CIRCLE ENDOMORPHISMS, DUAL CIRCLES AND THOMPSON'S GROUP

FREDERICK P. GARDINER AND YUNPING JIANG

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Abstract. We construct the dual Cantor set for a degree two expanding map f acting as cover of the circle $\mathbb{T}$ onto itself. Then we use the criterion for a continuous function on this Cantor set to be the scaling function of a uniformly asymptotically affine UAA expanding map to show that the scaling function for f descends to a continuous function on a dual circle $\mathbb{T}^*$. We use this representation to view the Teichmüller space UAA as the set of scaling continuous functions on this dual circle and to construct a natural action of Thompson's F-group as a group of geometrically realized biholomorphic isometries for UAA. Finally, we use the dual derivative $D^*(f)$ for f defined on $\mathbb{T}^*$ to obtain a generalized version of Rohlin's formula for metric entropy where we take an integral over the dual circle.

Introduction

The first idea presented in this paper is the construction of the dual circle for an expanding circle map and the use of the space of scaling functions defined on this circle to construct a faithful action of Thompson's group as a group of biholomorphic isometries of the associated Teichmüller space. The second idea is the use of the dual circle to give a generalized version Rohlin's formula that expresses the metric

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entropy of an invariant measure in terms of the derivative of the dual shift with respect to the dual measure with support on the dual circle.

To explain what is a dual circle, we first explain what is the dual Cantor set C^* of a linearly ordered Cantor set C marked by an heirarchical planar tree associated to its gaps. We view C with its order topology as being constructed by excising gaps from an interval following an inductive procedure that also determines an heirarchical planar tree. C is a compact Hausdorff space with a topology induced by a linear ordering of the interval, that is, the topology is generated by open sets of the form $\{x : a < x < b\}$. Since we are interested in the dynamics of the iteration of degree two expanding circle endomorphisms, we view the circle $\mathbb{T}$ as the unit interval $I = \{x : 0 \leq x \leq 1\}$ with 0 identified with 1. Of course, the neighborhoods of the point $p = \{0, 1\}$ are sets of the form $\{x : -\delta < x \leq 0\} \cup \{x : 0 \leq x < \delta\}$.

The gaps in the tree form a set of disjoint connected open subsets of I and the C is I minus the union of these gaps. We assume that each gap in the tree has two descendants and the chain of all descendancies describes the heirarchy of the tree. Also, the natural ordering of the real numbers in I induces a coding of the gaps by words, which we describe inductively in the following way.

(1) The top gap G is coded by the empty word. This gap forms the 0-th generation,
(2) The next two gaps, G_0 and G_1, lie on the left and the right sides G are coded by 0 and 1. These gaps form the 1-st generation.
(3) The next four gaps, $G_{00}, G_{01}, G_{10}, G_{11}$, lie successively in the four closed intervals comprising the set I minus the gaps of the 0-th and 1-st generations. These gaps form the second generation.
(4) Inductively, the 2^n gaps in the n-th generation lie successively in the 2^n closed intervals that comprise the set I minus the gaps in all of the previous generations.

So far we do not assume anything about the sizes of these gaps, and with respect to certain measures they may have size zero. But we put C equal to I minus the union of all of the gaps in all of the generations, and if C has no interior we call it a Cantor set. Note that all of the endpoints of gaps belong to C. Also, if we map every gap to its left endpoint, we obtain a one-to-one map from every vertex in the tree of gaps to binary numbers expressed as a finite sequence of 0-s and 1-s and adding $.00\ldots01$ using the ordinary rules of arithmetic modulo 1 moves in order from left to right across all of the gaps up to an $(n-1)$-st generation for the appropriate choice of n which is equal to the number of 0-s that appear after the decimal point and before the digit 1.

The Cantor set C^* dual to C has its own marked heirarchical tree, which can be obtained by the following procedure. The position of every gap G_w with code $w = i_0 \dots i_{n-1}$ is moved to the position of the gap G_{w^*}, where $w^* = i_{n-1} \dots i_0$. Here the symbols in the code for w^* are the same as the symbols in the code for w written in reverse order. Clearly, if the gaps have variable size, viewed as a subset of the same interval I the set C^* can differ from the set C, and certainly its marked heirarchical tree will be different. More importantly, the natural order topology on I differs completely from natural order topology induced by the dual codes w^* for C^*. However, C^{**} and its marked heirarchical tree will coincide C and its marked heirarchical tree.

The dual circle $\mathbb{T}^*$ is constructed from the dual Cantor set C^* by attaching every gap on its left and on its right to the gaps that now appear in adjacent positions and are in the same and previous generations. The attachments have the effect of identifying any two points with codes of the form

$$\dots 00001w. \text{ and } \dots 11110w.$$

To bring in dynamical systems to this discussion we view the gaps in the above construction as generated by a degree two expanding map f acting on the unit interval $I = [0, 1]$ with 0 identified with 1. The class of maps f we wish to consider take the form $f = h \circ f_0 \circ h^{-1}$ where $f_0(x) = 2x \mod 1$, h is an orientation preserving homeomorphism of I with some restrictions on h. The forward compositions f^n of f have degree 2^n and induce the combinatorial structure of the heirarchical tree of gaps that is conjugate to the combinatorial structure for the tree generated by $f_0 : x \mapsto 2x \mod 1$.

When the mappings f are $C^{1+\alpha}$, there are associated transfer operators and the theorems of thermodynamical formalism apply. In this setting scaling functions on the associated dual Cantor set are defined and these functions can be viewed as forming a parameter space whose elements represent deformations in the given smoothness class. But the dual circle is not introduced and its significance for Rohlin's formula and for a natural action of Thompson's group is not realized.

In this paper we bring these topics into focus by working with a larger class of mappings. We assume that the endomorphisms f are *uniformly asymptotically affine* (UAA) in the sense that each forward composition f^n of f viewed at fine scales is approximately affine, and degree of approximation does not depend on n.

The paper is organized into eight sections. In section 1 we give a canonical way to view a circle endomorphism as a homeomorphism of its universal covering by $x \mapsto e^{2\pi i x}$ and interpret various smoothness classes in this covering. In section 2 we give Rohlin's formula for the metric entropy of a smooth invariant measure for this system in the case that f is $C^{1+\alpha}$, which is a formula expressed as in integral over the circle $\mathbb{T}$. In section 3 we explain the codings for the Cantor set and the dual Cantor set that are generated by the forward powers f and the forward powers of f^*. In section 4 we introduce the dual dynamical system and the scaling function on the dual Cantor set and in section 5 we invoke the necessary and sufficient condition for a continuous function on the dual Cantor set to be the scaling function of a UAA system to show that this function descends to a continuous function on the dual circle $\mathbb{T}^*$. In sections 6 and 7 we define Teichmüller's metric on UAA and exhibit the action of Thompson's group on UAA. Finally, in section 8 we show how the dual circle $\mathbb{T}^*$ is the natural setting for Rohlin's formula.

1. Circle endomorphisms

Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle and $\pi : \mathbb{R} \to \mathbb{T}$ defined by

$$\pi(x) = e^{2\pi i x}$$

be the universal covering with covering group $\mathbb{Z}$. π determines an isomorphism from $\mathbb{R}/\mathbb{Z}$ onto $\mathbb{T}$, and any orientation preserving covering map from $\mathbb{T}$ onto itself lifts via π to an orientation-preserving homeomorphism of the real line.

Let d be the topological degree of the circle covering map. In this paper, we only consider two cases, namely $d = 1$ or $d = 2$. If the $d = 1$, the map is a circle homeomorphism, and we usually use h to denote this map and H to denote its lift. By assuming that $0 \leq H(0) < 1$, we obtain a one-to-one correspondence between circle homeomorphisms h and real line homeomorphisms H with $H(x + 1) = H(x) + 1$. Since either h or its lift H uniquely realize the circle homeomorphism, we refer to either as the circle homeomorphism.

Observe that any expanding circle endomorphism with degree 2 or greater has one fixed point. By conjugating the endomorphism with a Möbius automorphism of $\mathbb{T}$, we may assume without loss of generality that this fixed point lies at $z = 1$. We usually denote the endomorphism by f and its homeomorphic lift to $\mathbb{R}$ by F. By stipulating that $F(0) = 0$, we obtain a one-to-one correspondence between degree 2 circle endomorphisms f with $f(1) = 1$ and their lifts F which satisfy

$F(0) = 0$ and $F(x+1) = x+2$. We will refer either to f or to its unique corresponding lift F as the circle endomorphism.

We denote the $n-$fold composition of f with itself by f^n, and similarly, F^n is the $n-$fold composition of F with itself.

A circle endomorphism f is in the class C^k for $k \geq 1$ if its k^{th}-derivative $F^{(k)}$ is continuous and $C^{k+\alpha}$ for some $0 < \alpha \leq 1$ if, furthermore, $F^{(k)}$ is α-Hölder continuous, that is,

$$\sup_{x \neq y \in \mathbb{R}} \frac{|F^{(k)}(x) - F^{(k)}(y)|}{|x-y|^{\alpha}} < \infty.$$

A C^1 circle endomorphism f is called expanding if there are constants $C > 0$ and $\lambda > 1$ such that

$$(F^n)'(x) \geq C\lambda^n, \quad n = 1, 2, \cdots, \text{ for all } x.$$

A circle homeomorphism h is called quasisymmetric if there is a constant $K \geq 1$ such that

$$K^{-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq K, \quad \forall x \in \mathbb{R}, \ \forall t > 0,$$

and it is called symmetric (see [10]) if in addition there is a bounded positive function $\epsilon(t)$ with $\epsilon(t) \to 0^+$ as $t \to 0^+$ such that

$$1 - \epsilon(t) \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq 1 + \epsilon(t), \quad \forall x \in \mathbb{R}, \ \forall t > 0.$$

In this case f is called uniformly symmetric or uniformly asymptotically affine (UAA) if all its inverse branches for f^n, $n = 1, 2, \cdots$, are symmetric uniformly. More precisely, f^n is UAA if there is a bounded positive function $\epsilon(t)$ with $\epsilon(t) \to 0^+$ as $t \to 0^+$ such that

$$1 - \epsilon(t) \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq 1 + \epsilon(t), \quad \forall x \in \mathbb{R}, \ \forall t > 0, \quad \forall n = 1, 2, \cdots.$$

Proposition. *Suppose* $0 < \alpha \leq 1$. *Any* $C^{1+\alpha}$ *expanding circle endomorphism* f *is uniformly symmetric.*

Proof. This fact follows from the naïve distortion lemma. For the proof see [15]. □

Remark. The space UAA of uniformly symmetric endomorphisms is much larger than the space C^{1+} of all expanding endomorphisms which are $C^{1+\alpha}$ for some $0 < \alpha \leq 1$. On the other hand, it turns out that UAA is the completion in Teichmüller's metric (which we define in section 6) of the $C^{1+\alpha}$ endomorphisms for any $0 < \alpha < 1$. The UAA

Teichmüller space is even the completion of the real-analytic endomorphisms (see [15]). Another important issue about the space UAA is that it coincides with the space of all uniformly asymptotically conformal circle maps (see [8]). Here a circle endomorphism f is called uniformly asymptotically conformal if there is an extension $\widetilde{f}$ of f to a small neighborhood of $\mathbb{T}$ symmetric about $\mathbb{T}$ with respect to the involution $j(re^{i\theta}) = (1/r)e^{i\theta}$ such that the Beltrami coefficients $|\mu_{\widetilde{f}^{-n}}(z)| \to 0$ uniformly on $n > 0$ and $z \to \mathbb{T}$.

Suppose f is a circle endomorphism of degree $d \geq 2$. A measure m on $\mathbb{T}$ is called an f-invariant measure if

$$m(f^{-1}(A)) = m(A), \text{ for all } m - \text{measurable sets } A.$$

Given a measure m, the push-forward measure $f_* m$ is defined by

$$f_* m(A) = m(f^{-1}(A)), \text{ for all } m - \text{measurable set } A.$$

Thus m is an f-invariant measure if and only if $f_* m = m$.

For any f-invariant measure m, its metric entropy $h_m(f)$ is defined by the following procedure (see [2, 16, 19, 21]). Let $\mathcal{D} = \{D_1, \cdots, D_k\}$ be a finite partition of (T, m) by m-measurable sets. Define

$$H_m(\mathcal{D}) = \sum_{i=1}^{k} -m(D_i) \log m(D_i).$$

If $\mathcal{D}$ and $\mathcal{C}$ are two partitions, their common refinement is defined by

$$\mathcal{D} \vee \mathcal{C} = \{D \cap C \mid D \in \mathcal{D}, C \in \mathcal{C}\}.$$

The metric entropy of f with respect to a partition $\mathcal{D}$ is defined as

$$h_m(f, \mathcal{D}) = \lim_{n\to\infty} \frac{1}{n} H_m(\mathcal{D} \vee f^{-1}\mathcal{D} \vee \cdots \vee f^{-n+1}\mathcal{D}).$$

From the invariance of m one shows that the positive sequence

$$u_n = H_m(\mathcal{D} \vee f^{-1}\mathcal{D} \vee \cdots \vee f^{-n+1}\mathcal{D})$$

is subadditive, namely, that

$$u_{n+k} \leq u_n + u_k$$

From this it follows that

$$\frac{u_n}{n} \to \inf_n \frac{u_n}{n} \quad \text{as} \quad n \to \infty.$$

So this limit exists and is finite, and the metric entropy of f is

$$h_m(f) = \sup_{\mathcal{D}} h_m(f, \mathcal{D}),$$

where supremum runs over all finite partitions of $\mathbb{T}$ by m-measurable sets. Since $\mathbb{T}$ is a metric space, for computational purposes, we may use the following formula.

$$h_m(f) = \lim_{n\to\infty} h_m(f, \mathcal{D}_n)$$

where $\{\mathcal{D}_n\}_{n=1}^{\infty}$ is any sequence of finite partitions for which $\mathrm{diam}(\mathcal{D}_n) \to 0$ as $n \to \infty$.

2. Rohlin formula for smooth expanding circle endomorphisms

The Rohlin formula for metric entropy gives an important relation between several invariants of a dynamical system. For differentiable dynamical systems, it relates the smooth invariant measure (or Sinai-Rulle-Bowen measure), the metric entropy, and the derivative (or partial derivatives).

Suppose f is a $C^{1+\alpha}$ expanding circle endomorphism of degree $d \geq 2$ for some $0 < \alpha \leq 1$. Then it is differentiable and its derivative

$$D(f)(x) = f'(x)$$

is an α Hölder continuous function.

Let ν_0 be the Lebesgue probability measure on $\mathbb{T}$. Then for each integer $k > 0$ consider the push forward probability measure $\nu_k = (f^k)_*\nu_0$. Then

$$\nu_k(A) = \nu_0(f^{-k}(A))$$

The partial sums

$$m_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_k, \quad n > 0.$$

form a sequence of probability measures on $\mathbb{T}$ satisfying

$$f_* m_n = m_n + \frac{\nu_n - \nu_0}{n}.$$

Since the space of all measures on $\mathbb{T}$ is weakly compact, $\{m_n\}_{n=1}^{\infty}$ has a convergent subsequence

$$m_{n_i} \to m \text{ as } i \to \infty.$$

Clearly, $f_* m = m$ and m is an f-invariant probability measure. From the theory of transfer operators (see, for example, [13]), we know that $\{m_n\}_{n=1}^{\infty}$ itself is a convergent sequence and the limiting measure m is a smooth probability measure, that is,

$$m(A) = \int_A \rho(x)dx$$

where A is any Borel set of $\mathbb{T}$ and $\rho(x)$ is a positive C^α function on $\mathbb{T}$. The equation in the following theorem is the famous Rohlin formula for metric entropy.

Theorem 1 (The Rohlin Formula [18]). *Suppose f is a $C^{1+\alpha}$ expanding circle endomorphism of degree $d \geq 2$ for some $0 < \alpha \leq 1$. Suppose m is the smooth f-invariant probability measure. Then*

$$h_m(f) = \int_T \log D(f)(x) dm(x).$$

Proof. This formula is well-known and there are many proofs in the literature (see, for examples, [2, 16, 19, 21]). For the purpose of the completeness of this paper and for comparison to the proof of Theorem 5, here we give another proof.

Since $f(1) = 1$, the preimage $f^{-1}(1)$ consists of $d \geq 2$ points in $\mathbb{T}$. Let $\mathcal{D}_0$ be the collection of closures of intervals of $\mathbb{T}$ whose endpoints are the set $f^{-1}(1)$. These intervals form a partition of $\mathbb{T}$. The partitions

$$\mathcal{D}_n = f^{-n}\mathcal{D}_0$$

for integers $n \geq 0$ form a filtration in the sense that $\mathcal{D}_{n+1}$ is a refinement of $\mathcal{D}_n$. Since f is expanding, the diameter of D in $\mathcal{D}_n$ approaches 0 as $n \to \infty$. Thus $\{\mathcal{D}_n\}_{n=0}^\infty$ generates the Borel algebra $\mathcal{B}$ on $\mathbb{T}$, that is, the σ-algebra generated by the open sets of $\mathbb{T}$. So from Kolmogorov-Sinai Theorem (see [2,16,19,21]), the metric entropy

$$h_m(f) = h_m(f, \mathcal{D}_0) = \lim_{n\to\infty} \frac{1}{n} H_m(f, \mathcal{D}_n)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{D\in\mathcal{D}_n} -m(D)\log m(D).$$

Put

$$p_n = \sum_{D\in\mathcal{D}_n} -m(D)\log m(D),$$

and

$$a_n = p_n - p_{n-1},$$

and consider the Césaro means

$$\sigma_n = \frac{a_1 + \cdots + a_n}{n} = \frac{p_n - p_0}{n}.$$

Provided that the sequence a_n converges, the sequence σ_n has the same limit as a_n, and we are led to the equality

$$h_m(f) = \lim_{n\to\infty} (p_n - p_0)/n = \lim_{n\to\infty} a_n.$$

But

$$a_n = p_n - p_{n-1} = \sum_{D\in\mathcal{D}_n} -m(D)\log m(D) + \sum_{D\in\mathcal{D}_{n-1}} m(D)\log m(D)$$

$$= \sum_{D\in\mathcal{D}_n} m(D)\log\frac{m(f(D))}{m(D)}.$$

Thus,

$$h_m(f) = \lim_{n\to\infty}\sum_{D\in\mathcal{D}_n} m(D)\log\frac{m(f(D))}{m(D)}$$

provided that the last limit exists.

On the other hand, from the mean value theorem,

$$\frac{m(f(D))}{m(D)} = \frac{\int_{f(D)}\rho(x)dx}{\int_D \rho(x)dx} = \frac{\rho(f(\xi_1))f'(\xi_2)}{\rho(\xi_3)}, \quad \text{for some } \xi_1,\xi_2,\xi_3\in D.$$

We have that

$$\sum_{D\in\mathcal{D}_n} m(D)\log\frac{m(f(D))}{m(D)} = \sum_{D\in\mathcal{D}_n} m(D)\log f'(\xi_2)$$

$$+\sum_{D\in\mathcal{D}_n} m(D)\log\rho(f(\xi_1)) - \sum_{D\in\mathcal{D}_n} m(D)\log\rho(\xi_3).$$

Since $f'(x)$, $\rho(f(x))$, and $\rho(x)$ are all continuous functions on $\mathbb{T}$, the last three summations converge, respectively, to the integrals

$$\int_{\mathbb{T}}\log f'(x)dm(x), \quad \int_{\mathbb{T}}\rho(f(x))dm(x), \quad \int_{\mathbb{T}}\rho(x)dm(x).$$

But the statement that m is f-invariant is equivalent to the statement that

$$\int_{\mathbb{T}}\phi(f(x))dm(x) = \int_{\mathbb{T}}\phi(x)dm(x)$$

for every continous function $\phi(x)$ on $\mathbb{T}$. Therefore,

$$\int_{\mathbb{T}}\rho(f(x))dm(x) = \int_{\mathbb{T}}\rho(x)dm(x).$$

Thus $\lim_{n\to\infty} a_n$ exists and

$$h_m(f) = \lim_{n\to\infty}\sum_{D\in\mathcal{D}_n} m(D)\log\frac{m(f(D))}{m(D)} = \int_T \log Df(x)dm(x).$$

This is the Rohlin formula. □

3. The symbolic dynamical system

For the degree two endomorphism f with $f(1) = 1$, let

$$\eta_0 = \{I\} \text{ and } \eta_1 = \{I_0, I_1\},$$

where I_0 and I_1 are the closures of the two intervals in $\mathbb{T} \setminus f^{-1}(1)$. We label them in counter-clockwise order so that both I_0 and I_1 are joined at their common endpoints. These two intervals form a Markov partition in the sense that

(1) $\mathbb{T} = I_0 \cup I_1$,
(2) the restriction of f to the interior of each I_i for $i = 0$ and $i = 1$ is injective, and
(3) $f(I_i) = \mathbb{T}$ for every $i = 0$ and 1.

We also have a sequence of Markov partitions

$$\eta_n,$$

where the dividing points between the closed intervals of η_n are the 2^n points of $f^{-1}(1)$. Furthermore, we can label each interval in η_n in the following way. Let

$$g_i(x) = f^{-1}|(\mathbb{T} \setminus \{1\}) \to I_i, \qquad \text{for } i = 0, 1.$$

Each g_i is a homeomorphism from $\mathbb{T} \setminus \{1\}$ to the interior of I_i. Given a word of length n, $w_n = i_0 \cdots i_k \cdots i_{n-1}$ where each $i_k = 0$ or 1, define

$$g_{w_n} = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{n-1}}.$$

Let I_{w_n} be the closure of $g_{w_n}(\mathbb{T} \setminus \{1\})$. Then

$$\eta_n = \{I_{w_n} \mid w_n = i_0 \cdots i_k \cdots i_{n-1},\ i_k \in \{0, 1\}\}.$$

Note that g_{w_n} is the restriction of $g_{w_{n-1}}$ to $g_{i_{n-1}}(\mathbb{T} \setminus \{1\})$. Therefore, $I_{w_n} \subset I_{w_{n-1}}$ where $w_n = i_0 \ldots i_{n-1}$ is the n-th truncation of the word $w = i_0 \cdots i_{n-1} i_n \cdots$ of infinite length, we have the following chain of inclusions:

$$\cdots \subset I_{w_n} \subset I_{w_{n-1}} \subset \cdots I_{w_1} \subset \mathbb{T}.$$

Since each I_{w_n} is compact,

$$I_w = \cap_{n=1}^{\infty} I_{w_n} \neq \emptyset.$$

Consider the space

$$\Sigma = \prod_{n=0}^{\infty} \{0, 1\}$$
$$= \{w = i_0 i_1 \cdots i_k \cdots i_{n-1} \cdots \mid i_k \in \{0, 1\},\ k = 0, 1, \cdots\};$$

Σ is a compact topological space with the product topology.

If each $I_w = \{x_w\}$ contains only one point, then we define the projection π from Σ onto $\mathbb{T}$ by

$$\pi(w) = x_w.$$

The projection π is 1-1 except for a countable set B, which consists of all labelings w of all of the endpoints of all of the intervals in the partitions $\eta_n = \{I_{w_n}\}$, $n = 0, 1, \cdots$.

If f is a uniformly quasisymmetric circle endomorphism, then there is a fixed constant $K > 0$ such that

$$K^{-1} \leq \frac{|F^{-n}(x+t) - F^{-n}(x)|}{|F^{-n}(x) - F^{-n}(x-t)|} \leq K, \quad \forall x \in \mathbb{R}, \ \forall t > 0, \ \forall n = 1, 2, \cdots .$$

For any interval $I_{w_n} \in \eta_n$,

$$I_{w_n} = I_{w_n 0} \cup I_{w_n 1},$$

and there is an integer $m > 0$ such that

$$F^{n+1}(I_{w_n}) = [m, m+2], \quad F^{n+1}(I_{w_n 0}) = [m, m+1],$$
$$F^{n+1}(I_{w_n 1}) = [m+1, m+2].$$

So

$$K^{-1} \leq \frac{|I_{w_n 0}|}{|I_{w_n 1}|} \leq K.$$

This implies that

$$|I_{w_n 0}|, |I_{w_n 1}| \leq \frac{K}{K+1} |I_{w_n}|.$$

Thus, if $0 < \tau = K/(K+1) < 1$, we have that

$$\max_{w_n} |I_{w_n}| \leq \tau^n, \quad \forall n > 0,$$

and so in this case I_w consists of only one point for every $w \in \Sigma$.

The points of Σ have a natural ordering induced by the ordering of the digits, namely, $0 < 1$. For two different sequences w^1 and w^2, $w^1 < w^2$ precisely if the first digit where they differ, that digit for w^1 is less than the corresponding digit for w^2. To obtain the continuum from Σ one identifies all the points of the form $w_n 100 \cdots 000 \cdots$ with points of the form $w_n 0111 \cdots 111 \cdots$. These are different points of Σ, but if we make all of these identifications and also identify $111 \cdots$ with $000 \cdots$, we obtain an ordered topological space together with an order preserving homeomorphism to the unit circle $\mathbb{T}$ with its counterclockwise ordering.

If $w = i_0 i_1 \cdots i_{n-1} i_n \cdots$, let

$$\sigma(w) = i_1 \cdots i_{n-1} i_n \cdots .$$

Then the system (Σ, σ) is called the symbolic dynamical system and σ is the left shift map. From our construction, one can check that

$$\pi \circ \sigma(w) = f \circ \pi(w), \quad w \in \Sigma,$$

and after the above identifications are made, the system $(\mathbb{T}, f)$ induced by (Σ, σ) is a continuous 2-to-1 proper covering from $\mathbb{T}$ to itself.

4. Dual dynamical systems and scaling functions

For each interval I_{w_n} at level n we have been numbering the symbol w_n from the left to the right, that is,

$$w_n = i_0 i_1 \cdots i_{n-1}.$$

Now we are going to number this symbol from the right to the left, that is, for the same w_n we write it as

$$w_n^* = j_{n-1} \cdots j_1 j_0.$$

We call it the dual labeling.

The dual symbolic space is

$$\Sigma^* = \{w^* = \cdots j_{n-1} \cdots j_k \cdots j_1 j_0 \mid j_k \in \{0,1\},\ k = 0, 1, \cdots\}$$

with the topological basis of right cylinders

$$[w_n^*] = [j_{n-1} \cdots j_0] = \{\widetilde{w}^* = \cdots j'_n j'_{n-1} \cdots j'_0 \mid j'_{n-1} = j_{n-1}, \cdots, j'_0 = j_0\}.$$

For the symbols $w_n \in \Sigma_n$ the intervals I_{w_n} are laid out in order from left to right so that $I_{w_n add\ one}$ is the next interval on the right after I_{w_n}, where w_n *add one* means add one to w_n to the right hand digit of w_n and carry the addition to the left in the usual way that one does arithmetic. In the symbolism for $w_n^* \in \Sigma_n^*$ the same intervals I_{w_n} that appear for Σ_n are arranged in a different order. That is, if the digits in w_n and w_n^* are identical, the two intervals I_{w_n} and $I_{w_n^*}$ are the same but the appear in different position. The next interval appearing to the right of $I_{w_n^*}$ is $I_{w_n^* add\ one}$, where w_n^* *add one* means add one to the left hand digit of w_n^* and carry the addition to the right in the opposite way that one ordinarily does arithmetic.

Consider the dual shift map σ^* that drops the first symbol on the right of an element $w^* = \cdots j_{n-1} \cdots j_1 j_0$ in Σ^* :

$$\sigma^* : \cdots j_{n-1} \cdots j_1 j_0 \mapsto \cdots j_{n-1} \cdots j_1.$$

Then we call (Σ^*, σ^*) the dual symbolic dynamical system for f. For an element $w_n^* \in \Sigma_n^*$ dropping the symbol on the left of w_n^* gives the symbol of the interval in Σ_{n-1}^* that lies spatially above the interval $I_{w_n^*}$. The same interval $I_{w_n^*}$ is situated dynamically beneath $I_{w_{n-1}^*}$ in Σ_n.

There is a dual dynamical system f^* acting on symbols Σ_n^* corresponding to the partitions of intervals. We write

$$\cdots \Sigma_n^* \prec \cdots \prec \Sigma_1^* \prec \Sigma_0^*$$

if each partition η_n^* is a refinement of the partition η_{n-1}^*. f carries the first 2^{n-1} intervals in Σ_n^* in order to the intervals of Σ_{n-1}^*, and it carries the second 2^{n-1} intervals also in order to the Σ_{n-1}^*. If there is an invariant measure m for f, the statement that m is additive on adjacent intervals in these partitions is equivalent to the statement that m^* is invariant for f^*, that is, that $m^*(f^{*-1}(I_{w_n})) = m^*(I_{w_n})$, and the statement that m^* is invariant for f^* is equivalent to the statement that m is additive. Also, in an obvious sense, $f^{**} = f$, $\Sigma_n^{**} = \Sigma_n$ and $\Sigma^{**} = \Sigma$.

For any $w^* = \cdots j_{n-1} \cdots j_1 j_0 \in \Sigma^*$, let $w_n^* = j_{n-1} \cdots j_1 j_0$, then

$$I_{w_n^*} \subset I_{\sigma^*(w_n^*)}.$$

Define

$$D_n^*(f)(w_n^*) = \frac{|I_{\sigma^*(w_n^*)}|}{|I_{w_n^*}|}.$$

We make the assumption that for every w^* in Σ^* that the limit as $n \to \infty$ of $D_n^*(f)(w_n^*)$ exists, and with this assumption we make the following definition (refer to [15]):

Definition. The dual derivative of f at $w*$ is given by

$$D^*(f)(w^*) = \lim_{n\to\infty} D_n^*(f)(w_n^*).$$

And

$$S(f)(w^*) = \frac{1}{D^*(f)(w^*)}$$

is called the scaling function.

If f is a uniformly symmetric circle endomorphism of degree 2, then the dual derivative $D^*(f)$ exists and is continuous on Σ^*, moreover, if f is $C^{1+\alpha}$ expanding circle endomorphism for some $0 < \alpha \leq 1$, then $D^*(f)$ is a Hölder continuous (see [15] for the proof). By this we mean there are constants $C > 0$ and $0 < \tau < 1$ such that

$$|D^*(f)(w^*) - D^*(f)(\widetilde{w}^*)| \leq C\tau^n$$

whenever the first n digits on the right of w^* and $\widetilde{w}^*$ are identical. This is equivalent to the standard definition of a Hölder continuous map from a metric space into another metric space with respect to the metric $d(\cdot,\cdot)$ on Σ^* defined by

$$d(w^*, \widetilde{w}^*) = \sum_{k=0}^{\infty} \frac{|j_k - j'_k|}{2^{k+1}} \tag{1}$$

for any $w^* = \cdots j_{n-1} \cdots j_k \cdots j_1 j_0$ and $\widetilde{w}^* = \cdots j'_{n-1} \cdots j'_k \cdots j'_1 j'_0$. This metric (1) is the standard metric of Lebesgue measure if we assume that each of the intervals I_{w_n} has length $1/d^n$.

Two UAA circle expanding maps f_0 and f_1 are called Teichmüller equivalent if there exists a symmetric self map s of $\mathbb{T}$ such that $s \circ f_0 \circ s^{-1} = f_1$. In [4], it has been shown that the set of all eigenvalues of f determines the Teichmüller equivalent class of f. Using a relation between the set of all eigenvalues and the scaling function in [12] it is shown that the scaling function $S(f)$ on Σ^* determines the Teichmüller equivalence class of f, (see [5] and [14]).

5. Scaling functions on the dual circle

In [5, 6], it is shown that the *summation condition* and the *compatibility condition* on the scaling function of a degree two UAA circle endomorphism f are necessary and sufficient for a positive continuous function h defined on Σ^* to be the scaling function of f.

Definition. Suppose a degree two circle endomorphism f has a scaling function S defined on the dual Cantor set Σ^*. Then S satisfies the *summation condition* if every finite code ω in Σ^*,

$$S(\omega 0) + S(\omega 1) = 1.$$

S satisfies the *compatibility condition* if there exists a number α with $0 < \alpha < \infty$ such that for every code ω of finite length in Σ^*

$$C_N(\omega) = \prod_{n=0}^{N} \frac{S(\omega 10 \cdots 0)}{S(\omega 01 \cdots 1)}$$

approaches α as N approaches ∞, where the number of zeroes and ones in the codes of the numerator and denominator is equal to n.

The following theorem is proved in [5].

Theorem 2. *Suppose f is a degree 2 expanding circle endomorphism from $\mathbb{T}$ onto $\mathbb{T}$ and m is a smooth measure on $\mathbb{T}$ with respect to which f is invariant. Then a continuous non-negative function h is a scaling function for some UAA f if and only if f satisfies the summation and compatibility conditions.*

Suppose A_n and B_n are the lengths (measured with respect to the invariant measure for f) two neighboring intervals in the partition η_n^* at level n induced by the dual dynamical system f^* lying on left and right sides of a gap. Let a_{n+1} and b_{n+1} be the lengths of subintervals of A_n and B_n in the next partition η_{n+1}, also lying in the left and right hand sides of the same gap. Then the compatibility condition says that A_n/B_n and a_{n+1}/b_{n+1} approach the same limit α as $n \to \infty$. Let $\omega^*(R)$ be the code of the element of the dual Cantor set $\Sigma*$ representing the path that travels down the right hand side of this gap and $\omega^*(L)$ be the code of the element of the dual Cantor set representing the path that travels down the left side. In the dual circle $\mathbb{T}^*$ these two codes are identified, and to show that the scaling function is well defined on the dual circle, we must show that it takes the same value on both codes, $\omega^*(L)$ and $\omega^*(R)$.

The value on the left side is $\lim_{n\to\infty} a_{n+1}/A_n$ and the value on the right side is $\lim_{n\to\infty} b_{n+1}/B_n$. But $(a_{n+1}/A_n)(B_n/b_{n+1}) \to \alpha/\alpha = 1$, and therefore $\lim_{n\to\infty} a_{n+1}/A_n = \lim_{n\to\infty} b_{n+1}/B_n$.

Theorem 3. *The scaling function $S(f)$ of a UAA circle endomorphism descends from a continuous function on the dual Cantor set Σ^* to a continuous function on the dual circle $\mathbb{T}^*$.*

Proof. This follows because we have just proved that its values on either side of any gap coincide and we already know that $S(f)$ is continuous on Σ^*. □

6. Teichmüller's metric on UAA.

In [8] it is shown that any UAA degree 2 circle endomorphism f is extendable to a neighborhood in the complex plane of the unit circle $\mathbb{T}$ to a uniformly asymptotically conformal map $\tilde{f}$ defined in a neighborhood of $\mathbb{T}$.

Suppose we have two such UAA endomorphisms f_0 and f_1 with extensions $\tilde{f}_0$ and $\tilde{f}_1$. Let h be a quasiconformal map defined in a neighborhood of the unit circle with $h \circ \tilde{f}_0 \circ h^{-1} = \tilde{f}_1$. The boundary dilatation $BD(h)$ of a mapping h is defined in [9]. It is the limit of the maximal dilatation of the extension of h to open neighborhoods U of $\mathbb{T}$ as the neighborhoods U shrink to $\mathbb{T}$. The Teichmüller distance between f_0 and f_1 is the infimum of the numbers

$$\log BD(h)$$

where $h \circ \tilde{f}_0 \circ h^{-1} = \tilde{f}_1$ and where $\tilde{f}_0$ and $\tilde{f}_1$ are any UAC extensions of the UAA endomorphismsf_0 and f_1. This metric makes UAA into a complete metric space. The complex structure on UAA is induced by the complex structure on the Beltrami coefficients of the conjugacies h.

Definition. *The Teichmüller space UAA consists of uniformly asymptotically affine degree 2 circle endomorphisms factored by an equivalence relation. Two such endomorphisms f and g are equivalent if there is a symmetric homeomorphism h of $\mathbb{T}$ such that $h \circ f \circ h^{-1} = g$ on $\mathbb{T}$.*

Remark. Note that equivalency h necessarily carries the sequence of Markov partitions $\eta_n(f)$ induced by f to the sequence of Markov partitions $\eta_n(g)$ induced by g.

7. The action of Thompson's F-group.

Before describing Thompson's group we first introduce terminology for the successive Markov partitions η_n induced by f acting on $\mathbb{T}$. To simplify the exposition we assume f has degree 2. If $n = 0$ no points are marked and the first partition η_0 consists of just one interval $[0, 1]$. If $n = 1$ only one midpoint is marked, namely the point a in $f^{-1}(1)$ that is not equal to the fixed point 1. It determines a partition η_1 of $[0, 1]$ into two subintervals, $[0, a]$ and $[a, 1]$. The set $f^{-2}(1)$ consists of 4 points laid out in order on the unit interval, namely, $0, a(0), a, a(1), 1$, where 0 is identified with 1, and the associated partition η_2 consists of 4 intervals, $[0, a(0)], [a(0), a], [a, a(1)], [a(1), 1]$. We shall call $a(0)$ the *relative midpoint* of $[0, a]$ and $a(1)$ the relative midpoint of $[a, 1]$. The set $f^{-3}(1)$ consists of 8 points laid out in order on the unit interval, namely,

$$0, a(00), a(0), a(01), a, a(10), a(1), a(11), 1,$$

with 0 identified with 1, and these are the boundary points of a partition η_3 of $[0, 1]$ into 8 intervals. Thus each partition η_{k+1} is a refinement of the partition η_k. Similarly, we use the same notation for the successive refinements of the Markov partitions in the sets $f^{-n}(1)$. Note that in general each of the Markov partitions η_n consist of d^n intervals, where d is the degree of f.

Now we describe Thompson's F-group in much the same way that it is done by Greenberg in [11] and by Cannon, Floyd and Parry in [3]. First we define an *allowable partition* of order n. It is a partition of $[0, 1]$ into n intervals whose marked endpoints are obtained inductively by the following procedure. There is no choice for the first marked point; it is necessarily marked at a. There are two choices for the second marked point. It can be either at $a(0)$ or at $a(1)$, in other words, either at the relative midpoint of $[0, a]$ or at the relative midpoint of $[a, 1]$. Proceeding inductively the point marked at the n-th stage lies at the relative midpoint between any two of the points in the set consisting of 0 and 1 and the $n-1$ relative midpoints marked at the previous stages.

An element of Thompson's F-group is determined by two allowable partitions both of the same order n. We denote these partitions by (D, F), standing for domain and range. The element $h(D, F)$ corresponding to (D, F) maps $[0, 1]$ to itself by piecewise continuous increasing parts that map the intervals of D in order onto the intervals of R and that preserve all relative midpoints of all succeeding intervals. The piecewise continuous parts of h are patched together at the endpoints of n intervals that partition the circle. The composition of two such maps h_1 and h_2 of requires patching along an allowable partition into a number of intervals no more than the sum of the number of intervals required for h_1 and the number required for h_2.

To obtain the desired action of Thompson's group on the Teichmüller space UAA, we do something a little different. Suppose we are given an element f in UAA and a pair of allowable partitions (D, F) representing an element of Thompson's group. We let $h = h(D, F)$. The action of h on f, which we denote by $(h, f) \mapsto \hat{h}(f)$, requires three steps:

Step 1. First construct the dual dynamical system f^* acting on the dual circle $\mathbb{T}^*$.

Step 2. Then construct a quasisymmetric map $h = h(D, F)$ determined by the partitions η_k for the dual dynamical system f^*. f^* will usually be only by UQC (uniformly quasiconformal) but not be UAC (uniformly asymptotically conformal).

Step 3. Finally, dualize back to a system acting on $\mathbb{T}$ by putting $\hat{h}(f) = (h \circ f^* \circ h^{-1})^*$.

To show that these steps give an action we must show that h is quasisymmetric and that $\hat{h}(f)$ is in UAA. Here we use the same letter f to denote a uniformly asymptotically conformal extension of f to a neighborhood Ω of $\mathbb{T}$. So f is a proper 2 to 1 covering taking Ω to $f(\Omega)$ with $f(\Omega) - \Omega$ equal to an annulus with positive modulus and whose restriction to the inner boundary of $\Omega - \mathbb{T}$ is equal to f on $\mathbb{T}$. To show that h is quasisymmetric we need to show how to build a quasiconformal extension of h from a long composition of the branches of f^* and $(f^*)^{-1}$ together with a quasiconformal part determined by the partitions D and F. The action of $\hat{h}(f)$ at deep levels duplicates the action of f, so it is also UAC.

Step 1. This step is described in detail in [7, pages 178-180].

Step 2. A version of this step is described in [7] but the setting is different and so requires explanation. We need to show how the quasiconformal map $h(D, R)$ defined in a neighborhood of $\mathbb{T}^*$ is built from the dynamical pieces of the UAC map f and from the given element of Thompson's group determined by the tree diagrams in (D, R).

Imitating the trees D and R we draw two corresponding Riemann surfaces $\mathcal{D}$ and $\mathcal{R}$ with front to back symmetry as shown in figure 1. Both the trees and the surfaces have a marked "top." The trees have a certain number n of tips at the bottom and the surfaces have the same number of holes at the bottom. The two illustrations in figure 2 show examples with 3 and 4 holes at the bottom. Obviously there is only one topological way to draw n non-homotopic simple closed curves in $\mathcal{D}$ and $\mathcal{R}$ with front-to-back symmetry that separate the top from the bottom. These curves separate $\mathcal{D}$ and $\mathcal{R}$ into $n + 1$ pairs of pants. There is only one homotopy class of map from $\mathcal{D}$ to $\mathcal{R}$ with front to back symmetry that preserves the tops and corresponding separating curves and that carries pairs pants in $\mathcal{D}$ to corresponding pairs of pants in $\mathcal{R}$. We denote by h a quasiconformal representative of this class.

We now use the dynamical branches of f to define a continuous quasiconformal map h at all the other points of Ω by replicating the simplicial structure below the holes in $\mathcal{D}$ and $\mathcal{R}$.

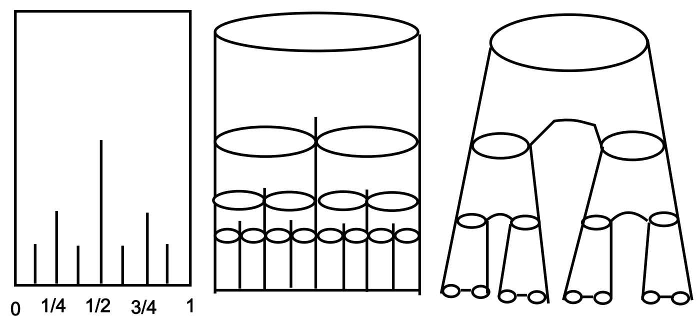
Step 3. Finally, $\hat{h}(f) = (h \circ f^* \circ h^{-1})^*$ is also a UAC system because at levels of the infinite trees lying below the holes on $\mathcal{D}$ and $\mathcal{R}$ the action of $\hat{h}(f)$ is the transported simplicial action of f to the different levels lying below these holes. These actions are asymptotically affine because by assumption f is asymptotically affine.

Theorem 4. *For a degree two UAA circle endomorphism f, the action of Thompson's F-group on the scaling functions of elements of UAA defined on the dual Cantor set represents F faithfully as a group of biholomorphic isometries in Teichmüller's metric.*

Proof. That the action is faithful follows from the same arguments that are given in [7, pages 182-185].

To see that the action of an element α in F yields an isometry, we first introduce the space UQS of all uniformly quasisymmetric circle expanding maps f of degree 2 acting on $\mathbb{T}$. This is also a complete metric space with the boundary dilatation metric and UAA is closed subspace of UQS. The map $f \mapsto f^*$ is an isometry of UQS onto itself with the boundary dilatation metric.

Since $\hat{\alpha}(f) = (\alpha \circ f^* \circ \alpha^{-1})^*$, it is the composition of three isometries and, therefore, an isometry. Since $(\alpha, f) \mapsto \hat{\alpha}(f)$ is a group action

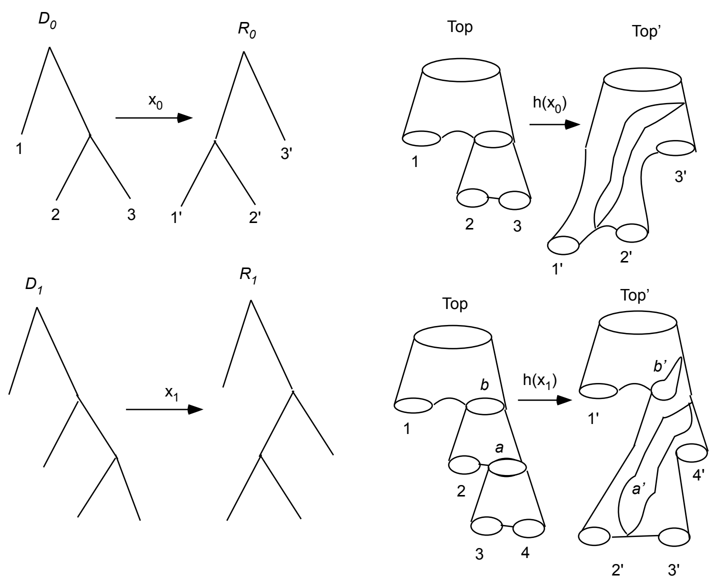


Making the tree of pairs of pants from two sheets of paper

Figure 1

on UAA, this isometry is invertible and, therefore, $\hat{\alpha}$ maps UAA onto itself.

The complex structure on UAA arises from its realization as all possible UAC deformations $h \circ f_0 \circ h^{-1}$ of the fixed UAC circle expanding map $f_0(z) = z^d$. Here $h = h^\mu$ is a quasiconformal conjugacy were μ is the Beltrami coefficient of h. The complex structure is induced by the complex structure on UQC (the uniformly quasiconformal endomorphisms of the unit disc). And this complex structure is induced by the complex structure on the open unit ball of Beltrami coefficients μ, with $||\mu||_\infty < 1$. Dualizing and precomposition are holomorphic operations. $\square$



The topology of the maps h(x_0) and h(x_1) induced by the Thompson generators x_0 and x_1

Figure 2

Conjecture. Every holomorphic automorphism of UAA is represented in this way by an element of Thompson's F-group.

8. The dual Rohlin formula

Just as in the $C^{1+\alpha}$ case described at the beginning of Section 2, we can construct an invariant probability measure m, that is, a measure for which $f_*m = m$. Such a measure m is obtained by taking a limit of any convergent subsequence m_{n_i} of the sequence

$$m_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k,$$

where $\nu_k = f_*^k \nu_0$, and ν_0 is Lebesgue measure. It is conceivable that different subsequences converge to different invariant measures. (See [15] for more details about the construction.)

Since f is a uniformly symmetric, the distribution functions

$$h_n(x) = m_n([0, x])$$

for the probability measures m_n are uniformly symmetric. Since the space of all $M-$symmetric homeomorphisms of $\mathbb{T}$ for a fixed constant M is weakly compact (see, for example, [1]), the distribution function $h(x) = m([1, x])$ of m is symmetric. We call m a symmetric f-invariant probability measure.

We now define the dual σ^*-invariant measure on Σ^* in the same way it was defined in [15]. For a cylinder $[w_n^*]$ let I_{w_n} be the interval in η_n with the labeling

$$w_n^* = j_{n-1} \cdots j_0 = i_0 \cdots i_{n-1} = w_n.$$

Define

$$m^*([w_n^*]) = m(I_{w_n}).$$

We must prove that m^* extends to a measure on Σ^*.

First, since

$$f^{-1}(I_{w_n}) = I_{0w_n} \cup I_{1w_n} \cup \cdots \cup I_{(d-1)w_n}$$

and since m is f-invariant,

$$m(I_{w_n}) = \sum_{i=0}^{d-1} m(I_{iw_n}).$$

This implies that

$$m^*([w_n^*]) = \sum_{i=0}^{d-1} m^*([iw_n^*]).$$

That is, m^* satisfies the finite summability condition. Since the distribution function of m is uniformly continuous on $\mathbb{T}$, if we have a sequence of cylinders $[w_n^*]$ of positive length $n > 0$, since the Lebesgue length $|I_{w_n}|$ tends to zero as n goes to ∞, we see that $m^*([w_n^*]) = m(I_{w_n})$ tends zero as n tends to infinity. This property together with the finite summability condition implies that the countable summability, and so m^* extends to a probability measure on Σ^*.

We claim that m^* is σ^*-invariant. For any cylinder $[w_n^*]$,

$$(\sigma^*)^{-1}([w_n^*]) = \cup_{i=0}^{d-1} [w_n^* i],$$

so

$$I_{w_n} = \cup_{i=0}^{d-1} I_{w_n i},$$

and we conclude that

$$m^*((\sigma^*)^{-1}([w_n^*])) = \sum_{i=0}^{d-1} m(I_{w_n i}) = m(I_{w_n}) = m^*([w_n^*]).$$

Therefore m^* is σ^*-invariant.

From the definition of the metric entropy, we see that

$$h_{m^*}(\sigma^*) = h_m(f)$$

and we denote it by $h_{m^*}(f)$.

Theorem 5 (The Dual Rohlin Formula)**.** *Suppose f is a uniformly symmetric circle endomorphism. Then we have the following equality*

$$h_{m^*}(f) = \int_{\mathbb{T}^*} \log D^*(f)(w^*) dm^*(w^*),$$

where $D^*(f)$ is the dual derivative of f defined in section 5.

Proof. The metric entropy

$$h_{m_*}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{w_n^*} -m^*([w_n^*]) \log m^*([w_n^*]).$$

Proceeding just as we did in section 2, this limit is equal to

$$\lim_{n\to\infty} \sum_{[w_n^*]} m^*([w_n^*]) \log \frac{m^*(\sigma^*([w_n^*]))}{m^*([w_n^*])},$$

provided the last limit exists.

Define

$$\widetilde{m}^*(B) = \sum_{j=0}^{d-1} m^*(\sigma^*(B \cap [j])),$$

where B is any Borel subset in Σ^*. Here $\widetilde{m}^*$ is a measure on $\mathbb{T}^*$ and on each cylinder $[w_n^*]$,

$$\widetilde{m}^*([w_n^*]) = m^*(\sigma^*([w_n^*])).$$

Note that one cannot define $\widetilde{m}^*(B) = m^*(\sigma^*(B))$ since the latter expression may not be a measure. The measure m^* is absolutely continuous with respect to $\widetilde{m}^*$. So the Radon-Nikodym derivative of m^* with respect to $\widetilde{m}^*$

$$RN(w^*) = \frac{dm^*}{d\widetilde{m}^*}(w^*), \quad \widetilde{m}^* - a.e.\ w^*,$$

exists and is a $\widetilde{m}^*$ measurable function. But since m is a symmetric measure, we have ,

$$RN(w^*) = \lim_{n\to\infty} \frac{m^*([w_n^*])}{m^*(\sigma^*([w_n^*]))} = \lim_{n\to\infty} \frac{|m(I_{w_n^*})|}{|m(I_{\sigma^*(w_n^*)})|}$$

$$= \lim_{n\to\infty} \frac{|I_{w_n^*}|}{|I_{\sigma^*(w_n^*)}|} = \Big(D^*(f)(w^*)\Big)^{-1}, \quad \widetilde{m}^* - a.e.\ w^*.$$

This implies that $RN(w^*)$ is a positive function on Σ^* for $\widetilde{m}^*-$ almost all w^*. Thus, it is a positive function for m^* almost all w^*. Therefore,

$$\frac{1}{RN(w^*)} = \frac{d\widetilde{m}^*}{dm^*} = D^*(f)(w^*)$$

is a positive function for m^* almost all w^* and it is a m^* measurable function equal to the dual derivative $D^*(f)(w^*)$ for m^* almost all w^*. Since $D^*(f)(w^*)$ is a positive continuous function on Σ^*, $\log D^*(f)(w^*)$ is a continuous function on Σ^* and is thus m^* integrable. Therefore,

$$\lim_{n\to\infty} \sum_{[w_n^*]} m^*([w_n^*]) \log \frac{m^*(\sigma^*([w_n^*]))}{m^*([w_n^*])} = \int_{\Sigma^*} \log D^*(f)(w^*) dm^*(w^*) < \infty.$$

This implies our dual Rohlin formula,

$$h_{m^*}(f) = \int_{\mathbb{T}^*} \log D^*(f)(w^*) dm^*(w^*),$$

and completes the proof. □

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Frederick P. Gardiner: Department of Mathematics, 2900 Bedford Avenue, Brooklyn, NY 11210-2889, and, Department of Mathematics, Graduate School of the City University of New York, 365 Fifth Avenue, New York, NY 10016

E-mail address: frederick.gardiner@gmail.com

Yunping Jiang: Department of Mathematics, Queens College of the City University of New York, Flushing, NY 11367-1597, and, Department of Mathematics, Graduate School of the City University of New York, 365 Fifth Avenue, New York, NY 10016

E-mail address: yunping.jiang@qc.cuny.edu