \{p_0\}$ except for possibly one additional point.

Our proof could be given in one or two pages but we take the opportunity to prove results with other applications and to develop the following ideas.

(1) weak measured foliations and their Dirichlet integrals,
(2) conjugate differentials and conformal metrics,
(3) pulling back measured foliations,
(4) intersection number and the Cauchy-Schwarz inequality,
(5) comparison of moduli of annuli to moduli of curve families,
(6) generalized locally L^2 derivatives and Weyl's lemma,
(7) integrable holomorphic quadratic differentials,
(8) Strebel's variational technique,
(9) extremal lengths as moduli for Teichmüller space,
(10) Rado's theorem on the separability of all Riemann surfaces.

We use a technique that involves an alternative expression for absolute intersection numbers of simple closed curves and prove several results that we believe have not been previously noticed. In particular, we introduce the idea of intersection number of weak measured foliations [1] and show how this kind of intersection applies to homotopy intersection of simple closed curves. For two closed curves α and β the intersection $i(\alpha, \beta)$ is the total minimal number of intersections, all counted positively, of the unoriented curves α' and β' where α' and β' are closed curves freely homotopic in X to α and β, respectively. If we let A and B be arbitrary annuli in X in the homotopy classes of α and β, we can define functions u_A and u_B that are harmonic on A and B and that are equal to 0 and 1 on the two sides of A and B, respectively. Then it turns out that

$$i(\alpha, \beta) = \inf \left\{ \int\int_{A \cap B} |du_A \wedge du_B| \right\}$$

where the infimum is taken over all such annuli A and B. This inequality combined with the Cauchy-Schwarz inequality

$$\left(\int\int_{A \cap B} |du_A \wedge du_B| \right)^2 \leq Dir(|du_A|) Dir(|du_B|)$$

gives a way to estimate extremal lengths of measured foliations in terms of Dirichlet integrals. It also brings in Riemann's idea for showing the existence of harmonic

[1] In earlier papers I have used the terminology partial measured foliation instead of weak measured foliation. The improved terminology was suggested by Dylan Thurston.

functions, which is to minimize the Dirichlet integral with side conditions. In our set-up we minimize the Dirichlet integral of a weak measured foliation subject to the side conditions on its heights. The significance of intersection numbers in Teichmüller theory was first developed in Thurston's theory of measured foliations, [**10**] [**27**]. See also [**21**].

The paper is organized into ten sections. In the first and second we define Riemann surfaces, weak measured foliations, differentials, the Dirichlet integral, conjugate differentials and conformal metrics. In section 3 we show that the Dirichlet integral of the pull back by a K-quasiconformal map of a weak measured foliation is K-invariant. In section 4 we give a method for computing intersection number and show how it can be estimated from above by Dirichlet integrals. In section 5 we define modulus of an annulus, observe that it is a conformal invariant and show that the modulus $M(A)$ of the closed curve family Γ_A in $A = \{z : 1 < |z| < R\}$ that separates its two boundary contours is $2\pi/\log R$. Section 6 presents several equivalent ways to define the modulus of a free homotopy class of simple closed curves on a Riemann surface. This is a key result that not only permits an elementary proof of Koebe's uniformization theorem but also provides a family of invariants that permit the construction the extremal length embedding of any Teichmüller space [**13**]. In section 7 we present Strebel's variational method which together with Weyl's lemma presented in section 9 provides the holomorphic quadratic differential whose unique existence is the key to the equivalence of the four different formulas for extremal length given in Theorem 1. In section 8 we prove the Riemann mapping theorem and the uniformization theorem by the same method. In section 9 we prove Weyl's lemma and finally, in section 10, we discuss the one-dimensional C^∞ manifold structure on the *long line* and the corresponding 2-dimensional manifold structure on the *long cylinder* and observe how our proof of uniformization shows why this 2-dimensional smooth surface cannot have subordinate complex structure. This example is featured to clarify a theorem of Rado that says any Riemann surface is metrizable by a metric that has a countable basis for its topology. Once the uniformization theorem is proved Rado's theorem is an obvious corollary. In studying classical proofs of uniformization it is often difficult to see why the existence of a complex structure forces the underlying topological space to be separable.

In the last analysis it turns out that the uniformizing coordinate for a simply connected Riemann surface X can be found by solving two differently stated extremal problems that apply to a given point on any Riemann surface X. These two problems both lead to an underlying cylindrical quadratic differential that has a double pole at one point with a quadratic residue equal to -1. One fixes a local parameter w that vanishes at a point $p_0 \in X$. The first extremal problem involves the family $\mathcal{F}$ of univalent functions that map the unit disc into X and that map 0 to p_0 and it is to find a function $f \in \mathcal{F}$ that realizes the infimum

$$U = \inf \left\{ \frac{1}{|(w \circ f)'(0)|} : f \in \mathcal{F} \right\}.$$

The second extremum problem involves something called the reduced extremal length Q at the point p_0 in X. One first finds the extremal distance in X_ϵ from the perimeter of a small disc of radius ϵ centered at p_0 to the ideal boundary of X_ϵ. The reduced extremal length Q is obtained by correcting this constant by subtracting the number $(1/2\pi)\log(1/\epsilon)$ and taking the limit as ϵ decreases to 0. By using Koebe's distortion theorem for univalent functions holomorphic in the unit disc [**3**] one can show that U and Q are related by the formula $U = \exp(-2\pi Q)$, [**20**]. Case 1 occurs when $U = 0$ and $Q = \infty$, and in this case X is conformal to the Riemann sphere or the complex plane. Case 2 occurs when $U > 0$ and $Q < \infty$, and in this case X is conformal to the unit disc. One sees that for a simply connected Riemann surface X finding the function inverse to the uniformizing function z amounts to finding a univalent function holomorphic in the unit disc that realizes the extremal value U. These ideas are developed in [**20**].

The uniformization theorem, a form of which is also known as Hilbert's 22nd problem [**6**], was developed over a period of time that starts with Riemann in the mid nineteenth century and goes to the present. A summary of contributions up to 1913 is given in Herman Weyl's book, *Die Idee der Riemannschen Fläche* and translated into English *The Concept of a Riemann Surface*, [**29**]. Koebe's original proof appears in three papers, the last one published in 1914, [**16**].

Most textbooks written for first semester courses in complex analysis contain the Riemann mapping theorem which is preliminary to uniformization, but few prove uniformization. Several of these are nicely summarized in Henry Kwan's Ph.D. 2004 thesis [**17**] written under the supervision of Yum-Tong Siu at Harvard University. There is also a book written in French and edited by de Saint-Gervais with chapters by fifteen mathemematicians published in 2010 [**7**].

Few expositions of Koebe's theorem address the possibility that the Riemann surface might be non-separable and none of them take advantage of the possibility of proving uniformization using extremal length as the main tool. The extremal length proof which can be described as a proof by exhaustion with annuli has at least two advantages. First, the core of the argument is brief. Second, the extremal length approach demonstrates a fundamental observation of Rado [**23**] that says if a surface is connected and if it has a complex structure it must necessarily have a countable basis for its topology. Essentally any piece-by-piece introduction of a complex coordinates on a surface with a non-countable basis for its topology forces the accumulation of nested annuli with moduli that increase to infinity. If the surface is non-separable and connected this forces the existence of nodal points. By definition nodal points do not admit local parameters.

The following authors are among those that include a statement of the theorem and sometimes a proof in notes or books: Abikoff [**1**], Bers [**5**], Ahlfors and Sario [**4**], Donaldson [**8**], Farkas and Kra [**9**], Imayoshi and Taniguchi [**15**], Kwan [**17**], Pommerenke [**22**], G. Springer [**25**], Schiffer and Spencer [**24**], Strebel [**26**], and Tsuji [**28**].

1. Riemann surfaces, weak measured foliations and the Dirichlet integral

Definition 1. A *Riemann surface* is a connected topological Hausdorff space X together with a system of local homeomorphisms (called local parameters) z_j defined on neighborhoods N_j contained in X and mapping into $\mathbb{C}$ such that

(1) the union of the neighborhoods N_j over j covers X,
(2) in any non-empty overlap $N_j \cap N_k$ the composition $z_k \circ (z_j)^{-1}$ is holomorphic.

Definition 2. A *chart* on a Riemann surface X is a homeomorphism w from an open set in X onto an open subset of $\mathbb{C}$ for which $w \circ z_j^{-1}$ is holomorphic for every local parameter z_j for which points in the domain of z_j overlap with points in the domain of w.

Note that with these definitions local parameters are also charts.

DEFINITION 3. A continuous function u defined on a set $\Omega \subset \mathbb{C}$ has generalized first partial derivatives u_x and u_y if for every C^∞ function ϕ with compact support

$$\int\int u\phi_x dxdy = -\int\int u_x \phi dxdy$$

and

$$\int\int u\phi_y dxdy = -\int\int u_y \phi dxdy.$$

DEFINITION 4. Let z_j be a set of charts defined on open subsets N_j of a Riemann surface X. A *weak measured foliation* $|du|$ is a system of continuous, real valued functions u_j defined on open sets $z_j(N_j)$ in $\mathbb{C}$ such that

(i) for any two of the sets N_j and N_k with $N_j \cap N_k$ non-empty,

$$u_j(z_j(p)) = \pm u_k(z_k(p)) + constant$$

for all $p \in N_j \cap N_k$,

(ii) in each open set N_j, u_j has generalized first partial derivatives with respect to x and y, $(u_j)_x$ and $(u_j)_y$, where $z = x + iy$, and

(iii) these generalized first partial derivatives are locally square integrable, that is, for each j,

$$\int\int_{z(N_j)} \left((u_j)_x^2 + (u_j)_y^2\right) dxdy < \infty.$$

Note that in this definition we do not assume $N = \bigcup_j N_j$ is equal to X.

DEFINITION 5. The *Dirichlet integral* of a weak measured foliation $|du|$ on a Riemann surface X is equal to

$$Dir(|du|) = \int\int \left(u_x^2 + u_y^2\right) dxdy.$$

Here, one considers the domain of integration to be either $N = \bigcup_k N_j$ or to be X, bearing in mind that by definition the integrand is identically zero in $X - N$.

Let $\gamma(t)$ be a continuous curve in X defined for $t \in [0,1]$ and $p_k = \gamma(t_k)$ where $0 = t_0 < t_1 < \cdots t_n = 1$. Let $mesh(\{p_k\}) = \max_{1 \leq k \leq n} t_k - t_{k-1}$.

DEFINITION 6. The height of γ with respect to $|du|$ is given by

$$ht(\gamma, |du|) = \limsup_{mesh(\{p_k\}) \to 0} \sum_{j=1}^{n} |u_k(p_j) - u_k(p_{j-1})| = \int_\gamma |du|$$

where we put $|u_k(p_j) - u_k(p_{j-1})| = 0$ if either p_j or p_{j-1} is not in N and where we assume that if both successive points p_{j-1} and p_j lie N then for sufficiently small value of $mesh(\{p_k\})$ there is a value of k for which they both lie in the same set N_k.

2. Conformal metrics and conjugate differentials

Definition 7. A conformal metric on a Riemann surface X is a system of non-negative measurable functions ρ_j defined on open sets $z_j(N_j)$ in $\mathbb{C}$ such that in each overlapping set $N_j \cap N_k$,

$$\rho_j(z_j) = \rho_k(z_k)\left|\frac{dz_k}{dz_j}\right| \text{ a.e.}$$

We note that in the definition of a Riemann surface we require that the domains of local parameters be defined on a family of open sets N_j that cover all of the Riemann surface X, but we do not make this requirement in the definitions of a weak measured foliation $|du|$ or of a metric $\rho(z)|dz|$. At any points where a conformal metric $\rho(z)|dz|$ or the weak measured foliation $|du|$ are not defined by convention we put ρ and $|du|$ equal to zero.

Given a conformal metric $\rho(z)|dz|$, where ρ is either upper semicontinuous or lower semicontinuous, we define the length of a piecewise differentiable arc $\gamma(t), t_0 \leq t \leq t_1$ to be the line integral

$$L(\rho,\gamma) = \int_{t_0}^{t_1} \rho(\gamma(t))|\gamma'(t))|dt$$

and we define the area of ρ to be the double integral

$$area(\rho) = \int\int_X \rho^2(z)dxdy.$$

Definition 8. Assume $z = x+iy$ is a local coordinate on a Riemann surface X. For any differential form $Pdx + Qdy$, where P and Q are locally square integrable, the conjugate differential form is

$$*(Pdx + Qdy) = -Qdx + Pdy.$$

Definition 9. A *quadratic differential* $q(z)(dz)^2$ on X is an assignment of a continuous function q_j on sets $z_j(N_j)$ on each open set N_j forming a covering of X such that for any $p \in N_j \cap N_k$,

$$q_j(z_j(p)) = q_k(z_k(p))\left(\frac{dz_k(p)}{dz_j(p)}\right)^2.$$

q is *holomorphic* if each of the functions q_j is holomorphic.

Lemma 1. *A weak measured foliation $|du|$ with finite Dirichlet integral on a Riemann surface X determines a conformal metric ρ by putting*

$$\rho(z)|dz| = |du + i*du| \tag{1}$$

and a quadratic differential ψ by putting

$$\psi = (du + i*du)^2,$$

where we put both ρ and ψ identically equal to 0 in $X - N$. Moreover,

$$area(\rho) = \int\int_X \rho^2 dxdy = Dir(|du|). \tag{2}$$

PROOF. Since

$$\begin{aligned}\rho^2|dz|^2 &= (du + i*du)(du - i*du)\\ &= u_x^2dx^2 + 2u_xu_ydxdy + u_y^2dy^2 + u_y^2dx^2 - 2u_xu_ydxdy + u_x^2dy^2\\ &= (u_x^2+u_y^2)(dx^2+dy^2),\end{aligned}$$

$\rho|dz| = (u_x^2+u_y^2)^{1/2}|dz|$ is an infinitesimal conformal metric. For such a metric we define its area to be

$$area(\rho) = \int\int_X \rho(z)^2 dx\wedge dy.$$

Then for $\rho(z)|dz| = |\psi(z)|^{1/2}|dz|$ defined by (1),

$$area(|\psi|^{1/2}) = \int\int_X |\psi(z)|dx\wedge dy.$$

Note that $dx\wedge dy = (1/2)|dz\wedge d\bar{z}|$. Finally, we have

$$\int\int |\psi|dx\wedge dy = (1/2)\int\int_X |(du+i*du)\wedge(du-i*du)| = \int\int_X du\wedge *du$$

$$= \int\int_X (u_xdx+u_ydy)\wedge(-u_ydx+u_xdy) = \int\int_X (u_x^2+u_y^2)dx\wedge dy = Dir(|du|).$$

□

The quadratic differential ψ is not necessarily holomorphic and by definition both ψ and ρ are identically equal to zero in the complement of $N = \bigcup_j N_j$.

LEMMA 2. *Any holomorphic quadradic differential q on X determines a one-parameter family of weak measured foliations by the formula*

$$v = \text{ the imaginary part of } \left(\int (e^{i\theta}q(z_j))^{1/2}dz_j\right) \tag{3}$$

where v is defined only on the complement of the set of zeros of q in X.

PROOF. Since we only consider points where q is nonzero, the only ambiguity in equation (3) involves the choice of a plus or minus sign in taking a square root and the choice of an additive constant in taking the antiderivative. Thus $|dv|$ is well defined on the complement in X of the set of zeros of q. □

When $\theta = 0$, $|dv|$ is called the vertical foliation of q and when $\theta = \pi$, $|dv|$ is called the horizontal foliation of q. We note that a parameterized differentiable arc $\gamma(t)\in X$ for which

$$q(\gamma(t))\gamma'(t)^2 > 0$$

is an arc contained in a horizontal leaf of the vertical foliation of q. Similarly if

$$q(\gamma(t))\gamma'(t)^2 < 0$$

then the image of γ is contained in a vertical leaf of the horizontal foliation. Segments of these types are called regular horizontal or regular vertical trajectories of q. Points where q is equal to zero are called critical points and at these points $|dv|$ is undefined. The trajectory structure in a neighborhood of a critical point p where $q = 0$ has a zero of order k is conformally equivalent to the trajectory structure of $z^k(dz)^2$. At these points q has $k+2$ trajectories that meet at p and the angle between any two neighboring trajectories is $2\pi/(k+2)$ radians. These are called $k+2$-pronged singularities.

3. Pulling back a weak measured foliaton

In this section we show how the Dirichlet integral of a weak measured foliation is changed when changing coordinates by a K-quasiconformal map.

Definition 10. Let $w = f(z) = u(z)+iv(z)$ be a homeomorphism from a Riemann surface X onto a Riemann surface Y and assume u and v have locally square integrable first partial distributional derivatives. Then f is called *quasiconformal* if there is a constant $k < 1$ such that

$$|f_{\overline{z}}(z)| \leq k|f_z(z)|, \ \textit{a.e.},$$

where

$$\frac{\partial}{\partial z} = (1/2)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and}$$

$$\frac{\partial}{\partial \overline{z}} = (1/2)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Lemma 3. *Let $f : X \to Y$ be quasiconformal and $|du|$ be a weak measured foliation on Y with finite Dirichlet integral. Then $|d(u \circ f)|$ is a weak measured foliation on X with finite Dirichlet integral and*

$$Dir_X(|d(u \circ f|)| \leq K Dir_Y(|du|),$$

where $K = \frac{1+k}{1-k}$.

Proof. We put $w = f(z)$ and note that

$$(u \circ f)_z = (u_w \circ f)f_z + (u_{\overline{w}} \circ f)\overline{f}_z.$$

Since $|\overline{f}_z| = |f_{\overline{z}}|$, u is real-valued and defined up to plus or minus sign and an additive constant, $|u_w| = |u_{\overline{w}}|$. Thus we can imitate the calculation given at the end of Chapter 1 in [**3**]. Putting $w = f(z)$ we have

$$|(u \circ f)_z| = |(u_w \circ f)f_z + (u_{\overline{w}} \circ f)\overline{f}_z|$$

$$\leq (|u_w| \circ f)(|f_z| + |\overline{f}_z|) = (|u_w| \circ f)(|f_z| + |f_{\overline{z}}|), \text{ and}$$

$$Dir_X(|d(u \circ f)|) = 4 \int\int_X |(u \circ f)_z|^2 \left|\frac{dz \wedge d\overline{z}}{2}\right|$$

$$\leq 2 \int\int_X (|u_w| \circ f)^2(|f_z| + |f_{\overline{z}}|)^2 \, |dz \wedge d\overline{z}|$$

$$= 2 \int\int_X (|u_w| \circ f)^2(|f_z|^2 - |f_{\overline{z}}|^2) \cdot \frac{(|f_z| + |f_{\overline{z}}|)^2}{|f_z|^2 - |f_{\overline{z}}|^2} \, |dw \wedge d\overline{w}|$$

$$\leq 4K \int\int_Y |u_w|^2 \frac{|dw \wedge d\overline{w}|}{2} = K \ Dir_Y(|du|).$$

□

4. Intersection number and the Cauchy-Schwarz inequality

If two curves on a surface intersect transversally at several points, their intersection number is the total number of essential intersections. This differs from the homology notion of intersection where the curves are oriented and where you count an intersection positively or negatively according to its orientation. The intersection $i(\alpha,\beta)$ of the homotopy classes of two closed curves α and β is the minimum number of intersections of α' and β' where α' and β' are freely homotopic to α and β.

There is a more devious way to view homotopical intersection of simple closed curves on oriented surfaces. For a closed curve α, first slightly fatten it so it becomes a ribbon (or annulus) A_α and then form a harmonic function u_α on A_α that is equal to zero on one side of A_α and equal to one on the other side. Then it turns out that the intersection number is equal to the infimum of the double integrals

$$\int\int |du_\alpha \wedge du_\beta|, \tag{4}$$

where the domain of integration is the common intersection of the two ribbons for α and β and the infimum is taken over all curves in the same free homotopy class and all possible ribbons. The absolute value is placed under the integral sign in (4) to ensure that all intersections are counted positively.

Here we show how this notion of intersection is related to Dirichlet integrals. First we note an application of the Cauchy-Schwarz inequality to the wedge product of two weak measured foliations.

Lemma 4. *Let $|du|$ and $|dv|$ be weak measured foliations on a Riemann surface X. Then*

$$\left(\int\int_X |du\wedge dv|\right)^2 \leq Dir(|du|)Dir(|dv|). \tag{5}$$

Proof. The following inequality for real numbers is elementary:

$$\left(det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)^2 \leq (a^2+b^2)(c^2+d^2).$$

For every point $p\in X$ it leads to

$$|du(p)\wedge dv(p)| \leq (u_x(p)^2+u_y(p)^2)^{1/2}(v_x(p)^2+v_y(p)^2)^{1/2}.$$

We obrtain (5) by applying Schwarz's inequality,

$$\left(\int fgdm\right)^2 \leq \left(\int f^2dm\right)\left(\int g^2dm\right).$$

□

Definition 11. A closed curve in a surface X is a continuous map γ from the unit circle into X. It is called simple if γ is injective. A simple closed curve contained in an annulus is called an essential curve or a core curve if it separates the two boundary components of the annulus.

Definition 12. For any two simple closed curves α and β on a Riemann surface X, $i(\alpha,\beta)$ is the number of essential intersections of α with β. That is, $i(\alpha,\beta)$ is the minimum number of intersections α' with β' where α' and β' are closed curves freely homotopic in X to α and β, respectively.

Lemma 5. **(The intersection inequality)** *Let A_α be an annulus with smooth boundary containing an essential simple closed curve α and u_α be a function harmonic in A_α with one-sided limiting values equal to* 0 *and* 1 *on the two sides of A_α. Let u_β have the same corresponding properties for an annulus A_β corresponding to a simple closed curve β. Finally, let $i(\alpha,\beta)$ be the number of essential intersections α and β. Then*

$$i(\alpha,\beta) \leq \int\int |du_\alpha \wedge du_\beta|. \tag{6}$$

Proof. Note that we can assume any simple closed curve α is an essential closed curve contained in some annulus $A_\alpha \subset X$. The fattening can be trimmed in such a way as to have a differentiable boundary; because X is orientable the boundary will have two components that are homotopic simple closed curves and there will be a function u_α that is equal to 0 on one of these boundary curves and equal to 1 on the other.

The integrand $|du_\alpha \wedge du_\beta|$ is nonnegative where the annulus A_α intersects the annulus A_β and zero elswhere. To estimate the integral in (6) note that since the level curves of u_α are homotopic to α and similarly those of u_β are homotopic to β, they necessarily intersect at least $i(\alpha,\beta)$ times. Thus the double integral can be estimated by integrating by parts and we obtain

$$\int\int_X |du_\alpha \wedge du_\beta| \geq \left| i(\alpha,\beta) \int_0^1 [u_\alpha \wedge du_\beta]_{u_\alpha=0}^{u_\alpha=1} \right| = i(\alpha,\beta) \left| \int_0^1 du_\beta \right| = i(\alpha,\beta).$$

□

5. Extremal length of annuli

In this section we assume A is an annulus in a given homotopy class on a Riemann surface X. Also, except for Lemma 8, we assume X contains only one free homotopy class of simple closed curve. We let γ be one of the closed curves in this class and Γ be the set of all such curves in this class.

Definition 13. The extremal length of a curve family Γ is given by

$$M(\Gamma) = \sup_\rho \frac{L(\rho,\Gamma)^2}{area(\rho)}, \tag{7}$$

where $\rho(z)|dz|$ is a continuous conformal metric on X and where $L(\rho,\Gamma)$ is the infimum of lengths of closed curves γ in Γ and $area(\rho) = \int\int_X \rho^2$.

The Riemann surface X is called annular if it has only one such class Γ. In this case we usually write A instead of X and since there is only one such class we write $M(A)$ instead of $M(\Gamma)$. If A is an annular Riemann surface we also define the conjugate curve family Γ^* and the conjugate modulus $M^*(A)$. Γ^* is the family of arcs in A that join its two boundary components and $M^*(A) = M(\Gamma^*)$. This number is often called the modulus of A and is given by the following definition.

Definition 14.

$$M^*(A) = \sup_\rho \frac{L(\rho,\Gamma^*)^2}{area(\rho)}. \tag{8}$$

In both (7) and (8) the supremum is over all conformal metrics ρ defined on X.

LEMMA 6. *If* $A = \{z : 1 < |z| < R\}$ *then* $M(A) = 2\pi/\log R$ *and* $M^*(A) = (\log R)/2\pi$.

PROOF. Any closed curve γ in A that winds once around the origin satisfies

$$\pm 1 = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

and so any closed curve $\gamma \in \Gamma(A)$ satisfies

$$1 \leq \int_\gamma \frac{1}{2\pi r}|dz| = \int_\gamma \rho_0(z)|dz|,$$

where $\rho_0(z)|dz| = 1/(2\pi r)|dz|$. This inequality is an equality for any of the concentric closed curves $re^{it}, 0 \leq t \leq 2\pi$ so $L(\rho_0) = 1$. On the other hand,

$$area(\rho_0) = (1/2\pi)^2 \int\int_A (1/r^2) r dr d\theta = (\log R)/2\pi$$

and so $M(A) \geq 2\pi/\log R$.

For the reverse inequality, observe that for any allowable metric $\rho|dz|$,

$$L(\rho) \leq \int_0^{2\pi} \rho(re^{it}) r dt,$$

$$L(\rho)/r \leq \int_0^{2\pi} \rho(re^{it}) dt.$$

Integrating both sides from $r = 1$ to $r = R$ gives

$$L(\rho)\log R \leq \int\int_A \rho dr dt = \int\int_A \rho r^{1/2} \cdot \frac{1}{r^{1/2}} dr dt$$

and applying Schwarz's inequality we get

$$L(\rho)^2(\log R)^2 \leq \int\int_A \rho^2 r dr dt \int\int_A \frac{1}{r} dr dt,$$

$$L(\rho)^2/area(\rho) \leq 2\pi/(\log R) \text{ for every allowable } \rho.$$

We conclude ρ_0 realizes the value $M(A)$ which is equal to $2\pi/(\log R)$.

For the second equality, the same metric ρ_0 is extremal, in the sense that it realizes the value $M^*(A)$. To see this note that for the radial arcs $\gamma_0^*(t) = te^{i\theta}, 1 \leq t \leq R$,

$$\int_{\gamma_0^*} \rho_0|dz| = (1/2\pi)\int_1^R (1/t)dt = \frac{1}{2\pi}\log R$$

and for any arc γ^* in A joining its inner to outer boundary,

$$(1/2\pi)\int_{\gamma^*} (1/t)dt \geq \frac{1}{2\pi}\log R$$

so $L(\rho_0, \Gamma^*) = \frac{1}{2\pi}\log R$. Also $area(\rho_0) = \left(\frac{1}{2\pi}\right)\log R$ so

$$L(\rho_0, \Gamma^*)^2/area(\rho_0) = (\log R)/(2\pi) \text{ and}$$

$$M^*(A) \geq (\log R)/(2\pi). \tag{9}$$

To see that ρ_0 realizes the maximal value for this curve family note that any arc $\gamma^*(t)$ joining $|z|=1$ to $|z|=R$, the definition of $L(\rho)$ implies

$$L(\rho) \leq \int_{\gamma^*} \rho(z)|dz|.$$

Integrating along the arc $te^{i\theta}, 1 \leq t \leq R$ for each θ and then integrating with respect to θ from 0 to 2π gives

$$2\pi L(\rho) \leq \int_{\theta=0}^{\theta=2\pi} \int_{|z|=1}^{|z|=R} \rho(z) dr d\theta.$$

Thus,

$$2\pi L(\rho) \leq \int\int_A \rho(z) r^{1/2} \cdot \frac{1}{r^{1/2}} dr d\theta, \text{ and by Schwarz's inequality,}$$

$$(2\pi L(\rho))^2 \leq \int\int_A \rho(z)^2 r dr d\theta \cdot \int\int_A \frac{1}{r} dr d\theta,$$

$$(2\pi L(\rho))^2 \leq area(\rho) 2\pi \log R \text{ and so}$$

$$\frac{L(\rho)^2}{area(\rho)} \leq \frac{\log R}{2\pi}.$$

we conclude that $M^*(A) \leq \log R/2\pi$ and so from (9) $M^*(A) = \log R/2\pi$. □

There are several observations we wish to make about the result of this lemma:

(1) the same metric ρ_0 is extremal for the supremum problems involving both Γ and Γ^*, and $\rho_0 = |q|^{1/2}$ where
$$q(z)(dz)^2 = -\left(\frac{1}{2\pi} \cdot \frac{dz}{z}\right)^2$$
is a holomorphic quadratic differential on A,
(2) the curve families Γ and Γ^* contain subfamilies Γ_0 and Γ_0^*, which are the horizontal and vertical trajectories of q,
(3) Γ_0 and Γ_0^* are minimally extremal for ρ_0 in the sense that ρ_0 is extremal for the extremal problems $M(\Gamma_0)$ and $M(\Gamma_0^*)$. This observation is called Beurling's lemma, (see [**2**], chapter 4, section 7).
(4) from Lemma 3 and a change of variable it follows that $M(A)$ is a conformal invariant,
(5) $q^{1/2} dz = \pm(i/2\pi)(d \log z) = \pm(i/2\pi)(d \log |z| + i d\theta) = i(du + i dv)$, where $u = \log |z|$ is a harmonic function in A equal to 0 on $|z| = 1$ and equal to $\log R$ on $|z| = R$,
(6) the level lines defined by $u = a$ *constant* and $v = a$ *constant* comprise the curve families Γ_0 and Γ_0^*,
(7) similar observations hold for the vertical and horizontal foliations of any integrable holomorphic quadratic differential on any Riemann surface,
(8) for the extremal problem $M(\Gamma)$ on any Riemann surface X if $M(\Gamma) > 0$ then the quadratic differential q is integrable and holomorphic on X and q expresses the first variation of $\log M(\Gamma)$ when the complex structure on X is allowed to vary.

Since it is not necessary for the proof of the uniformization theorem we do not include justification of statement (8) in this article. It can be found in the last pages of [**12**].

LEMMA 7. *For any annular Riemann surface A conformal to $\{z : 1 < |z| < R\}$, $M(A) = 2\pi/\log R$ and there is a unique harmonic function u defined on A with the property that $u = 0$ on one boundary component of A and is equal to $(1/2\pi)\log R$ on the other boundary component.*

PROOF. u is unique by the maximum principle. If $z : A \to \{z : 1 < |z| < \log R\}$ is conformal, then $u = (1/2\pi)\log|z|$ is harmonic and has the prescribed boundary values. □

LEMMA 8. *Suppose there is a homeomorphism h from $\{z : 1 \leq |z| \leq 2\}$ onto a domain A in a Riemann surface X and*

$$\gamma_0 = h(\{z : |z| = 1\}) \text{ and } \gamma_1 = h(\{z : |z| = 2\})$$

are its two boundary contours. Then there exists a unique harmonic function u defined on A which is equal to 0 on γ_0 and equal to 1 on γ_1. Moreover, u determines a conformal map $z = \exp\left(2\pi \int (du + i * du)\right)$ from A to a region of the form $\{z : 1 < |z| < R\}$ and a quadratic differential

$$q_A = -\left(\frac{1}{2\pi} \cdot \frac{dz}{z}\right)^2 \tag{10}$$

defined on A whose horizontal and vertical trajectories are, respectively, the concentric circles $|z| = \text{a constant}$ and the rays $\arg z = \text{a constant}$.

PROOF. One considers the family of all C^1 functions defined on A with the given boundary values on the two simple closed curves that form the boundary of A and minimizes the Dirichlet integral. One uses L^2 estimates on first partial derivatives and Lemma 12, Weyl's lemma, to show that the minimum is realized for some function u and this minimum is harmonic. Moreover, if p_n is a convergent sequence of points in A and if $u(p_n) \to 1$, then p_n must converge to a point γ_1. Also, $u(p_n)$ converging to 0 implies p_n converges to a point in γ_0. This is because otherwise the harmonic function u would take its minimum or maximum value at an interior point of A, which contradicts the maximum principle for harmonic functions.

After constructing u we form $U = (\log R)u$ where R is chosen so that $M(A) = 2\pi/\log R$. Then U is harmonic in A with boundary value 0 on γ_0 and $\log R$ on γ_1 and $\zeta = U + i^*U$ maps A to a rectangle in the ζ-plane with width $\log R$ and height 2π and $z = \exp(i\zeta)$ maps A to the region between two concentric circles in the z-plane with inner radius 1 and outer radius R. □

The essence of Lemma 8 is the statement that every topological annulus embedded in a Riemann surface is conformal to a round annulus and the horizontal and vertical trajectories of the quadratic differential q in (10) are realized by concentric circles and radial lines emanating from the origin in the z-plane.

6. Moduli of curve families on a Riemann surface

This section concerns moduli of homotopy classes of simple closed curves on an arbitrary Riemann surface X. Although it has several equivalent definitions, we choose to define the modulus of the homotopy class of Γ in X in the following way.

DEFINITION 15. The modulus of the curve family Γ in X is equal to

$$M(\Gamma) = \sup_{\rho} \left\{ \frac{L(\Gamma,\rho)^2}{\int\int_A \rho^2 dxdy} \right\}, \tag{11}$$

where the supremum is taken over all conformal metrics $\rho(z)|dz|$ on X and

$$L(\rho,\Gamma) = \inf_{\gamma\in\Gamma} \int_\gamma \rho(z)|dz|.$$

In this definition of $M(\Gamma)$ there is no particular annulus on which to apply the Grötzsch length-area argument given in Lemma 6. However, because of the next theorem, one can single out a unique annulus on which the argument can be applied. It is the annulus embedded in X in the same homotopy class for which $M(\Gamma)$ is as small as possible. Moreover, when $0 < M(\Gamma)$ this annulus is uniquely associated to a particular integrable quadratic differential q defined and holomorphic on X. Even when $M(\Gamma) = 0$, there is also a corresponding domain A and a corresponding holomorphic quadratic differential but which fails to be integrable. In the non-integrable case the domain A is conformal either to a punctured disc or a doubly punctured Riemann sphere. All of the regular horizontal trajectories of q are closed and in the homotopy class of Γ. Moreover, except for the critical trajectories of q which comprise a union of analytic arcs of zero area, these regular closed horizontal trajectories fill X.

We make the following definitions:

(i) $M_1 = \inf_A \{M(A)\}$, where the infimum is over all annuli A homotopic to γ.

(ii) $M_2 = \inf_{|du|} \{Dir(|du|)\}$, where the infimum is over all C^1 functions u defined on an arbitrary annulus A homotopic to γ in X and equal to 1 on one of its boundary contours and equal to 0 on the other.

(iii) $M_3 = \sup_{|dv|} \left\{ \frac{1}{Dir(|dv|)} \right\}$, where the supremum is over all weak measured foliations $|dv|$ for which $\int_\gamma |dv| \geq 1$ for all $\gamma \in \Gamma$.

THEOREM 1. *Suppose $\gamma \in \Gamma$ and there is a closed curve β contained in X such that $i(\gamma,\beta) > 0$. Then*

$$M(\Gamma) = M_1 = M_2 = M_3.$$

Moreover, there is a unique annulus A in X that realizes the infimum for M_1. A has the following properties:

(i) It is the characteristic annulus of a quadratic differential

$$q_A(z)(dz)^2 = -\left(\frac{dz}{z}\right)^2$$

on A.

(ii) q_A is defined and holomorphic on X and A is equal to the union of the closed horizontal trajectores of $q_A(z)(dz)^2$.

(iii) The square root $q_A(z)^{1/2}dz$ is equal to $\pm(du + iM(\Gamma)dv)$, where $|du|$ and $|dv|$ are the two unique foliations that realize the extremal values M_2 and M_3.

(iv) Up to scalar multiple the metric ρ that uniquely realizes the supremum for $M(\Gamma)$ is given by $\rho = |q_A|^{1/2}$.

(v) *Finally, if* $A = X$ *then* z *is a univalent map the maps* A *to a round annulus* $\{z : 1/R < |z| < 1\}$ *where* $M(\Gamma) = 2\pi/\log R$.

REMARK. For a family Γ of closed curves on a surface the quantities M_2 and M_3 give effective ways to estimate from below and from above the extremal length $M(\Gamma)$ defined in Definition 15. This observation was pointed out to me by Dylan Thurston.

PROOF. For a given annulus A in X with core curves in the homotopy class of Γ we choose the harmonic function u_A given by Lemma 8. For this u_A the Dirichlet integral $Dir(|du_A|)$ is equal to $2\pi/\log R$ which is equal to $M(A)$ so $M_1 \leq M_2$. On the other hand, every such harmonic function is by definition supported on an annulus A and so $M_2 \leq M_1$.

From Lemmas 4 and 5 and the definitions of $|du|$ and $|dv|$ in M_2 and M_3 we get

$$\frac{1}{Dir(|dv|)} \leq Dir(|du|), \tag{12}$$

which shows that $M_3 \leq M_2$. More generally, let A be an annulus with core curve in the homotopy class of Γ and u a harmonic function of the type described in the definition of M_2. If there is a curve β that intersects γ, then $i(\beta, \gamma) > 0$ and

$$0 < \frac{i(\gamma, \beta)^2}{Dir(|dv|)} \leq Dir(|du|), \tag{13}$$

where v is any C^1 function on an annulus B in the homotopy class of β which is equal to 0 and 1 on the two sides of B. Thus by (13) the infimum taken in the definition of M_1 is the infimum of a set which is bounded below. Given an annulus in the set we know by Lemma 8 there is a univalent function $1/z_A$ from A onto $\{z : 1/R < |z| < 1\}$. If we let c_A be the univalent function which is inverse to z_A then as $M(A)$ decreases to the infimum M_1 the functions c_A have increasing domains of definition. The functions z_A determine quadratic differentials

$$q_A = -\left(\frac{1}{2\pi}\right)^2 \cdot \left(\frac{dz_A}{z_A}\right)^2, \tag{14}$$

which have L_1-norm bounded by $1/M_1$. Since the L_1-norms of q_A are bounded we can select a sequence A_n such that q_{A_n} converge uniformly on compact subsets. The limit of such a sequence determines an annulus A that realizes the extremal value M_1. By Strebel's variational technique, Lemma 9, and Weyl's Lemma, Lemma 12, for this A the quadratic differential q_A given by formula (14) is holomorphic on X and its L_1-norm is equal to $1/M_1$.

The closed trajectories for this q_A sweep out a unique extremal annulus in X. Any closed curve $\gamma \in \Gamma$, even if while staying in X it veers outside of A, measured in the metric $|q_A|^{1/2}$ it is at least as long as any of the closed trajectories of q_A. This is because q is holomorphic and closed horizontal trajectories of a holomorphic quadratic differential minimize length in the metric $|q|^{1/2}$ among all closed curves in the same homotopy class. For this reason, one can apply Grötzsch's argument on the annulus A to show that $M(\Gamma) = M_1$.

But then from Lemma 1 the supremum in M_3 is realized by a measured foliation equal to a constant times the conjugate measured foliation $|*du|$. That is, the $|dv|$

that realizes this supremum is given by $(1/M(\Gamma))|*du|$. The holomorphic quadratic differential

$$q_A = (du + i * du)^2 = (du + iM(\Gamma)dv)^2$$

where $|du|$ realizes M_2 and $|dv|$ realizes M_3 gives the metric $|q_A|^{1/2}$ extremal metric for $M(\Gamma)$. $\square$

7. Strebel's variational technique

Lemma 9. *Suppose $M(\Gamma) > 0$. Then there is a unique annulus A in X homotopic to Γ for which $M(A)$ is minimum. This annulus is is filled by closed horizontal trajectories of a quadratic differential q_γ holomorphic on X and the complement of A in X consists of the critical trajectories of q and*

$$\int\int_X |q_\gamma| = (2\pi)^2/M(\Gamma).$$

Finally, for this particular annulus A, $M(A) = M(\Gamma)$.

Proof. Let z be a local parameter that vanishes at p and assume

$$\{p \in X : |z(p)| < 1\}$$

is a simply connected neighborhood N of p. Put

$$w = \begin{cases} z + \epsilon h(z) & \text{in } N \\ z & \text{outside of } N, \end{cases}$$

where $h(z)$ is continuous, is identically equal to zero outside of $\{p : |z(p)| < 1/2\}$ and has continuous first partial derivatives. Then the Beltrami coefficient $\nu(z)$ of w is given by

$$\nu(z) = \frac{w_{\overline{z}}(z)}{w_z(z)} = \frac{\epsilon h_{\overline{z}}(z)}{1 + \epsilon h_z(z)}. \tag{15}$$

Provided ϵ is small enough, for values of z in the support of $h(z)$ the sum $z + \epsilon h(z)$ will lie in N. By taking ϵ even smaller if necessary the bounds on the derivatives $h_{\overline{z}}$ and h_z imply that ν in equation (15) satisfies $||\nu||_\infty < 1$. This bound implies $z \mapsto w$ is a quasiconformal self mapping of X which is homotopic to the identity.

From Lemma 8 we can assume $\zeta = \xi + i\eta$ maps A to a rectangle

$$R = \{0 \le \xi \le a, 0 \le \eta \le b\}.$$

This map carries any horizontal segment α in R defined by η equal to a constant to closed curve in the class Γ. Also, let $A' = w(A)$ and assume ζ' is a conformal parameter. Note that if $\zeta' = \xi' + i\eta'$ is a conformal parameter then for any non-zero constant λ the product $\lambda \cdot \zeta'$ is also a conformal parameter. Thus by choosing λ appropriately we may assume the height of R' is b. That is, the conformal parameter ζ' takes A' to the rectangle with height b,

$$R' = \{0 \le \xi' \le a', 0 \le \eta' \le b\}.$$

Then

$$a' \le \int_{w(\alpha)} |d\zeta'| = \int_\alpha |w_\zeta d\zeta + w_{\overline{\zeta}} d\overline{\zeta}| = \int_\alpha |w_\zeta||1 + \nu(\zeta)|d\xi.$$

Thus,

$$a'b \le \int\int_R |w_\zeta||1 + \nu|d\xi d\eta, \text{ and}$$

by introducing factor of $(1-|\nu|^2)^{1/2}$ in the numerator and denominator and applying Schwarz's inequality we obtain

$$(a'b)^2 \le \int\int_R |w_\zeta|^2(1-|\nu|^2)d\xi d\eta \int\int_R \frac{|1+\nu|^2}{1-|\nu|^2}d\xi d\eta.$$

Noting that by the Jacobian change of variable formula the first integral is equal to the area of R', which is equal to $a'b$. So we get

$$a'b \le \int\int_R \frac{|1+\nu|^2}{1-|\nu|^2}d\xi d\eta. \tag{16}$$

We put $q = (d\zeta)^2$ in R and identically equal to zero on $X - R$ and similarly we put $q' = (d\zeta')^2$ in R' and identically equal to zero on $X - R'$. Also, we view ν, q and q' as invariant differential forms. Then equation (16) becomes

$$\int\int_X |q'|dxdy \le \int\int_X \frac{\left|1+\nu q/|q|\right|^2}{1-|\nu|^2}|q|dxdy. \tag{17}$$

Since $M(A)$ realizes the smallest possible value of moduli of annuli in the homotopy class Γ, q has the smallest norm in this class and we have $\int\int_X |q|dxdy \le \int\int_X |q'|dxdy$, from (17) we obtain

$$\int\int_X |q|dxdy \le \int\int_X \frac{\left|1+\nu q/|q|\right|^2}{1-|\nu|^2}|q|dxdy. \tag{18}$$

Squaring the numerator on the right hand side of (18) and dividing by 2 yields

$$0 \le \mathrm{Re} \int\int_X \frac{\nu q}{1-|\nu|^2}dxdy + \int\int_X \frac{|\nu|^2|q|}{1-|\nu|^2}dxdy. \tag{19}$$

The first order consequence of (15) and (19) is that for every C^1 function h with compact support in $N \subset X$,

$$\mathcal{R}e \int\int_N h_{\overline{z}}(z)q(z)\ dxdy \ge 0.$$

Since the same argument also applies when h is replaced by $-h$ we conclude that

$$\mathcal{R}e \int\int_N h_{\overline{z}}(z)q(z)\ dxdy = 0. \tag{20}$$

Similarly, by replacing h by ih and $-ih$, the same reasoning implies

$$\mathcal{I}m \int\int_N h_{\overline{z}}(z)q(z)\ dxdy = 0 \tag{21}$$

and putting (20) and (21) together we get

$$\int\int_N h_{\overline{z}}(z)q(z)\ dxdy = 0.$$

By Weyl's lemma, Lemma 12 in section 9, this implies q is holomorphic at every point $p \in X$ up to the addition of a function that is almost everywhere equal to zero. And Grötszch's argument [**14**] implies that the quadratic differential with these properties is unique.

In summary, there is a global holomorphic quadratic differential q defined on X and an extremal annulus A contained in X and a univalent holomorphic function

z mapping A to the region between two concentric circles $|z| = R_1$ and $|z| = R_2$ in the complex plane such that the restriction of q to A satisfies

$$q(z)(dz)^2 = -(1/2\pi)^2(dz/z)^2.$$

□

From the minimum norm principle in [**11**], [**12**] and [**18**] we know that up to a multiplicative constant with absolute value equal to one q is unique among holomorphic quadratic differentials q' that minimize $\int\int_X |q'|dxdy$ and satisfy $\int_\gamma |q'|^{1/2} \geq 1$ for all $\gamma \in \Gamma$. This important consequence is not necessary for the proof of Theorem 1.

8. Uniformization

We need a topological lemma that applies to a doubly connected Riemann surface X that has at least one boundary contour.

LEMMA 10. *Suppose X is a doubly connected Hausdorff space that has a Riemann surface structure and suppose it is bounded on one side by an analytic closed curve C. Suppose also that β is a simple piecewise analytic arc in X that joins two points p_1 and p_2 in X and also that p_1 can be joined to C by a simple arc $\tilde{\gamma}$. Then there exists a simple piecewise analytic arc γ in X with both endpoints on C such that $i(\beta, \gamma) \geq 1$.*

PROOF. First note that by a simple piecewise analytic arc joining p_1 to p_2 we mean a piecewise real analytic function $\gamma(t)$ that maps the interval $[0, 1]$ into X, that is one-to-one and such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$. If p_1 and p_2 are in the same compact subset of X, then one can find a finite number of local parameters that cover any arc that joins p_1 to p_2. Also, one can use these local parameters to join p_1 to p_2 by a curve that consists of successive horizontal and vertical intervals in the successive coordinate charts.

By hypothesis there is a simple arc $\tilde{\gamma}$ that joins C to p_1. To find γ one first constructs the tubular domain of points that have distance less than δ from $\tilde{\gamma}$ for sufficiently small $\delta > 0$. Then one lets γ be the part of the boundary of the tube excluding that part which lies on C. For sufficiently small δ this provides the curve γ for which $i(\beta, \gamma) \geq 1$. □

In the next lemma we assume X is any simply connected Riemann surface, z is a local parameter vanishing at $p_0 \in X$, $D_\epsilon(p_0) = \{p \in X : |z(p)| \leq \epsilon\}$ and

$$X_\epsilon = X - D_\epsilon(p_0) \text{ and}$$

$$C_\epsilon = \{p : |z(p)| = \epsilon\}.$$

Finally, we let Γ_ϵ be the family of simple closed curves in X_ϵ that are homotopic to C_ϵ. We also let A_n be any annular domain contained in X_ϵ with one boundary component coinciding with C_ϵ and the other boundary component a piecewise real analytic closed curve in X_ϵ homotopic to C_ϵ.

LEMMA 11. *Assume the extremal length $M(\Gamma_\epsilon) = 0$. Then there exists a sequence of annuli $A_n \subset X_\epsilon$ in the homotopy class of C_ϵ with the following properties:*

1) each A_n is bounded by C_ϵ on one of its sides and a piecewise real analytic closed curve on its other side,

2) A_n is ascending in the sense that $A_k \subset A_{k+1}$ for each integer k, and

3) the sequence $M(A_n)$ decreases to 0 as $n \to \infty$.
Moreover, for any such sequence of annuli A_k, $X_\epsilon \setminus \bigcup_k A_k$ contains at most one point.

PROOF. Consider the family $\mathcal{F}$ of simple closed curves α contained in $A_k \cup A_{k+1}$ that are homotopic to C_ϵ in X_ϵ. Note that every simple closed curve $\alpha \in \mathcal{F}$ separates X_ϵ into two components. We refer to the component adjacent to the perimeter of $D_\epsilon(p_0)$ as the inside and to the other component as the outside. Form the function v_α equal 0 on $\{p : |z(p) - z(p_0)| = \epsilon\}$ and equal to 1 on α. Then let $A_\alpha = \{p : v(p) < 1\}$ and

$$\tilde{A}_{k+1} = \{p : \text{there exists } \gamma \in \mathcal{F} \text{ such that } p \in A_\gamma\}.$$

Inductively we construct the new sequence $\tilde{A}_n$ of annuli with property 2). Since $A_n \subset \tilde{A}_n$, the sequence $\tilde{A}_n$ also has property 1). By induction we can assume that the sequence A_n has property 2). The set $A = \bigcup_n A_n$ is open and doubly connected and also the hypothesis $M(\Gamma_\epsilon) = 0$ implies $M(A) = 0$.

Now let β be an arc joining p_1 to p_2 and γ be the arc given by Lemma 10. Fatten γ to a narrow strip B both ends of which are situated on the circle C_ϵ. Make the strip B so narrow that every arc in B that connects its two ends separates p_1 from p_2. Since p_1 and p_2 are in X_ϵ but not in A, any core curve α in A must satisfy

$$i(\alpha, \gamma) \geq 2.$$

Now let u_{A_n} a function harmonic in A_n, equal to 1 on C_ϵ and equal to 0 on the other boundary of A_n. Also let v_B be the function with minimum Dirichlet integral which is equal to 1 on one side of the strip B and zero on its other side.

So from Lemmas 4 and 5

$$4/Dir(|dv_B|) \leq Dir(|du_{A_n}|). \tag{22}$$

Then by (22) and Theorem 1 it is impossible for $M(A)$ to be equal to 0.

From this we conclude there cannot be more than one point of $X_\epsilon - A$ that can be joined to C_ϵ by an arc. Since A is conformal to a once punctured disc, the complement of A in X_ϵ is either empty or consists of just one point. If there is such a point, there is a unique way the complex structure of A extends to this point and that way must coincide with the complex structure of X_ϵ. □

The next theorem is the uniformization theorem and its corollary is the Riemann mapping theorem.

THEOREM 2. **(Uniformization)** *Any simply connected Riemann surface X is conformal either to the Riemann sphere $\overline{\mathbb{C}}$ or to the complex plane $\mathbb{C}$ or to the unit disc $\mathbb{D} = \{z : |z| < 1\}$.*

PROOF. Take a point $p_0 \in X$ and a local parameter z defined in an ϵ_0 neighborhood of p_0 and a value of ϵ_0 small enough so that $D_{\epsilon_0}(p_0) = \{p : |z(p) - z(p_0)| \leq \epsilon_0\} \subset X$. Then consider the doubly connected Riemann surface $X_\epsilon = X - D_\epsilon(p_0)$ where $0 < \epsilon \leq \epsilon_0$. Put Γ_ϵ equal to the family of closed curves in X_ϵ homotopic to the boundary of $D_\epsilon(p_0)$ and divide the proof into two cases,

Case 1: $M(\Gamma_\epsilon) = 0$, and

Case 2: $M(\Gamma_\epsilon) > 0$.

In Case 1 we construct the squence of annuli A_n with the properties of Lemma 11 and consider the subset $\bigcup_n A_n$ of X_ϵ. It can omit at most one point of X_ϵ because otherwise the extremal length of the family of the family Γ_ϵ would be positive. If it has exactly one element then X_ϵ is a conformal to the exterior of $D_\epsilon(p_0)$ in $\mathbb{C}$ by a mapping z_ϵ. As ϵ converges to 0 the conformal mappings z_ϵ converge uniformly to a limit that is defined on a doubly connected domain A for which $M(A) = 0$. In this case the complement of A in X_ϵ can contain at most one point. If it contains no points then X is conformal to $\mathbb{C}$ and if it contains one point then X is conformal to $\overline{\mathbb{C}}$.

In Case 2 by Theorem 1 there is an annulus A_ϵ with $M(A_\epsilon) > 0$ with the property that $D_{\epsilon_0}(p_0) \cup A_\epsilon$ fills X. Letting $\epsilon \to 0$, one can obtain a subsequence of these uniformizing parameters for these A_ϵ that converges to a conformal mapping from X onto a disc of finite radius. $\square$

Corollary 1. **(The Riemann mapping theorem)** *Any simply connected domain X contained in $\overline{\mathbb{C}}$ is conformal either to the Riemann sphere $\overline{\mathbb{C}}$ or to the complex plane $\mathbb{C}$ or to the unit disc $\mathbb{D} = \{z : |z| < 1\}$.*

Proof. The conclusion of the corollary is exactly the same as the conclusion of Theorem 2 but the corollary has stronger hypotheses. $\square$

Many presentations of this corollary start with an arbitrary simply connected domain in $X \subset \overline{\mathbb{C}}$ omitting two or more points. The further assumption that it is simply connected implies that one can apply a branch of a square root to obtain a conformal image of X that lies inside the complement of an open set in $\overline{\mathbb{C}}$. By applying a Möbius transformation that has a simple pole inside that open set, one obtains conformal image of X that is a bounded domain in $\mathbb{C}$. After this step the extremal length method of Theorem 1 yields the Riemann mapping theorem immediately and one does not need the intersection inequality. The square root trick is not available for Theorem 2 because one cannot assume in advance that X is realizable conformally as a subset of $\overline{\mathbb{C}}$.

9. Weyl's lemma

The following result is called Weyl's lemma,

Lemma 12. *Suppose q is a complex valued function defined on a plane domain D and q is in $L^1(D)$. Suppose further that for every C^1 function h with compact support in D*

$$\int\int h_{\overline{z}} q(z) dx dy = 0. \tag{23}$$

Then there is an analytic function $\tilde{q}$ defined on D such that

$$q(z) = \tilde{q}(z)$$

for almost all $z \in D$.

Proof. In the $z = x + iy$-chart, let m be the Lebesgue measure, so that

$$m(S) = \int\int_S |dx \wedge dy|,$$

for every measurable set S. Put $J_\epsilon(z) = 1/(2\pi\epsilon^2)$ in $\{|z| < \epsilon\}$ and equal to 0 elsewhere. J_ϵ is called an approximate identity because for every continuous function

f defined on a domain in D, the convolution

$$(J_\epsilon * f)(z) = \int\int J_\epsilon(w-z)f(w)dm(w)$$

approaches $f(z)$ as $\epsilon \to 0$.

Note that $J_\epsilon(z)$ is not continuous, but the convolution $j_\epsilon = J_\epsilon * J_\epsilon$ is and for any function q in $L^1(D)$, $j_\epsilon * q$ has continuous first partial derivatives. Thus,

$$(j_\epsilon * q)_{\overline{z}}(z) = \int\int_D q(w)(j_\epsilon(w-z))_{\overline{z}}dm(w) = \int\int_D q(w)(j_\epsilon(w-z))_{\overline{w}}dm(w).$$

Substituting $h(w) = j_\epsilon(w-z)$ into the hypothesis (23) we see that $(j_\epsilon * q)_{\overline{z}}(z) = 0$ and so $j_\epsilon * q$ is holomorphic.

Note that $||j_\epsilon * q||_1 \leq ||q||_1$ and $||q||_1 = Dir(|du|)$, which is bounded. By normal families and the next lemma we know that there is a normal limit $\tilde{q}$ holomorphic in D such that $j_\epsilon * q \to \tilde{q}$ uniformly on compact subsets of D and also $j_\epsilon * q$ converges to q in L^1. This implies that $q = \tilde{q}$ almost everywhere. □

LEMMA 13. *Suppose u_n is a sequence of harmonic functions defined in a bounded plane domain U and*

$$\int\int_U |u_n|dxdy \leq B.$$

Then u_n has a uniformly convergent subsequence and the limit of any such convergent subsequence is harmonic.

PROOF. Because of the mean value property for any disc $D_{r_0}(p_0)$ of radius r_0 centered at p_0 and contained in U,

$$|u_n(p_0)| \leq \frac{1}{\pi r_0^2}\int\int_D |u_n|dxdy \leq B/(\pi r_0^2). \tag{24}$$

Since the circumference of any disc with center p and radius r contained in U can be covered by a finite number of discs contained in U, there is a constant M such that $|u_n(z)| \leq M$ for all z with $|z-p| = r$. By (24) the sequence u_n is equicontinuous on this circle and must have a subsequence that converges uniformly. Since each of the functions u_n satisfies the mean value property,

$$u_n(p) = \frac{1}{2\pi}\int_0^{2\pi} u_n(p + re^{i\theta})d\theta,$$

the uniform limit u also satisfies this property and is therefore harmonic. □

10. The long ray, the long line and the long cylinder

An immediate consequence of uniformization is Rado's theorem which says that any Riemann surface X is separable, that is, it has countable basis for its topology. The proof is simple: X must be conformal to either the Riemann sphere or the complex plane or the unit disc or a quotient space of one of the two latter mentioned surfaces by the action of a fixed point free, discontinuous group. Consequently, in the middle of any proof of uniformization there should appear a reason why any attempt to put a complex structure on any nonseparable surface will necessarily fail. In this section we construct a *long cylinder* which is a connected surface with differentiable structure and show how our proof of uniformization eliminates the possibility that it could have a subordinate complex structure and at the same time be "*long*."

Consider an uncountable well-ordered set ω_1 with the order topology and the half closed unit interval $[0,1)$ and form the *long ray*

$$L = \omega_1 \times [0,1).$$

The *long line* is obtained by patching together two copies of the long ray but since we are concerned only with what at first appears to be a paradox concerning Rado's theorem we only need the long ray.

From the long ray we can construct the *long cylinder* by putting

$$X = L \times \mathbb{S}^1.$$

Points of X have coordinates (β, x, θ) where $\beta \in \omega_1, x \in [0,1)$ and $\theta \in \mathbb{S}^1$. The standard Euclidean structure on the product of the unit circle $\mathbb{S}^1$ with the open unit interval $(0,1)$ induces a complex structure at all points of X except those with coordinates $(\beta, 0, \theta)$ in X.

We may attempt to extend this to a complex structure on all of the long cylinder X, that is, including those points with coordinates $(\beta, 0, \theta)$ in the following way. Any element β of the well-ordered set ω_1 has an immediate successor, namely, the smallest element of the set of elements of ω_1 strictly larger than β. We denote this element by $\beta + 1$. Now we join together the two half-open annuli $\beta \times [0,1) \times \mathbb{S}^1$ and $(\beta+1) \times [0,1) \times \mathbb{S}^1$ by putting the first cylinder on the left and the second on the right and by identifying the point $(\beta, 1, \theta)$ on the left with the point $(\beta+1, 0, \theta)$ on the right.

If β_0 is the smallest element of ω_1 we can delete from X its initial circular boundary, namely, the copy of the unit circle which consists of the points with coordinates $(\beta_0, 0, \theta)$ to obtain a surface Y. At the interior points of the cyclindrical components

$$\beta \times [0,1) \times \mathbb{S}^1$$

of Y there is the standard complex structure. Moreover, at the points $(\beta, 1) \times \mathbb{S}^1$ attached to the points $(\beta+1, 0) \times \mathbb{S}^1$ there is also a standard complex structure.

Since there must be some points $\alpha \in \omega_1$ for which there is no β with the property that $\beta + 1 = \alpha$, the definitions given so far do not introduce a complex coordinate structure at every point of Y. In fact, because of Rado's theorem there cannot be an extension of this complex structure or in fact any complex structure to all of Y.

Definition 16. We say that two points P and Q in X are joined by a chain of parametric discs $D_1, \ldots, D_n$ if $P \in D_1$ and $Q \in D_n$ and each successive intersection $D_j \cap D_{j+1}$, $1 \le j \le n-1$ is non-empty.

Proposition 1. *No matter what complex structure is introduced on the component cylinders of the long ray, there are points α and β in ω_1 with $\alpha < \beta$ such the cylinder joining $\alpha \times \mathbb{S}^1 \times [0,1]$ and $\beta \times \mathbb{S}^1 \times [0,1]$ in Y cannot have finite modulus.*

Proof. Let $\mathcal{F} = \mathcal{F}(\alpha, \beta)$ be the family of elements of the long ray that lie between α and β. The union of the component cylinders starting at α and coming before β form a long component. For each γ in $\mathcal{F}$ let $M(\gamma)$ be the modulus of the component cylinder which has coordinate γ in this component must have positive modulus. From the comparison inequality

$$\frac{1}{M(A)} \geq \sum_{\gamma \in \mathcal{F}} \frac{1}{M(\gamma)},$$

this implies $M(A) = 0$ where A is the cylinder starting at α, and so it cannot be attached to the component cylinder corresponding to β. $\square$

References

[1] W. Abikoff. *The Real Analytic Theory of Teichmüller space*, volume 820 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, 1980.
[2] L. V. Ahlfors. *Conformal Invariants: Topics in Geometric Function Theory*. McGraw-Hill, New York, 1973.
[3] L. V. Ahlfors. *Lectures on Quasiconformal Mapping,*, volume 38 of *University Lecture Series*. Amer. Math. Soc., 2006.
[4] L. V. Ahlfors and L. Sario. *Riemann Surfaces*. Princeton University Press, Princeton, N. J., 1960.
[5] L. Bers. *Riemann Surfaces, Lecture Notes*. Inst. Math. Sci. Lect. Notes, NYU, New York, 1957-58.
[6] L. Bers. On Hilbert's 22nd problem. *Proc. of Symposia in Pure Mathematics*, 28:559–609, 1976.
[7] H. P. de Saint-Gervais. *Uniformisation des surface de Riemann, retour sur un théorème centenaire (French)*. ENS Éditions, Lyons, 2010.
[8] S. K. Donaldson. *Riemann Surfaces*. Graduate Texts in Mathematics, 22. Oxford University Press, Oxford, 2011.
[9] H. Farkas and I. Kra. *Riemann Surfaces*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1980.
[10] A. Fathi, F. Laudenbach, and V. Poénaru. *Travaux de Thurston sur les Surfaces*, volume 66-67 of *Asterisque*. Soc. Math. de France, Seminaire Orsay, 1979.
[11] F. P. Gardiner. Measured foliations and the minimal norm property for quadratic differentials. *Acta Math.*, 152:57–76, 1984.
[12] F. P. Gardiner. *Teichmüller Theory and Quadratic Differentials*. John Wiley & Sons, New York, 1987.
[13] F. P. Gardiner and H. Masur. Extremal length geometry of Teichmüller space. *Complex Variables*, 16:209–237, 1991.
[14] H. Grötszch. Über einige Extremalprobleme der konformen Abbildung I, II. *Acad. Wiss.*, 80:367–376, 497–502, 1918.
[15] Y. Imayoshi and M. Taniguchi. *An Introduction to Teichmüller Spaces*. Springer-Verlag, Tokyo, 1992.
[16] P. Koebe. Über die Uniformizierung der algebraische Kurven. *Ann. Math.*, 67, 69, 75:145–224, 1–81, 42–129, 1909, 1910, 1914.
[17] H. Kwan. Uniformization of Riemann Surfaces. *Ph.D. Thesis, Harvard University*, pages 1–29, 2004.
[18] A. Marden and K. Strebel. The heights theorem for quadratic differentials on Riemann surfaces. *Acta Math.*, 153:153–211, 1984.
[19] C. T. McMullen. Notes. *Harvard University*, pages 1–17, 2005.
[20] C. D. Minda. The Hahn metric on Riemann surfaces. *Kodai Math. J.*, 6:57–69, 1983.
[21] Y. N. Minsky. Harmonic maps, length and energy in Teichmüller space. *J. Differential Geom.*, 35:151–217, 1992.
[22] C. Pommerenke. *Boundary Behaviour of Conformal Maps*. Springer-Verlag, New York, Berlin, 1992.
[23] T. Rado. Über den Begriff der Riemannschen Fläche. *Acta Szeged*, 2:101–121, 1925.
[24] M. Schiffer and D. C. Spencer. *Functionals of Finite Riemann Surfaces*. Princ. Univ. Press, Princeton, N. J., 1954.
[25] G. Springer. *Introduction to Riemann Surfaces*. Addison-Wesley, Reading, Mass., 1957.
[26] K. Strebel. *Quadratic Differentials*. Springer-Verlag, Berlin & New York, 1984.

[27] W. Thurston. *The Geometry and Topology of Three Manifolds*, volume I. Princeton Lecture notes, 1979.
[28] M. Tsuji. *Potential Theory in Modern Function Theory Manifolds*. Maruzen, 1959.
[29] H. Weyl. *Die Idee der Riemannschen Fläche*. 1st ed. Teubner, 2nd ed. Teubner, 3rd ed. Addison-Wesley, 1913, 1923, 1955.

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