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Carathéodory's and Kobayashi's metrics on Teichmüller space

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Abstract. Carathéodory's and Kobayashi's infinitesimal metrics on Teichmüller spaces of dimension two or more are never equal in the direction of any tangent vector defined by a separating cylindrical differential.

Introduction

The infinitesimal form of Poincaré's metric ρ on the unit disc $\Delta = \{z : |z| < 1\}$ satisfies the equation

$$\rho(p, V) = \frac{|dp(V)|}{1 - |p|^2},$$

where V is a tangent vector at a point p with $|p| < 1$. Integration of this form yields the global metric

$$\rho(p, q) = \frac{1}{2} \log \frac{1 + r}{1 - r}, \tag{1}$$

where $r = \frac{|p-q|}{|1-\overline{p}q|}$. ρ has constant curvature and the choice of the coefficient $1/2$ makes the curvature equal to -4. At $p = 0$, it coincides with the Euclidean metric $|dp|$ and it is defined at other points of Δ so as to be invariant by Möbius automorphisms of Δ.

In dual ways the infinitesimal form ρ on Δ induces infinitesimal forms on any complex manifold M. The first way uses the family $\mathcal{F}$ of all holomorphic functions f from Δ to M. It is commonly called Kobayashi's metric and we denote it by K. The second way uses of the family $\mathcal{G}$ of all holomorphic functions g from M to Δ. It is commonly called Carathéodory's metric and we denote it by C.

For a point $\tau \in M$ we let $\mathcal{G}(\tau)$ be the subset of $\mathcal{G}$ consisting of those functions $g \in \mathcal{G}$ for which $g(\tau) = 0$. Similarly, we let $\mathcal{F}(\tau)$ be the subset of $f \in \mathcal{F}$ for which $f(0) = \tau$. Given a point $\tau \in M$ and a vector V tangent to M at τ, the infinitesimal form K is defined by the infimum problem

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$$K_M(\tau, V) = \inf_{f \in \mathcal{F}(\tau)} \left\{ \frac{1}{|a|} : df_0(1) = aV \text{ and } f(0) \in \mathcal{F}(\tau) \right\} \tag{2}$$

and the infinitesimal form C is defined by the supremum problem

$$C_M(\tau, V) = \sup_{g \in \mathcal{G}(\tau)} \{|b| : dg_\tau(V) = b \text{ and } g \in \mathcal{G}(\tau)\}. \tag{3}$$

When $M = \Delta$ the metrics K_Δ, C_Δ and ρ coincide. Schwarz's lemma applied to $g \circ f : \Delta \to \Delta$ where $f \in \mathcal{F}(\tau)$ and $g \in \mathcal{G}(\tau)$ implies

$$|b| \leq 1/|a|$$

and so the formulas (2) and (3) together imply for all complex manifolds M

$$C_M(\tau, V) \leq K_M(\tau, V). \tag{4}$$

In the case M is a Teichmüller space we can use the Bers' embedding, [**2, 5**]. For any point $\tau \in Teich(R)$ that embedding is a biholomorhic map Φ_τ from $Teich(R)$ onto a bounded simply connected domain in the Banach space of bounded cusp forms B_τ. Moreover, the image of

$$\Phi_\tau : Teich(R_\tau) \to B_\tau$$

in B_τ contains the open ball of radius 2 and is contained in the ball of radius 6, [**2**]. Assume V is a tangent vector $||V||_{B_\tau} = 1$. Then the complex linear map $V \mapsto (1/6)V$ extends by the Hahn-Banach theorem to a complex linear map $L : B_\tau \to \mathbb{C}$ with $||L|| \leq 1/6$. Then because $\Phi_\tau(Teich(R)$ is contained in the ball of radius 6, $L(\Phi_\tau(Teich(R))$ is contained in the unit disc and $L \in \mathcal{G}$. Thus from definition (3)

$$C_{Teich(R)}(\tau, V) \geq 1/6.$$

On the other hand, put $f(t) = 2tV$. Then since the image of Φ_τ contains $f(\Delta)$ from definiton (2)

$$K_{Teich(R)}(\tau, V) \leq 1/2.$$

Putting the preceeding two inequalities together we find

$$(1/3)K_{Teich(R)}(\tau, V) \leq C_{Teich(R)}(\tau, V), \tag{5}$$

and so we obtain from (4) and (5) the double inequality

$$(1/3)K_{Teich(R)}(\tau, V) \leq C_{Teich(R)}(\tau, V) \leq K_{Teich(R)}(\tau, V). \tag{6}$$

Inequality (5) has been pointed out to me by Stergios Antonokoudis and the same argument is given by Miyachi in [**27**] to prove the parallel result for asymptotic Teichmüller space.

If h is a holomorphic function from a complex manifold M to a complex manifold N, the derivative dh_τ of h at $\tau \in M$ is a complex linear map from the tangent space to M at τ to the tangent space to N at $h(\tau)$. From Schwarz's lemma it follows both C_M and K_M have the pull-back contracting property, namely, for any point $\tau \in M$ and tangent vector V at τ, C and K satisfy

$$K_N(h(\tau), dh_\tau(V)) \leq K_M(\tau, V) \tag{7}$$

and

$$C_N(h(\tau), dh_\tau(V)) \leq C_M(\tau, V). \tag{8}$$

Any infinitesimal form on a complex manifold satisfying this pull-back contracting property for all holomorphic functions h is called a Schwarz-Pick metric by Harris in [**17**] and by Harris and Earle in [**7**]. They also observe that any infinitesimal form with this property must necessarily lie between $C_M(\tau, V)$ and $K_M(\tau, V)$.

A global metric d satisfying mild smoothness conditions has an infinitesimal form given by the limit:

$$d(\tau, V) = \lim_{t \searrow 0} d(\tau, \tau + tV).$$

(See [**8**].) When this is the case the integral of the infinitesimal form $d(\tau, V)$ gives back a global metric $\overline{d}$. By definition the metric $\overline{d}(\tau_1, \tau_2)$ on pairs of points τ_1 and τ_2 is the infimum of the arc lengths of arcs in the manifold that join τ_1 to τ_2. In general $\overline{d}$ is symmetric, satisfies the triangle inequality, determines the same topology as d and one always has the inequality $\overline{d} \geq d$. However, in many situations $\overline{d}$ is larger than d. It turns out that when M is a Teichmüller space and d is Teichmüller's metric, then d and $\overline{d}$ coincide, [**9**] [**28**]. Moreover, for all Teichmüller spaces Teichmüller's and Kobayashi's metrics coincide (see [**30**] for finite dimensional cases and [**12, 13**] for infinite dimensional cases).

The main result of this paper is the following theorem.

THEOREM 1. *Assume $Teich(R)$ has dimension more than 1 and V is a tangent vector corresponding to a separating cylindrical differential. Then there is strict inequality;*

$$C_{Teich(R)}(\tau, (V) < Kob_{Teich(R)}(\tau, V). \tag{9}$$

To explain this theorem we must define a "separating cylindrical differential." From the Jordan curve theorem any simple closed curve γ embedded in a Riemann surface divides it into one or two components and we call γ separating if it divides R into two components. In that case we can try to maximize the modulus of a cylinder in R with core curve homotopic to γ. If this modulus is bounded it turns out that there is a unique embedded cylinder of maximal modulus. By a theorem of Jenkins ([**19**],[**20**]) and Strebel [**33**] it corresponds to a unique quadratic differential q_γ which is holomorphic on R with the following properties. Its noncritical horizontal trajectories all have length 2π in the metric $|q_\gamma|^{1/2}$, these trajectores fill the interior of the cylinder and no other embedded cylinder in the same homotopy class has larger modulus. If such a cylinder exists and if $Teich(R)$ has dimension more than 1 we call q_γ a separating cylindrical differential.

Inequality (9) is a feature that distinguishes between different tangent vectors. In fact by a theorem of Kra [**22**] for some tangent vectors one can have equality

$$C(\tau, V) = K(\tau, V). \tag{10}$$

Kra's theorem states (10) holds when V has the form $\overline{\partial}V = |q|/q$ where q is an integrable holomorphic quadratic differential and where q is equal to the square of an abelian differential;

$$q(z)(dz)^2 = (\omega(z)dz)^2.$$

If q is such a quadratic differential, then we call the locus of points

$$\{[t|q|/q] : |t| < 1\} \subset Teich(R)$$

an abelian Teichmüller disc. The study of abelian Teichmüller discs is equivalent to the study of translation surfaces which is a large topic, see for example [**18**] and papers referenced there.

Kra's result is obtained by using the Riemann period relations and Rauch's formula for the variation of the entries in the period matrix [**29**] induced by a Beltrami differential μ, where $\overline{\partial}V = \mu$. A consequence of this infinitesimal result is that for any two points τ_1 and τ_2 in the same abelian Teichmüller disc, $C(\tau_1, \tau_2) = K(\tau_1, \tau_2)$. However, this leaves the possibility that $C(\tau_1, \tau_2) < K(\tau_1, \tau_2)$ when τ_1 and τ_2 lie in distinct abelian Teichmüller discs. In contrast to abelian tangent vectors, any tangent vector V corresponding to a separating cylindrical differential must originate from what are sometimes called $\mathcal{F}$-structures [**10**] and sometimes called half-translation structures [**11**]. In particular, when the dimension of the Teichmüller space is more than 1, an abelian tangent vector can never correspond to a separating cylindrical differential. Note that the usual simple closed curves α_j and β_j, $1 \leq j \leq g$ taken as a basis for the homology group of a surface of genus g are all non-separating and can in fact correspond to abelian differentials $\omega_j(z)dz$.

The rough idea for the proof of Theorem 1 is to move in the direction of a separating cylindrical differential towards thinner parts of $Teich(R)$, that is, in the direction where a separating cylinder becomes taller while constantly having waist length equal to 2π. If there is a finite set of such nonintersecting cylinders then in a part of Teichmüller space where all of these cylinders become thin Teichmüller's metric resembles a supremum metric and the metric does not have negative sectional curvature, (see [**26**]). Also these are the parts where the anamolous aspects of the extremal length boundary become interesting, see for example [**24**] and [**34**]. For Theorem 1 we only need one such curve. By deforming in this direction we are able to construct a canonical curve sequence of waist curves $\beta_n(t)$ in the plane. For fixed n, t with $0 \leq t \leq 2\pi n$ parametrizes a waist curve on an unrolled surface. This curve induces a plane curve β_n that covers the core curve of a cylinder in the unrolled surface. The modulus cylinder increases proportionately with n. By computing the length of each β_n with respect to the Teichmuller density λ, where

$$(1/2)\rho \le \lambda \le \rho$$

and ρ is the Poincaré density we arrive at a contradiction. On the one hand the computation shows that this length is proportional to $n/\log n$, a number which is unbounded as $n \to \infty$. On the other hand, the assumption that

$$C_{Teich(R)}(\tau, V) = K_{Teich(R)}(\tau, V), \tag{11}$$

where V is the tangent vector corresponding to a separating cylindrical differential, implies that this length is bounded independently of n.

The assumption (11) provides the existence of a certain nonconstant function holomorphic on Bers' fiber space with special properties. This function restricts to a non-constant holomorphic function on a Teichmueller disc that is embedded in that space. A sequence of holomorphic functions constructed in this way provides functions defined on the maximal cylinder which in the limit as n approaches ∞ realize the Carathéodory extremal value for the cylinder. On the core curve of the cylinder that extremal value has order $2/r$ which, for large values of r, is much smaller than $2/\log r$. It turns out to be so much smaller that the length of the plane curve β_n would be bounded independently of n. This estimate gives the contradiction.

The sections of this paper are organized as follows. Section 1 gives an important consequence of equality in (11). In section 2 Bers' fiber space is defined and it is shown that if $C_{Teich(R)} = K_{Teich(R)}$ on a Teichmüller disc in $Teich(R)$ then the corresponding equality holds in the fiber space $F(Teich(R))$. Section 3 defines the Teichmüller density λ_U of a plane domain U and proves the metric equivalence of λ_U and Poincaré's metric ρ_U, [**16**]. Section 4 gives an estimate for the ratio of Carathéodory's and Kobayashi's metrics for an annulus A along its core curve. Section 5 explains the spinning map on an annulus and section 6 compares $C_{Teich(A)}$ with $K_{Teich(A)}$. Section 7 gives a review of key properties of maximal embedded cylinders and section 8 explains the spinning map on a surface and gives an estimate of how spinning changes the modulus of the maximal embedded cylinder. Section 9 describes how the process of unrolling a maximal cylinder induces a sequence of embeddings of $Teich(R)$ into higher dimensional Teichmüller spaces. At this point one uses in essential way Teichmüller theory for surfaces with boundary, which is a large topic of current research, see for example [**3**]. Section 10 gives two estimates of the lengths of the sequence of waist curves. The assumption that Carathéodory's and Kobayashi's metric are equal in the tangent direction of a separating cylinder leads to the contradiction.

Since writing this paper I have learned of a related paper by V. Markovic that proves a related result [**25**] concerning the ratio of the global Carathéodory's and Kobayashi's metrics.

1. Teichmüller discs

Definition 1. A Teichmüller disc is the image $\mathbb{D}(q)$ of any map $f : \Delta \to Teich(R)$ of the form $f(t) = [t|q|/q]$ for $|t| < 1$ where q is an integrable, holomorphic, non-zero quadratic differential on R.

By Teichmüller's theorem this map is injective and is isometric with respect to the Poincaré's metric on Δ and Teichmüller's metric on $Teich(R)$. It is interesting to note that any disc isometrically embedded in a finite dimensional Teichmüller space in this way necessarily has the form $f(t)$ or $f(\bar{t})$ where f has the form described in this definition, [**4**].

Turning our focus to the relationship between C and K, we see that if $C(\tau, V) = K(\tau, V)$ for a particular tangent vector V, then from the sufficiency part of Schwarz's lemma there must be a holomorphic function $g : Teich(R) \to \Delta$ for which

$$(g \circ f)(t) = t$$

for all $|t| < 1$.

Lemma 1. Suppose V is a vector tangent to $Teich(R)$. Then $C(\tau, V) = K(\tau, V)$ if and only if there exist holomorphic functions $f \in \mathcal{F}(0, \tau)$ and $g \in \mathcal{G}(\tau, 0)$ for which $(g \circ f)(t) = t$ for all $|t| < 1$ and for which $df_0(1)$ is a complex multiple of V.

Proof. By Schwarz's lemma and a normal families argument the infimum in (2) can be equal to the supremum in (3) if an only if such mappings f and g exist. $\square$

2. Bers' fiber space

Over every Teichmüller space $Teich(S)$ of a hyperbolic quasiconformal surface S there is a canonical fiber space [**6**]

$$\Psi : F(Teich(S)) \to Teich(S),$$

called the Bers' fiber space. To describe this space assume S is a fixed quasiconformal surface in the sense that S is a topological Hausdorff space equipped with a finite system of charts z_j mapping into the complex plane defined on domains $U_j \subset S, 1 \leq j \leq n$, such that

$$\bigcup_{j=1}^{n} U_j = S,$$

and for which there is an $M > 0$ such that for each j and k the dilatation K of $z_j \circ z_k^{-1}$ satisfies

$$K(z_j \circ z_k^{-1}) \leq M.$$

The Teichmüller equivalence relation is an equivalence relation on the set of quasiconformal maps f from S to variable Riemann surfaces $f(S)$. This equivalence relation makes quasisconformal maps f_0 and f_1 mapping to Riemann surfaces $R_0 = f_0(S)$ and $R_1 = f_1(S)$ equivalent if and only if f_0 and f_1 have isotopy representatives $\tilde{f}_0$ and $\tilde{f}_1$ for which the dilatation $K(\tilde{f}_0 \circ \tilde{f}_1^{-1}) = 1$.

$$\begin{tikzcd} & f_1(S) \arrow[d, "c"] \\ S \arrow[ur, "f_1"] \arrow[r, "f_0"'] & f_0(S) \end{tikzcd}$$

FIGURE 1. Equivalence of f_0 and f_1

To define Bers' fiber space $F(Teich(S))$ one picks an arbitrary point p in S and considers quasiconformal maps f to variable Riemann surfaces $f(S-p)$. The equivalence relation has a similar description: two maps f_0 and f_1 from $S - p$ to $f_0(S-p)$ and to $f_1(S-p)$ are equivalent if there is an isotopy g_t connecting f_0 to f_1 as described above. But now the isotopy g_t must pin down the point p. Thus the composition $f_1 \circ f_0^{-1}$ keeps track of the variable conformal structure of the marked Riemann surface $f(S)$ and as well as the movement of the point $f(p)$ within that surface.

Since by definition a representative f of a class $[[f]] \in Teich(R-p)$ is quasiconformal on $S-p$, it extends uniquely to a quasiconformal map defined on S. Moreover two such reprentatives isotopic through quasiconformal maps on $S-p$ are also isotopic on S. Therefore the

$$\Psi : F(Teich(S)) \to Teich(S)$$

which forgets that the isotopy must fix the point at p is well-defined and by definition $\Psi([[f]]) = [f]$.

In general $F(Teich(S))$ is a complex manifold and for each point $\tau \in Teich(S)$, and

$$\mathbb{K}_\tau = \Psi^{-1}(\tau) \tag{12}$$

is a one-dimensional properly embedded submanifold of $F(Teich(S))$ which is conformal to a disc. Bers explicitly describes $F(Teich(S))$ as a moving family of normalized quasicircles. One side of each of these quasicircles determines a point in $Teich(S)$ and the other side gives a conformal image of the fiber $\mathbb{K}_\tau$ which is conformally and properly embedded in $F(Teich(S))$.

Now assume S is given an underlying complex structure which makes it into a Riemann surface R. Then denote by $\dot{R}$ the punctured Riemann surface $R-p$ where p is one point of R. Bers shows that the fibration Ψ is isomorphic to the fibration

$$\Psi_p : Teich(\dot{R}) \to Teich(R),$$

where $\dot{R} = R - p$.

One system of global holomorphic charts for $Teich(\dot{R})$ is given by the Bers' embedding of a Fuchsian universal covering group $\dot{G}$ that covers $R(\tau) - p$ at any point $\tau \in Teich(R)$ and any point $p \in R$. Bers' main theorem gives a second global holomorphic chart for $Teich(\dot{R})$. Assume $\tau = \Psi(\hat{\tau})$ where $\hat{\tau} \in Teich(\dot{R})$. One takes a Fuchsian universal cover $\Pi : \Delta \to \Delta/G \cong R$ with covering group G that maps 0 to p and then one forms the quasi-Fuchsian group $G_\tau = w^{\Psi(\tau)} \circ G \circ (w^{\Psi(\tau)})^{-1}$. Here w^τ has Beltrami coefficient μ supported on the exterior of the unit disc and μ represents the point $\tau \in Teich(R)$. One uses the fact that two Beltrami coefficients μ_1 and μ_2 are Teichmüller equivalent if and only if $w^{\mu_1}(z) = w^{\mu_2}(z)$ for all $z \in \Delta$. Note that μ is identically equal to 0 in Δ and μ represents the marked Riemann surface $R(\tau)$ in the complement $\Delta^* = \overline{\mathbb{C}} \setminus \Delta$.

Let $QF(G)$ be the space of quasiconformal conjugacies $w^\mu \circ G \circ (w^\mu)^{-1}$, where it is assumed that

$$\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z)$$

for all $\gamma \in G$. This implies that $w^\mu \circ G \circ (w^\mu)^{-1}$ is a group of Möbius transformations. We say such a Beltrami coefficient μ is equivariant for G. If we take $||\mu||_\infty = 1$, the mappings $w^{t\mu}$ for $|t| < 1$ form a holomorphic motion of Δ parameterized by $\{t : |t| < 1\}$ and this motion is equal to the identity when $t = 0$. Bers shows that the map

$$Teich(\dot{R}) \ni \tau \mapsto (w^{\Psi(\tau)}, w^{\Psi(\tau)}(0)) \in QF(G) \tag{13}$$

is a global holomorphic chart for $Teich(\dot{R})$.

By Bers' theorem for a fixed point $\tau \in Teich(R)$ the range of values

$$\{w : w = \infty \text{ or } w = w^\tau(z) \text{ where } |z| > 1\} \tag{14}$$

is conformal to $\mathbb{K}_\tau$ defined in (12), the quasi-Fuchsian group $G^\tau = w^\tau \circ G \circ (w^\tau)^{-1}$ acts discontinuhously on $\mathbb{K}_\tau$ and the quotient space is conformal to R_τ.

In the following lemma we assume V is a tangent vector to $Teich(R)$ of the form $\overline{\partial} V = |q|/q$, where q is a quadratic differential form supported in the complement of Δ and where $\overline{\partial} V = 0$ in Δ. Note that if q is integrable and holomorphic on R then it is also integrable and holomorphic on $\dot{R}$. Without changing the notation we view V as representing a vector tangent to $F(Teich(R))$ and at the same time a vector tangent to $Teich(R)$.

Lemma 2. Suppose $\tau \in Teich(R-p)$ and $\overline{\partial} V = |q|/q$ where q is a holomorphc quadratic differential on R. In the setting just described suppose

$$C_{Teich(R)}(\Psi(\tau), V) = K_{Teich(R)}(\Psi(\tau), V).$$

Then $C_{F(Teich(R))}(\tau, V) = K_{F(Teich(R))}(\tau, V)$.

Proof. First observe that since q is holomorphic and integrable on R, it is also holomorphic and integrable on $R - p$. This implies that V can be viewed both as a tangent vector to $Teich(R)$ and to $Teich(R-p)$. Let $\hat{g} = g \circ \Psi$ where Ψ is the projection from Bers' fiber space $F(Teich(R))$ to $Teich(R)$. We obtain

$$\hat{f} : \Delta \to F(Teich(R)) \text{ and } \hat{g} : F(Teich(R)) \to \Delta$$

where both $\hat{f}$ and $\hat{g}$ are holomorphic and

$$\hat{f} \circ \hat{g}(t) = t. \tag{15}$$

The result follows from Lemma 1. □

From this Lemmas 1 and 2 each point $\tau \in Teich(R)$ induces two kinds of conformal discs in $F(Teich(R))$. The first is the fiber $\mathbb{K}_\tau$ of the forgetful map Ψ over the point τ. The second is a bundle of Teichmüller discs, $\mathbb{D}_q$, defined in Definition 1, one for each projective class of non-zero quadratic differential q holomorphic on R_τ.

3. Comparing metrics on surfaces

For very $\tau \in Teich(R)$ the fiber $\mathbb{K}_\tau = \Psi^{-1}(\tau)$ of the forgetful map,

$$\Psi : F(Teich(R)) \to Teich(R), \tag{16}$$

carries two metrics that are induced by inherent geometry. The first is the Poincaré metric ρ defined in (1). It is natural because $\mathbb{K}_\tau$ is conformal to a disc. When viewed as a metric on $\mathbb{K}_\tau$ we denote it by ρ_τ and when viewed as a metric on R we denote it by ρ_R. The second metric is the restriction of the infinitesimal form of Teichmüller's metric on $F(Teich(R))$ to $\mathbb{K}_\tau$. Similarly, when viewed as a metric on $\mathbb{K}_\tau$ we denote it by λ_τ, and when viewed as a metric on $R = R(\tau)$ we denote it by λ_R.

Definition 2. $\lambda_\tau(z)|dz|$ is the pull-back by a conformal map $c : \Delta \to \mathbb{K}_\tau$ of the infinitesimal form of Teichmüller's metric on $F(Teich(R))$ restricted to $\mathbb{K}_\tau$.

From the previous section we know that $F(Teich(R))$ has a global coordinate given by $Teich(R-p)$, where R is a marked Riemann surface and p is a point on R. For any point $[g] \in Teich(R-p)$ the evaluation map Ev is defined by $Ev([g]) = g(p)$ is holomorphic and, since the domain of Ev is simply connected, Ev lifts to Δ by the covering map Π_p to $\widetilde{Ev}$ mapping $Teich(R-p)$ to Δ. For this reason we can interpret the inequality of the next theorem as an inequality of metrics on $R = R(\tau)$.

Theorem 2. For any surface R with marked conformal structure τ whose universal covering is conformal to Δ, the conformal metrics ρ_τ and λ_τ satisfy

$$(1/2)\rho_\tau(p)|dp| \leq \lambda_\tau(p)|dp| \leq \rho_\tau(p)|dp|. \tag{17}$$

Proof. For a point in $F(Teich(R))$ corresponding to a Riemann surface R and a point $p \in R$, we choose the Bers coordinate with base point at the identity $[id] \in Teich(R-p)$. For each point $\tau \in Teich(R-p)$ there is a quasiconformal selfmapping of the Riemann sphere w^t that maps Δ holomorphically to a quasidisc $w^t(\Delta)$ and that conjugates the covering for R by the complement of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ to the covering of $R(\tau)$ by the complement of $\overline{w^t(\Delta)}$.

Denote by Π^t the universal covering for the group

$$G^t = w^t \circ G \circ (w^t)^{-1}$$

where G is the covering group of R and let W be the tangent vector to the motion w^t at $t = 0$. This implies that $\Pi^t = w^t \circ \Pi \circ (w^t)^{-1}$.

This means that if

$$w^t_{\bar{z}} = \mu_t w^t_z,$$

then $\overline{\partial} W = \mu_0$ where $\mu_0 = \lim_{t \to 0}(1/t)\mu_t$.

We know by Slodkowski's extension theorem that w^t extends to a holomorphic motion of all of Δ and

$$w^t_{\bar{z}} = \mu_t w^t_z.$$

Since Kobayashi's infinitesimal form coincides with Teichmüller's infinitesimal form, we know from Schwarz's lemma that

$$||W||_{Teich(R-p)} = 1.$$

That is,

$$W = \frac{d}{dt}\,|_{t=0}\; w^t = \left(\frac{d}{dt}\,|_{t=0}\; \Pi^{\mu_t}(t)\right) = \Pi'(0)\mu_0.$$

If we put

$$W_0 = \frac{W}{\Pi'(0)}$$

then

$$||W_0||_{Teich(R-p)} = \rho(0)||W||_{Teich(R-p)} = \rho(0). \tag{18}$$

Fixing the quadratic differential q which is holomorphic on R, we let v^t be any normalized holomorphic selfmapping of the Riemann sphere with Beltrami coefficient identically zero in Δ with the property that v^t has Beltrami coefficient μ_t in the complement of Δ and $v^t(0) = p(t)$. That is, v^t is required to duplicate the motion w^t only infinitesimally at the point p and at the points of the complement of Δ. Otherwise, we select v^t so that the norm

$$||V||_{Teich(R-p)}$$

of its tangent vector V at $t = 0$ is as small as possible.

Putting

$$V_0 = \frac{V}{\Pi'(0)}$$

and carrying through the same reasoning we obtain

$$||V_0||_{Teich(R-p)} = \rho(0)||V||_{Teich(R-p)} \leq \rho(0). \tag{19}$$

On the other hand
$$||V_0||_{Teich(R-p)} \geq \frac{|V_0(\hat{\varphi})|}{||\hat{\varphi}||}$$
for every integrable holomorphic quadratic differential on $R-p$.

Put
$$\psi_p(z) = \frac{1}{\pi} \sum_{\gamma \in G} \frac{\gamma'(z)^2}{\gamma(z)},$$
where G is the covering group for a universal covering Π of R for which $\Pi(0) = p$.

By the Ahlfors-Bers density theorem ψ_p is the L_1-limit of integrable holomorphic functions ψ_p in the plane with a finite number of simple poles located at p and located on the boundary of the unit disc. Thus to evaluate $V(\psi_p)$ we may assume it has only a finite number of poles. Since $\tilde{V}_0$ is tangent to a holomorphic motion of the entire disc, it has bounded $\overline{\partial}$-derivative and
$$V_0(\psi_p) = \iint \overline{\partial}\tilde{V}_0 \psi_p dxdy = \iint \overline{\partial}\tilde{V}_0 \psi_p \frac{d\overline{z}dz}{2i}$$
$$= \iint d\left(\tilde{V}_0 \psi_p \frac{dz}{2i}\right) = -\pi \; res(\psi_p, p), \tag{20}$$
The only residue of $\left(\tilde{V}_0 \psi_p dz\right)$ is at p and this residue is equal to $1/\pi\Pi'(0)$. Since $\rho(p) = 1/|\Pi'(0)|$,
$$|V_0(\psi_p)| = \rho(p), \tag{21}$$
and we get
$$\rho(p) \geq ||V_0||_T \geq \frac{\rho(p)}{||\psi_p||}$$
which yields $||\psi_p|| \geq 1$.

On the other hand,
$$||\psi_p|| = \left| \iint_\omega \sum \frac{\gamma'(z)^2}{\pi\gamma(z)} \right| \leq \iint_\omega \sum \left| \frac{\gamma'(z)^2}{\pi\gamma(z)} \right| = \iint_{|z|<1} \frac{1}{|\pi z|} dxdy = 2,$$
where ω is a fundamental domain in Δ for the covering group. This yields
$$\lambda_R(t)|dt| = ||V_0||_{Teich(R-p)} \geq \frac{\rho(t)|dt|}{2}.$$
□

The following alternative description of λ_R appears in [**16**].

THEOREM 3. For $\varphi \in Q(R-p)$ be the space of integrable holomorphic quadratic differentials on $R-p$ let $||\varphi|| = \iint_R |\varphi|$. Then
$$\lambda_R(t)|dt| = \sup_{\varphi \in Q(R-p), ||\varphi|| \leq 1} \left\{ |\pi \; \text{ times the residue of } \varphi \text{ at } p| \right\}.$$

4. Comparing C and K on an annulus A

Let K_r and C_r be the Kobayashi and Carathéodory infinitesimal forms on the annulus

$$A = A_r = \{w : (1/r) < |w| < r\},$$

which has core curve $\gamma = \{w : |w| = 1\}$. Since rotations through any angle θ are conformal automorphisms of A_r, $C_r(1) = C_r(e^{i\theta})$ and $A_r(1) = A_r(e^{i\theta})$ for every point $e^{i\theta}$ in the unit circle. We first prove the following lemma which gives an asymptotic estimate for $C_r(1)$ for large values of r. Afterwards we give an exact formula which is informative but which we do not need in our subsequent application.

Lemma 3. Let C_r be Carathéodory's metric on A_r. Then

$$C_r(e^{i\theta}) < \frac{2}{r} \cdot \frac{1}{1 - 9/r^2}. \tag{22}$$

Proof. Since $z \mapsto e^{i\theta}z$ acts biholomorphically on A_r, $C_r(e^{i\theta}) = C_r(1)$ for every point $e^{i\theta}$. By definition the infinitesimal form $C_r(1)$ is equal to

$$\sup\{|g'(1)| : g \text{ maps } \mathrm{A_r} \text{ into } \Delta, \mathrm{g} \text{ is holomorphic and } \mathrm{g}(1) = 0\}.$$

Let D be the disc contained in A_r of radius $(1/2)(r - 1/r)$ and centered at the point $(1/2)(r + 1/r)$ and let C_D be Carathéodory's metric on D. Since Carathéodory's metric has the Schwarz-Pick property [**8**] and $D \subset A_r$,

$$C_r < C_D$$

and since D is a disc $C_D = K_D$. But for a disc of radius R centered at the origin

$$K_R = \frac{R}{R^2 - |z|^2}.$$

Translating this disc to the disc D centered at $(1/2)(r + 1/r)$ and making rough simplifications one obtains inequality (22). □

The exact formula for $C_r(1)$ is given [**32**].

$$C_r(w)(V)|dw(V)| = \frac{1}{r} \frac{\Pi_1^\infty (1 + r^{-4n})^2 (1 - r^{-4n})^2}{\Pi_1^\infty (1 + r^{-4n+2})^2 (1 - r^{-4n+2})^2} |dw(V)|. \tag{23}$$

It is interesting to compare this to $K_r(1)$ which can be more easily shown by using the logarithmic change of coordinates $\zeta = \xi + i\eta, \zeta = i \log w$. Then the formula for K_r is

$$K_r(\zeta, V) = \frac{\pi |dw(V)|}{(2 \log r) \cos(\frac{\eta\pi}{\log r})}. \tag{24}$$

The annulus A_r corresponds conformally to the rectangle

$$\log(1/r) < \eta < \log r, -\pi \leq \xi < \pi i.$$

The core curve γ in A_r corresponds in the ζ-coordinate to $\eta = 0$, and we have

$$K_r(|w| = 1, V) = \frac{\pi}{2\log r}. \tag{25}$$

Of course from formula (4) for any tangent vector V and any point P in any complex manifold M, the ratio $C_M(P,V)/K_M(P,V) \leq 1$. From formulas (25) and (23) when M is the annulus A_r we have a stronger inequality for any point P along the core curve, namely, the ratio

$$C_r(|w| = 1, V)/K_r(|w| = 1, V) = \frac{4}{\pi} \cdot \frac{\log r}{r} \cdot \frac{\Pi_1^\infty (1 + r^{-4n})^2 (1 - r^{-4n})^2}{\Pi_1^\infty (1 + r^{-4n+2})^2 (1 - r^{-4n+2})^2}. \tag{26}$$

In particular, the ratio decreases as r increases and it approaches 0 as r approaches ∞.

LEMMA 4. For $r > 16$, the ratio

$$C_r(|w| = 1, V)/K_r(|w| = 1, V) < 1/2.$$

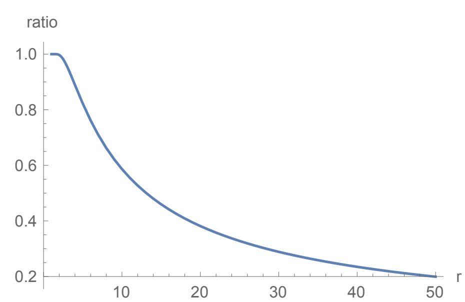

FIGURE 2. metric ratio

PROOF. This follows from formula (26). The Mathematica plot in Figure 5 (which was worked out for me by Patrick Hooper and Sean Cleary) shows that when $r > 16$ the ratio is less than 1/2.

$\square$

5. Spinning an annulus

We are interested in a particular selfmap $Spin_t$ of the annulus

$$A_r = \{z : 1/r < |z| < r\}.$$

$Spin_t$ spins counterclockwise the point $p = 1$ on the unit circle $\{z : |z| = 1\} \subset A_r$ to the point $p = e^{it}$ for real numbers t while keeping points on the boundary of A_r fixed. The formula for $Spin_t$ is easier to write in the rectangular coordinate ζ where we put

$$\zeta = \xi + i\eta = -i \log z.$$

We call ζ rectangular because the strip

$$\{\zeta = \xi + i\eta :, -\log r \leq \eta \leq +\log r\} \tag{27}$$

factored by the translation $\zeta \mapsto \zeta + 2\pi$ is a conformal realization A_r with a fundamental domain that is the rectangle

$$\{\zeta : 0 \leq \xi \leq 2\pi \text{ and } -\log r \leq \eta \leq +\log r\}.$$

The covering map is $\Pi(\zeta) = z = e^{-i\zeta}$ and the points $\zeta = 2\pi n$ for integers n cover the point $z = 1$. In the ζ coordinate we make the following definition.

Definition 3. For $\zeta = \xi + i\eta$ in the strip $-\log r \leq \eta \leq \log r$, let

$$Spin_t(\zeta) = \xi + t\left(1 - \frac{|\eta|}{\log r}\right) + i\eta. \tag{28}$$

Note that $Spin_t(\zeta + 2\pi) = Spin_t(\zeta) + 2\pi$ and $Spin_t$ shears points with real coordinate ξ and with imaginary coordinate $\pm\eta$ along horizontal lines to points with real coordinate $\xi + t(1 - |\eta|/\log r)$ while fixing points on the boundary of the strip where $\eta = \pm \log r$.

We let μ_t denote the Beltrami coefficient of $Spin_t$ and let $A_r(\mu_t)$ be the annulus A_r with conformal structure determined by μ_t. By the uniformization theorem applied to the annulus $A(\mu_t)$ we know that there is a positive number $r(t)$ such that $A(\mu_t)$ is conformal to $A_{r(t)}$.

Theorem 4. Assume $t > 2\pi$. Then the unique real number $r(t)$ with the property that $A_r(\mu_t)$ is conformal to $A_{r(t)}$ satisfies

$$t \cdot \frac{(1 - 2\pi/t)^2 + ((\log r)/t)^2}{(\sqrt{1 + ((\log r)/t)^2})} \leq \frac{\log r(t)}{\log r} \leq t \cdot \left(\frac{1}{t^2} + \frac{1}{(\log r)^2}\right)^{1/2}. \tag{29}$$

Proof. We begin the proof with a basic inequality valid for any pair of measured foliations $|du|$ and $|dv|$ on any orientable surface S with a marked conformal structure τ. Actually, this inequality is true for any pair of weak measured foliations,(see [**14**]). We denote by $S(\tau)$ the surface S with the conformal structure τ. For any smooth surface the integral $\int\int_S |du \wedge dv|$ is well defined for any pair of weak measured foliations $|du|$ and $|dv|$ on defined S. The Dirichlet integral of a measured foliation $|du|$ defined by

$$Dir_\tau(|du|) = \int\int_{S(\tau)} (u_x^2 + u_y^2)dxdy,$$

is well defined only after a complex structure τ has been assigned to S. τ gives invariant meaning to the form $(u_x^2 + u_y^2)dx \wedge dy$ where $z = x + iy$ is any holomorphic local coordinate. The following is a version of the Cauchy-Schwarz inequality.

Lemma 5. With this notation

$$\left(\int\int_S |du \wedge dv|\right)^2 \leq \left(\int\int_{S(\tau)} (u_x^2 + u_y^2)dxdy\right) \cdot \left(\int\int_{S(\tau)} (v_x^2 + v_y^2)dxdy\right). \tag{30}$$

PROOF. Begin with the elementary inequality

$$\left(det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^2 \leq (a^2+b^2)(c^2+d^2).$$

By adding the terms indexed by pairs of integers j and k this leads to the generalization

$$\left(\sum_j \sum_k |\alpha_j \delta_k - \beta_j \gamma_k| \right)^2 \leq \left(\sum_j \alpha_j^2 + \beta_j^2 \right) \left(\sum_k \gamma_k^2 + \delta_k^2 \right). \tag{31}$$

By taking limits of elementary functions to obtain Lebesgue integrals of L^1-functions, (31) implies (30). □

The measured foliations $|d\eta|$ and $|d\xi|$ on the strip (27) induce horizontal and vertical foliations whose corresponding trajectories are concentric circles and radial lines on A_r. Correspondingly, $|d\eta \circ (Spin_t)^{-1}|$ and $|d\xi \circ (Spin_t)^{-1}|$ are horizontal and vertical foliations on $A_r(\mu_t)$. We apply Lemma 5 to these foliations with $S = A_r(\mu_t)$ and $S(\tau) = A_r$ and we get

$$(4\pi \log r(t))^2 \leq [4\pi \log r] \left[(4\pi \log r)(1 + \frac{t^2}{(\log r)^2} \right],$$

which leads to the right hand side of (29).

To obtain the left hand side we work with the conformal structure on $A_r(\mu_t)$ where the slanting level lines of $\xi \circ (Spin_t)^{-1}$ are viewed as orthogonal to the vertical lines of η. By definition the number $r(t)$ is chosen so that $A_{r(\mu_t)}$ is conformal $A_{r(t)}$ and this modulus is equal to

$$\frac{2 \log r(t)}{2\pi}.$$

On the other hand this modulus is bounded below by any of the fractions

$$\frac{(\inf_\beta \int_\beta \sigma)^2}{area(\sigma)},$$

where β is any arc that joins the two boundary contours of the annulus and σ is any metric on $A_{r(t)}$.

From the definition of extremal length one obtains

$$\frac{\left(2\sqrt{(t-2\pi)^2 + (\log r)^2} \right)^2}{4\left(\sqrt{t^2 + (\log r)^2} \right)} \leq \frac{\log r(t)}{\log r},$$

This lower bound leads to the left hand side of (41). □

6. Comparing C and K on $Teich(A_r)$

THEOREM 5. For every point $\tau \in Teich(A_r)$ and for the tangent vector V with $\overline{\partial} V = |q|/q$ where $q = \left(\frac{dz}{z}\right)^2$,

$$C_{Teich(A)}(\tau, V) < K_{Teich(A)}(\tau, V). \tag{32}$$

PROOF. By Teichmüller's theorem the embedding

$$\{s : |s| < 1\} \ni f(s) = [s|q|/q|] \in Teich(A_r)$$

is isometric in Teichmüller's metric and it lifts to an isometric embedding

$$\{s : |s| < 1\} \ni f(s) = [[s|q|/q|]] \in Teich(A_r - \{1\}).$$

$$\begin{tikzcd}
 & Teich(A_r - \{1\}) \arrow[d, "\Psi"] \arrow[dr, "\hat{g}"] & \\
\{s : |s| < 1\} \arrow[ur, "\hat{f}"] \arrow[r, "f"'] & Teich(A_r) \arrow[r, "g"'] & \{s : |s| < 1\}
\end{tikzcd}$$

FIGURE 3. $\hat{f}, \hat{g}, f$ and g

The map f in Figure 3 is defined by $f(s) = [s|q|/q]$ and $\hat{f}(s) = [[s|q|/q]]$ where the single and double brackets denote equivalence classes of Beltrami coefficients representing elements of $Teich(A_r)$ and $Teich(A_r - \{1\})$, respectively. The map g is any holomorphic map from $Teich(A_r)$ into Δ and $\hat{g} = g \circ \Psi$ where Ψ is the forgetful map. That is, Ψ applied to an equivalence class $[[\mu]]$ forgets the requirement that the isotopy in the equivalence that determines elements $Teich(A_r - \{1\})$ pin down the point 1 in A_r and looks only at the requirement that the isotopy pin down every point of the innner and outer boundaries of A_r.

For the proof we need to derive a contradiction to the assumption that there is equality in (32). From Lemma 1 equality implies there exists a holomorphic function g mapping $Teich(A_r)$ onto Δ with

$$g \circ f(s) = s \tag{33}$$

for all $s \in \Delta$.

Let $\zeta \mapsto s(\zeta)$ be the conformal map from the strip in (27) that carries $\zeta = 0$ to $s = 0$, the horizontal line $\{\zeta = \xi + i\eta : \eta = \log r\}$ to the upper half semicircle on the boundary of $\{|s| < 1\}$ and the horizontal line $\{\zeta = \xi + i\eta : \eta = -\log r\}$ to the lower half semicircle. Then (33) yields

$$g \circ f(s(\zeta)) = s(\zeta) \tag{34}$$

for every ζ in the strip. In particular ,

$$g \circ f(s(t)) = s(t)$$

for every $t \in \mathbb{R}$.

Since q is also integrable and holomorphic on $A_r - \{1\}$, f lifts to $\hat{f}$ with $\Psi \circ \hat{f} = f$. If we put $\hat{g} = g \circ \Psi$, then all of the mappings in Figure 3 commute and so we also have $\hat{g} \circ \hat{f}(s) = s$ for every $|s| < 1$. From the chain rule applied at any point $s \in \Delta$,

$$(dg)_{f(s)} \circ (df)_s(1) = 1 \text{ and } (d\hat{g})_{f(s)} \circ (d\hat{f})_s(1) = 1. \tag{35}$$

Equation (33) implies the restriction of g to the image of f is the inverse mapping of f. Thus for γ an element of the fundamental group of A_r, g restricted to $f(\Delta)$ satisfies $g \circ \gamma = g$ and so g factors to a holomorphic function on the factor space, which is conformal to A_r. Moreover, noting that $\frac{\partial}{\partial t}$ is the vector field on A_r that is equal to 1 at the point e^{it} and zero elsewhere, we have

$$dg_{f(s)}\left(\frac{\partial}{\partial t}\right) = \frac{dg}{dt}$$

and so g restricted to $f(\Delta)$ realizes the supremum in the extremal problem that defines the Carathéodory metric on the annulus $A_{r(t)}$. That extremal problem is to find the supremum in

$$C_{A_{r(t)}}(1) = \sup_g \{|\frac{dg}{dt}(t)|\}, \tag{36}$$

where g ranges over all holomorphic functions from $A_{r(t)}$ onto Δ with $g(1) = 0$. Since Carathéodory's metric on $A_{r(t)}$ is less than or equal to

$$\frac{2}{r(t)} \cdot \frac{1}{1 - 9/(r(t))^2}, \tag{37}$$

we obtain $\left|\frac{dg}{dt}\right|$ is bounded by the same quantity. But $\lambda_{A_r(s)}(s) \geq (1/2)\rho_{A_r(s)}(s)$ and

$$\rho_{A_r(s)}(t) = \frac{\pi}{2} \cdot \frac{1}{\log r(t)}. \tag{38}$$

The inequalities obtained from (37) and (38) are contradictory for sufficiently large t. $\square$

It is of interest to note that the linear maps $(dg)_{f(s)}$ and $(d\hat{g}_{\hat{f}(s)})$ are represented by different quadratic differentials $q_s \in Q(A_r)$ and $\hat{q}_s \in Q(A_r - \{1\})$. From Schwarz's lemma we have

$$||q_s|| = 1 \text{ and } ||\hat{q}_s|| = 1.$$

Thus the value of the supremum in (36) can be calculated by

$$dg_{f(s)}\left(\frac{\partial}{\partial t}\right) = \frac{dg}{dt} = \pi \text{ times the residue of } \hat{q}_s \text{ at } t. \tag{39}$$

7. Maximal separating cyclinders

Let R be a Riemann surface for which $Teich(R)$ has dimension at least 2 and assume R is of finite analytic type, by which we mean that the fundamental group of R is finitely generated. Consider a simple closed curve γ that divides R into two connected components R_1 and R_2 and assume that both R_1 and R_2 have non-trivial topology in the sense that each of them has a fundamental group generated by two or more elements. From a theorem of Jenkins [**19**] and also of Strebel [**33**], there exists on R an integrable, holomorphic, quadratic differential $q_\gamma(z)(dz)^2$ that realizes R in cylindrical form, in the following sense. A cylinder realized as the factor space of a horizontal strip in the ζ-plane

$$\mathcal{C} = \{\zeta = \xi + i\eta : 0 - \log r \leq \eta \leq \log r\}/(\zeta \mapsto \zeta + 2\pi) \tag{40}$$

with $r > 1$ is embedded in R with the following properties:

i) $(d\zeta)^2 = -(d \log z)^2$ is the restriction of a global holomorphic quadratic differential $q_\gamma(z)(dz)^2$ on R with $\int\int_R |q|dxdy = 2\pi \log(1/r)$,
ii) the horizontal line segments in the ζ-plane that fill the rectangle $\{\zeta = \xi + i\eta : 0 \leq \xi < 2\pi, -\log r < \eta < \log r\}$ comprise all of the regular horizontal closed trajectories of q on R,
iii) the core curve $\gamma = \{\zeta = \xi + i\eta : \eta = 0, 0 \leq \xi \leq 2\pi\}$ separates R into two components R_1 and R_2,
iv) the remaining horizontal trajectories of q on R, that is, those which run into singular points of q, comprise two critical graphs, one lying in the subsurface R_1 and the other lying in the subsurface R_2,
v) the critical graphs $G_1 \subset R_1$ and $G_2 \subset R_2$ include as endpoints all the punctures of R as simple poles of q, and
vi) each of the graphs G_1 and G_2 is connected,
vii) for every closed curve $\tilde{\gamma}$ homotopic in R to the core curve γ $\int_{\tilde{\gamma}} |q_\gamma|^{1/2} \geq \int_\gamma |q_\gamma|^{1/2}$.

Theorem 6. [Jenkins and Strebel] The condition that the embedded cylinder described above has maximal modulus in its homotopy class implies that q_γ is a global holomorphic quadratic differential on R whose restriction to the interior of the cylinder is equal to $(d\zeta)^2$. q_γ is maximal in the sense that any quadratic differential q holomorphic on R satisfying $||q|| = ||q_\gamma||$ and the inequality $\int_{\tilde{\gamma}} |q|^{1/2} \geq \int_\gamma |q_\gamma|^{1/2}$ for every $\tilde{\gamma}$ homotopic to the core curve γ is identically equal to q_γ. Moreover, the critical graphs G_1 and G_2 are connected.

Proof. To see that G_1 is connected take any two points in G_1. Since R is arcwise connected, by assumption they can be joined by an arc in R. If that arc crosses the core curve γ, cuf off all the parts that lie in R_2 and join the endpoints by arcs appropriate subarcs of the core curve γ to obtain a closed curve α in $R_1 \cup \gamma$ that joins the two points. Then use the vertical trajectories of the cylinder to project every point of α continuously to a curve that lies in the critical graph

G_1 that joins the same two points. This shows that G_1 is arcwise connected. Of course, the same is true of G_2.

That the two conditions $||q|| = ||q_\gamma||$ and the inequality $\int_{\tilde{\gamma}} |q|^{1/2} \geq \int_\gamma |q_\gamma|^{1/2}$ imply that $q = q_\gamma$ follows from Grötzsch's argument or from the minimum norm principle proved in [**13**]. Finally, if $(d\zeta)^2$ failed to be the restriction of a global holomorphic quadratic differential on R, it would be possible to deform the embedding of the cylinder inside R in such a way as to increase the modulus of the cylinder. □

Definition 4. We call a cyclindrical differential q_γ with the properties of the previous theorem *separating* and any simple closed curve homotopic to γ is a *separating curve.*

8. Spinning

We extend the estimates for moduli of annuli given in Theorem 4 to moduli of a maximal embedded separating cylinder $\mathcal{C} \subset R$. Just as in Theorem 4 we let μ_t be the Beltrami coefficient of $Spin_t$. It determines a new conformal structure on $R - p$ and measured foliatons $|d\xi \circ Spin_t^{-1}|$ and $|d\eta \circ Spin_t^{-1}|$ on $R - p(t)$, where $p(t)$ is a point on the core curve of a maximal cylinder $\mathcal{C} \subset R$ represented by the point t on the real axis in the rectangular coordinate that represents the cylinder defined in (27).

Theorem 7. Assume $t = 2\pi n$ where $n \geq 1$. Then the unique real number $r(t)$ with the property that $A_r(\mu_t)$ is conformal to $A_{r(t)}$ satisfies

$$(1/4) \cdot t \cdot \frac{(1 - 2\pi/t)^2 + ((\log r)/t)^2}{(\sqrt{1 + ((\log r)/t)^2})} \leq \frac{\log r(t)}{\log r} \leq t \cdot \left(\frac{1}{t^2} + \frac{1}{(\log r)^2}\right)^{1/2}. \tag{41}$$

Proof. We apply Lemma 5 to the measured foliations $|d\eta|$ and $|d\xi|$ on the strip (27). Correspondingly, $|d\eta \circ (Spin_t)^{-1}|$ and $|d\xi \circ (Spin_t)^{-1}|$ are horizontal and vertical foliations on $A_r(\mu_t)$. Applying Lemma 5 to these foliations with $S = A_r(\mu_t)$ and $S(\tau) = A_r$ and we get

$$(4\pi \log r(t))^2 \leq [4\pi \log r] \left[(4\pi \log r)(1 + \frac{t^2}{(\log r)^2}\right],$$

which leads to the right hand side of (41).

To obtain the left hand side we work with the conformal structure on $A_r(\mu_t)$ where the slanting level lines of $\xi \circ (Spin_t)^{-1}$ are viewed as orthogonal to the vertical lines of η. By definition the number $r(t)$ is chosen so that $A_{r(\mu_t)}$ is conformal $A_{r(t)}$ and this modulus is equal to

$$\frac{2 \log r(t)}{2\pi}.$$

This is bounded below by any of the fractions

$$\frac{1}{4}\cdot\frac{(\inf_\beta \int_\beta \sigma)^2}{area(\sigma)},$$

where β is any homotopy class of closed curve on R that joins the two boundary contours of the annulus and has intersection number 2 with the core curve of $A_{r(t)}$ and σ is the flat $\sigma(\zeta)|d\zeta|$ on $A_{r(t)}$ given by $\sigma(\zeta)\equiv 1$. Then

$$\int_\beta |d\zeta| \geq 4\log r(t).$$

From the definition of extremal length one obtains

$$\frac{(2\sqrt{(t-2\pi)^2+(\log r)^2})^2}{4(\sqrt{t^2+(\log r)^2})}\leq \frac{\log r(t)}{\log r},$$

This lower bound leads to the left hand side of (41). □

9. Unrolling

Just as in the previous sections we let $\mathcal{C}_\gamma$ be the maximal closed cylinder corresponding to a separating simple closed curve γ contained in R. Then

$$Strip_\gamma/(\zeta\mapsto\zeta+2\pi)\cong\mathcal{C}_\gamma.$$

Moreover the interval $[0,2\pi]$ along the top perimeter of $\mathcal{C}_\gamma$ partitions into pairs of subintervals of equal length that are identified by translations or half translations of the form

$$\zeta\mapsto\pm\zeta+constant$$

that realize the conformal structure of R along the top side of $\mathcal{C}_\gamma$.

Also the bottom perimeter of $\mathcal{C}_\gamma$ partitions into pairs of intervals similarly identified that realize the local consformal structure of R on the bottom side.

We can use the same strip to define a sequence of Riemann surfaces R_n for any integer $n\geq 1$ where by definition $R_1 = R$. We use the same identifications along the top and bottom horizontal sides of

$$Strip_\gamma$$

and denote by $\mathcal{C}_{\gamma_n}$ the strip $Strip_\gamma$ factored by the translation $\zeta\mapsto\zeta+2\pi n$.

Let A_n be the interior of $\mathcal{C}_{\gamma_n}$ so A_n is a conformal annulus of maximum modulus contained in R_n.

Lemma 6. Assume γ is a separating closed curve in R. and let q_n be the quadratic differential $(d\zeta)^2$ on $\mathcal{A}_n$ and $\Delta=\{t:|t|<1\}$. Then

$$\Delta\ni t\mapsto [f^{t|q_n|/q_n}]_{R_n}\in Teich(R_n)$$

is an injective.

PROOF. Since the domain A_n has extremal modulus in R_n the quadratic differential q_n on A_n is the restriction of an integrable holomorphic quadratic differential on R_n. Thus the statement follows from Teichmüller's theorem. $\square$

We let p be a point on the core curve γ of $\mathcal{C}_\gamma$ corresponding to $\zeta = 0$ in the strip. Similarly, we let p_n corresponding to the same point $\zeta = 0$ in the strip but in the factor space

$$\mathcal{C}_{\gamma_n} \cong Strip_\gamma/(\zeta \mapsto \zeta + 2\pi n).$$

With these choices by Bers' fiber space theorem we identify the fiber spaces $F(Teich(R_n))$ and $F(Teich(R))$ with Teichmüller spaces $Teich(R_n - p_n)$ and $Teich(R - p)$, that is, the Teichmüller spaces of the punctured Riemann surfaces $R - p$ and $R_n - p_n$.

LEMMA 7. Assume q_n is the holomorphic quadratic differential on R whose restriction to the maximal annulus A_n is equal to $(d\zeta)^2$. Also assume V_n is a tangent vector to $Teich(R_n)$ for which $\overline{\partial} V_n = |q_n|/q_n$ and R_n is Riemann surface with marked complex structure τ_n. If $Car_{Teich(R)}(\tau, V) = Kob_{Teich(R)}(\tau, V)$ then for every integer $n \geq 1$

$$Car_{F(Teich(R_n))}(\tau_n, V_n) = Kob_{F(Teich(R_n))}(\tau_n, V_n).$$

PROOF. Choose the point p_n on R_n which corresponds in the ζ coordinate to $\zeta = 0$. The integrable holomorphic quadratic differential q_n on R_n is also integrable and holomorphic on $R_n - p_n$ and the Teichmüller disc $[t|q_n|/q_n]$ in $Teich(R_n)$ lifts to the Teichmüller disc $[[t|q_n|/q_n]]$ in $Teich(R_n - p_n)$. $\square$

LEMMA 8. Assume $Teich(R)$ has dimension more than 1 and p covers a point on the separating core curve of $\mathcal{C}_n$. Then $Spin_{2\pi n}$ represents an element of the mapping class group of $Teich(R_n - p)$ that fixes the Teichmüller disc

$$\mathbb{D}(q_n) = \{[[s|q_n|/q_n]] \in Teich(R_n - p) : |s| < 1\}$$

and $Spin_{2\pi n}$ restricted to $\mathbb{D}(q_n)$ is a hyperbolic transformation.

PROOF. $Spin_{2\pi n}$ preserves $\mathbb{D}(q_n)$ because on R_n it is homotopic to the translation $\zeta \mapsto \zeta + 2\pi n$ and $q_n(\zeta)(d\zeta)^2 = (d\zeta)^2$ is automorphic for this translation. Since the dimension of $Teich(R)$ is more than 1 the translation length of this transformation is positive. $\square$

10. The waist sequence

Fix the point p in a Riemann surface R lying on the core curve of a separating cylindrical differential q_γ. Assume p corresponds to $\zeta = 0$ in the maximal cylindrical strip $\mathcal{C}_\gamma$ which is covered by the horizontal strip $Strip(q_\gamma)$.

DEFINITION 5. Let $\mathcal{C}_n$ be the cylinder conformal to

$$Strip(q_n)/(\zeta \mapsto \zeta + 2\pi n)$$

and $[[\mu_t]]$ the equivalence class of Beltrami coefficient of $Spin_t$ in $F(Teich(R_n))$. Then we call

$$\alpha_n(t) = [[\mu_t]] \in Teich(R_n - p)$$

for $\mu_t, 0 \le t \le 2\pi n$ the n-th spin curve.

The spin curve α_n determines a waist curve $\beta_n(t)$ in $\mathbb{C}$. To obtain β_n we use the Bers' isomorphism that realizes $Teich(R_n - p)$ as a fiber space over $Teich(R_n)$. In the normalization used by Bers R_n is covered by Fuchsian group acting on the upper half plane and one assumes the point p on the surface R_n is covered by the point i in the upper half plane. Then Bers isomorphism of $Teich(R_n - p_n)$ with $F(Teich(R_n))$ is realized by the map

$$Teich(R_n - p_n) \ni [[\mu]] \mapsto \Big([\mu], w^{[\mu]}(i)\Big) \in Teich(R_n) \times \Delta_\mu. \tag{42}$$

Here Δ_μ is by definition the unit disc Δ with moving complex structures given by the Beltrami coefficients μ that represent elements $[\mu]$ of $Teich(R_n)$.

Definition 6. [**the waist sequence**] The n-th waist curve is the plane curve

$$\beta_n(t) = w^{[\mu_t]}(i)$$

given by the second entry in (42), where $[\mu_t] = \Psi(\alpha_n(t))$ and $\alpha_n(t) = [[\mu_t]] \in Teich(R_n - p)$.

We call β_n the n-th waist. It represents the curve that passes once around the waist of annulus A_n which is the interior of the cylinder $\mathcal{C}_n$. Note that A_n is maximally embedded in R_n. We will use the upper and lower bounds from Theorem 2 to estimate the length of $\beta_n(t), 0 \le t \le 2\pi n$ with respect to the Teichmüller density λ_{A_n} on the annulus A_n in two ways.

From Theorem 2 of section 3 we have for all Riemann surfaces

$$(1/2)\rho_R \le \lambda_R \le \rho_R. \tag{43}$$

Although they are unnecessary for our proof of the main theorem we point out the following facts. Except in cases when both metrics are identically zero, the left hand side of (43) is equality only if R is simply connected and the right hand side only if R is the four times punctured sphere, [**16**]. In the case where R is the complex plane or a surface covered by the plane both metrics are identically equal to zero.

We now apply (43) in the case that R is any of the annuli A_n. We use the left hand side of (43) to estimate the asymptotic lengths $\lambda_{A_n}(\beta_n)$ of β_n from below in their dependence on n. We have

$$\lambda_{A_n}(\beta_n) \ge (1/2)\rho_{A_n}(\beta_n) \ge \frac{n}{2\log n}, \tag{44}$$

because the annulus $A_n = \{z : 1/r_n < |z| < r_n\}$ where from (37) and (38) r_n is asymptotic to $\log n$.

We use the right hand side to the estimate the lengths $\lambda_{A_n}(\beta_n)$ from above in their dependence on n under the asssumption that

$$Car_{Teich(R_n-p_n)}(\tau_n, V_n) = Kob_{Teich(R_n-p_n)}(\tau_n, V_n).$$

This assumption implies the existence of a holomorphic function $\hat{g}_n$ defined on $Teich(R_n - p_n)$ such that

$$\hat{g}_n \circ \hat{f}_n(s) = s \text{ for } |s| < 1. \tag{45}$$

Note that $\hat{g}_n$ is defined on $Teich(R_n - p_n)$ and holomorphic with respect to the parameter ζ in the strip. It is also automorphic for $\zeta \mapsto \zeta + 2\pi n$. Thus $\hat{g}_n$ restricts to a holomorphic function on the fiber over the identity of the forgetful map

$$\Phi : Teich(R_n - p_n) \to Teich(R_n).$$

In particular, $\hat{g}_n$ restricts to a function holomorphic on the annulus A_n which maps into the unit disc. Together with (45) this implies that for points p on the core curve β_n

$$c_{A_n}(p) = \rho_{A_n}(p),$$

where c_{A_n} is the Carathéodory metric of A_n. So from the right hand side of (43)

$$\lambda_{A_n}(p) \le c_{A_n}(p)$$

and from section 4 c_{A_n} is asymptotic to $2/r_n$, which is asymptotic to $1/n$. Since β_n is comprised of n intervals of equal length in the interval $[0, 2\pi n]$, this estimate implies $\lambda_{A_n}(\beta_n)$ is bounded independently of n, which contradicts (44) and proves Theorem 1.

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