A Short Course on Teichmüller's Theorem

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Abstract

We present a brief exposition of Teichmüller's theorem.

Introduction

An orientation preserving homeomorphism f from a Riemann surface X onto a Riemann surface Y is given. Teichmüller's problem is to find a quasiconformal homeomorhism in the homotopy class of f with minimal maximal dilatation, that is, to find a homeomorphism f_0 whose maximal dilatation $K(f_0)$ is as small as possible in its homotopy class.

Teichmüller's theorem states that the problem has a unique extremal solution provided that X is compact or compact except for a finite number of punctures, namely, a Riemann surface of finite analytic type. Moreover, except when f_0 is conformal, f_0 is equal to a stretch mapping along the horizontal trajectories of some uniquely determined holomorphic quadratic differential $\varphi(z)(dz)^2$, with $\int\int_X |\varphi| dxdy = 1$, postcomposed by a conformal map. It turns out that even for arbitrary Riemann surfaces, whether or not they are of finite analytic type, this statement is generically true (see [20], [27]).

The goal of this course is to present a brief proof of the original Teichmüller theorem in a series of lectures and exercises on the following topics:

1. conformal maps and Riemann surfaces,
2. quasiconformal maps, dilatation and Beltrami coefficients,
3. extremal length,
4. the Beltrami equation,
5. the Reich-Strebel inequality and Teichmüller's uniqueness theorem,
6. the minimum norm principle,
7. the heights argument,
8. the Hamilton-Krushkal condition and Teichmüller's existence theorem,

9. trivial and infinitesimally trivial Beltrami differentials,
10. Teichmüller space and Teichmüller's metric,
11. infinitesimal Teichmüller's metric,
12. tangent vectors to Teichmüller spaces.

The course is organized as follows. The minimal necessary background and defining concepts in the statement of the theorem are introduced in the first four lectures. The uniqueness part of Teichmüller's Theorem is presented in lectures 5, 6 and 7 and the existence part and the expression for extremal mapping in terms of a unique holomorphic quadratic differential are presented in lectures 8 and 9. Finally Teichmüller space, Teichmüller's metric, the infinitesimal form of Teichmüller's metric, and the structure of the tangent space to the Teichmüller space of an n-punctured Riemann sphere are explained in the lectures 10, 11 and 12.

There are many expositions of Teichmüller's theorem in the literature. However, our approach differs from others in that it highlights the minimum norm principle for measured foliations. [1, 19, 20, 23, 33, 35, 36]

These notes are meant merely to serve as a first introduction to the subject. There are many topics we have omitted which, if put together, make a unified subject. These include the equality of Kobayashi's and Teichmüller's metrics [37] [18], a result which shows that the complex structure of Teichmüller space determines its metric. There is also Royden's result that for compact surfaces of genus bigger than 2 the mapping class group is the full group of biholomorphic isomorphisms of $T(R)$. This result was extended by Earle and Kra to surfaces of finite analytic type [15] and [16]. Though still unknown for many infinite dimensional dynamical Teichmüller spaces, it has also been extended to Teichmüller spaces of Riemann surfaces of infinite type in varying degrees of generality by Earle, Gardiner [14] and Lakic [26] and by Markovic [17], [32].

One should add to these topics the theory of holomorphic motions, that is, the theorems of Mañé, Sad and Sullivan [30], Bers-Royden [11], Sullivan-Thurston [39] and Slodkowski [38]. There are also the beautiful results of Reich, Strebel [35] [36] and Hamilton [21] which relate infinitesimal extremality to global extremality and further result by Bozin, Lakic, Markovic and Mateljevic [13] that relates infinitesimal unique extremality to global unique extremality.

Unfortunately, it would take a whole book to explore all of these topics. We recommend [33], [23], [19] and [20].

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1 Conformal Maps and Riemann Surfaces

A map $z \mapsto w = f(z) = u(z)+iv(z)$ from a domain U in $\mathbb{C}$ onto another domain $f(U)$ in $\mathbb{C}$ is conformal if it is a holomorphic homeomorphism. Thus we assume conformal maps are one-to-one and orientation preserving. In particular the first order approximation of such a map at a point $z_0 \in U$ has the form

$$f(z) = a_0 + a_1(z - z_0) + o(|z - z_0|),$$

with $a_1 \neq 0$. This means that in a neighborhood of z_0 the mapping f is approximated infinitesimally by the complex linear map $\Delta z \mapsto \Delta w = a_1 \Delta z$, where $a_1 = re^{i\theta}$, and $e^{i\theta}$ effects a rotation by angle θ and $r \neq 0$ effects a magnification by r. Such a linear map does not change shape. For example, the image of an equilateral triangle by $z \mapsto a_1 z$ is still an equilateral triangle.

Definition. A Riemann surface is a connected Hausdorff topological space X together with a system of charts z_j mapping open sets U_j of X into the complex plane. The system of charts is assumed to cover X and on overlapping charts $z_j(U_i \cap U_j)$ the transition functions $z_i \circ (z_j)^{-1}$ are conformal.

Any chart z is a holomorphic map from an open subset of X into $\mathbb{C}$ and therefore $dz = dx + idy$ is a linear map from the tangent bundle to X into the tangent bundle to $\mathbb{C}$. From this viewpoint dx and dy are real-valued linear functionals on the tangent bundle to X, and thus they are elements of its cotangent bundle. Choose a basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ dual to the basis $\{dx, dy\}$ in the sense that

$$\begin{pmatrix} (\frac{\partial}{\partial x})(dx) & (\frac{\partial}{\partial x})(dy) \\ (\frac{\partial}{\partial y})(dx) & (\frac{\partial}{\partial y})(dy) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 1. *By definition, the partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are the basis dual to $dz = dx + idy$ and $d\overline{z} = dx - idy$. Show that*

$$\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}). \end{cases}$$

Exercise 2. *Show that if $f = u + iv$, then $\frac{\partial}{\partial \overline{z}} f = 0$ is equivalent to $u_x = v_y$ and $u_y = -v_x$.*

Exercise 3. *Using the notations $\frac{\partial}{\partial z} f = f_z$ and $\frac{\partial}{\partial \overline{z}} f = f_{\overline{z}}$, show that $u_x v_y - u_y v_x = |f_z|^2 - |f_{\overline{z}}|^2$.*

Exercise 4. *Show that $u_{xx} + u_{yy} = 4u_{z\overline{z}}$. By definition the Laplacian of u is $\Delta u = u_{xx} + u_{yy}$. A C^2 real-valued function $f(x, y)$ is called harmonic, subharmonic or superharmonic if $\Delta f = 0$, ≥ 0, or ≤ 0, respectively. Show that* *the notions of harmonic, subharmonic and superharmonic are well-defined on a Riemann surface. That is, show that if z and w are two charts defined on overlapping domains in a Riemann surface X and if Δ_z is the Laplacian with respect to x and y with $z = x + iy$ and Δ_w is the Laplacian with respect to u and v with $w = u + iv$, then the Laplacian of f with respect to z or with respect to w are simultaneously, zero, positive or negative.*

2 Quasiconformal Maps, Dilatation and Beltrami Coefficients

If one thinks of a conformal structure as a shape, on an infinitesimal level conformal maps do not distort shape. Thus, in order to study maps that permit distortion of shape on an infinitesimal level, one must extend the discussion to maps that are not conformal. If at the same time one keeps the amount of distortion bounded, one is lead to the notion of a quasiconformal map.

We shall first consider what it means for a real linear map to be quasiconformal. By definition quasiconformal maps are locally approximable almost everywhere by orientation preserving real linear maps which that distort shape by a uniformly bounded amount. To begin our discussion we consider a real linear map given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$T = df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since we assume quasiconformal maps are orientation preserving, $ad - bc > 0$. The circle $|z|^2 = x^2 + y^2 = 1$ is mapped by T to the ellipse whose equation in the $w = u + iv-$plane is $|T^{-1}w|^2 = 1$. Expressed in terms of the inner product, this ellipse is the set of points satisfying

$$\left(T^{-1}\begin{pmatrix} u \\ v \end{pmatrix}, T^{-1}\begin{pmatrix} u \\ v \end{pmatrix}\right) = \left((T^{-1})^t T^{-1}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = 1.$$

The distortion K of T is the eccentricity of this ellipse, that is, the length of its major axis divided by the length of its minor axis. This ratio is the same as the square root of the ratio of the larger eigenvalue λ_2 of $(T^{-1})^t T^{-1}$ to its smaller eigenvalue λ_1. Since $\left((T^{-1})^t T^{-1}\right)^{-1} = TT^t$, this ratio is the same as the corresponding ratio for

$$TT^t = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}.$$

But $K = \left(\frac{\lambda_2}{\lambda_1}\right)^{1/2}$ implies $K + 1/K = \frac{\lambda_1 + \lambda_2}{\sqrt{\lambda_1 \lambda_2}}$. The numerator is the trace of TT^t and the denominator is the square root of the determinant of TT^t, which is the same as the determinant of T. Thus

$$K + 1/K = \frac{a^2 + b^2 + c^2 + d^2}{ad - bc}.$$

The complex distortion of a two-dimensional linear map is determined by the eccentricity K and by an angle and these two pieces of information are conveniently expressed in complex notation. Any real linear map from $\mathbb{C}$ to $\mathbb{C}$ has the complex form

$$w = T(z) = Az + B\overline{z},$$

with complex constants A and B. For orientation preserving T, the determinant is $|A|^2 - |B|^2 > 0$ and the formula can be written as

$$w = T(z) = A(z + \mu\overline{z}), \tag{1}$$

where $\mu = B/A$ and $|\mu| < 1$. In this form T is the stretch map $S(z) = z + \mu\overline{z}$ postcomposed by a multiplication, which is conformal and consists of a rotation through angle $\arg A$ and magnification by the factor $|A|$. Thus all of the distortion of shape caused by T is expressed by the complex number μ. Here μ is called the complex dilatation or the Beltrami coefficient. From it one can find angles of directions of maximal magnification and maximal shrinking, as well as the distortion. If we let $\mu = ke^{i\alpha}$ and $z = re^{i\theta}$, then

$$S(z) = re^{i\theta} + kre^{i\alpha - \theta} = re^{i\theta}(1 + ke^{i(\alpha - 2\theta)})$$

and as θ moves from 0 to 2π, $S(z)$ sweeps out an ellipse whose maximum distance from the origin occurs when $\theta = \alpha/2$ or $\alpha/2 + \pi$ and whose minimum distance from the origin occurs when $\theta = \alpha/2 - \pi/2$ or $\alpha/2 + \pi/2$. The eccentricity of the ellipse is equal to $\frac{1+|\mu|}{1-|\mu|}$, we denote it by $K(T)$. Since it depends only on μ, we also denote it by $K(\mu)$.

After applying the stretch map, multiplication by A rotates the direction of maximal stretching by the angle $\arg A$. Consider the composition $T_2 \circ T_1$ of two such real-linear maps $T_1 = A_1 \circ S_1$ and $T_2 = A_2 \circ S_2$. The image by T_1 of the direction of maximal stretching of S_1 may coincide with the direction of maximal stretching of S_2. Because the direction of maximal shrinking is orthogonal to the direction of maximal stretching, when the image by T_1 of the direction of maximal stretching of S_1 coincides with the direction of maximal stretching of S_2, the similar statement is true for the image of the direction of maximal shrinking. Thus these directions coincide precisely when $K(T_2 \circ T_1) = K(T_2)K(T_1)$ and otherwise $K(T_2 \circ T_1) < K(T_2)K(T_1)$.

Exercise 5. *Let μ_1 and μ_2 be Beltrami coefficients of $T_1(z) = A_1 z + B_1\overline{z}$ and $T_2(z) = A_2 z + B_2\overline{z}$. Show that the Beltrami coefficient of $T_2 \circ (T_1)^{-1}$ is*

$$\left(\frac{\mu_2 - \mu_1}{1 - \overline{\mu_1}\mu_2}\right)\frac{1}{\theta_1}, \tag{2}$$

where $\theta_1 = \overline{A_1}/A_1$.

Definition. A C^1 orientation preserving homeomorphism $w = f(z)$ of a Riemann surface X onto another Riemann surface $f(X)$ is quasiconformal if

$$K(f) = \sup_{z \in X} K_z(f) < \infty,$$

where $K_z(f) = \frac{1+k(z)}{1-k(z)}$ and $k(z) = |f_{\overline{z}}/f_z|$. If $K = K(f)$, then f is called K-quasiconformal.

Exercise 6. *For a mapping f from a Riemann surface X to a Riemann surface Y, show that $K_z(f)$ and $K(f)$ are well-defined independently of choice of coordinate charts.*

3 Extremal Length

Let Γ be a family of curves on a Riemann surface. Every γ in Γ is assumed to be a countable union of open arcs or closed curves. The extremal length of Γ, $\Lambda(\Gamma)$, is a sort of average minimum length of the curves in Γ. It is an important quantity because it is invariant under conformal mappings and quasi-invariant under quasiconformal mappings, in the sense to be described precisely in Proposition 2. First we define the set of allowable metrics. A metric $\rho(z)|dz|$ is allowable if

- it is defined independently under choice of chart, that is, $\rho_1(z_1)|dz_1| = \rho_2(z_2)|dz_2|$, where ρ_1 and ρ_2 are representatives of ρ in terms of the charts z_1 and z_2.
- ρ is measurable and ≥ 0 everywhere, and
- $A(\rho) = \int\int_X \rho^2 dxdy \neq 0$ or ∞.

For such an allowable ρ, define

$$L_\gamma = \int_\gamma \rho |dz|$$

if ρ is measurable along γ; otherwise define $L_\gamma = +\infty$. Let $L(\rho, \Gamma) = \inf L_\gamma(\rho)$, where the infimum is taken over all curves γ in Γ. The extremal length of the curve family Γ is

$$\Lambda(\Gamma) = \sup_\rho \frac{L(\rho, \Gamma)^2}{A(\rho)}, \tag{3}$$

where the supremum is taken over all allowable metrics ρ. Notice that the ratio in the surpremum is invariant if ρ is multiplied by a positive constant. Thus in attempting to evaluate $\Lambda(\Gamma)$ we may normalize in different ways. For example by putting $L(\rho)$ or $A(\rho)$ equal to 1, or by putting $L(\rho) = A(\rho)$. A metric ρ is said to be extremal if it realizes the supremum in (3).

Lemma 1. *Let X be a rectangle with width a and height b and Γ be the family of arcs in X that join the left side of X to the right side of X. Then $\Lambda(\Gamma) = a/b$ with extremal metric $\rho_0 = 1$.*

Proof. We show that $\rho_0 = 1$ is an extremal metric. Note that $L(\rho_0, \Gamma) = a$ and $A(\rho_0) = ab$. Therefore $\Lambda(\Gamma) \geq a^2/ab = a/b$. Let ρ be any competing metric and scale ρ so that $L(\rho, \Gamma) = a$. Then for every horizontal line segment γ crossing X, $a \leq \int_\gamma \rho(z)|dz| = \int_\gamma \rho(z)dx$, so $ab \leq \int\int_X \rho(z)dxdy$. By Schwarz's inequality, $(ab)^2 \leq \left(\int\int_X \rho^2 dxdy\right)\left(\int\int_X dxdy\right)$. Hence $ab \leq A(\rho)$ and $a/b = a^2/ab \geq L(\rho)^2/A(\rho)$, which implies $\Lambda(\Gamma) \leq a/b$. Thus $\Lambda(\Gamma) = a/b$. □

Lemma 2. *Let $X = \{z : 1 < |z| < R\}$ and Γ be the family of closed curves in X homotopic in X to a core curve of X. By definition a core curve is any circle centered at the origin and lying in the interior of X. Then $\Lambda(\Gamma) = 2\pi/\log R$ with extremal metric $\rho_0(z) = 1/r$.*

Proof. We will show that $\rho_0 = 1/r$ is an extremal metric. Clearly $A(\rho_0) = 2\pi \log R$ and $L(\rho_0) = \int_0^{2\pi}(1/r)rd\theta = 2\pi$. Therefore,

$$\Lambda(\Gamma) \geq 2\pi/\log R.$$

For any allowable metric $\rho(z)|dz|$, $L(\rho) \leq \int_0^{2\pi} \rho(re^{i\theta})rd\theta$,

$$\frac{L(\rho)}{r} \leq \int_0^{2\pi} \rho(re^{i\theta})d\theta,$$

$$L(\rho)\log R \leq \int\int_X \rho drd\theta = \int\int \frac{1}{\sqrt{r}}\sqrt{r}\rho drd\theta.$$

So by Schwarz's inequality

$$(L(\rho)\log R)^2 \leq \int\int \frac{1}{r}drd\theta \int\int \rho^2 rdrd\theta,$$

$$\frac{L(\rho)^2}{A(\rho)} \leq 2\pi/\log R.$$

Thus $\Lambda(\Gamma) \leq 2\pi/\log R$ and $\Lambda(\Gamma) = 2\pi/\log R$. □

Exercise 7. *Let $X = \{z : 1 < |z| < R\}$ and Γ^t be the family of arcs in X that join $|z| = 1$ to $|z| = R$. Show that $\Lambda(\Gamma^t) = \log R/(2\pi)$ also with extremal metric $\rho_0(z) = 1/r$.*

Now let $w = f(z)$ be a continuously differentiable and quasiconformal map from an annulus $X = \{z : 1 < |z| < R\}$ onto another annulus $f(X) = \{z : 1 < |z| < R'\}$ with $R' > R$ and assume μ is the Beltrami coefficient of f, that is,

$$f_{\overline{z}}(z) = \mu(z)f_z(z).$$

Then if Γ is the family of arcs in X joining its inner and outer boundary contours, $\Gamma^\mu = f(\Gamma)$ will be the family of arcs joining the inner and outer boundary contours of $f(A)$.

Note that $f_0 : z = re^{i\theta} \mapsto r^{K_0}e^{i\theta}$ maps $X = \{z : 1 < |z| < R\}$ onto $X' = \{\tilde{z} : 1 < |\tilde{z}| < R'\}$ provided that $R' = R^{K_0}$, and $\zeta = \log z$ and $\tilde{\zeta} = \log \tilde{z}$ are natural parameters for the quadratic differentials $(dz/z)^2$ and $(d\tilde{z}/\tilde{z})^2$ on X and X'. The mapping f_0 expressed in terms of ζ and $\tilde{\zeta}$ is given by $\zeta = \xi + i\eta \mapsto \tilde{\zeta} = K_0\xi + i\eta$. We let f be an arbitrary quasiconformal map from X onto X' also mapping the inner and outer contours of X onto the inner and outer contours of X'.

Proposition 1. *In the notation described above*

$$K_0 = \frac{\Lambda(\Gamma^\mu)}{\Lambda(\Gamma)} \leq \int\int_X \frac{|1 + \mu\frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2}|\varphi|dxdy.$$

Moreover, if μ is the Beltrami coefficient of f, then $||\mu||_\infty > k_0$ unless

$$\mu = k_0|\varphi|/\varphi \quad a.e.,$$

where $k_0 = \frac{K_0-1}{K_0+1}$ and $\varphi(z) = \frac{1}{(2\pi \log R)z^2}$.

Proof. Through the correspondence $z = \exp(\zeta)$, f determines a mapping from the rectangle $\{\zeta : 0 \leq \xi \leq \log R, 0 \leq \eta \leq 2\pi\}$ onto the annulus $\{z : 1 < |z| < R'\}$. Let γ be a radial arc in Γ joining the inner and outer boundary contours of X. Then

$$\log R' \leq \int_{f(\gamma)} |dz| = \int_\Gamma |df| = \int_0^{\log R} |f_\zeta||1 + \tilde{\mu}|d\xi,$$

where $\tilde{\mu} = f_{\overline{\zeta}}/f_\zeta$. Integrating from $\eta = 0$ to $\eta = 2\pi$ we obtain

$$2\pi \log R' \leq \int\int |f_\zeta||1 + \tilde{\mu}|d\xi d\eta,$$

where the integral is over the rectangle $\{\zeta : 0 \leq \xi \leq \log R, 0 \leq \eta \leq 2\pi\}$. Introducing a factor of $\sqrt{1 - |\mu|^2}$ in both numerator and denominator and applying Schwarz's inequality together with the Jacobian change of variable formula (see exercise 3) yields

$$(2\pi \log R')^2 \leq \int\int |f_\zeta|^2(1 - |\tilde{\mu}|^2)d\xi d\eta \int\int \frac{|1 + \tilde{\mu}|^2}{1 - |\tilde{\mu}|^2}d\xi d\eta.$$

Since the first integral on the right hand side is the area of the rectangle in the $\tilde{\zeta}-$plane, we get

$$2\pi \log R' \leq \int\int \frac{|1 + \tilde{\mu}|^2}{1 - |\tilde{\mu}|^2}d\xi d\eta.$$

Since $\zeta = \log z$, $\tilde{\mu} = \mu\overline{z}/z$, where $\mu = f_{\overline{z}}/f_z$. Noting that $\varphi(z) = \frac{1}{(2\pi \log R)z^2}$, the last inequality can be rewritten as

$$K_0 = \frac{\Lambda(\Gamma^\mu)}{\Lambda(\Gamma)} \leq \int\int_{1<|z|<R} \frac{|1 + \mu\frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2}|\varphi|dxdy. \tag{4}$$

Since

$$\frac{|1+\mu\frac{\varphi}{|\varphi|}|^2}{1-|\mu|^2} \leq \frac{(1+k(f))^2}{1-k(f)^2} = \frac{1+k(f)}{1-k(f)}, \tag{5}$$

from (4) and (5) we obtain

$$\frac{1+k_0}{1-k_0} \leq \frac{|1+\mu\varphi/|\varphi||^2}{1-|\mu|^2} \leq \frac{1+k(f)}{1-k(f)}.$$

If $k(f) = k_0$, then these inequalities are equalities and hence $\mu\varphi/|\varphi| = k_0$, that is, $\mu = k_0|\varphi|/\varphi$ almost everywhere. $\square$

Exercise 8. *By the same technique as used in the previous theorem, except by using concentric circles centered at the origin in place of radial line segments, show that*

$$\frac{\Lambda(\Gamma)}{\Lambda(\Gamma^\mu)} \leq \int\int_{1<|z|<R} \frac{|1-\mu\varphi/|\varphi||^2}{1-|\mu|^2}|\varphi|dxdy.$$

Conclude that for small complex numbers t,

$$\log\Lambda(\Gamma^{t\mu}) = \log\Lambda(\Gamma) + 2\mathrm{Re}\left(t\int\int_{1<|z|<R}\mu\varphi dxdy\right) + o(t),$$

where $\Gamma^{t\mu} = f_t(\Gamma)$ and f_t satisfies the Beltrami equation $(f_t)_{\overline{z}} = t\mu(f_t)_z$ (see the next section).

Exercise 9. *Consider a complex number μ in the unit disc and a number $M >$ 0. Show that*

$$\frac{|1-\mu e^{-i\theta}|^2}{1-|\mu|^2} = M$$

is the equation of a circle inside the unit disc and tangent to the unit circle with diameter $2M/(M+1)$ and tangent to the unit circle at the point $e^{i\theta}$.

Exercise 10. *Consider the map f mapping the unit square onto a figure distorted by dyadic translations along subrectangles of the unit square whose vertical sides are located on the gaps of a standard middle-thirds Cantor set on the unit interval of the imaginary axis. On the largest subrectangle f is defined by the translation $z \mapsto z+1/2$ and on the two rectangles at the next two levels of the Cantor set f is defined by $z \mapsto z+1/4$ and $z \mapsto z+3/4$. The definition of f should be clear from the following picture:*

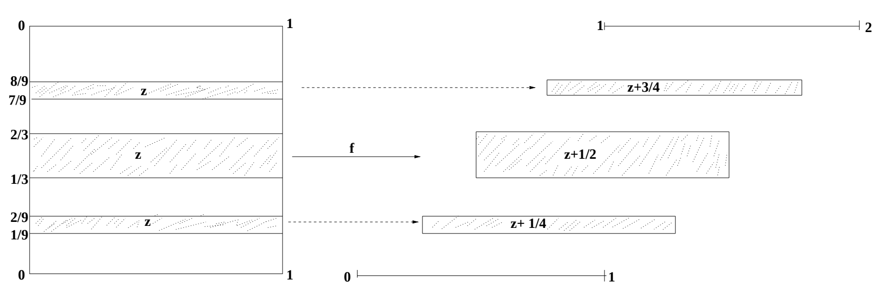

Fig 1

In figure 1 the scaling on the y-axis is different from the scaling on the x-axis so the unit square appears as a rectangle. Show that f is a homeomorphism onto its image, and in fact, f is Hölder continuous with Hölder exponent $\alpha = \log 2/\log 3$. Furthermore, show that f is not quasiconformal by finding a sequence of curve families Γ_n such that there is no finite value K for which

$$\Lambda(f(\Gamma_n)) \leq K\Lambda(\Gamma_n).$$

Hint: Let Γ_1 be the curve family of arcs lying in the rectangle $\{0 \leq x \leq 1, 2/9 \leq y \leq 1/3\}$ with initial point at $(x, 2/9)$ and endpoint at $(x, 1/3)$. If β is such an arc, then $f(\beta)$ has initial point $(x, 2/9)$ and endpoint $(x + 1/2, 1/3)$. Consider the analogous curve families Γ_n lying in the rectangles

$$\{0 \leq x \leq 1, 1/3 - 1/(3^{n+1}) \leq y \leq 1/3\}.$$

Proposition 2. *Suppose f is a C^1-quasiconformal mapping with dilatation K taking a Riemann surface X onto a Riemann surface X' and taking a curve family Γ onto a curve family Γ'. Then $K^{-1}\Lambda(\Gamma) \leq \Lambda(f(\Gamma)) \leq K\Lambda(\Gamma)$.*

Proof. Let w be a local parameter on X' and z be a local parameter on X and assume the mapping f takes z to w. Then for any allowable metric ρ on X,

$$\tilde{\rho}(w) = \left| \frac{\rho(z)}{|w_z| - |w_{\overline{z}}|} \right|$$

defines an allowable metric on X'. Then for $\gamma' = w(\gamma)$ and $w = u + iv = w(z)$, we have

$$\int_{\gamma'} \tilde{\rho}(w)|dw| = \int_{\gamma} \left| \frac{\rho(z)}{|w_z| - |w_{\overline{z}}|} \right| |w_z dz + w_{\overline{z}}\overline{dz}|$$

$$= \int_{\gamma} \left| \frac{\rho(z)}{|w_z| - |w_{\overline{z}}|} \right| |w_z + w_{\overline{z}}\frac{\overline{dz}}{dz}||dz| \geq \int_{\gamma} \rho(z)|dz|$$

and

$$\int\int_{X'} \tilde{\rho}(w)^2 dudv = \int\int_{X} \rho(z)^2 \frac{|w_z| + |w_{\overline{z}}|}{|w_z| - |w_{\overline{z}}|} dxdy \leq KA(\rho).$$

This means that we can transport any metric ρ on X to a metric $\tilde{\rho}$ on X' in such a way that curves γ are transported to curves γ' that are at least as long, and such that the area of X' with respect to $\tilde{\rho}$ is no larger than K times the area of X with respect to ρ. Therefore, $\Lambda(\Gamma) \leq K\Lambda(f(\Gamma))$ and the reverse inequality follows by applying the same argument to the inverse mapping. □

4 The Beltrami equation

Suppose that Ω is a plane domain and $\mu(z)$ is a continuous complex valued function defined on Ω with $|\mu(z)| < 1$. Consider the equation

$$f_{\overline{z}}(z) = \mu(z) f_z(z). \tag{6}$$

(6) is called the Beltrami equation, a mapping f satisfying (6) is called a solution, and μ is called the Beltrami coefficient or the complex dilatation of f. We now suppose local homeomorphic solutions are known to exist in a neighborhood of every point z in Ω.

Exercise 11. *Show that the set of such solutions define a Riemann surface structure on Ω. That is, show that transition functions are holomorphic.*

We remark that for each t with $|t| < 1/||\mu||_\infty$, the construction of the previous exercise creates a one-parameter family of Riemann surface structures on X which yield the original structure when $t = 0$.

Now, we consider the following special case of (6). The Beltrami coefficient has the special form $\mu(z) = k|\varphi(z)|/\varphi(z)$ where $\varphi(z)$ is a holomorphic function defined on a plane domain Ω and $0 < k < 1$.

Exercise 12. *Assume that $\varphi(z)$ is a holomorphic function defined on Ω. Show that (6) has a local homeomorphic solution when*

$$\mu(z) = k \frac{|\varphi(z)|}{\varphi(z)}$$

in a neighborhood of every point where $\varphi(z) \neq 0$ and that μ defines a Riemann surface structure on Ω.

Hint: Let $\zeta(z) = \int \sqrt{\varphi(z)} dz$, in a neighborhood of any point z where $\varphi(z) \neq 0$. ζ is defined by taking an anti-derivative of a power series for $\sqrt{\varphi(z)}$. Note that ζ is defined ambiguously up to a plus or minus sign and an additive constant, but $\zeta = \xi + i\eta$ is a local homeomorphism. The function ζ is called a natural parameter with respect to the quadratic differential $\varphi(z)(dz)^2$. Then form

$$f(z) = \zeta + k\overline{\zeta} = \int \sqrt{\varphi(z)} dz + k \overline{\int \sqrt{\varphi(z)} dz}.$$

Show that $f_{\overline{\zeta}}/f_\zeta = k$, that $f_{\overline{z}}/f_z = k\frac{|\varphi|}{\varphi}$ and that f is $K-$quasiconformal with $K = \frac{1+k}{1-k}$.

Exercise 13. *For a holomorphic quadratic differential $\varphi(z)(dz)^2$ defined on a Riemann surface X, the conditions $\varphi(z)(dz)^2 > 0$ and $\varphi(z)(dz)^2 < 0$ define parameterized curves on X called horizontal and vertical trajectories which are perpendicular. These trajectories are mapped to horizontal or vertical line segments in the $\zeta-$plane for any natural parameter ζ.*

Exercise 14. *In a neighborhood of $z = 0$ draw the critical horizontal trajectories of $z(dz)^2$, $z^2(dz)^2$, and $z^3(dz)^2$. In addition, draw regular horizontal trajectories which are near to $z = 0$. Tell how many prongs there are for the critical horizontal trajectories of $\varphi(z)(dz)^2 = z^n(dz)^2$ near to $z = 0$. Putting $\zeta = d\xi + id\eta = \int \sqrt{\varphi(z)}dz$, tell why it is not possible to find η as a single valued function of z in a neighborhood of $z = 0$ when n is an odd integer.*

Ultimately you will learn that equation (6) admits global homeomorphic solutions on the Riemann sphere even if $\mu(z)$ is assumed only to be measurable. That is, if $\Omega = \mathbb{C} \cup \{\infty\}$ and $\mu(z)$ is measurable and $||\mu||_\infty < 1$, then there is a global homeomorphism f of $\mathbb{C} \cup \{\infty\}$ satisfying (6), where the partial derivatives with respect to z and $\overline{z}$ are taken in the generalized sense. (See [29], [3],[23]). Moreover, if the solution $f^\mu(z)$ is normalized to fix three points on the Riemann sphere, say $0, 1$ and ∞, then it is unique and $w = f^\mu(z)$ is a holomorphic function of μ, ([12],[4]). This version of the existence theorem for equation (6) (with holomorphic dependence of the solution on the Beltrami coefficient) is sometimes called the *measurable Riemann mapping theorem* (denoted by MRMT). In this phrase the adjective measurable refers not to the theorem but to the coefficient in the Beltrami equation. Moreover, a central aspect of the theorem is not mentioned in its title, namely, the fact that dependence of the solution $w = f^\mu(z)$ on the coefficient μ is holomorphic.

Here we briefly outline the proof of MRMT with holomorphic dependence. It is based on properties of the following two singular integral operators:

$$P\mu(z) = -\frac{1}{\pi} \int \int \mu(\zeta) \left\{ \frac{1}{\zeta - z} - \frac{1}{\zeta} \right\} d\xi d\eta$$

and

$$T\mu(z) = -\frac{1}{\pi} \int \int \mu(\zeta) \left\{ \frac{1}{(\zeta - z)^2} - \frac{1}{\zeta^2} \right\} d\xi d\eta.$$

P is a smoothing operator in the sense that if $\mu(\zeta)$ has compact support and is in L^p with $p > 2$, then

$$|P\mu(z_1) - P\mu(z_2)| \leq C|z_1 - z_2|^{1-2/p},$$

and T preserves smoothness in the sense that for μ in $L^p, (p > 2)$ and with compact support

$$||T\mu||_p \leq C_p||\mu||_p,$$

with $C_p \to 1$ as p decreases to 2. Moreover, if partial derivatives are interpreted in the generalized sense, then

$$(P\mu(z))_{\overline{z}} = \mu(z) \tag{7}$$

and

$$(P\mu(z))_z = T\mu(z). \tag{8}$$

Then a (non-normalized) solution to (6) is given by

$$f^\mu(z) = z + P\mu(z) + P\mu T\mu(z) + P\mu T\mu T\mu(z) + \cdots . \tag{9}$$

We leave it to the reader to check that formal term-by-term differentiation of this infinite sum yields

$$(f^{\mu}(z))_{\overline{z}} = \mu(z) + \mu T\mu(z) + \mu T\mu T\mu(z) + \cdots =$$

$$\mu(z)\left(1 + T\mu(z) + T\mu T\mu(z) + T\mu T\mu T\mu(z) + \cdots\right),$$

and

$$(f^{\mu}(z))_{z} = 1 + T\mu(z) + T\mu T\mu(z) + T\mu T\mu T\mu(z) + \cdots .$$

Here, we make a few comments about the meaning of (9) and the structure of the proof. We assume that $||\mu||_{\infty} = k < 1$ and that μ has compact support. In (9) we view μT as the operator given by first applying the linear map T and afterwards multiplying by μ. Thus, for example, the expression

$$\mu T\mu T\mu(z)$$

means first apply T to μ, then multiply the resulting L_p function by μ, then apply T again, and finally multiply by μ. Moreover, the norm of μT as an operator satisfies $||\mu T||_p \leq C_p k$. Since $k < 1$ and for p larger than 2 but sufficiently close to 2, C_p approaches 1, $C_p k$ is less than 1. Therefore, μT is an operator with norm less than 1, and $(I - \mu T)^{-1}$ is a bounded operator on L_p. So the solution (9) can be rewritten as a perturbation of the identity:

$$f^{\mu}(z) = z + P((I - \mu T)^{-1})(\mu).$$

Note that the final application of P smooths the L_p-function $(I - \mu T)^{-1}(\mu)$ and so the resulting solution is Hölder continuous with Hölder exponent $1 - 2/p$. Several more steps are involved in showing that the series yields a homeomorphism of the Riemann sphere for arbitrary μ in L_∞ with $||\mu||_\infty < 1$, and that the operators P and T satisfy properties (7) and (8) for generalized partial derivatives. For these steps we refer to [3] and [29].

We point out that if μ is replaced by $t\mu$ for a complex number t, then the solution $f^{t\mu}(z)$ is exhibited as a power series in t convergent for $|t| < 1/||\mu||_\infty$, its derivative at $t = 0$ is the first term of the series, namely, the vector field $V(z)\frac{\partial}{\partial z}$. When the solution is normalized so that $f^{t\mu}(z)$ fixes the points $0, 1$ and ∞, then this vector field vanishes at those points and is given by the formula

$$\begin{aligned} V(z) &= -\frac{1}{\pi}\int\int \mu(\zeta)\left\{\frac{1}{\zeta - z} - \frac{z}{\zeta - 1} + \frac{z-1}{\zeta}\right\} d\xi d\eta \\ &= -\frac{z(z-1)}{\pi}\int\int \mu(\zeta)\left\{\frac{1}{\zeta(\zeta-1)(\zeta-z)}\right\} d\xi d\eta. \qquad (10) \end{aligned}$$

5 The Reich-Strebel Inequality and Teichmüller's Uniqueness Theorem

To generalize the example of the previous section, we assume $f_0 : X \to X^\mu$ is a mapping satisfying the Beltrami equation

$$f_{\overline{z}}(z) = \mu(z) f_z(z),$$

where $\mu = k_0|\varphi_0|/\varphi_0$ and φ_0 is a holomorphic quadratic differential on X with $\int\int_X |\varphi_0| dxdy = 1$. Such a Beltrami coefficient is said to be of Teichmüller form. We also assume that f is another mapping from X to X^μ which is homotopic to f_0.

For non-compact Riemann surfaces X the form of homotopy we require needs to be clarified. We require that there exists a map $f_t(z)$ continuous simultaneously in both variables such that it is equal to f_0 for $t = 0$ and equal to f for $t = 1$, and $f_0(p) = f_t(p) = f(p)$ for every t with $0 \leq t \leq 1$ and every boundary point p of X. For now we will assume that X is compact (and so has no boundary) or a punctured compact surface with only finitely many punctures. In the later case, the boundary of X consists of finitely many points.

Just as in the case of the annulus, we will obtain the following Reich-Strebel inequality

$$K_0 \leq \int\int_X \frac{|1 + \nu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\nu|^2} |\varphi_0| dxdy, \tag{11}$$

where ν is the Beltrami coefficient of any competing map f. In this section, we first show how this inequality implies the next theorem.

Theorem 1 (**Teichmüller's Uniqueness Theorem).** *If a quasiconformal mapping f_0 from a Riemann surface X to another Riemann surface X' has Beltrami coefficient of Teichmüller form $k_0|\varphi_0|/\varphi_0$ for some holomorphic quadratic differential φ_0 with $||\varphi_0|| = 1$ and $0 < k_0 < 1$, then any other quasiconformal mapping f^μ in the same homotopy class either has the same Beltrami coefficient or $||\mu||_\infty > k_0$. Moreover, if f_0 and f_1 are two mappings with Beltrami coefficients of Teichmüller form $k_0|\varphi_0|/\varphi_0$ and $k_1|\varphi_1|/\varphi_1$, then $k_0 = k_1$ and $\varphi_0 = \varphi_1$.*

Proof. Clearly,

$$\frac{|1 + \mu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\mu|^2} \leq \frac{(1 + k(f))^2}{1 - k(f)^2} = \frac{1 + k(f)}{1 - k(f)}. \tag{12}$$

By the Reich-Strebel inequality (11) and the above estimate, we obtain

$$K_0 \leq \int\int_X \frac{|1 + \mu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\mu|^2} |\varphi_0| dxdy \leq K(f^\mu), \tag{13}$$

that is,

$$\frac{1 + k_0}{1 - k_0} \leq \int\int_X \frac{|1 + \mu \frac{\varphi_0}{|\varphi_0|}|^2}{1 - |\mu|^2} |\varphi_0| dxdy \leq \frac{1 + k(f)}{1 - k(f)}.$$

Therefore, if f^μ is also extremal in the same class, then $k(f) = k_0$ and hence these inequalities are equalities, which forces $\mu\varphi_0/|\varphi_0| = k_0$, that is, $\mu = k_0|\varphi_0|/\varphi_0$ almost everywhere. $\square$

Before starting out the work of proving (11), we take note of two closely related inequalities.

Theorem 2. *Suppose f is a quasiconformal self map of X homotopic to the identity on X and φ is any holomorphic quadratic differential on X with*

$$||\varphi|| = \int\int_X |\varphi| dxdy = 1.$$

Then

$$1 \leq \int\int_X \frac{|1 + \mu \frac{\varphi}{|\varphi|}|^2}{1 - |\mu|^2} |\varphi| dxdy. \tag{14}$$

We note that (14) is the special case of (11) when $K_0 = 1$, and we postpone the proof until the end of section 7. For now we show only how Theorem 2 implies another inequality, which is a version of *The Main Inequality of Reich and Strebel* and which implies (11).

Theorem 3. *Let f and g be two quasiconformal maps from X to Y, with Beltrami differential coefficients μ_f and μ_g. Assume $g \circ f^{-1}$ is homotopic to the identity on Y, $w = u + iv = f(z)$, and let $\psi(w)(dw)^2$ be an integrable, holomorphic, quadratic differential on Y with $\int\int_Y |\psi| dudv = 1$. Then*

$$1 \leq \int\int_Y \frac{\left|1 + \mu_g \frac{1}{\theta} \frac{\psi}{|\psi|}\right|^2}{1 - |\mu_g|^2} \frac{\left|1 - \mu_f \frac{1}{\theta} \frac{\psi}{|\psi|} \left(\frac{1 + \mu_g \frac{\overline{\mu_f}}{\mu_f} \frac{\overline{\psi}}{|\psi|}}{1 + \mu_g \frac{1}{\theta} \frac{\psi}{|\psi|}} \right)\right|^2}{1 - |\mu_f|^2} |\psi(w)| dudv. \tag{15}$$

where $\theta = \overline{p}/p$ and $p = f_z$.

Proof. Note that

$$\mu_{g \circ f^{-1}} = \left[\frac{\mu_g - \mu_f}{1 - \overline{\mu_f}\mu_g} \cdot \frac{1}{\theta} \right] \circ f^{-1}. \tag{16}$$

On using (14) for the surface Y and replacing μ by (16), for $||\psi|| = 1$ we obtain

$$1 \leq \int\int_Y \frac{|1 + \sigma \frac{\psi}{|\psi|}|^2}{1 - |\sigma|^2} |\psi(w)| dudv. \tag{17}$$

where

$$\sigma(w) = \left[\frac{\mu_g - \mu_f}{1 - \overline{\mu_f}\mu_g} \cdot \frac{1}{\theta} \right] \circ f^{-1}(w).$$

(17) simplifies to

$$1 \leq \int\int_Y \frac{\left|(1 - \overline{\mu_f}\mu_g) + (\mu_g - \mu_f) \frac{1}{\theta} \frac{\psi}{|\psi|}\right|^2}{|1 - \overline{\mu_f}\mu_g|^2 - |\mu_g - \mu_f|^2} |\psi(w)| dudv. \tag{18}$$

The part inside the absolute value in the numerator of this integrand can be rewritten in the following way

$$(1-\overline{\mu_f}\mu_g)+(\mu_g-\mu_f)\frac{1}{\theta}\frac{\psi}{|\psi|}=\left[1+\mu_g\frac{1}{\theta}\frac{\psi}{|\psi|}\right]\left[1-\mu_f\frac{1}{\theta}\frac{\psi}{|\psi|}\left(\frac{1+\mu_g\frac{\overline{\mu_f}}{\mu_f}\frac{\overline{\psi}}{|\psi|}}{1+\mu_g\frac{1}{\theta}\frac{\psi}{|\psi|}}\right)\right]$$

Therefore (18) can be rewritten as

$$1\leq\int\int_Y\frac{\left|1+\mu_g\frac{1}{\theta}\frac{\psi}{|\psi|}\right|^2}{1-|\mu_g|^2}\cdot\frac{\left|1-\mu_f\frac{1}{\theta}\frac{\psi}{|\psi|}\left(\frac{1+\mu_g\frac{\overline{\mu_f}}{\mu_f}\frac{\overline{\psi}}{|\psi|}}{1+\mu_g\frac{1}{\theta}\frac{\psi}{|\psi|}}\right)\right|^2}{1-|\mu_f|^2}\cdot|\psi(w)|dudv. \tag{19}$$

□

We now go to the proof of (11). We assume $w=f(z)$ is of Teichmüller form, with Beltrami coefficient $k\frac{|\varphi_0(z)|}{\varphi_0}$. This means that in terms of $\zeta=\int\sqrt{\varphi_0(z)}dz$, $f(\zeta)=\tilde{\zeta}=\tilde{\xi}+i\tilde{\eta}=K^{1/2}\xi+iK^{-1/2}\eta$ and the push-forward of φ_0 under f is a holomorphic quadratic differential on X' of the form $\psi_0(\tilde{\zeta})(d\tilde{\zeta})^2=(d\tilde{\zeta})^2$ with norm also equal to 1. Note that in terms of the natural parameters ζ, and $\tilde{\zeta}$, $\theta\equiv 1,\psi\equiv 1$ and $\mu_f\equiv k$. Applying (19) to this mapping f and to the quadratic differential $\psi_0(\tilde{\zeta})(d\tilde{\zeta})^2$, (19) becomes

$$1\leq\int\int_Y\frac{\left|1+\mu_g\circ f^{-1}(\tilde{\zeta})\right|^2}{1-|\mu_g\circ f^{-1}(\tilde{\zeta})|^2}\cdot\frac{(1-k)^2}{1-k^2}d\tilde{\xi}d\tilde{\eta}$$

$$=\frac{1}{K}\int\int_X\frac{|1+\mu_g(\zeta)|^2}{1-|\mu_g(\zeta)|^2}d\xi d\eta=\frac{1}{K}\int\int_X\frac{\left|1+\mu_g(z)\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|^2}{1-|\mu_g|^2}|\varphi_0(z)|dxdy,$$

and then we obtain (11).

6 The Minimum Norm Principle

Now suppose we are given two integrable quadratic differentials φ and ψ. Also suppose φ is holomorphic and ψ is locally L_1 with respect to $d\xi d\eta$, where $\zeta=\xi+i\eta$, the natural parameter measure for φ. For any vertical trajectory β of φ, we can form $h_\psi(\beta)=\int_\beta|Im(\sqrt{\psi(z)}dz)|$, where the integral is unoriented and taken in the positive sense regardless of the orientation of β. In the unusual circumstance that β runs along a closed trajectory, by $h_\psi(\beta)$ we mean the arclength integral of $|Im(\sqrt{\psi(z)}dz)|$ along the entire parametrization of β.

Theorem 4 (**The Minimum Norm Principle**). *Assume X is a Riemann surface of finite analytic type and φ is a holomorphic quadratic differential on X with $||\varphi||=\int\int_X|\varphi|dxdy<\infty$. Let ψ be another locally integrable quadratic* *differential and assume there is a constant M such that for every non-critical vertical segment β for φ, one has $h_\varphi(\beta) \leq h_\psi(\beta) + M$. Then*

$$||\varphi|| \leq \int\int_X |\sqrt{\varphi}||\sqrt{\psi}|dxdy.$$

Proof. As usual, we let $\zeta = \xi + i\eta$ be a natural parameter for φ, that is, $(d\zeta)^2 = \varphi(z)(dz)^2$. Put

$$g(p) = \int_{\beta_p} |Im(\sqrt{\psi(\zeta)}d\zeta)|,$$

where β_p is a vertical segment for φ with height b and midpoint p. Since the number of zeroes of φ on X is finite, the number of vertical trajectories of φ which lead into these zeroes is countable. Therefore, the set of regular vertical trajectories of φ is of full measure and g is defined on a set of full measure in X. We wish to establish the following inequality.

$$\int\int_X g(p)d\xi d\eta \leq b\int\int_X |\sqrt{\psi(\zeta)}|d\xi d\eta. \tag{20}$$

All that is required is a change of order of integration with respect to $d\eta$ along any regular vertical trajectory of φ. Since the transformation $\zeta = \xi + i\eta \mapsto \xi + i(\eta + t)$ has Jacobian identically equal to 1, this change of variable is justified so long as we can view this transformation as an almost everywhere defined self-mapping of X. However, since trajectories of a quadratic differential do not in general have globally defined orientation on X, it is not possible to distinguish between the maps $\zeta = \xi + i\eta \mapsto \xi + i(\eta + t)$ and $\zeta = \xi + i\eta \mapsto \xi - i(\eta + t)$. To avoid this problem, we pass to the double cover $\tilde{X}$ of X where $\sqrt{\varphi}$ can be globally defined, and where the norm of the lifted quadratic differential $|\tilde{\varphi}|$ on $\tilde{X}$ is exactly twice norm of φ on X. After these steps we end up with (20).

To go on with the proof, we now invoke the hypothesis that $b - M \leq \int_{\beta_p} |Im(\sqrt{\psi}dz)|$, and so $b - M \leq g(p)$ for almost all p. From (20) this implies

$$(b - M)\int\int_X d\xi d\eta \leq b\int\int_X |\sqrt{\psi(\zeta)}|d\xi d\eta,$$

and dividing both sides by b and taking the limit as $b \to \infty$, we obtain

$$\int\int_X d\xi d\eta \leq \int\int_X |\sqrt{\psi(\zeta)}|d\xi d\eta,$$

where $\psi(\zeta)$ is the expression for the quadratic differential ψ in terms of the local parameter ζ. Since $d\zeta = \sqrt{\varphi(z)}dz$,

$$d\xi d\eta = \frac{1}{2}|d\zeta d\bar{\zeta}| = \frac{1}{2}|\varphi dz d\bar{z}| = |\varphi|dxdy,$$

and

$$\sqrt{\psi(\zeta)} = \frac{\sqrt{\psi(z)}}{\sqrt{\varphi(z)}},$$

by changing from the natural parameter ζ to the local parameter z, we obtain the desired inequality, namely,

$$||\varphi|| \leq \int\int_X |\sqrt{\varphi}||\sqrt{\psi}|dxdy.$$

□

7 The Heights Argument

Lemma 3. *Let φ be a holomorphic quadratic differential on X with $||\varphi|| < \infty$. Let f be a quasiconformal self-mapping of X which is homotopic to the identity. Then there exists a constant M such that for every non-critical vertical segment β of φ, one has*

$$h_\varphi(\beta) \leq h_\varphi(f(\beta)) + M.$$

the constant M depends on φ and f but not on β.

Proof. Let $\overline{X}$ be the completion of X with the n punctures filled in. So $\overline{X}$ is compact with no punctures, and since f is quasiconformal, it extends to a quasiconformal map fixing the punctures. By hypothesis this extension is homotopic to the identity by a homotopy f_t which fixes the punctures pointwise. The infinitesimal form $|\varphi|^{1/2}|dz|$ determines a finite valued metric on X. To see that the distance from a point in X to a puncture is finite, one observes that φ has at most simple poles and so to find the length of a short arc ending at a puncture, one has to calculate an integral of the form $\int_0^a t^{-1/2}dt$, and this clearly converges.

Let f_t be the homotopy connecting f to the identity, so $f_0(p) = p$ and $f_1(p) = f(p)$. Let $l(p)$ be the infimum of the $\varphi-$lengths of all curves which go from p to $f(p)$ and which are homotopic with fixed endpoints to the curve $t \mapsto f_t(p)$. Clearly, $l(p)$ is a continuous function on the compact set $\overline{X}$. Let M_1 be the maximum of this function.

Let β be a noncritical vertical segment for φ with endpoints p and q. The segment β and the curve which consists of a curve homotopic to $t \mapsto f_t(p)$ followed by $f(\beta)$ and then followed by a curve homotopic to $t \mapsto f_{1-t}(q)$ is homotopic to β with fixed endpoints. From the following lemma it will follow that

$$h_\varphi(\beta) \leq h_\varphi(f_t(p)) + h_\varphi(f(\beta)) + h_\varphi(f_{1-t}(q)).$$

Since the first and third terms of this inequality are each bounded by M_1, the lemma follows with $M = 2M_1$. □

Lemma 4. *Let φ be a holomorphic quadratic differential on X. Suppose β is a segment of a vertical trajectory of φ and that γ is an arc in X with the same endpoints as β such that the arc β followed by the arc γ^{-1} forms a closed curve homotopic to a point. Then $h_\varphi(\beta) \leq h_\varphi(\gamma)$.*

Proof. It is sufficient to prove this inequality for lifts of the arcs β and γ in the universal cover $\tilde{X}$. If β followed by γ^{-1} forms a closed curve in X homotopic to a point, the lifts of these curves form a closed curve in $\tilde{X}$ contractible to a point in $\tilde{X}$. Because there can be only finitely many singularities of φ in a bounded simply connected domain containing this curve, β may be subdivided into a finite number of subintervals such that horizontal strips emanating from these subintervals must be crossed by γ. Since the height of the strip is constant, the height of β along each strips is less than or equal to the height of γ. By adding up over all of the strips, the lemma follows. $\square$

Now our approach to obtain Theorem 2 is to apply the minimum norm principle of the previous section to the quadratic differential

$$\psi(z) = \varphi(f(z))f_z^2(z)\left(1 - \mu\varphi/|\varphi|\right)^2 . \tag{21}$$

We leave it to the reader to check that this is a quadratic differential, and go on to show that for any vertical segment β relative to φ,

$$h_\psi(\beta) = h_\varphi(f(\beta)).$$

By definition,

$$h_\varphi(f(\beta)) = \int_{f(\beta)} |Im(\sqrt{\varphi(f)}df)| = \int_\beta |Im\left(\sqrt{\varphi(f(z))}f_z(1 + \mu(z)(d\overline{z}/dz))dz\right)|.$$

Note that for a natural parameter ζ, $d\zeta = \sqrt{\varphi(z)}dz$, and for a parametrization $\zeta = it$ of the vertical trajectory β, $\frac{\overline{d\zeta}}{d\zeta} = -1$. Therefore, $d\overline{z}/dz = -\varphi/|\varphi|$ along β. The final result is that $h_\varphi(f(\beta)) = h_\psi(\beta)$. Therefore, we obtain the following lemma.

Lemma 5. *There exists a constant M such that for all regular vertical trajectories β of φ,*

$$h_\varphi(\beta) \leq h_\psi(\beta) + M.$$

Proof. Merely apply the previous lemma and the previously derived equality $h_\varphi(f(\beta)) = h_\psi$. $\square$

Now we are ready to prove inequality (14) in Theorem 2. From the minimum norm principle,

$$||\varphi|| \leq \int\int |\varphi(f(z))|^{1/2}|f_z| \;\; |1 - \mu\varphi/|\varphi|| \;\; |\varphi|^{1/2}dxdy.$$

Introducing a factor of $(1 - |\mu|^2)^{1/2}$ into the numerator and denominator and applying Schwarz's inequality yields

$$||\varphi|| \leq \left(\int\int |\varphi(f(z))||f_z|^2(1 - |\mu|^2)dxdy\right)^{1/2} \left(\int\int \frac{|1 - \mu\varphi/|\varphi||^2}{1 - |\mu|^2}|\varphi|dxdy\right)^{1/2}.$$

Since the first factor on the right hand side of this expression is simply equal to $||\varphi||^{1/2}$, we obtain

$$1 \leq \int\int \frac{|1-\mu\varphi/|\varphi||^2}{1-|\mu|^2}|\varphi|dxdy.$$

Replacing φ by $-\varphi$, we obtain (14).

8 The Hamilton-Krushkal condition and Teichmüller's Existence Theorem

Suppose $f : X \to Y$ is a quasiconformal map between Riemann surfaces and $f_{\overline{z}}(z)/f_z(z) = \mu(z)$ is its Beltrami differential coefficient. We shall say f is extremal in its homotopy class if no other mapping in the same class has smaller dilatation. If we let

$$K = \inf\{K(g) : \text{ where } g \text{ is in the homotopy class of } f\},$$

then by definition there is a sequence of mappings g_n in the homotopy class of f with $K(g_n) < K + 1/n$. The sequence of the maps g_n is uniformly Hölder continuous, and therefore it is a normal family. Let g be a limit of a subsequence of $\{g_n\}_{n=1}^{\infty}$. Then g, is also Hölder continuous and $K(g) \leq K$. Moreover, g is in the same homotopy class as that of f because homotopy equivalence is determined by the images of closed curves γ on X. That is to say, two maps g_0 and g_1 are homotopic if g_0 and g_1 take the same values at the punctures and if for every closed curve γ on X, $g_0(\gamma)$ is freely homotopic to $g_1(\gamma)$ in Y. Thus the limit g must be in the same homotopy class of f. We conclude that $K(g) = K$ and every homotopy class of a quasiconformal homeomorphism f has an extremal representative, which is the existence for Teichmüller's Theorem.

The more interesting and useful part of Teichmüller's Theorem is to find an explicit description of the unique extremal mapping, this is the interpretation through stretching and shrinking of the horizontal and vertical trajectories of a holomorphic quadratic differential. In other words, the Beltrami coefficient of the extremal mapping is of Teichmüller form. By pairing Beltrami differentials with holomorphic quadratic differential, Hamilton and Krushkal obtained the following useful condition.

Theorem 5. [21][25] [The Hamilton-Krushkal Condition] *If f^{μ} is extremal in its homotopy class, then*

$$k = (K-1)/(K+1) = ||\mu||_{\infty} = \sup Re \int\int_X \mu\varphi dxdy, \tag{22}$$

where the supremum is taken over all holomorphic quadratic differentials φ with $||\varphi|| = \int\int_X |\varphi|dxdy = 1$.

It is clear that for any μ,

$$\sup_{||\varphi||=1} Re \int\int_X \mu\varphi dxdy \leq ||\mu||_\infty.$$

We say that μ satisfies the Hamilton-Krushkal condition if the equality holds.

Proof. Start by assuming $k = ||\mu||_\infty > k_0$, where k_0 is the supremum in (22). By the Hahn-Banach and the Riesz representation theorems, there exists an $L_\infty-$complex valued function ν such that

$$\int\int_X \mu\varphi dxdy = \int\int_X \nu\varphi dxdy$$

such that $||\nu||_\infty = k_0$, the supremum in (22). Hence $\mu - \nu$ is infinitesimally trivial, and we shall show in the next section that this implies there exists a smooth curve σ_t of Beltrami coefficients such that

$$||\sigma_t - t(\mu - \nu)||_\infty = O(t^2),$$

and such that for each t, f^{σ_t} is a self-map of X homotopic to the identity. For brevity let $\sigma_t = \sigma$ and form $f^\tau = f^\mu \circ (f^\sigma)^{-1}$. Clearly, f^τ is in the same homotopy class as μ, and we will show that, for sufficiently small $t > 0$, $||\tau||_\infty < ||\mu||_\infty$, which contradicts with the assumption that μ is extremal in its class and completes the proof of the theorem. Note that

$$\tau(f^\sigma(z)) = \frac{\mu - \sigma}{1 - \overline{\sigma}\mu} \cdot \frac{1}{\theta}, \tag{23}$$

where $\theta = \overline{p}/p$ and $p = f^\sigma_z$. This implies

$$|\tau \circ f^\sigma|^2 = \frac{|\mu|^2 - 2 \text{ Re } \mu\overline{\sigma} + |\sigma|^2}{1 - 2 \text{ Re } \mu\overline{\sigma} + |\mu\sigma|^2},$$

which gives

$$|\tau \circ f^\sigma| = |\mu| - \frac{1 - |\mu|^2}{|\mu|} \text{ Re } (\mu\overline{\sigma}) + O(t^2).$$

Replacing σ by $t(\mu - \nu)$, we obtain

$$|\tau \circ f^\sigma| = |\mu| - t\frac{1 - |\mu|^2}{|\mu|} \text{ Re } (|\mu|^2 - \mu\overline{\nu}) + O(t^2). \tag{24}$$

Since $k_0 = ||\nu||_\infty < k = ||\mu||_\infty$, by putting

$$S_1 = \{z \in X : |\mu(z)| \leq (k + k_0)/2\}, \text{ and}$$

$$S_2 = \{z \in X : (k + k_0)/2 < |\mu(z)| \leq k\},$$

we obtain $S_1 \cup S_2 = X$ and (23) implies there exist $\delta_1 > 0$ and $c_1 > 0$ such that for $0 < t < \delta_1$,

$$|\tau \circ f^\sigma(z)| \leq k - c_1 t \quad \text{for } z \text{ in } S_1.$$

For z in S_2 the coefficient of t in (24) is bounded below by

$$\frac{1-k^2}{k} \cdot \left[\left(\frac{k+k_0}{2} \right)^2 - k_0 k \right] = \frac{1-k^2}{k} \cdot \left(\frac{k-k_0}{2} \right)^2 > 0.$$

Therefore, (24) implies there exist $\delta_2 > 0$ and $c_2 > 0$ such that for $0 < t < \delta_2$,

$$|\tau \circ f^\sigma| \leq k - c_2 t \text{ for } z \text{ in } S_2.$$

Putting these two statements for S_1 and for S_2 together, we find that $||\tau||_\infty < k$ for sufficiently small $t > 0$, and this proves the theorem. □

Theorem 6. *Suppose X is a Riemann surface of finite analytic type and assume that it is given the finite dimensionality of the linear space holomorphic quadratic differentials on X. Let f be a quasiconformal map from X to another Riemann surface Y. Then in the homotopy class of f there exists an extremal representative f_0 homotopic to f and its Beltrami coefficient is of the form $k_0|\varphi_0|/\varphi_0$. The constant k_0 is uniquely determined by f and if $k_0 > 0$, then φ is uniquely determined up to positive multiple.*

Proof. By the remark preceding Theorem 5 we know that in any homotopy class of quasiconformal mappings from one surface to another there exists an extremal representative. We also know from Theorem 5 that the extremality of f in its Teichmüller class forces the linear condition

$$||\mu||_\infty = \sup Re \int\int_X \mu\varphi dxdy,$$

where the supremum is taken over all holomorphic quadratic differentials with $\int\int |\varphi| dxdy = 1$. By the finite dimensionality of the space of integrable quadratic differentials on X, there is a quadratic differential φ_0 with $||\varphi_0|| = 1$ such that

$$\sup Re \int\int_X \mu\varphi dxdy = Re \int\int_X \mu\varphi_0 dxdy,$$

and then

$$\int\int [||\mu||_\infty |\varphi_0| - Re(\mu\varphi_0)] = 0. \tag{25}$$

Since

$$|\mu\varphi_0| \leq ||\mu||_\infty |\varphi_0|,$$

the equality (25) forces that $\mu = ||\mu||_\infty \frac{|\varphi_0|}{\varphi_0}$ almost everywhere. □

9 Trivial and Infinitesimally Trivial Beltrami Differentials

Let $M(X)$ be the space of Beltrami coefficients μ with $||\mu||_\infty < 1$. Since $M(X)$ is the open unit ball in a complex Banach space, it is also a complex manifold.

The natural map Φ from $M(X)$ onto $T(X)$ takes a Beltrami coefficient μ onto its Teichmüller equivalence class $\Phi(\mu)$, and the complex structure on $T(X)$ is the one which makes Φ holomorphic.

The preimage under Φ of the Teichmüller class of the identity corresponds to those mappings f from X to X' for which there is a conformal map c from X' to X such that $c \circ f$ is a quasiconformal self-mapping of X homotopic to the identity. We denote by $M_0(X)$ the subset in $M(X)$ consisting of of Beltrami coefficients of such mappings. It is called the space of trivial Beltrami coefficients. By Theorem 2 every Beltrami coefficient in $M_0(X)$ satisfies inequality (14). Note that by changing φ to $-\varphi$, the numerator in the integrand of the integral in the right hand side of (14) (at the end of Section 7) can be replaced by

$$\left|1 - \mu \frac{\varphi}{|\varphi|}\right|^2,$$

and multiplying this by $|\varphi|$, we obtain

$$\left|1 - \mu \frac{\varphi}{|\varphi|}\right|^2 |\varphi| = |\varphi|(1 - |\mu|^2) - 2\mathrm{Re}\ \mu\varphi + 2|\mu|^2|\varphi|.$$

Since $\int\int_X |\varphi| dxdy = 1$, from (14) we obtain

$$\mathrm{Re} \int\int_X \frac{\mu\varphi}{1 - |\mu|^2} dxdy \leq \int\int_X \frac{|\mu|^2|\varphi|}{1 - |\mu|^2} dxdy. \tag{26}$$

Since this form of the inequality is unchanged if φ is multiplied by a positive number, in (26) we only need to impose the condition that $||\varphi|| = \int\int_X |\varphi| dxdy < \infty$. By reversing these steps, it is also clear that (26) implies (14).

Now suppose μ_t is a smooth curve of Beltrami coefficients passing through the zero Beltrami coefficient at time $t = 0$ with tangent vector ν, that is, suppose

$$||\frac{\mu_t - t\nu}{t}||_\infty \to 0 \tag{27}$$

as $t \to 0$. By letting t be a real parameter and a purely imaginary parameter respectively and calculating the first order variation in inequality (26), we obtain that if ν is a tangent vector to a μ_t lying in $M_0(X)$, then

$$\int\int_X \nu\varphi dxdy = 0 \tag{28}$$

for all holomorphic quadratic differentials φ with $||\varphi|| < \infty$.

Definition. A Beltrami differential ν is called infinitesimally trivial if it satisfies (28) for all integrable holomorphic quadratic differentials φ.

We have already seen that the burden of proof of the Hamilton-Krushkal condition in Theorem 5 is to show the converse of (27), that is, we need the following theorem.

Theorem 7. *If a Beltrami differential ν is infinitesimally trivial, then there exists a curve of trivial Beltrami coefficients μ_t passing through the origin at time $t = 0$ with the property that*

$$||\frac{\mu_t - t\nu}{t}||_\infty \to 0$$

as $t \to 0$.

Proof. There are many references for this well-known theorem, [1, 3, 6, 9, 19, 20, 33, 34]. We first give a proof in the case where X is the Riemann sphere with n points removed. After treating this case, we outline the proof for any surface of genus g with n punctures, provided that $3g - 3 + n > 0$.

The case X has genus zero:

Note that in the genus zero case, the Möbius transformations are triply-transitive, and so any two such surfaces are conformal if $n \leq 3$, and in that case the Teichmüller space of X consists of just one point, $M_0(X) = M(X)$, and there is nothing to prove. If n is 4 or more, by the action of a Möbius transformation we can move three of the points to $0, 1$ and ∞, and label the remaining $n - 3$ points by $p_1, \ldots, p_{n-3}$. Thus $X = \mathbb{C} \setminus \{0, 1, p_1, \ldots, p_{n-3}\}$, and an arbitrary integrable, holomorphic quadratic differential on X has the form

$$\varphi(z)(dz)^2 = \frac{a_0 + a_1 z + \cdots + a_{n-4} z^{n-4}}{z(z-1)(z-p_1)\cdots(z-p_{n-3})}(dz)^2, \tag{29}$$

where $a_0, \ldots, a_{n-4}$ are arbitrary complex numbers. This quadratic differential has at most simple poles at n points, namely, $0, 1, p_1, \ldots, p_{n-3}$ and at ∞. To see the order of the pole at ∞ we need to express $\varphi(z)(dz)^2$ in terms of the local parameter $w = 1/z$ which vanishes at $z = \infty$. Using the equation $\varphi^z(z)(dz)^2 = \varphi^w(w)(dw)^2$, we obtain

$$\varphi^w(w) = \frac{1}{w^4} \cdot \frac{a_0 + a_1(1/w) + \cdots + a_n(1/w)^{n-4}}{1/w((1/w)-1)((1/w)-p_1)\cdots((1/w)-p_{n-3})}.$$

Multiplying both the numerator and denominator by w^{n-4}, we see that

$$\varphi^w(w) = \frac{1}{w} \cdot \frac{a_0 w^{n-4} + a_1 w^{n-5} + \cdots + a_n}{1(1-w)(1-wp_1)\cdots(1-wp_{n-3})},$$

which has a simple pole at $w = 0$ unless $a_n = 0$, in which case it is holomorphic at $w = 0$. Since $\varphi(z)(dz)^2$ has at most a simple zero at any one of these n points, it is integrable in a neighborhood of each of these points and therefore integrable over $\mathbb{C} \cup \{\infty\}$.

Consider a family of disjoint discs D_j centered at $p_j, 1 \leq j \leq n-3$, with positive radii, and let $\Psi : M(X) \to (\mathbb{C} \setminus \{0,1\})^{n-3}$ be the map which assigns to μ in $M(X)$ the vector $(f^\mu(p_1), \ldots, f^\mu(p_{n-3}))$, where f^μ is the quasiconformal self-mapping of $\mathbb{C} \cup \{\infty\}$ normalized to fix $0, 1$ and ∞. Because of the theory of the Beltrami equation [4], each coordinate $f^\mu(p_j)$ depends continuously and holomorphically on μ. In particular, there exists a number $\delta > 0$, such that if $||\mu||_\infty < \delta$, then $f^\mu(p_j)$ is in D_j for each j. The map Ψ is also locally surjective, holomorphic and $\Psi^{-1}(p_1, \ldots, p_{n-3}) \cap \{\mu \in M(X) \text{ with } ||\mu||_\infty < \delta\} = \{\mu \in M_0(X) \text{ with } ||\mu||_\infty < \delta\}$. Therefore, $M_0(X)$ has a manifold structure at $\mu = 0$, and a vector ν is a tangent vector to $M_0(X)$ at $\mu = 0$ if $D\Psi_0(\nu) = (0, \ldots, 0)$. But from the theory of the Beltrami equation

$$f^{t\nu}(p_j) = p_j - t\frac{1}{\pi} \int\int_{\mathbb{C}} \frac{p_j(p_j-1)}{z(z-1)(z-p_j)} \nu(z) dxdy + O(t^2).$$

Since we are assuming that ν is infinitesimally trivial and since for each j, with $1 \leq j \leq n-3$, $\frac{p_j(p_j-1)}{z(z-1)(z-p_j)}(dz)^2$ is an integrable holomorphic quadratic differential on X, we conclude that $D\Psi_0(\nu) = (0, \ldots, 0)$. Therefore, there is a curve μ_t in $M_0(X)$ tangent to ν.

The case X:

We briefly explain this proof in several steps. For more complete explanations we refer to any of the following references: [1], [5], [7], [19], [22], [23], [24].

Step 1 (*Uniformization and Poincaré series*). The first step is to realize the Riemann surface X as the upper half plane $\mathbb{H}$ factored by a Fuchsian group Γ, that is, a discrete group of Möbius transformations preserving $\mathbb{H}$. Then the Riemann surface X is conformal to the factor space $\mathbb{H}/\Gamma$ and integrable quadratic differentials φ on X can be realized as automorphic functions φ in $\mathbb{H}$ satisfying the following properties:

1. $\varphi(z)$ is holomorphic in $\mathbb{H}$,
2. φ is automorphic in the sense that

$$\varphi(A(z))A'(z)^2 = \varphi(z) \tag{30}$$

   for all A in Γ,

3. $\varphi(z)$ is integrable over X in the sense that

$$\int\int_{X/\Gamma} |\varphi(z)| dxdy = \int\int_{\omega} |\varphi(z)| dxdy < \infty,$$

   where ω is a fundamental domain for Γ in $\mathbb{H}$.

Now we begin with the elementary identity

$$\int\int_{\mathbb{H}} \frac{d\xi d\eta}{|\zeta - z|^4} = \frac{\pi}{4} y^{-2}, \tag{31}$$

where $\zeta = \xi + i\eta$ and $z = x + iy$ are in the upper and lower half planes, respectively. The integral in (31) is invariant under the translation $\zeta \mapsto \zeta - x$, and so is equal to

$$\int\int_{\mathbb{H}} \frac{d\xi d\eta}{|\zeta - iy|^4}.$$

Make the substitution $w = \frac{\zeta+iy}{\zeta-iy,}$ and observe that $\zeta \mapsto w$ is a conformal mapping carrying $\mathbb{H}$ onto the unit disc $|w| < 1$. Moreover, if $w = u + iv$, then since

$$\frac{dw}{d\zeta} = \frac{-2iy}{(\zeta - iy)^2},$$

$$\int\int_{\mathbb{H}} \frac{4y^2 d\xi d\eta}{|\zeta - iy|^4} = \int\int_{|w|<1} dudv = \pi,$$

and (31) follows.

From (31) we see that if $\varphi_z(\zeta) = \sum_{A\in\Gamma} \frac{A'(\zeta)^2}{(A(\zeta)-z)^4}$, then $\varphi_z(\zeta)$ is given by an absolutely convergent series because

$$||\varphi_z|| = \int\int_{X/\Gamma} \left| \sum_{A\in\Gamma} \frac{A'(\zeta)^2}{(A(\zeta) - z)^4} \right| d\xi d\eta \leq$$

$$\int\int_{\omega} \sum_{A\in\Gamma} \left| \frac{A'(\zeta)^2}{(A(\zeta) - z)^4} \right| d\xi d\eta = \int\int_{\mathbb{H}} \frac{d\xi d\eta}{|\zeta - z|^4}.$$

Since the hypothesis of Theorem 7 says that $\int\int_X \nu\varphi dxdy = 0$ for every integrable holomorphic quadratic differential, on replacing φ by φ_z we conclude that

$$0 = \int\int_{\omega} \nu\varphi_z d\xi d\eta = \int\int_{\mathbb{H}} \frac{\nu(\zeta)}{(\zeta - z)^4} d\xi d\eta, \tag{32}$$

for every z in the lower half plane.

Step 2 (*The Bers' embedding*). We may identify a Beltrami coefficient μ on X with a measurable complex valued function μ defined on $\mathbb{H}$ satisfying $||\mu||_\infty < 1$, and

$$\mu(A(\zeta)) \frac{\overline{A'(\zeta)}}{A'(\zeta)} = \mu(\zeta). \tag{33}$$

Such a coefficient represents a point $[\mu]$ in $T(X)$ and by putting μ identically equal to zero in the lower half plane, we may associate to μ the unique quasiconformal homeomorpism of $\mathbb{C}$ that solves the Beltrami equation

$$f^\mu_{\bar{z}}(z) = \mu(z) f^\mu_z(z)$$

and fixes the points $0, 1$ and ∞. This homeomorphism has the following properties.

1. It is holomorphic in the lower half plane.

2. It yields by conjugation an isomorphism χ of Γ into a discrete subgroup of $PSL(2, \mathbb{C})$ by the equation

$$f^\mu(A(z)) = \chi(A)(f^\mu(z)).$$

3. Since Γ is the covering group of a Riemann surface of finite analytic type its limit set is the entire real axis and the values of f^μ on the real axis normalized to fix $0, 1$ and ∞ determine and are determined by the Teichmüller equivalence class of μ.

4. Since f^μ applied to the lower half plane is a simply connected region and a Jordan domain, $f^\mu(z)$ realizes the Riemann mapping from the lower half plane to its image and this Riemann mapping determines and is determined by the Teichmüller equivalence class of μ.

Because of these properties, it is natural to form the Schwarzian derivative $\{f, z\}$ of $f = f^\mu$ for values for z in the lower half plane. The Schwarzian derivative of a function f is defined by

$$S_f(z) = N_f(z)' - \frac{1}{2}N_f(z)^2 = 6 \lim_{w \to z} \frac{\partial^2}{\partial w \partial z} \log \frac{f(w) - f(z)}{w - z}, \tag{34}$$

where

$$N_f(z) = \frac{f''(z)}{f'(z)}.$$

The map Φ taking the Teichmüller equivalence class of μ in $M(X)$ to the Schwarzian derivative of f^μ in $B(X)$ is called the Bers' embedding. Φ maps an equivariant Beltrami coefficient, that is, a Beltrami coefficient μ satisfying $||\mu||_\infty < 1$ and (33), to a holomorphic quadratic differential φ satisfying (30) and

$$||y^2\varphi(z)||_\infty \leq 3/2. \tag{35}$$

In fact, Φ is a one-to-one holomorphic mapping from $T(X)$ onto an open set in the Banach space $B(X)$ of functions $\varphi(z)$ holomorphic in the lower half plane satisfying (30) and

$$||y^2\varphi(z)||_\infty < \infty.$$

For this result we cite [3]. Although we do not need this result in full generality, the first step in proving it is the next step of the argument we are currently following.

Step 3 (*A section for* Φ). If a map Φ from $T(X)$ onto B is given and $\Phi(0) = 0$, s is called a local section of Φ at 0 if s maps an open neighborhood U of 0 in B onto an open neighborhood of 0 in $T(X)$ and $\Phi \circ s = I$, the identity on U.

Lemma 6. **[Ahlfors-Weill]** *The map* $s : \varphi \mapsto -2y^2\varphi(\overline{z})$ *provides a section for* $\Phi : M(X) \to B$. *That is, if* $||2y^2\varphi(z)||_\infty < 1$, *then* $\Phi \circ s(\varphi) = \varphi$, *or what is the same,*

$$\{f^{2y^2\varphi(\overline{z})}, z\} = \varphi(z).$$

Proof. If φ is given and you want to solve for f the equation

$$\{f, z\} = \varphi(z),$$

you can follow the following procedure. You solve for two linearly independent solutions η_1 and η_2 of the equation $\eta'' = -\frac{1}{2}\varphi\eta$ normalized by $\eta_1'\eta_2 - \eta_2'\eta_1 \equiv 1$ and then form

$$\hat{f}(z) = \begin{cases} \frac{\eta_1(\overline{z})+(z-\overline{z})\eta_1'(\overline{z})}{\eta_2(\overline{z})+(z-\overline{z})\eta_2'(\overline{z})} & \text{for z in the upper half plane,} \\ \frac{\eta_1(z)}{\eta_2(z)} & \text{for } z \text{ in the lower half plane.} \end{cases} \tag{36}$$

In the case that $||y^2\varphi(z)||_\infty < 1$, $\hat{f}$ is a quasiconformal self mapping of the entire plane, holomorphic in the lower half plane and for z in the lower half plane $\{f, z\} = \varphi(z)$. All of the details of this calculation are given in [3] and in [19], pages 97-102. $\square$

Step 4 (*Construction of* μ_t *in Theorem 7*).

Let ν be a Beltrami coefficient satisfying (32). We need to calculate $\dot{\Phi} = \lim_{t\to 0}(1/t)\Phi(t\nu) = 0$, and in order to do this we view $\Phi(t\nu)$ as the composition of the maps $t\nu \mapsto f^{t\nu}$ and the map $f^{t\nu} \mapsto \{f^{t\nu}, z\}$. From the power series solution to the Beltrami equation the derivative of the first map is given by (10) with μ replaced by ν. Note that if $f^t(z) = z + t\dot{f} + o(t)$, then since $\{f, z\} = \frac{f'''}{f'} - (3/2)\left(\frac{f''}{f'}\right)^2$, $\lim_{t\to 0}(1/t)\{f^t, z\} = \dot{f}'''$, where

$$\dot{f}(z) = -\frac{1}{\pi}\int\int_{\mathbb{C}} \nu(\zeta)\left\{\frac{1}{\zeta - z} - \frac{z}{\zeta - 1} + \frac{z-1}{\zeta}\right\} d\xi d\eta.$$

Since ν is identically equal to zero in the lower half plane,

$$\dot{\Phi}(t\nu)|_{t=0} = \dot{f}''' = \frac{6}{\pi}\int\int_{\mathbb{H}} \frac{\nu(\zeta)}{(\zeta - z)^4} d\xi d\eta$$

and by (32) this is equal to zero for all z in the lower half plane.

Since $\Phi(t\nu) = \varphi^t = t\varphi_1 + t^2\varphi_2 + t^3\varphi_3 + \cdots$ is a convergent power series in the Banach space B, we see from the previous paragraph and formula (32) that $\varphi_1 \equiv 0$. This implies that the holomorphic curve $\nu_t = -2y^2\varphi^t(\overline{z})$ has vanishing first order term, that is, $||\nu_t(z)||_\infty = ||2y^2\varphi^t(\overline{z})||_\infty \leq Ct^2$.

Now form $f^{\mu_t} = (f^{\nu_t})^{-1} \circ f^{t\nu}$. This composition is well-defined because each of the mappings $(f^{\nu_t})^{-1}$ and $f^{t\nu}$ are normalized self-mappings of the complex plane. If we let $\tilde{\nu}$ be the Beltrami coefficient of $(f^{\nu_t})^{-1}$ and $\hat{\nu} = \tilde{\nu}\theta$, where $\theta = \overline{p}/p$ and $p = f_z^{t\nu}$, then $||\hat{\nu}||_\infty = ||\nu_t||_\infty \leq Ct^2$. But from the composition formula for Beltrami coefficients, we have

$$\mu_t = \frac{t\nu + \hat{\nu}}{1 + \overline{t\nu}\hat{\nu}} = t\nu + \hat{\nu}\left(\frac{1 - |t\nu|^2}{1 + \overline{t\nu}\hat{\nu}}\right) = t\nu + O(t^2).$$

But since $t\nu$ and ν_t have the same image under Φ, they are equivalent Beltrami coefficients and so μ_t is in $M_0(X)$. $\square$

10 Teichmüller Space and Teichmüller's Metric

For any Riemann surface X, Teichmüller's space $T(X)$ consists of the quasiconformal maps f from X to variable Riemann surfaces $f(X)$ factored by an equivalence relation. Two such maps f_0 and f_1 are considered to be equivalent if there is a conformal map c from $f_0(X)$ to $f_1(X)$ and a homotopy g_t such that $g_0 = c \circ f_0$, $g_1 = f_1$, $g_t(z)$ is a jointly continuous map from X to $c \circ f_0(X)$, and $g_t(p) = c \circ f_0(p) = f_1(p)$ for every p in the ideal boundary of X.

The reader may not know what we mean by the ideal boundary of X, and if so we refer to any of the various books on Teichmüller theory in the bibliography. Since these notes attempt to treat only the case that X is of finite analytic type, X has no ideal boundary if X is compact or its ideal boundary consists of finitely many punctures.

For a given map f from X to a variable Riemann surface Y, we denote its equivalence class by $[f, Y]$.

Definition. The Teichmüller metric $d : T(X) \times T(X) \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$d([f_0, Y_0], [f_1, Y_1]) = \inf \frac{1}{2} \log K(g),$$

where the infimum is taken over all g in the equivalence class of $f_1 \circ (f_0)^{-1}$.

Exercise 15. *Show that d provides a complete metric on $T(X)$.*

Hint: The triangle inequality follows from properties of the infimum and the fact that dilatation satisfies the inequality $K(f \circ g) \leq K(f)K(g)$. Symmetry follows from the fact that $K(g) = K(g^{-1})$. It is obvious that $d \geq 0$ because by definition $K \geq 1$. Finally, you need to show that if $d([f_0, Y_0], [f_1, Y_1]) = 0$, then $[f_0, Y_0]$ and $[f_1, Y_1]$ are the same equivalence class. That is, you must show that there is a conformal map c from $f_0(Y_0)$ to $f_1(Y_1)$ such that $c \circ f_0$ is homotopic to f_1 through a homotopy that fixes boundary points. The completeness of d follows from properties of quasiconformal mappings.

Exercise 16. *Show that*

$$d([f_0, Y_0], [f_1, Y_1]) = \inf \frac{1}{2} \log \frac{1 + ||\frac{\mu_1 - \mu_0}{1 - \overline{\mu_0}\mu_1}||_\infty}{1 - ||\frac{\mu_1 - \mu_0}{1 - \overline{\mu_0}\mu_1}||_\infty},$$

where the infimum is taken over Beltrami coefficients of all possible maps in the equivalence classes $[f_0, Y_0]$ and $[f_1, Y_1]$.

Theorem 8. *The Teichmüller space of a Riemann surface of finite analytic type of genus g with n punctures is homeomorphic to a cell of dimension $3g - 3 + n$ provided that this number is positive.*

Proof. From Teichmüller's theorem we know that each equivalence class $[f, Y]$ except the equivalence class of the identity is represented by a Beltrami coefficient of the form $k|\varphi|/\varphi$, where k and φ are uniquely determined with $0 < k < 1$

and $||\varphi|| = 1$. We let $Q(X)$ be the Banach space of integrable holomorphic quadratic differentials on X and consider the map Ψ from the open unit ball of $Q(X)$ into $T(X)$ given by

$$\varphi \mapsto [f^{k|\varphi|/\varphi}, f^{k|\varphi|/\varphi}(X)],$$

where $k = ||\varphi||$. This map is injective and surjective by Teichmüller's uniqueness and existence theorems. To show that it is continuous consider a sequence φ_n in the open unit ball of $Q(X)$ converging in norm to an element φ of this ball. If we put $k_n = ||\varphi_n||$ and $k = ||\varphi||$, then k_n converges to k and $0 \leq k < 1$. If $k = 0$, then the Beltrami coefficients $k_n|\varphi_n|/\varphi_n$ converge uniformly to 0, and the Teichmüller distance from $[f^{k_n|\varphi_n|/\varphi_n}, Y_n]$ to $[identity, X]$ is no greater than $(1/2)\log\frac{1+k_n}{1-k_n}$. This shows that Ψ is continuous at 0. If $k > 0$, then the Beltrami coefficients $\mu_n = k_n|\varphi_n|/\varphi_n$ converge to $\mu = k|\varphi|/\varphi$ in the bounded pointwise sense. From the theory of quasiconformal mapping, this is enough to guarantee that the maps f^{μ_n} converge normally to the map f^μ, and because $T(X)$ is locally compact, this is enough to guarantee that $[f^{\mu_n}, Y_n]$ converges to $[f^\mu, Y]$ in the Teichmüller metric.

Finally, we must show that Ψ applied to a compact set is compact. A compact set C in the open ball B is closed and bounded away from the unit sphere. If $\varphi_n \in C$ converges to $\varphi \in C$, then there exists $r < 1$ such that $||\varphi_n|| \leq r < 1$. Thus $K(f_n) \leq \frac{1+r}{1-r}$, where f_n is a normalized family of quasiconformal mappings with Beltrami coefficients of the form

$$||\varphi_n||\frac{|\varphi|}{\varphi}.$$

Thus $\Psi(C)$ forms a bounded set in $T(X)$. By the same argument as given in the previous paragraph, this set is also closed. □

11 Infinitesimal Teichmüller's Metric

Let X be a Riemann surface of finite analytic type, f be a quasiconformal map from X to another Riemann surface Y, and μ be the Beltrami coefficient of f. Let K_0 be the extremal dilatation of a mapping in the class $[f, Y]$. We have already seen that the main inequality gives the following upper bound for K_0.

$$K_0 \leq \sup \int\int_X \frac{\left|1 + \mu\frac{\varphi}{|\varphi|}\right|^2}{1 - |\mu|^2}|\varphi|dxdy, \tag{37}$$

where the supremum is taken over holomorphic quadratic differentials φ with $||\varphi|| = 1$. Moreover, if $K_0 > 1$ and if f^μ is extremal in its class, then the supremum is realized by a unique quadratic differential φ and the Beltrami coefficient of any extremal mapping f_0 in the class of f is equal to $k_0|\varphi|/\varphi$.

Exercise 17. *Show that the main inequality also gives a lower bound for K_0, namely,*

$$\frac{1}{K_0} \leq \int\int_X \frac{\left|1 - \mu\frac{\varphi}{|\varphi|}\right|^2}{1 - |\mu|^2} |\varphi| dxdy, \tag{38}$$

for every holomorphic quadratic φ with $||\varphi|| = 1$.

Exercise 18. *Replace μ by $t\mu$ in (37) and in (38) and show that for any Beltrami coefficient μ,*

$$d([id, X], [f^{t\mu}, f^{t\mu}(X)]) = t\left(\sup\left| \text{ Re } \int\int_X \mu\varphi dxdy\right|\right) + O(t^2), \tag{39}$$

where the supremum is taken over all holomorphic quadratic differentials φ with $||\varphi|| = 1$.

The coefficient of t in the expression on the right hand side of (39) is called the infinitesimal form of Teichmüller's metric at the base point. It transports to every point $[f^\mu, f^\mu(X)]$ of Teichmüller space with tangent vector ν at that point and is given by the global form expressed by the following formula:

$$F([\mu], \nu) = \sup\left| \text{ Re } \int\int_X \varphi(w)\left[\frac{\nu}{1 - |\mu|^2} \cdot \frac{1}{\theta}\right] dudv\right|, \tag{40}$$

where $w = f^\mu(z)$, $\theta = \overline{p}/p$, $p = \frac{\partial}{\partial z} f^\mu$, $w = u + iv$, and the supremum is taken over all holomorphic quadratic differentials on the Riemann surface $f^\mu(X)$ with norm equal to 1.

It is not difficult to see from these formulas that Teichmüller's metric is equal to the integral of its infinitesimal form. That is, the Teichmüller distance between any two points is equal to the infimum of the arc length integrals with respect to F in (40), where the infimum is taken over all differentiable curves joining the two points, and any curve of the form $\gamma(t) = t\nu$, $0 \leq t \leq 1$, will realize this infimum.

Instead of pursuing this calculation we prefer to go on to an interpretation of the Teichmüller infinitesimal norm of a tangent vector to Teichmüller space in the case that the Riemann surface X is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with $n \geq 4$ points removed.

12 Tangent Vectors to Teichmüller Space

In this section we study the infinitesimal Teichmüller norm on vector fields tangent to Teichmüller space. The most direct way to illustrate the theory is to consider only the case when X is a Riemann surface equal to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ minus $n \geq 4$ points. For simplicity we temporarily assume that the n points are in the finite plane, $\mathbb{C}$, and that they are the finite set

$E = \{a_k\}_{k=1}^n$. In that case the quadratic differentials can be identified with holomorphic functions of z of the form

$$\varphi(z) = \sum_{k=1}^{n} \frac{\rho_k}{z - a_k},$$

where the residues ρ_k satisfy the equations

$$\begin{aligned} \textstyle\sum_{k=1}^n \rho_k &= 0, \\ \textstyle\sum_{k=1}^n \rho_k a_k &= 0, \\ \textstyle\sum_{k=1}^n \rho_k a_k^2 &= 0. \end{aligned} \tag{41}$$

Exercise 19. *Let $w = 1/z$ be a local parameter at $z = \infty$. Verify that the assumption that $\varphi^w(w)\left(\frac{dw}{dz}\right)^2 = \varphi(z)$ is a holomorphic quadratic differential at $w = 0$ forces the side conditions (41). Also verify that*

$$\varphi(z) = \frac{p(z)}{\prod_{k=1}^n (z - a_k)},$$

where $p(z)$ is a polynomial of degree less than or equal to $n - 4$.

Given a Beltrami differential μ, its infinitesimal Teichmüller norm is the smallest possible value of $||\nu||_\infty$ where ν is a Beltrami differential with the property that

$$\int\int_X \nu\varphi dxdy = \int\int_X \mu\varphi dxdy$$

for all φ in $Q(X)$. By the Hahn-Banach and Riesz representation theorems such a ν exists.

Definition. We denote this number $||\nu||_\infty$ by $||\mu||_T$ and call it the Teichmüller infinitesimal norm of μ. It is the norm of μ considered as a linear functional on the Banach space $Q(X)$.

If we let $L_\infty(X)$ be the vector space of measurable Beltrami differentials $\mu(z)\frac{\overline{dz}}{dz}$ defined on X, then clearly

$$(\varphi, \mu) = \int\int_X \mu\varphi dxdy$$

defines a pairing. On putting $N = \{\mu : (\varphi, \mu) = 0 \text{ for all } \varphi \in Q(X)\}$, this pairing establishes an isomorphism between the $Q(X)^*$, the dual space to $Q(X)$, and $L_\infty(X)/N$.

Another way to construct $Q(X)^*$ is to introduce the vector space Z of vectors v_j assigned to the points a_j of E factored by an equivalence relation. Two vectors $V = (v_1, ..., v_n)$ and $W = (w_1, ..., w_n)$ are considered equivalent if there is a quadratic polynomial $p(z) = az^2 + bz + c$ such that $v_j - w_j = p(a_j)$ for every j.

Exercise 20. *Use (41) to show that the pairing*

$$(V, \varphi) = \sum_{j=1}^{n} v_j \rho_j,$$

where $V = (v_1, ..., v_n)$ *and* $\varphi(z) = \sum_{j=1}^{n} \frac{\rho_j}{z - a_j}$, *is well-defined and non-degenerate, and establishes an isomorphism between* Z *and* $Q(X)^*$.

Now we set up an isomorphism between $L_\infty(X)/N$ and Z which satisfies the identity

$$-\frac{\pi}{2}(V, \varphi) = \int\int_X \mu\varphi dxdy. \tag{42}$$

The formula for V in terms of μ is

$$V(z) = -\frac{(z-a_1)(z-a_2)(z-a_3)}{\pi} \int\int \frac{\mu(\zeta)}{(\zeta - a_1)(\zeta - a_2)(\zeta - a_3)(\zeta - z)} d\xi d\eta,$$

and the important properties are that

1. V is continuous and has generalized first partial derivatives,
2. $V_{\overline{z}}(z) = \mu$ and
3. $V(z) = O(|z|^2)$ as $z \to \infty$.

In order to do the following exercise you do not need this formula; all you need are these three properties.

Exercise 21. *Prove equation (42) by using the identity*

$$d(V(z)\varphi(z)dz) = V_{\overline{z}}(z)\varphi(z)\overline{dz}dz$$

and Stokes' theorem in the region consisting of the complement of n very small discs of radius ϵ *centered at the points of* E.

The point of this exercise is that the linear functional determined by μ in $L_\infty(X)/N$ depends precisely on the values of the vector field V at the points of E, up to the addition of a quadratic polynomial. This is reflected in the infinitesimal form of Teichmüller's metric. According to (39) the infinitesimal displacement in Teichmüller's metric caused by a tangent vector μ is

$$\sup_{||\varphi||=1} \left| \int\int_X \mu\varphi dxdy \right| = \sup_{||\varphi||=1} \left| \frac{\pi}{2} \sum_{j=1}^{n} \rho_j v_j \right|.$$

In this way the velocity of the moving Riemann surface in $T(X)$ is expressed in terms of the velocities of the points in E and the quadratic differentials of norm one on $X = \mathbb{C} \cup \{\infty\} \setminus E$.

References

[1] W. Abikoff. *The Real Analytic Theory of Teichmüller space*, volume 820 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, 1980.

[2] L. V. Ahlfors. On quasiconformal mappings. *J. Anal. Math.*, 4:1–58, 1954.

[3] L. V. Ahlfors. *Lectures on Quasiconformal Mapping*, volume 38 of *University Lecture Series*. Amer. Math. Soc, 2006.

[4] L. V. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Annals of Math.*, 72:385–404, 1961.

[5] L. V. Ahlfors and L. Sario. *Riemann Surfaces*. Princeton University Press, Princeton, 1960.

[6] L. Bers. Quasiconformal mappings and Teichmüller's theorem, in Analytic Functions. *Princeton University Press, Princeton, N. J.*, pages 89–119, 1960.

[7] L. Bers. Automorphic forms and general Teichmüller spaces. In *Proceedings of the Conference on Complex Analysis (Minneapolis 1964)*, pages 109–113, Berlin, 1965. Springer.

[8] L. Bers. A non-standard integral equation with applications to quasiconformal mapping. *Acta Math.*, 116:113–134, 1966.

[9] L. Bers. Extremal quasiconformal mappings. *Annals of Math. Studies*, 66:27–52, 1971.

[10] L. Bers. Fibre spaces over Teichmüller spaces. *Acta Math.*, 130:89–126, 1973.

[11] L. Bers and H. Royden. Holomorphic families of injections. *Acta Math.*, 157:259–286, 1986.

[12] B. Bojarski. Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients. *Math. Sbornik*, 85:451–503, 1957.

[13] V. Bozin, N. Lakic, M. Markovic, and M. Mateljevic. Unique extremality. *J. d'Analyse Math.*, 75:299–337, 1998.

[14] C. J. Earle and F. P. Gardiner. Geometric isomorphisms between infinite dimensional Teichmüller spaces. *Trans. Am. Math. Soc.*, 348(3):1163–1190, 1996.

[15] C. J. Earle and I. Kra. On holomorphic mappings between Teichmüller spaces. In L. V. Ahlfors et al., editor, *Contributions to Analysis*, pages 107–124, New York, 1974. Academic Press.

[16] C. J. Earle and I. Kra. On isometries between Teichmüller spaces. *Duke Math. J.*, 41:583–591, 1974.

[17] C. J. Earle and V. Markovic. Isometries between the spaces of L^1 holomorphic quadratic differentials on Riemann surfaces of finite type. *Duke Math. J.*, 120(2):433–440, 2003.

[18] F. P. Gardiner. Approximation of infinite dimensional Teichmüller spaces. *Trans. Amer. Math. Soc.*, 282(1):367–383, 1984.

[19] F. P. Gardiner. *Teichmüller Theory and Quadratic Differentials*. John Wiley & Sons, New York, 1987.

[20] F. P. Gardiner and N. Lakic. *Quasiconformal Teichmüller Theory*. AMS, Providence, Rhode Island, 2000.

[21] R. S. Hamilton. Extremal quasiconformal mappings with prescribed boundary values. *Trans. Amer. Math. Soc.*, 138:399–406, 1969.

[22] W. J. Harvey. *Discrete Groups and Automorphic Functions*. Academic Press, 1977.

[23] Y. Imayoshi and M. Taniguchi. *An Introduction to Teichmüller Spaces*. Springer-Verlag, Tokyo, 1992.

[24] I. Kra. *Automorphic Forms and Kleinian Groups*. Benjamin, Reading, Mass., 1972.

[25] S. L. Krushkal. Extremal quasiconformal mappings. *Sib. Math. J.*, 10:411–418, 1969.

[26] N. Lakic. An isometry theorem for quadratic differentials on Riemann surfaces of finite genus. *Trans. Amer. Math. Soc.*, 349:2951–2967, 1997.

[27] N. Lakic. Strebel points. *Contemporary Math.*, 211:417–431, 1997.

[28] O. Lehto. *Univalent Functions and Teichmüller Spaces*. Springer-Verlag, New York, 1987.

[29] O. Lehto and K. I. Virtanen. *Quasiconformal Mapping in the Plane*. Springer-Verlag, New York, Berlin, 1965.

[30] R. Màñé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Èc. Norm. Sup.*, 96:193–217, 1983.

[31] A. Marden and K. Strebel. The heights theorem for quadratic differentials on Riemann surfaces. *Acta Math.*, 153:153–211, 1984.

[32] V. Markovic. Biholomorphic maps between Teichmueller spaces. *Duke Math. J.*, 120(2):405–431, 2003.

[33] S. Nag. *The Complex Analytic Theory of Teichmüller Spaces*. Canadian Mathematical Society, Wiley-Interscience, New York, 1987.

[34] E. Reich. On the decomposition of a class of plane quasiconformal mappings. *Comment. Math. Helv.*, 53:15–27, 1978.

[35] E. Reich and K. Strebel. Teichmüller mappings which keep the boundary pointwise fixed. *Ann. Math. Studies*, 66:365–367, 1971.

[36] E. Reich and K. Strebel. Extremal quasiconformal mappings with given boundary values. *Contributions to Analysis*, pages 375–392, 1974.

[37] H. Royden. Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces*, pages 369–384, 1971.

[38] Z. Slodkowski. Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.*, 111:347–355, 1991.

[39] D. P. Sullivan and W. P. Thurston. Extending holomorphic motions. *Acta Math.*, 157:243–257, 1986.

[40] O. Teichmüller. Untersuchungen über konforme und quasikonforme Abbildungen. *Deutsche Math.*, 3:621–678, 1938.

[41] O. Teichmüller. Eine Verschärfung des Dreikreisesatzes. *Deutsche Math.*, 4:16–22, 1939.

[42] O. Teichmüller. Extremale quasikonforme Abbildungen und quadratische Differentiale. *Abh. Preuss. Akad.*, 22:3–197, 1939.

[43] O. Teichmüller. Über extremalprobleme der konformen Geometrie. *Deutsche Math.*, 6:50–77, 1941.

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