Notes on Dynamic Optimization

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Abstract

The aim of this lecture notes is to provide a self-contained introduction to the subject of "Dynamic Optimization" for the MSc course on "Mathematical Economics", part of the MSc on Economics and the MSc in Financial Mathematics in ISEG, the Economics and Business School of the Technical University of Lisbon. It is assumed that the students have a good working knowledge of calculus in several variables, linear algebra. as well as difference and differential equations.

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1 Introduction

The purpose of this notes is to provide a first elementary introduction to the subject of Dynamic Optimization, also known as Optimal Control Theory. This theory addresses the problem faced by a decision maker on a evolving "environment". The decision maker must come up with decisions affecting the evolution with time of a given dynamical systems in order to achieve a desired goal. Since the systems under consideration evolve with time, any decision (or control!) must be based on the latest information collected, and thus, must also evolve with time. The decision makers goal is then to select an optimal decision among all possible ones in order to achieve the better possible final result, with respect to some predefined performance criteria. Such optimization problems are therefore called *Optimal control problems*. In this lecture notes, we consider mainly the following two types of dynamical systems: continuous time dynamical systems defined by differential equations and discrete time dynamical systems defined by difference equations or iterations of maps.

There are two mainstream methods developed to tackle Optimal control problems. One is based is based on variational methods, and leads to Pontryagin's maximum principle. The second is based on the dynamic programming principle, leading to the Hamilton-Jacobi-Bellman equation. Section 2 provides a brief introduction to the Calculus of variations. In section 3 we introduce the setup we will work with and provide a rigorous definition for the Optimal control problem. Section 4 deals with the variational approach to the Optimal Control problem and the Pontryagin's maximum principle. Section 5 introduces the dynamic programming principle and the Hamilton-Jacobi-Belman equation.

2 Calculus of variations

The main problem in the calculus of variations is that of determining a function maximizing or minimizing a given functional. This is clearly an infinite dimensional problem. The finite dimensional analogue is the standard calculus problem of determining a point at which a specific function attains its maximum or minimum value. The typical approach to solve such problem is to compute the first derivative of the function in order to find its zeros, leading to the first-order condition for the existence of an extreme point for the function. A similar procedure will be used to obtain the first-order condition for the variational problem. The analogy with classical optimization extends to the second-order maximization condition of calculus.

2.1 The continuous-time case

Let $x : [0,T] \to \mathbb{R}^n$ be a C^1 function defined over the interval $[0,T]$, i.e. $x(\cdot)$ is continuously differentiable in $[0,T]$. We say that $x(\cdot) \in C^1([0,T],\mathbb{R}^n)$ is an *admissable function* if it satisfies the following boundary conditions:

$$x(0) = x_0 \ , \qquad x(T) = x_T \ ,$$

where $x_0, x_T \in \mathbb{R}^n$. We denote by $\mathcal{A}[0,T]$ the set of such admissible functions.

Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. The problem of the calculus of variations is to find the admissible function $x^*(\cdot) \in \mathcal{A}[0,T]$ for which the functional $J : C^1([0,T],\mathbb{R}^n) \to \mathbb{R}$ given by

$$J[x(\cdot)] = \int_0^T L(t,x,\dot{x}) \, \mathrm{d}t \tag{1}$$

has a relative maximum. We note that this is one of the most simple problems in the Calculus of Variations. Several modification of this problem have been studied throughout the last three centuries. We will study some of the easier problems below.

2.1.1 Euler-Lagrange equation

We will derive a necessary condition for an admissable function to be an extremal of the functional (1): the Euler-Lagrange equation.

Theorem 2.1 (Euler-Lagrange equation). *Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and let $\mathcal{A}_1[0,T]$ be the set*

$$\mathcal{A}_1[0,T] = \{x(\cdot) \in C^2([0,T],\mathbb{R}^n) : x(0) = x_0 \ , x(T) = x_T\} \ .$$

If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_1[0,T]$, then $x^*(\cdot)$ is a solution of the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} D_{\dot{x}} L = D_x L \ . \tag{2}$$

with boundary conditions given by $x(0) = x_0$ and $x(T) = x_T$.

Before proceeding with the proof of the theorem above, we state and prove the following auxiliary result:

Lemma 2.2 (The fundamental lemma of the Calculus of Variations). *Let $h \in C^0([0,T],\mathbb{R})$ be such that*

$$\int_0^T h(t)\phi(t)\mathrm{d}t = 0$$

for every function $\phi \in C^0([0,T],\mathbb{R})$ with $\phi(0) = \phi(T) = 0$. Then $h(t) = 0$ for all $t \in [0,T]$.

Proof. Let $h(t^*) > 0$ for some $t^* \in (0, T)$. Since h is continuous, we have that $h(t) > c$ in some neighborhood U_1 of t^*, i.e. there exists $\delta > 0$ such that $h(t) > c$ for every t such that $0 < t^* - \delta < t < t^* + \delta < T$. Let $\phi(t)$ be a continuous function such that $\phi(t) = 0$ outside $(t^* - \delta, t^* + \delta)$, $\phi(t) > 0$ in $(t^* - \delta, t^* + \delta)$, and $\phi(t) = 1$ in $(t^* - \delta/2, t^* + \delta/2)$. Then,

$$\int_0^T h(t)\phi(t)\mathrm{d}t \geq \delta c > 0.$$

Thus, we must have that $h(t^*) = 0$ for all $t^* \in [0, T]$. □

Proof of theorem 2.1. Let $x(\cdot) \in \mathcal{A}_1[0, T]$ be a critical point of the functional $J[\cdot]$ and let $y(\cdot) \in \mathcal{A}_1[0, T]$ be such that

$$y(t) = x(t) + \epsilon\eta(t) \ , \qquad t \in [0, T] \ ,$$

where ϵ is a small real number and $\eta(\cdot) \in C^2([0, T], \mathbb{R}^n)$ is such that $\eta(0) = \eta(T) = 0$.

Let ϵ_0 be a small positive real number and define the function $V : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ by

$$V(\epsilon) = J[x(\cdot) + \epsilon\eta(\cdot)] = \int_0^T L(t, x(t) + \epsilon\eta(t), \dot{x}(t) + \epsilon\dot{\eta}(t)) \ \mathrm{d}t \ . \tag{3}$$

Recalling that $x(\cdot)$ is a critical point of $J[\cdot]$ and noting that V is a continuously differentiable function, we obtain that $V'(0) = 0$. Differentiating V with respect to ϵ and setting $\epsilon = 0$, we get

$$\delta J = \frac{\mathrm{d}V}{\mathrm{d}\epsilon}_{\epsilon=0} = \int_0^T D_x L(t, x(t), \dot{x}(t)) \cdot \eta(t) + D_{\dot{x}} L(t, x(t), \dot{x}(t)) \cdot \dot{\eta}(t) \ \mathrm{d}t = 0$$

Integrating by parts the second term in the integral above yields:

$$\begin{aligned} \delta J = \frac{\mathrm{d}V}{\mathrm{d}\epsilon}_{\epsilon=0} &= \int_0^T \left(D_x L(t, x(t), \dot{x}(t)) - \frac{\mathrm{d}}{\mathrm{d}t} D_{\dot{x}} L(t, x(t), \dot{x}(t)) \right) \cdot \eta(t) \ \mathrm{d}t \\ &\quad + [D_{\dot{x}} L(t, x(t), \dot{x}(t)\eta(t)]_{t=0}^{t=T} = 0 \ . \end{aligned}$$

Since $\eta(0) = \eta(T) = 0$, we get

$$\delta J = \frac{\mathrm{d}V}{\mathrm{d}\epsilon}_{\epsilon=0} = \int_0^T \left(D_x L(t, x(t), \dot{x}(t)) - \frac{\mathrm{d}}{\mathrm{d}t} D_{\dot{x}} L(t, x(t), \dot{x}(t)) \right) \cdot \eta(t) \ \mathrm{d}t = 0 \ .$$

By lemma 2.2, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} D_{\dot{x}} L = D_x L \ ,$$

as required. □

The Euler-Lagrange equation is in general a second order differential equation, thus its solution will depend on two arbitrary constants, determined either from the boundary conditions $x(0) = x_0$ and $x(T) = x_T$, or from some transversality conditions in the cases where boundary conditions are not imposed. The problem usually considered in the theory of differential equations is that of finding a solution which is defined in the neighbourhood of some point and satisfies given initial conditions. However, when solving the Euler-Lagrange equation, we are looking for a solution which is defined on the whole interval $[0, T]$, whose existence is not guaranteed.

The next couple of results concern second-order condition for minimizers of the functional $J[\cdot]$ in $\mathcal{A}[0, T]$. It should be remarked that such conditions are only necessary.

Theorem 2.3 (Jacobi's necessary condition)**.** *Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Suppose $x^*(\cdot) \in \mathcal{A}[0,T]$ is a maximizer (resp. minimizer) of the functional $J[\cdot]$ in $\mathcal{A}[0,T]$. Then, for each $\eta(\cdot) \in C^1([0,T],\mathbb{R}^n)$ such that $\eta(0) = \eta(T) = 0$ the integral*

$$\int_0^T \eta^T(t)D^2_{xx}L(t,x^*,\dot{x}^*)\eta(t) + 2\eta(t)D^2_{x\dot{x}}L(t,x^*,\dot{x}^*)\dot{\eta}(t) + \dot{\eta}^T(t)D^2_{x\dot{x}}L(t,x^*,\dot{x}^*)\dot{\eta}(t)\mathrm{d}t$$

is negative (resp. positive).

Proof. We deal here with the case where $x^*(\cdot) \in \mathcal{A}[0,T]$ is a maximizer of the functional $J[\cdot]$ in $\mathcal{A}[0,T]$, the case of a minimizer being analogous.

Since $x^*(\cdot)$ is a maximizer, then the function $V(\epsilon) = J[x^*(t) + \epsilon\eta(t)]$ defined by (3) in the proof of theorem (2.1) has a maximum at $\epsilon = 0$. The result then follows by computing the second derivative of V at $\epsilon = 0$. Details are left as an exercise. □

Corollary 2.4 (Legendre's necessary condition)**.** *Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Suppose $x^*(\cdot) \in \mathcal{A}[0,T]$ is a maximizer (resp. minimizer) of the functional $J[\cdot]$ in $\mathcal{A}[0,T]$. Then, $D_{\dot{x}\dot{x}}L(t,x^*,\dot{x}^*)$ is negative-semidefinite (resp. positive-semidefinite).*

Proof. Use theorem 2.3 with $\eta(t) = \epsilon\xi(t)\sin(t/\epsilon)$, for some $\xi(\cdot) \in C^1([0,T],\mathbb{R}^n)$, and let $\epsilon \to 0$. Details are left as an exercise. □

The Euler-Lagrange equation is generally a very complicated nonlinear differential equation of second order. To solve it explicitly in terms of elementary functions and their integrals is typically impossible. However, for a number of special cases explicit solutions can be derived. We consider some of these cases below.

(A) *L does not depend on $\dot{x}$.*
Since $D_{\dot{x}}L = 0$, the Euler equation reduces to the static optimization condition:

$$D_xL(t,x) = 0 \ , \quad t \in [0,T]$$

The previous equation is not even a differential equation, its solution not involving any arbitrary constants. Hence, the problem of maximizing $J[\cdot]$ in $\mathcal{A}[0,T]$ will usually have no solution. However, the optimality condition above is still interesting necessary condition for the problem of maximizing the function $J[\cdot]$ with no boundary conditions.

(B) *L does not depend on x.*
In this case $D_xL = 0$, so the Euler equation tells us that the time derivative of $D_{\dot{x}}L(t,\dot{x})$ is equal to 0 for all $t \in [0,T]$. Hence, we have that

$$D_{\dot{x}}L(t,x) = c$$

for some constant c, and for every $t \in [0,T]$. This is a first-order differential equation, which may eventually be possible to solve explicitly. One strategy would be to solve it for $\dot{x}$ and integrate with respect to t to obtain the required solution.

2.1.2 Alternative boundary conditions

The restrictions on the admissable functions $x(\cdot)$ may vary. We summarize some simple variations of theorem 2.1 in the proposition below.

Proposition 2.5. *Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and define the sets*

$$\begin{aligned}
\mathcal{A}_2[0,T] &= \{x(\cdot) \in C^2([0,T],\mathbb{R}^n) : x(0), x(T) \in \mathbb{R}^n\} \\
\mathcal{A}_3[0,T] &= \{x(\cdot) \in C^2([0,T],\mathbb{R}^n) : x(0) = x_0 \ , x(T) \in \mathbb{R}^n\} \\
\mathcal{A}_4[0,T] &= \{x(\cdot) \in C^2([0,T],\mathbb{R}^n) : x(0) = x(T) \in \mathbb{R}^n\} \ .
\end{aligned}$$

If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_i[0,T]$, $i \in \{2,3,4\}$, then $x^*(\cdot)$ is a solution of the Euler Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}} = L_x \ .$$

Moreover, the following transversality conditions must be satisfied:

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_2[0,T]$, then*

$$L_{\dot{x}}(0, x(0), \dot{x}(0)) = L_{\dot{x}}(T, x(T), \dot{x}(T)) = 0 \ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_3[0,T]$, then*

$$L_{\dot{x}}(T, x(T), \dot{x}(T)) = 0 \ , \qquad x(0) = x_0 \ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_4[0,T]$, then*

$$L_{\dot{x}}(0, x(0), \dot{x}(0)) = L_{\dot{x}}(T, x(T), \dot{x}(T)) \ , \qquad x(0) = x(T) \ .$$

Proof. Left as an exercise. □

2.1.3 Infinite horizon

One of the main areas for application of variational methods in economics is growth theory, where many models have the feature that the planning period is infinite. This feature leads to variational problems with unbounded domains of integration. Two mathematical problems arise: (i) to find some natural class of admissible functions; and (ii) to find correct transversality conditions at infinity. For properly specified problems of this sort, the Euler equation is still a necessary condition for optimality.

Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and let $J : C^1(\mathbb{R}_0^+, \mathbb{R}^n) \to \mathbb{R}$ be the functional

$$J[x(\cdot)] = \int_0^\infty L(t, x, \dot{x}) \ \mathrm{d}t \ .$$

Define the set $\mathcal{A}_\infty$ of *admissable functions* as

$$\mathcal{A}_\infty = \{x(\cdot) \in C^1(\mathbb{R}_0^+, \mathbb{R}^n) : \text{there exists} \lim_{t\to+\infty} x(t) \text{ and } |J[x(\cdot)]| < +\infty\} \ .$$

We will now consider the following problem: to find criteria for an admissible function $x^*(\cdot) \in \mathcal{A}_\infty$ to be a relative maximum of the functional $J[\cdot]$. The following proposition states that if $x^*(\cdot) \in \mathcal{A}_\infty \cap C^2(\mathbb{R}_0^+, \mathbb{R}^n)$ is a critical point of $J[\cdot]$, then it must satisfy the Euler-Lagrange equation and some transversality conditions associated with some eventual boundary restrictions imposed on $x^*(\cdot)$.

Proposition 2.6. *Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Define the following sets:*

$$\begin{aligned}
\mathcal{A}_\infty^1 &= \{x(\cdot) \in \mathcal{A}_\infty \cap C^2(\mathbb{R}_0^+, \mathbb{R}^n) : x(0) = x_0 \ , \lim_{t\to+\infty} x(t) = x_\infty\} \\
\mathcal{A}_\infty^2 &= \{x(\cdot) \in \mathcal{A}_\infty \cap C^2(\mathbb{R}_0^+, \mathbb{R}^n) : x(0) \ , \lim_{t\to+\infty} x(t) \ \mathit{free}\} \\
\mathcal{A}_\infty^3 &= \{x(\cdot) \in \mathcal{A}_\infty \cap C^2(\mathbb{R}_0^+, \mathbb{R}^n) : x(0) = x_0 \ , \lim_{t\to+\infty} x(t) \ \mathit{free}\} \ .
\end{aligned}$$

If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_\infty^i$, $i \in \{1,2,3\}$, then $x^*(\cdot)$ is a solution of the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}} = L_x \ .$$

Moreover, the following transversality conditions must be satisfied:

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}^1_\infty$, then*

$$x(0) = x_0 \text{ and } \lim_{t\to+\infty} x(t) = x_\infty \ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}^2_\infty$, then*

$$L_{\dot{x}}(0, x(0), \dot{x}(0)) = \lim_{T\to+\infty} L_{\dot{x}}(T, x(T), \dot{x}(T)) = 0 \ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}^3_\infty$, then*

$$\lim_{T\to+\infty} L_{\dot{x}}(T, x(T), \dot{x}(T)) = 0 \ , \qquad x(0) = x_0 \ .$$

Proof. Left as an exercise. □

2.1.4 A sufficient condition

We will now state a theorem providing a sufficient condition for a solution of the Euler-Lagrange equation to be a maximizer (resp. minimizer) of the functional $J[\cdot]$) for every $t \in [0, T]$. The key condition in the theorem is that $L(t, x, \dot{x})$ must be a concave (resp. convex) function of $(x, \dot{x})$. Such condition considerably restrict the class of functions L for which the theorem can be applied. However, the concavity (or convexity) requirement is often a natural one in economic problems, and the theorem is therefore a very important result.

Theorem 2.7. *Suppose that the conditions in theorem 2.1 are satisfied and let $x^*(\cdot) \in \mathcal{A}_1[0, T]$ be a solution of the Euler-Lagrange equation (2). Then:*

(i) if the map $(x, \dot{x}) \to L(t, x, \dot{x})$ is concave (resp. convex) for every $t \in [0, T]$, then $x^*(\cdot)$ is a maximizer (resp. minimizer) of $J[\cdot]$ in $\mathcal{A}[0, T]$.

(ii) if the map $(x, \dot{x}) \to L(t, x, \dot{x})$ is strictly concave (resp. convex) for every $t \in [0, T]$, then the maximizer (resp. minimizer) of $J[\cdot]$, if it exists, is unique.

Proof. We deal with the case of a maximizer $x^*(\cdot) \in \mathcal{A}_1[0, T]$ in this proof, the case of a minimizer being analogous.

Let $x^*(\cdot) \in \mathcal{A}_1[0, T]$ be a solution of the Euler-Lagrange equation (2) with $x^*(0) = x_0$, $x^*(T) = x_T$. Since we assume that the $(x, \dot{x}) \to L(x, x, \dot{x})$ is concave for every $t \in [0, T]$, we get that

$$L(t, x, \dot{x}) \leq L(t, x^*, \dot{x}^*) + L_x(t, x^*, \dot{x}^*) \cdot (x - x^*) + L_{\dot{x}}(t, x^*, \dot{x}^*) \cdot (\dot{x} - \dot{x}^*)$$

for every $x(\cdot) \in \mathcal{A}[0, T]$. Integrating the above inequality we get

$$J[x(\cdot)] \leq J[x^*(\cdot)] + \int_0^T L_x(t, x^*, \dot{x}^*) \cdot (x - x^*) + L_{\dot{x}}(t, x^*, \dot{x}^*) \cdot (\dot{x} - \dot{x}^*) \mathrm{d}t$$

Integrating by parts the second term in the integral, and noting that $x(0) - x^*(0) = x(T) - x^*(T) = 0$, we get

$$J[x(\cdot)] \leq J[x^*(\cdot)] + \int_0^T \left(L_x(t, x^*, \dot{x}^*) - \frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{x}}(t, x^*, \dot{x}^*) \right) \cdot (x - x^*) \mathrm{d}t$$

Since $x^*(\cdot)$ is a solution of the Euler-Lagrange equation (2), we get that $J[x(\cdot)] \leq J[x^*(\cdot)]$, which ends the proof of item (i).

We will now prove item (ii). Let $x_1(\cdot), x_2(\cdot) \in \mathcal{A}[0, T]$ be two maximizers of $J[\cdot]$ and denote by m the maximum value. We will show that $x_1(\cdot)$ and $x_2(\cdot)$ are necessarily equal. Define

$$y(t) = \frac{1}{2} x_1(t) + \frac{1}{2} x_2(t)$$

and observe that $y(\cdot) \in \mathcal{A}[0,T]$. Using the concavity of the map $(x, \dot{x}) \to L(t, x, \dot{x})$, we obtain

$$\begin{aligned}\frac{1}{2}L(t, x_1(t), \dot{x}_1(t)) + \frac{1}{2}L(t, x_2(t), \dot{x}_2(t)) \quad &\leq \quad L\left(t, \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t), \frac{1}{2}\dot{x}_1(t) + \frac{1}{2}\dot{x}_2(t)\right) \\ &= L(t, y(t), \dot{y}(t))\end{aligned}$$

and hence

$$m = \frac{1}{2}J[x_1(t)] + \frac{1}{2}J[x_1(t)] \leq J[y(t)] \leq m \ .$$

Therefore, we get

$$\int_0^T \frac{1}{2}L(t, x_1(t), \dot{x}_1(t)) + \frac{1}{2}L(t, x_2(t), \dot{x}_2(t)) - L\left(t, \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t), \frac{1}{2}\dot{x}_1(t) + \frac{1}{2}\dot{x}_2(t)\right) \mathrm{d}t = 0$$

Since the integrand is, by strict concavity of L, positive unless $x_1(\cdot) = x_2(\cdot)$ and $\dot{x}_1(\cdot) = \dot{x}_2(\cdot)$, we deduce that $x_1(\cdot) = x_2(\cdot)$ as wished. □

2.1.5 Example

We will now discuss a well known model due to F. Ramsey concerning Optimal Growth. Consider an economy developing over time where $K = K(t)$ denotes the accumulated capital, $C = C(t)$ the consumption and $Y = Y(t)$ the net national production at time t. Suppose that the national production is a function of the accumulated capital alone:

$$Y = f(K) \ ,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing concave function of the accumulated capital. For each t we have the relation

$$C(t) = f(K(t)) - \dot{K}(t) \ , \tag{4}$$

meaning that production, given by $f(K(t))$, is divided between consumption, $C(t)$, and investment, $\dot{K}(t)$.

Let $K(0) = K_0$ be the accumulated capital at some instant of time $t = 0$, and suppose that we are considering a fixed planning period $[0, T]$, for some $T > 0$. For each choice of investment function $\dot{K}(t)$ on $[0, T]$, the capital accumulation function is fully determined by

$$K(t) = K_0 + \int_0^t \dot{K}(s) \ \mathrm{d}s \ ,$$

while the identity (4) determines the consumption function $C(t)$ associated with such investment.

The problem faced by the planning authorities is to choose the investment function. High consumption at the present time is in itself preferable, but leads by (4) to a low rate of investment which results in a lower capital accumulation in the future, reducing future consumption. The planner needs to find a way to reconcile the conflict between providing for the present and taking care of the future.

To address such problem, we assume that the society preferences concerning consumption may be described by a utility function $U : \mathbb{R} \to \mathbb{R}$. We assume that U is a strictly increasing and strictly concave function. Moreover, we introduce a discount factor ρ and assume that the investment criterion is as follows: Choose $\dot{K}(t)$ for $t \in [0, T]$ such that the total discounted utility for the country in the period $[0, T]$ is maximal, i.e. find the capital accumulation function $K = K(t)$, with $K(0) = K_0$ and $K(T) = K_T$, maximizing the functional

$$J[K(t)] = \int_0^T \mathrm{e}^{-\rho t} U(C(t)) \mathrm{d}t = \int_0^T \mathrm{e}^{-\rho t} U(f(K(t)) - \dot{K}(t)) \mathrm{d}t \ .$$

Note that the problem to be solved is a variational one. Denote by L the integrand function:

$$L(t, K, \dot{K}) = \mathrm{e}^{-\rho t} U(f(K) - \dot{K}) \ .$$

Theorem 2.1 provides a necessary condition to be satisfied by the solutions of this variational problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{K}} = \frac{\partial L}{\partial K} \ .$$

Computing the derivatives of L with respect to K and $\dot{K}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[-\mathrm{e}^{-\rho t} U'(f(K) - \dot{K}) \right] = \mathrm{e}^{-\rho t} U'(f(K) - \dot{K}) f'(K) \ .$$

We now compute the derivative with respect to t in the equation above to get

$$\rho \mathrm{e}^{-\rho t} \left[U'(f(K) - \dot{K}) \right] - \mathrm{e}^{-\rho t} U''(f(K) - \dot{K})(f'(K)\dot{K} - \ddot{K}) = \mathrm{e}^{-\rho t} U'(f(K) - \dot{K}) f'(K) \ .$$

Dividing the previous equality by $\mathrm{e}^{-\rho t} U''(f(K) - \dot{K})$ and rearranging terms, we obtain

$$\ddot{K} - f'(K)\dot{K} + (\rho - f'(K)) \frac{U'(f(K) - \dot{K})}{U''(f(K) - \dot{K})} = 0 \ . \tag{5}$$

A solution for the variational problem under consideration must then satisfy the second order differential equation above, with boundary conditions $K(0) = K_0$ and $K(T) = K_T$. It should be noted that explicit solutions cannot be obtained for this differential equation except in some very special cases. However, we can obtain extra information from the equation without having to actually solve it.

We will now compute the relative growth rate of C, given by the ratio $\dot{C}/C$. Recall that $C = f(K) - \dot{K}$ to obtain

$$\frac{\dot{C}}{C} = \frac{f(K)\dot{K} - \ddot{K}}{C} = (\rho - f'(K)) \frac{U'(C)}{CU''(C)}$$

Since we are assuming that $U'(C) > 0$ and $U''(C) < 0$, we obtain that the relative growth rate of C is positive if $f'(K)$ (the marginal productivity of capital) is larger then the discount rate ρ.

We will now see that a solution of (5) is indeed a maximizer of $J[\cdot]$. Note that: (i) f is a concave function; (ii) $f(K) + (-\dot{K})$ is the sum of two concave functions, hence concave in $(K, -\dot{K})$; (iii) U is increasing and concave; we obtain that $L(t, K, \dot{K}) = \mathrm{e}^{-\rho t} U(f(K) - \dot{K})$ is concave in $(K, \dot{K})$ for every fixed t. Thus, by theorem 2.7 a solution of (5) is a maximizer of $J[\cdot]$.

2.1.6 Exercises

Exercise 1. Prove proposition 2.5.
Hint: use the proof of theorem 2.1.

Exercise 2. Prove proposition 2.6.
Hint: use the proof of theorem 2.1.

Exercise 3. Complete the details in the proof of theorem 2.3.

Exercise 4. Complete the details in the proof of corollary 2.4.

Exercise 5. Consider again the example of section 2.1.5 with $U(C) = C^{1-\nu}/(1-\nu)$, $\nu \in (0, 1)$ and $f(K) = bK$, $b > 0$. Assume that $K_0 > 0$ and $K_T > 0$.

(i) Find the Euler-Lagrange equation in this case. Prove that the general solution for $b \neq a$, where $a = (b - \rho)/\nu$, is of the form:

$$K(t) = A\mathrm{e}^{bt} + B\mathrm{e}^{at} \ .$$

(ii) Find the corresponding solution for $C(t)$.

(iii) Prove that the function obtained in (i) solves the variational problem.

(iv) Find the condition for a positive relative growth rate of consumption.

Exercise 6. Consider again the example of section 2.1.5 with $Y(t) = f(K(t), t)$, $C(t) = f(K(t), t) - \dot{K}(t)$, and $U(C, t)$ instead of $\mathrm{e}^{-\rho t}U(C)$:

$$\max \int_0^T U(f(K(t), t) - \dot{K}(t), t)\mathrm{d}t \ , \qquad K(0) = K_0 \ , \qquad K(T) = K_T \ .$$

Compute the Euler-Lagrange equation associated with this variational problem and find an expression for the relative growth rate of C.

Exercise 7. A monopolist offers a certain commodity for sale on a market. If the production per unit of time is x, let $b(x)$ denote the associated total cost. Suppose that the demand at time t for this commodity depends not only on the price $p(t)$, but also $\dot{p}(t)$, so that the demand at time t is $D(p(t), \dot{p}(t))$, where D is some given function. If production is adjusted to the demand at each time t, the total profit for the monopolist in the time interval $[0, T]$ is given by

$$\int_0^T D(p, \dot{p})p - b(D(p, \dot{p}))\mathrm{d}t.$$

Suppose that $p(0)$ and $p(T)$ are given. The problem for the monopolist is to find the price function $p(t)$ maximizing his profit.

(i) Find the Euler-Lagrange equation for this problem.

(ii) Let b be given by $b(x) = \alpha x^2 + \beta x + \gamma$, $x = D(p, \dot{p}) = Ap + B\dot{p} + C$ where α, β, γ, B and C are positive constants, while A is negative. Find the Euler equation and its solution in this case.

(iii) Prove that the function obtained in (ii) is a solution for the corresponding variational problem.

Exercise 8. Let $w(t)$ denote the total wealth of a certain person at time t, i his constant income rate, $c(t)$ is consumption rate at time t, and suppose that he can borrow or lend at a constant interest rate r. The value of his total wealth varies according to the equation:

$$\dot{w}(t) = i + rw(t) - c(t) \ .$$

Suppose that he plans his consumption from $t = 0$ until his expected death date T so as to maximize

$$\int_0^T \mathrm{e}^{-\rho t}U(c(t))\mathrm{d}t \ ,$$

where U is a utility function such that $U' > 0$ and $U'' < 0$, and $\rho > 0$ is a discount factor. Assume that his initial wealth $w(0)$ is equal to w_0 and that he wants to leave his heirs a fixed amount w_T.

(i) Find the Euler-Lagrange equation associated with this variational problem.

(ii) Let $U(c) = a - \mathrm{e}^{-bc}$, where a and b are positive constants. Solve the Euler-Lagrange equation in this case.

(iii) Prove that the function obtained in (ii) is a solution for the corresponding variational problem.

2.2 The discrete-time case

Let $\Theta = \{0, 1, \ldots T\}$ be a set of indexes and $x = \{x_t\}_{t\in\Theta}$ be a sequence in $\mathbb{R}^n$ indexed by the elements of Θ. Note that x can also be thought as a function $x : \Theta \to \mathbb{R}^n$. We say that $x(\cdot) \in \ell^\infty(\Theta, \mathbb{R}^n)$ is an *admissable sequence* if it satisfies the following boundary conditions:

$$x(0) = x_0 \ , \qquad x(T) = x_T \ ,$$

where $x_0, x_T \in \mathbb{R}^n$. We denote by $\mathcal{A}_1^\Theta$ the set of such admissible sequences.

Let $L : \Theta \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function for every fixed $t \in \Theta$. The problem of the calculus of variations is to find an admissible sequence $x^*(\cdot) \in \mathcal{A}_1^\Theta$ for which the functional $J : \mathcal{A}_1^\Theta \to \mathbb{R}$ given by

$$J[x(\cdot)] = \sum_{t=0}^{T-1} L(t, x_t, x_{t+1}) \tag{6}$$

has a relative maximum.

2.2.1 Euler-Lagrange equation

Similarly to the continuous-time case, a necessary condition for an admissable function to be an extremal of the functional (6) is provided by the (discrete-time) Euler-Lagrange equation.

Theorem 2.8 (Euler-Lagrange equation)**.** *Let $L : \Theta \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 function for every fixed $t \in \Theta$. If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_1^\Theta$, then $x^*(\cdot)$ is a solution of the Euler-Lagrange equation*

$$\frac{\partial L}{\partial x_t}(t-1, x_{t-1}, x_t) + \frac{\partial L}{\partial x_t}(t, x_t, x_{t+1}) = 0, \qquad t = 1, 2, \ldots, T-1 \tag{7}$$

with boundary conditions given by $x(0) = x_0$ and $x(T) = x_T$.

Proof. Let $x(\cdot) \in \mathcal{A}_1^\Theta$ be a critical point of the functional $J[\cdot]$ and let $y(\cdot) \in \mathcal{A}_1^\Theta$ be such that

$$y(t) = x(t) + \epsilon\eta(t) \ , \qquad t \in \Theta \ ,$$

where ϵ is a small real number and $\eta(\cdot) \in \ell^\infty(\Theta, \mathbb{R}^n)$ is such that $\eta(0) = \eta(T) = 0$.

Let ϵ_0 be a small positive real number and define the function $V : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ by

$$V(\epsilon) = J[x(\cdot) + \epsilon\eta(\cdot)] = \sum_{t=0}^{T-1} L(t, x_t + \epsilon\eta_t, x_{t+1} + \epsilon\eta_{t+1}) \ .$$

Recalling that $x(\cdot)$ is a critical point of $J[\cdot]$ and noting that V is a continuously differentiable function, we obtain that $V'(0) = 0$. Differentiating V with respect to ϵ and setting $\epsilon = 0$, we get

$$\delta J = \frac{\mathrm{d}V}{\mathrm{d}\epsilon}_{|\epsilon=0} = \sum_{t=1}^{T-1} \left(\frac{\partial L}{\partial x_t}(t-1, x_{t-1}, x_t) + \frac{\partial L}{\partial x_t}(t, x_t, x_{t+1}) \right) \eta_t = 0$$

Since the equality above holds for every sequence $\eta \in \ell^\infty(\Theta, \mathbb{R}^n)$ such that $\eta(0) = \eta(T) = 0$, we get

$$\frac{\partial L}{\partial x_t}(t-1, x_{t-1}, x_t) + \frac{\partial L}{\partial x_t} L(t, x_t, x_{t+1}) = 0 \ , \qquad t = 1, 2, \ldots T-1 \ ,$$

as required. □

2.2.2 Alternative boundary conditions

As in the continuous-time case, the restrictions on the admissable functions $x(\cdot)$ may vary. We summarize some simple variations of theorem 2.8 in the proposition below.

Proposition 2.9. *Let $L : \Theta \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 function for every fixed $t \in \Theta$ and define the sets*

$$\begin{aligned}\mathcal{A}_2^{\Theta} &= \{x(\cdot) \in \ell^{\infty}(\Theta, \mathbb{R}^n) : x(0), x(T) \in \mathbb{R}^n\}\\ \mathcal{A}_3^{\Theta} &= \{x(\cdot) \in \ell^{\infty}(\Theta, \mathbb{R}^n) : x(0) = x_0\ , x(T) \in \mathbb{R}^n\}\\ \mathcal{A}_4^{\Theta} &= \{x(\cdot) \in \ell^{\infty}(\Theta, \mathbb{R}^n) : x(0) = x(T) \in \mathbb{R}^n\}\ .\end{aligned}$$

If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_i^{\Theta}$, $i \in \{2, 3, 4\}$, then $x^*(\cdot)$ is a solution of the Euler Lagrange equation

$$\frac{\partial L}{\partial x_t}(t-1, x_{t-1}, x_t) + \frac{\partial L}{\partial x_t}(t, x_t, x_{t+1}) = 0, \qquad t = 1, 2, \ldots, T-1\ .$$

Moreover, the following transversality conditions must be satisfied:

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_2^{\Theta}$, then*

$$\frac{\partial L}{\partial x_0}(0, x_0, x_1) = \frac{\partial L}{\partial x_T}(T-1, x_{T-1}, x_T) = 0\ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_3^{\Theta}$, then*

$$\frac{\partial L}{\partial x_T}(T-1, x_{T-1}, x_T) = 0\ , \qquad x(0) = x_0\ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}_4^{\Theta}$, then*

$$\frac{\partial L}{\partial x_0}(0, x_0, x_1) + \frac{\partial L}{\partial x_T}(T-1, x_{T-1}, x_T)\ , \qquad x(0) = x(T)\ .$$

Proof. Left as an exercise. □

2.2.3 Infinite horizon

We will now briefly discuss the discrete-time variational problem with infinite horizon, of great relevance to topics such as Optimal Growth Theory. Let $T = \infty$ and $\Theta = \mathbb{N}_0$. Consider the functional $J : \ell^{\infty}(\Theta, \mathbb{R}^n) \to \mathbb{R}$ given by

$$J[x(\cdot)] = \sum_{t=0}^{\infty} L(t, x_t, x_{t+1})$$

and define the set $\mathcal{A}_{\infty}$ of *admissable functions* as

$$\mathcal{A}_{\infty}^1 = \{x(\cdot) \in \ell^{\infty}(\Theta, \mathbb{R}^n) : \text{there exists} \lim_{t\to+\infty} x(t) \text{ and } |J[x(\cdot)]| < +\infty\}\ .$$

We will now look for criteria for an admissible function $x^*(\cdot) \in \mathcal{A}_{\infty}^1$ to be a relative maximum of the functional $J[\cdot]$.

Proposition 2.10. *Let $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Define the following sets:*

$$\begin{aligned}\mathcal{A}_{\infty}^2 &= \{x(\cdot) \in \mathcal{A}_{\infty} \cap C^2(\mathbb{R}_0^+, \mathbb{R}^n) : x(0)\ , \lim_{t\to+\infty} x(t) \textit{ free}\}\\ \mathcal{A}_{\infty}^3 &= \{x(\cdot) \in \mathcal{A}_{\infty} \cap C^2(\mathbb{R}_0^+, \mathbb{R}^n) : x(0) = x_0\ , \lim_{t\to+\infty} x(t) \textit{ free}\}\ .\end{aligned}$$

If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}^i_\infty$, $i \in \{2,3\}$, then $x^*(\cdot)$ is a solution of the Euler-Lagrange equation

$$\frac{\partial L}{\partial x_t}(t-1, x_{t-1}, x_t) + \frac{\partial L}{\partial x_t}(t, x_t, x_{t+1}) = 0, \qquad t \in \mathbb{N} \ .$$

Moreover, the following transversality conditions must be satisfied:

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}^2_\infty$, then*

$$\frac{\partial L}{\partial x_0}(0, x_0, x_1) = \lim_{T \to +\infty} \frac{\partial L}{\partial x_T}(T-1, x_{T-1}, x_T) = 0 \ .$$

- *If $x^*(\cdot)$ is a critical point of the functional $J[\cdot]$ in $\mathcal{A}^3_\infty$, then*

$$\lim_{T \to +\infty} \frac{\partial L}{\partial x_T}(T-1, x_{T-1}, x_T) = 0 \ , \qquad x(0) = x_0 \ .$$

2.2.4 A sufficient condition

We now provide a sufficient condition for a solution of the Euler-Lagrange equation to be a maximizer (resp. minimizer) of the functional $J[\cdot]$) for every $t \in \Theta$.

Theorem 2.11. *Suppose that the conditions in theorem 2.8 are satisfied and let $x^*(\cdot) \in \mathcal{A}^\Theta_1$ be a solution of the Euler-Lagrange equation* (7)*. Then:*

(i) if the map $(x, y) \to L(t, x, y)$ is concave (resp. convex) for every $t \in [0, T]$, then $x^*(\cdot)$ is a maximizer (resp. minimizer) of $J[\cdot]$ in $\mathcal{A}^\Theta$.

(ii) if the map $(x, y) \to L(t, x, y)$ is strictly concave (resp. convex) for every $t \in [0, T]$, then the maximizer (resp. minimizer) of $J[\cdot]$, if it exists, is unique.

Proof. Left as an exercise. □

2.2.5 Example

Consider the problem of a consumer facing a T-period planning period, where T is some positive integer. We assume that this consumer has initial wealth $w_0 > 0$. If at some time t the consumer has wealth w_t and consumes c_t of his wealth, then at the beginning of the next period the consumer will have wealth $w_{t+1} = (1 + r)(w_t - c_t)$, where r is a constant positive interest rate.

Let $U : \mathbb{R} \to \mathbb{R}$ be an utility function describing the consumer's preferences (with respect to the amount of consumption. Assume that the function U is twice continuously differentiable with positive first derivative and negative second derivative.

Let $\beta \in (0, 1)$ is a discount factor. The consumer goal is to maximize the functional

$$J[w] = \sum_{t=0}^{T-1} \beta^t U(c_t) \ ,$$

under fixed boundary conditions w_0 and w_T.

Rewrite the functional J as

$$J[w] = \sum_{t=0}^{T-1} \beta^t U\left(w_t - \frac{1}{1+r} w_{t+1}\right) \ .$$

The corresponding Euler-Lagrange problem is given by

$$U'\left(w_t - \frac{1}{1+r} w_{t+1}\right) - \frac{1}{\beta(1+r)} U'\left(w_{t-1} - \frac{1}{1+r} w_t\right) = 0 \ , \qquad t = 1, 2, \ldots, T-1 \ ,$$

with boundary conditions w_0 and w_T at $t = 0$ and $t = T$, respectively.

2.2.6 Exercises

Exercise 1. Prove proposition 2.9.
Hint: use the proof of theorem 2.8.

Exercise 2. Prove theorem 2.11.
Hint: adjust the proof of theorem 2.7 to discrete-time.

Exercise 3. Consider again the example of section 2.2.5 with the period utility function given by $U(c) = \ln(c)$.

(i) Write down the calculus of variations problem.
(ii) Determine the Euler–Lagrange condition.
(iii) Solve the Euler-Lagrange equation.
(iv) Prove that the function obtained in (iii) solves the variational problem.

Exercise 4. Consider again the example of section 2.2.5 with the period utility function given by $U(c) = \frac{c^{1-\nu}}{1-\nu}$, with $\nu \in (0, 1)$.

(i) Write down the calculus of variations problem.
(ii) Determine the Euler–Lagrange condition.
(iii) Solve the Euler-Lagrange equation.
(iv) Prove that the function obtained in (iii) solves the variational problem.

Exercise 5. Consider the problem of firm with cash flow in period t given by

$$\pi_t = AK_t - I_t\left(1 + \xi I_t\right) \ ,$$

where K_t denotes its capital stock at period t, I_t its investment at period t, and $A > 0$ and $\xi > 0$ are the productivity and investment cost parameters, respectively. Let $\delta \in [0, 1)$ denote the rate of depreciation of capital, and assume that the capital accumulation dynamics are given by

$$K_{t+1} = I_t + (1 - \delta)K_t \ ,$$

and that the initial capital stock is fixed and equal to $K_0 > 0$. Assume that firm's goal is to maximize the value functional

$$\sum_{t=0}^{\infty}(1 + r)^{-t}\pi_t \ ,$$

where $r > 0$ is the market interest rate. Assume that $A > r + \delta$,

(i) Write down the calculus of variations problem.
(ii) Determine the optimality conditions.
(iii) Determine the optimality conditions.
(iv) Solve the Euler-Lagrange equation to find an explicit solution for K_t.
(v) Determine the optimal investment sequence I_t.

3 The Optimal Control Problem

In this section we introduce the Optimal Control problem for both discrete-time and continuous-time deterministic dynamical systems.

3.1 The continuous-time case

Let T be such that $0 < T \leq +\infty$, $x_0 \in \mathbb{R}^n$ and Γ be a metric space. Consider the dynamical system defined by the ordinary differential equations

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \ , & t \in [0, T] \\ x(0) = x_0 \ , \end{cases} \tag{8}$$

where $f : [0, T] \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$ is a given map. A measurable map $u : [0, T] \to \Gamma$ is called a *control*, x_0 is called the *initial state*, T is called the *terminal time*, and a solution $x(\cdot)$ of (8), if it exists, is called a *state trajectory* corresponding to $u(\cdot)$. In the case where the function f is such that for any $x_0 \in \mathbb{R}^n$ and any $u(\cdot)$ there exists a unique solution $x(\cdot) = x(\cdot, u(\cdot))$ of (8) we obtain an input-output relation with input $u(\cdot)$ and output $x(\cdot)$.

There may be several constraints associated with the *state variable* $x(t)$ and the *control variable* $u(t)$. For instance, the *state constraint* may be given by

$$x(t) \in S(t) \ , \quad \text{for all } t \in [0, T] \tag{9}$$

and the *control constraint* may be given by

$$u(t) \in U(t) \ , \quad \text{for a.e. } t \in [0, T] \ ,$$

where $S : [0, T] \to \mathcal{P}(\mathbb{R}^n)$ and $U : [0, T] \to \mathcal{P}(\Gamma)$ are some given correspondences, i.e. for each $t \in [0, T]$, $S(t) \subseteq \mathbb{R}^n$ and $U(t) \subseteq \Gamma$. Some other types of constraints are also possible such as, for instance, constraints in an integral form. We will only consider the case where the control constraint is time invariant, i.e. $U(t) \equiv U \subseteq \Gamma$. Recall that U is itself a metric space. Thus, from now on we will replace Γ by U.

We define the set $\mathcal{V}[0, T]$ as

$$\mathcal{V}[0, T] = \{u : [0, T] \to U : u \text{ is measurable}\}$$

and call to any element $u(\cdot) \in \mathcal{V}[0, T]$ a *feasible control*.

The next ingredient is the *cost functional* measuring the performance of the controls

$$J(u(\cdot)) = \int_0^T L(t, x(t), u(t)) \mathrm{d}t + \Phi(x(T)) \ , \tag{10}$$

for given maps $L : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ called *running cost* and *terminal cost*, respectively. Note that in the case where $T = +\infty$, one must check that the right hand side of the functional defined in (10) is well-defined.

Definition 3.1. *A control $u(\cdot)$ is said to be an* admissible control*, and $(x(\cdot), u(\cdot))$ is called an* admissible pair*, if*

(i) $u(\cdot) \in \mathcal{V}[0, T]$*;*

(ii) $x(\cdot)$ *is the unique solution of the initial value problem* (8) *under* $u(\cdot)$*;*

(iii) the state constraint (9) *is satisfied;*

(iv) the map $t \to L(t, x(t), u(t))$ *is in* $L^1[0, T]$.

We will denote the set of admissible controls by $\mathcal{V}_{ad}[0, T]$. The optimal control problem can then be stated as follows:

Problem 3.2. *Maximize* (10) *over* $\mathcal{V}_{ad}[0, T]$.

If T is finite, we say that the optimal control problem has *finite horizon*, while in the case where $T = +\infty$, we say it has *infinite horizon*. We say that problem 3.2 is *finite* if (10) has a finite upper bound, and that it is *uniquely solvable* if there is a (unique) $\bar{u}(\cdot) \in \mathcal{V}_{ad}[0,T]$ such that

$$J(\bar{u}(\cdot)) = \sup_{u(\cdot)\in\mathcal{V}_{ad}[0,T]} J(u(\cdot)) \ . \tag{11}$$

Any control $\bar{u}(\cdot)$ satisfying (11) is said to be an *optimal control*, and the corresponding state trajectory $\bar{x}(\cdot) \equiv x(\cdot, \bar{u}(\cdot)$ and pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ are called *optimal state trajectory* and *optimal pair*, respectively.

An optimal control problem 3.2 with terminal cost $\Phi = 0$ is called a *Lagrange problem*, that with running cost $L = 0$ is called a *Mayer problem*, and that with $L \neq 0$ and $\Phi \neq 0$ is called a *Bolza problem*. It is well known that these three problems are mathematically equivalent.

We will now provide some examples of typical optimal control problems.

Example 1. Production planning
A machine is producing one type of product. The raw materials are processed by the machine, and the finished products are stored in a buffer. Suppose that at time t the production rate is $u(t)$ and the inventory level in the buffer is $x(t)$. If the demand rate for this product is a know function $z(t)$ and the inventory is x_0 at time $t = 0$, then the relationship between these quantities may be described by

$$\begin{cases} \dot{x}(t) = u(t) - z(t) \ , & t \geq 0 \\ x(0) = x_0 \ . \end{cases} \tag{12}$$

The variable $x(t)$ can take both positive and negative values. The product has a surplus if $x(t) > 0$, and a backlog if $x(t) < 0$. Suppose that the cost of having the inventory x and production rate u per unit time is $h(x, u)$. A typical example of h is given by

$$h(x, u) = c^+x^+ + c^-x^- + pu \ ,$$

where $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$, $c^+, c^- \geq 0$ are the marginal cost and penalty for surplus and backlog, respectively, and p is the unit cost of production. The production management goal is to choose some $u(\cdot)$ so as to minimize the total discounted cost over some planning horizon $[0, T]$. Namely, the following functional is to be minimized

$$J(u(\cdot)) = \int_0^T \mathrm{e}^{-\gamma t} h(x(t), u(t))\mathrm{d}t \ , \tag{13}$$

where $\gamma > 0$ is the discount rate. The decision or control $u(\cdot)$ is a function on $[0, T]$, called the production plan. If the machine has a maximum production rate k, the production capacity, then any production plan must satisfy the constraints

$$0 \leq u(t) \leq k \ , \qquad \text{for all } t \in [0, T] \ . \tag{14}$$

If the buffer size is $b > 0$, then the state variable must satisfy the constraint

$$x(t) \leq b \ , \qquad \text{for all } t \in [0, T] \ . \tag{15}$$

Any production plan satisfying (12), (14) and (15) is an admissible production plan. The problem is then to minimize the cost functional (13) over all admissible production plans.

Example 2. Life-cycle saving
Consider a worker with a known span of life $T > 0$, over which he will earn wages at a constant rate w and receive interest at a constant rate r on accumulated savings, or pay the same rate on accumulated debts. Thus, if the value of his accumulated assets (or debt if

negative) is $k(t)$ at time t, his income at such time equals $w+rk(t)$. Denoting his consumption rate at time t by $c(t)$, his capital accumulation is then described by the differential equation

$$\dot{k}(t) = w + rk(t) - c(t) \;, \qquad t \in [0,T] \;. \tag{16}$$

Assume that there are no inheritances or bequests, i.e. the following boundary conditions are satisfied

$$k(0) = k(T) = 0 \;. \tag{17}$$

The worker's goal is to choose his consumption plan $c(\cdot)$ so as to maximize the following discounted utility over some planning horizon $[0,T]$:

$$J(c(\cdot)) = \int_0^T \mathrm{e}^{-\gamma t} U(c(t))\mathrm{d}t \;, \tag{18}$$

where $\gamma > 0$ is the discount rate and $U(\cdot)$ is the instantaneous utility function for the worker's consumption, which is usually assumed to be an increasing and concave function. In this case, the consumption rate $c(\cdot)$ is the control variable, while the capital accumulation $k(\cdot)$ is the state variable. If, for instance, the worker can not take a debt load larger some negative real number d at any given time $t \in [0,T]$, the state variable must satisfy the constraint

$$k(t) \geq d \;, \qquad \text{for all } t \in [0,T] \;. \tag{19}$$

Any consumption plan $c(\cdot)$ satisfying (16), (17) and (19) is an admissible consumption plan. The problem is to maximize the functional (18) over all admissible consumption plans.

Example 3. Optimum growth
Like the previous example, this one is also concerned with the problem of optimal saving, but now from the point of view of the whole economy, not just the individual worker. This change brings with it two new features. The first one is that the rate of return on savings can not be taken as an exogenous market rate of interest as it would be for an individual, but rather as the endogenous marginal product of capital. The second feature is that there is no natural finite planning horizon, and thus we consider the problem to be of infinite horizon $T = +\infty$.

We assume for simplicit that the accumulated capital is a real valued function of time $k(\cdot)$, and that the rate of output $F(k)$ is given by a production function $F : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

(i) $F(0) = 0$;

(ii) $\lim_{k\to+\infty} F'(k) = 0$.

Moreover, we assume that capital depreciates at a constant rate $\delta > 0$. If the consumption rate is given by some function $c(\cdot)$, then the capital accumulation is determined by the differential equation

$$\dot{k}(t) = F(k) - \delta k(t) - c(t) \;, \qquad t \geq 0 \;. \tag{20}$$

Assume that the initial capital $k(0)$ is given and that there are no other constraints on either the control variable $c(t)$ or the state variable $k(t)$. Let $U(c(\cdot))$ denote the utility derived from the consumption plan $c(\cdot)$, where $U : \mathbb{R} \to \mathbb{R}$ is an increasing and concave function. Assuming a discount rate $\gamma > 0$, we can describe the planner's goal as the one of choosing the consumption plan $c(\cdot)$ of the economy so as to maximize the following discounted utility

$$J(c(\cdot)) = \int_0^{+\infty} \mathrm{e}^{-\gamma t} U(c(t))\mathrm{d}t \;, \tag{21}$$

Any consumption flow satisfying (20) with the fixed initial condition $k(0)$ is an admissible consumption flow. The problem is to maximize the functional (21) over all admissible consumption flows. One final comment is in order in what concerns the integral in the definition of the functional (21): this is convergent only for values of the discount rate γ sufficiently large.

3.2 The discrete-time case

The discrete-time optimal control problem is analogous to the continuous-time problem discussed above. We describe it below for the sake of completeness. Let $T \in \mathbb{N} \cup \{\infty\}$, $\Theta = \{0, 1, \ldots, T-1\}$, $x_0 \in \mathbb{R}^n$ and Γ be a metric space. Consider the dynamical system defined by the difference equation

$$\begin{cases} x(t+1) = f(t, x(t), u(t)) \ , & t \in \Theta \\ x(0) = x_0 \end{cases} \tag{22}$$

where $f : \Theta \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^n$ is a given map. As before, a measurable map $u : \Theta \to \Gamma$ is called a *control*, x_0 is called the *initial state*, T is called the *terminal time*, and a solution $x_(\cdot)$ of (22), if it exists, is called a *state trajectory* corresponding to $u(\cdot)$.

Constraints for the state variable may be given by

$$x(t) \in S(t) \ , \quad \text{for all } t \in \Theta \tag{23}$$

and for the control variable may be given by

$$u(t) \in U(t) \ , \quad \text{for a.e. } t \in [0, T] \ ,$$

where $S : \Theta \to \mathcal{P}(\mathbb{R}^n)$ and $U : \Theta \to \mathcal{P}(\Gamma)$ are some given correspondences. As with the time-continuous case, we will only consider the case where the control constraint is time invariant, i.e. $U(t) \equiv U \subseteq \Gamma$, and from now on we will replace Γ by U.

We define the set $\mathcal{V}^\Theta$ as

$$\mathcal{V}^\Theta = \{u : \Theta \to U : u \text{ is measurable}\}$$

and call to any element $u(\cdot) \in \mathcal{V}^\Theta$ a *feasible control*.

The next ingredient is the *cost functional* measuring the performance of the controls

$$J(u(\cdot)) = \sum_{t=0}^{T-1} L(t, x(t), u(t)) + \Phi(x(T)) \ , \tag{24}$$

for given maps $L : \Theta \times \mathbb{R}^n \times U \to \mathbb{R}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ called *running cost* and *terminal cost*, respectively. In the case $T = +\infty$, one must check that the right hand side of (10) is well-defined.

Definition 3.3. *A control $u(\cdot)$ is said to be an* admissible control*, and $(x(\cdot), u(\cdot))$ is called an* admissible pair*, if*

(i) $u(\cdot) \in \mathcal{V}^\Theta$*;*

(ii) $x(\cdot)$ *is the unique solution of the initial value problem* (22) *under* $u(\cdot)$*;*

(iii) the state constraint (23) *is satisfied;*

(iv) the map $t \to L(t, x(t), u(t))$ *is in* $\ell^1(\Theta)$.

We will denote the set of admissible controls by $\mathcal{V}^\Theta_{ad}$. The optimal control problem can then be stated as follows:

Problem 3.4. *Maximize* (24) *over* $\mathcal{V}^\Theta_{ad}$.

If T is finite, we say that the optimal control problem has *finite horizon*, while in the case where $T = +\infty$, we say it has *infinite horizon.* We say that problem 3.4 is *finite* if (24) has a finite upper bound, and that it is *uniquely solvable* if there is a (unique) $\bar{u}(\cdot) \in \mathcal{V}^\Theta_{ad}$ such that

$$J(\bar{u}(\cdot)) = \sup_{u(\cdot) \in \mathcal{V}^\Theta_{ad}} J(u(\cdot)) \ . \tag{25}$$

Any control $\bar{u}(\cdot)$ satisfying (25) is said to be an *optimal control*, and the corresponding state trajectory $\bar{x}(\cdot) \equiv x(\cdot, \bar{u}(\cdot)$ and pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ are called *optimal state trajectory* and *optimal pair*, respectively.

Example 4. Multiperiod consumption-saving
A consumer faces a T-period planning period, where T is some positive integer. We assume that this consumer has initial wealth $w > 0$. If at some time t the consumer has wealth w_t and consumes c_t of his wealth, then at the beginning of the next period the consumer will have wealth

$$w_{t+1} = (1 + r)(w_t - c_t) \ , \tag{26}$$

where r is the constant interest rate r.

Assume that the consumer has some positive initial wealth w_0 and plans to reach the planning horizon T with some fixed positive wealth w_T, i.e.

$$w(0) = w_0 \ , \qquad w(T) = w_T \ . \tag{27}$$

The consumer's goal is to choose his consumption plan $c(\cdot)$ so as to maximize the following discounted utility over some planning horizon $[0, T]$:

$$J(c(\cdot)) = \sum_{t=0}^{T-1} \beta^t U(c_t) \ , \tag{28}$$

where $\beta \in (0, 1)$ is the discount rate and $U(\cdot)$ is the instantaneous utility function for the agent's consumption, which is usually assumed to be an increasing and concave function. The consumption sequence $c(\cdot)$ is the control variable, while wealth $w(\cdot)$ is the state variable.

Any consumption plan $c(\cdot)$ satisfying (26) and (27) is an admissible consumption plan. The problem is to maximize the functional (28) over all admissible consumption plans.

4 Pontryagin's Maximum Principle

Optimal control problems may be seen as optimization problems in infinite-dimensional spaces. For this reason, such problem are substantially more difficult to solve. The maximum principle, formulated and derived by Pontryagin and his group in the 1950s, is one of the key results in optimal control theory. It states that any optimal control along with the optimal state trajectory must be a solution for an (extended) Hamiltonian system, which is a two-point boundary value problem, together with a maximum condition for the Hamiltonian function.

The strength of the maximum principle relies on the fact that maximizing the Hamiltonian function, being a finite dimensional problem, is much easier than the infinite-dimensional original control problem.

4.1 The continuous-time case

In this section we discuss Pontryagin's maximum principle in the setup of continuous-time dynamical systems, as introduced in section 3.1. Recall the formulation of the optimal control problem in section 3.1. Given a dynamical system defined by the ordinary differential equations

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \ , & t \in [0, T] \\ x(0) = x_0 \ , \end{cases} \tag{29}$$

maximize the functional

$$J(u(\cdot)) = \int_0^T L(t, x(t), u(t)) \mathrm{d}t + \Phi(x(T)) \ , \tag{30}$$

over all admissible controls $u(\cdot) \in \mathcal{V}_{ad}[0, T]$ (see definition 3.1.)

4.1.1 Maximum principle with no state constraints

We will now introduce a formulation of Pontryagin's maximum principle with no state constraints. Consider the following assumptions:

(A1) $S(t) = \mathbb{R}^n$ for every $t \in [0, T]$.

(A2) U is a separable metric space (i.e. it contains a countable dense subset).

(A3) The maps $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $L : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.

Under the assumptions (A2) and (A3), for any $u(\cdot) \in \mathcal{V}[0, T]$, equation (29) admits a unique solution $x(\cdot; u(\cdot))$ and thus, (30) is well-defined. The reason is that the state trajectory $x(\cdot)$ obtained from (29) is bounded as long as f has linear growth.

It should be remarked that assumption (A3) can be weakened: it is enough to assume that f, L and Φ are continuously differentiable with respect to x, and uniformly continuous in t and u. Moreover, L and Φ may be allowed to have quadratic growth in x. Finally a comment concerning assumption (A2) to remark that any subset of $\mathbb{R}^m$ endowed with the Euclidean metric will satisfy this assumption.

The following result gives a set of first-order necessary condition for the existence of optimal pairs.

Theorem 4.1 (Pontryagin's maximum principle)**.** *Let (A1)-(A3) hold and let* $(\overline{x}(\cdot), \overline{u}(\cdot))$ *be an optimal pair for the optimal control problem under consideration. Then there exists a function* $p(\cdot) : [0, T] \to \mathbb{R}^n$ *satisfying the following differential equations*

$$\begin{cases} \dot{p}(t) = -f_x(t, \overline{x}(t), \overline{u}(t)) \cdot p(t) - L_x(t, \overline{x}(t), \overline{u}(t)) \ , & a.e.\ t \in [0, T], \\ p(T) = \Phi_x(\overline{x}(T)) \ , \end{cases} \tag{31}$$

and

$$H(t,\overline{x}(t),\overline{u}(t),p(t)) = \max_{u\in U} H(t,\overline{x}(t),u,p(t)) \ , \quad a.e.\ t\in[0,T], \tag{32}$$

where

$$H(t,x,u,p) = L(t,x,u) + f(t,x,u)\cdot p \ , \qquad (t,x,u,p)\in[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n. \tag{33}$$

We call $p(\cdot)$ the *adjoint variable* and (31) the *adjoint equation* (corresponding to the pair $(\overline{x}(\cdot),\overline{u}(\cdot))$). The function H defined by (33) is called *Hamiltonian* and the identity (32) is called the *maximum condition*.

The state equation (29), the corresponding adjoint equation (31), along with the maximum condition (32), can be written as:

$$\begin{cases} \dot{x}(t) = H_p(t,x(t),u(t),p(t)), & \text{a.e. } t\in[0,T], \\ \dot{p}(t) = -H_x(t,x(t),u(t),p(t)), & \text{a.e. } t\in[0,T], \\ x(0)=x_0,\ p(T)=\Phi_x(x(T)), & \\ H(t,x(t),u(t),p(t)) = \max_{u\in U} H(t,x(t),u,p(t)) & \text{a.e. } t\in[0,T]. \end{cases} \tag{34}$$

The above system of differential equations is called an *extended Hamiltonian system.* In what follows, if $(x(\cdot),u(\cdot))$ is an optimal pair and $p(\cdot)$ is the corresponding adjoint function, then $(x(\cdot),u(\cdot),p(\cdot))$ will be called an *optimal triple*.

The main ingredients for the proof of Pontryagin's maximum principle 4.1 are Taylor expansions of the state trajectory and the cost functional with respect to a perturbation of the control variable, and the duality between the variational equation (or linearized state equation) and the adjoint equation. The type of perturbation used is called the *spike variation technique.* We omit the details of the proof here. We provide only a sketch of the proof below.

Sketch of the proof of theorem 4.1. Consider the modified functional

$$\overline{J}[u(\cdot)] = J[u(\cdot)] - \int_0^T (\dot{x}(t) - f(t,x(t),u(t)))\cdot p(t)\mathrm{d}t$$

and note that the integrand function above is identically zero for any trajectory of (29). We consider the problem of maximizing the modified functional $\overline{J}$ instead of maximizing J. For convenience, define the Hamiltonian function

$$H(t,x,u,p) = L(t,x,u) + f(t,x,u)\cdot p$$

and note that $\overline{J}$ may be written as

$$\overline{J}[u(\cdot)] = \int_0^T H(t,x(t),u(t),p(t)) - \dot{x}(t)\cdot p(t)\ \mathrm{d}t + \Phi(x(T)) \ .$$

Let $(\overline{x},\overline{u})$ denote the optimal pair and let v be a perturbation of the optimal control $\overline{u}$ such that for each component of $\overline{u}-v$ satisfies

$$\int_0^T |\overline{u}_i(t) - v_i(t)|\mathrm{d}t < \epsilon \ ,$$

for some small $\epsilon>0$. The control v determines a new state trajectory $x^\epsilon(t) = \overline{x}(t) + \delta x(t)$. Note that the change $\delta x(t)$ is small for all $t\in[0,T]$.

Define the variation $\delta\overline{J}$ as

$$\delta\overline{J} = \overline{J}[v(\cdot)] - \overline{J}[\overline{u}(\cdot)]$$

and note that optimality of $\overline{u}$ ensures that $\delta\overline{J}$ is negative for all v close enough to $\overline{u}$. Clearly, we have that

$$\begin{aligned}\delta\overline{J} &= \int_0^T H(t,x^\epsilon(t),v(t),p(t)) - H(t,\overline{x}(t),\overline{u}(t),p(t)) - \dot{\delta x}(t)\cdot p(t)\ \mathrm{d}t \\ &\quad +\Phi(\overline{x}(T)+\delta x(t)) - \Phi(\overline{x}(T))\ .\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\delta\overline{J} &= \int_0^T H(t,x^\epsilon(t),v(t),p(t)) - H(t,\overline{x}(t),\overline{u}(t),p(t)) + \delta x(t)\cdot \dot{p}(t)\ \mathrm{d}t \\ &\quad +\Phi(\overline{x}(T)+\delta x(t)) - \Phi(\overline{x}(T)) - p(T)\delta x(T) + p(0)\delta x(0)\ .\end{aligned} \tag{35}$$

For convenience of notation, let us drop the dependence of the functions on the time variable t. Using the first order Taylor expansion, one gets

$$\begin{aligned}&\int_0^T H(t,x^\epsilon,v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t = \\ &\quad = \int_0^T H(t,\overline{x}+\delta x,v,p) - H(t,\overline{x},v,p) + H(t,\overline{x},v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t \\ &\quad = \int_0^T H_x(t,\overline{x},v,p)\delta x + H(t,\overline{x},v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t + O(\epsilon) \\ &\quad = \int_0^T H_x(t,\overline{x},u,p)\delta x + H(t,\overline{x},v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t + O(\epsilon)\ .\end{aligned} \tag{36}$$

Combining (35) with (36), we get

$$\begin{aligned}\delta\overline{J} &= \int_0^T (H_x(t,\overline{x},u,p) + \dot{p}(t))\delta x\ \mathrm{d}t + \int_0^T H(t,\overline{x},v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t \\ &\quad +\Phi(\overline{x}(T)+\delta x(t)) - \Phi(\overline{x}(T)) - p(T)\delta x(T) + p(0)\delta x(0) + O(\epsilon)\ .\end{aligned} \tag{37}$$

Recalling that the initial condition is fixed, and thus $\delta x(0) = 0$, and using a first order Taylor expansion once again, we obtain:

$$\begin{aligned}\delta\overline{J} &= \int_0^T (H_x(t,\overline{x},u,p) + \dot{p}(t))\delta x\ \mathrm{d}t + \int_0^T H(t,\overline{x},v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t \\ &\quad +(\Phi_x(\overline{x}(T)) - p(T))\delta x(T) + O(\epsilon)\ .\end{aligned} \tag{38}$$

Fixing the dynamics of the adjoint variable $p(t)$ to be given by

$$\begin{cases}\dot{p}(t) = -H_x(t,\overline{x}(t),u(t),p(t)) \\ p(T) = \Phi_x(x(T))\ ,\end{cases}$$

one gets from (38) that

$$\delta\overline{J} = \int_0^T H(t,\overline{x},v,p) - H(t,\overline{x},\overline{u},p)\ \mathrm{d}t + O(\epsilon)\ .$$

Since $p(t)$, $\overline{x}(t)$ and $\overline{u}(t)$ are known and are independent of $v(t)$, the condition above provides a necessary condition for optimality. It is possible to check that if the control function $\overline{u}(t)$ is optimal, then for any $t \in [0,T]$, one must have that

$$H(t,\overline{x}(t),v(t),p(t)) \leq H(t,\overline{x}(t),\overline{u}(t),p(t))\ ,$$

and thus, $\overline{u}(t)$ maximizes the Hamiltonian function, as required. $\square$

4.1.2 Alternative constraints

Note that in the formulation of theorem 4.1, no terminal condition was specified for the state variable, only an initial condition. Instead, the Pontryagin maximum principle provides a terminal condition on the adjoint variable p. Terminal conditions on the adjoint variables, also known as transversality conditions, are extremely important in optimal control theory. We will discuss some very simple, yet very important cases in this section.

I: Free endpoint. In this case we do not put any constraint on the terminal state $x(T)$. As we have already seen above, the transversality condition is

$$p(T) = \Phi_x(\bar{x}(T)) \ .$$

Note that this includes the condition $p(T) = 0$ in the special case of $\Phi(x) = 0$.

II: Fixed end point. This is the other extreme case from the free endpoint case. The terminal condition is

$$\bar{x}(T) = k \ .$$

and the transversality condition in theorem 4.1 does not provide any information for $p(T)$, which is a constant to be determined by solving the boundary value problem, where the differential equations system consists of the state equations with both initial and terminal conditions, and the adjoint equations with no boundary conditions.

III: One-sided constraint. If the ending value of the state variable is restricted to be in a one-sided interval such as

$$\bar{x}(T) \geq k \ ,$$

then the transversality conditions are of the form

$$p(T) \geq \Phi_x(\bar{x}(T)) \ ,$$

and

$$(p(T) - \Phi_x(\bar{x}(T)))(\bar{x}(T) - k) = 0 \ .$$

In the case where $\Phi = 0$, the transversality conditions above reduce to

$$p(T) \geq 0 \quad \text{and} \quad p(T)(\bar{x}(T) - k) = 0 \ .$$

IV: One-sided constraint. If the ending value of the state variable is restricted to be in a one-sided interval such as

$$\bar{x}(T) \leq k \ ,$$

then the transversality conditions are of the form

$$p(T) \leq \Phi_x(\bar{x}(T)) \ ,$$

and

$$(p(T) - \Phi_x(\bar{x}(T)))(\bar{x}(T) - k) = 0 \ .$$

In the case where $\Phi = 0$, the transversality conditions above reduce to

$$p(T) \leq 0 \quad \text{and} \quad p(T)(\bar{x}(T) - k) = 0 \ .$$

4.1.3 Infinite horizon

Up until this point, we have only considered problems whose horizon is finite. We will now briefly discuss the case where $T = \infty$ in the objective function. This case is especially important in many economics and management science problems.

Since the class of problems under consideration may be extremely hard to handle, we will introduce some simplifications. Namely, we will consider the case where:

(i) the function L depends on the time variable t only through a discounting factor, i.e.

$$L(t, x, u) = \exp(-\rho t)\ell(x, u) \ , \quad \rho > 0 \ .$$

(ii) the function f does not dependent on the time variable t.

(iii) the function Φ is identically zero.

Then, the optimal pair must satisfy Hamilton's equations

$$\begin{cases} \dot{x}(t) = H_p(t, x(t), u(t), p(t)) \\ \dot{p}(t) = -H_x(t, x(t), u(t), p(t)) \\ x(0) = x_0 \\ H(t, x(t), u(t), p(t)) = \max_{u \in U} H(t, x(t), u, p(t)) \end{cases}$$

together with some transversality conditions. We list a couple of simple and useful cases below:

I: Free endpoint. In this case the transversality condition is

$$\lim_{T \to +\infty} \exp(-\rho t)p(T) = 0 \ .$$

II: One-sided constraint. If the ending value of the state variable is of the form

$$\lim_{T \to +\infty} \bar{x}(T) \geq 0 \ ,$$

then the transversality conditions are of the form

$$\lim_{T \to +\infty} \exp(-\rho t)p(T) = 0 \quad \text{and} \quad \lim_{T \to +\infty} \exp(-\rho t)p(T)\bar{x}(T) = 0 \ .$$

4.1.4 Sufficient condition

The extended Hamiltonian system (34) partially characterizes the optimality of the optimal control problem of section 4.1.1. In cases where certain convexity condition are presented, the Hamiltonian system of the above form fully characterizes an optimal control.

The above maximum principle gives only necessary conditions for the optimal controls. We now provide a *sufficient condition* for optimality. Consider the following assumption.

(A4) The control domain U is a is convex subset of $\mathbb{R}^k$ with nonempty interior, and the maps b and f are locally Lipschitz in the control variable u.

Theorem 4.2 (Sufficient Conditions of Optimality)**.** *Let the conditions (A1)-(A4) hold. Let $(\overline{x}(\cdot), \overline{u}(\cdot))$ be an admissible pair and let $p(\cdot)$ be the corresponding adjoint variable. Assume that $\Phi(\cdot)$ is concave and that $H(t, \cdot, \cdot, p)$ is a concave function of (x, u) for every $(t, p) \in [0, T] \times \mathbb{R}^n$. Then $(\overline{x}(\cdot), \overline{u}(\cdot))$ is optimal if*

$$H(t, \overline{x}(t), \overline{u}(t), p(t)) = \max_{u \in U} H(t, \overline{x}(t), u, p(t)), \ \ a.e.\ t \in [0, T]. \tag{39}$$

The concavity conditions in the theorem above can be used to provide sufficient conditions for the existence of a solution to the Optimal Control problem if there exist constraints of the form discussed in section 4.1.2 or the horizon is infinite as in section 4.1.3.

4.1.5 Example

Consider the example 2 in section 3.1. We will use the maximum principle to find the optimal control and the corresponding state trajectory. Start by defining the Hamiltonian function

$$H(t, k, c, p) = L(t, k, c) + f(t, k, c) \cdot p \ ,$$

where

$$\begin{aligned} L(t,k,c) &= \mathrm{e}^{-\gamma t}U(c) \\ f(t,k,c) &= w + rk - c \ , \end{aligned}$$

The first order condition for c to maximize H is given by

$$\frac{\partial H}{\partial c} = 0 \ .$$

Computing the derivative above, one gets

$$\mathrm{e}^{-\gamma t}U'(c) - p = 0 \ ,$$

providing the optimal consumption $c^*(\cdot)$:

$$c^*(t) = I(p\mathrm{e}^{\gamma t}) \ ,$$

where $I(\cdot)$ denotes the inverse of $U'(\cdot)$. Therefore, the maximized Hamiltonian function is given by

$$H^*(t,k,c^*,p) = \mathrm{e}^{-\gamma t}U\left(I(p\mathrm{e}^{\gamma t})\right) + \left(w + rk - I(p\mathrm{e}^{\gamma t})\right)p \ .$$

The differential equations for k and p are then

$$\begin{aligned} \dot{k} &= \frac{\partial H^*}{\partial p} = U'\left(I(p\mathrm{e}^{\gamma t})\right)I'(p\mathrm{e}^{\gamma t}) + w + rk - I(p\mathrm{e}^{\gamma t}) - \mathrm{e}^{\gamma t}I'(p\mathrm{e}^{\gamma t})p \\ \dot{p} &= -\frac{\partial H^*}{\partial k} = -rp \ , \end{aligned}$$

with boundary conditions $k(0) = k(T) = 0$. Since I is the inverse function of U', the equations above simplify to

$$\begin{aligned} \dot{k} &= w + rk - I(p\mathrm{e}^{\gamma t}) \\ \dot{p} &= -rp \ , \end{aligned}$$

Note that the second differential equation has solutions of the form

$$p(t) = p_0 \exp(-rt) \ ,$$

where p_0 is a constant to be determined from the boundary conditions. Substituting $p(t)$ in the differential equation for $k(t)$, we obtain

$$\dot{k} - rk - w + I\left(p_0\mathrm{e}^{(\gamma-r)t}\right) = 0 \ .$$

Multiplying by the integrating factor $\mu(t) = \exp(-rt)$, one gets

$$\mu\dot{k} - r\mu k - \mu\left(w - I\left(p_0\mathrm{e}^{(\gamma-r)t}\right)\right) = 0 \ ,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu k) = \mu\left(w - I\left(p_0\mathrm{e}^{(\gamma-r)t}\right)\right) \ .$$

The solution $k(t)$ to the ordinary differential equation above is given by

$$k(t) = \exp(rt)k(0) + \exp(rt)\int_0^t \exp(-ru)\left(w - I\left(p_0\mathrm{e}^{(\gamma-r)u}\right)\right)\mathrm{d}u \ ,$$

where $k(0)$ is given, and p_0 is determined using the boundary condition $k(T)$.

4.1.6 Exercises

Exercise 1. Consider again example 2 in section 3.1 with $U(c) = \ln(c)$. Assume that $k(0) = 0$ and $k(T) = 0$.

(i) Use Pontryagin's maximum principle to obtain a necessary condition for optimality.
(ii) Find the optimal consumption $c^*(t)$.
(iii) Solve Hamilton's equations to obtain the optimal state trajectory.
(iv) Show that the pair obtained in items (ii) and (iii) is indeed a solution for the optimal control problem.

Exercise 2. Consider again example 2 in section 3.1 with $U(c) = c^{1-\nu}/(1-\nu)$, $\nu \in (0, 1)$. Assume that $k(0) = 0$ and $k(T) = 0$.

(i) Use Pontryagin's maximum principle to obtain a necessary condition for optimality.
(ii) Find the optimal consumption $c^*(t)$.
(iii) Solve Hamilton's equations to obtain the optimal state trajectory.
(iv) Show that the pair obtained in items (ii) and (iii) is indeed a solution for the optimal control problem.

Exercise 3. Consider the previous problem with $k(0) = k_0$ and $k(T) = k_T$. How large can k_T be before the problem has no feasible solution.

Exercise 4. Consider the optimal control problem with infinite horizon of example 3 in section 3.1 with $U(c) = \ln(c)$ and $F(k) = \alpha k$, with $\alpha > 0$. Assume that $k(0) = k_0$.

(i) Use Pontryagin's maximum principle to obtain a necessary condition for optimality.
(ii) Find the optimal consumption $c^*(t)$.
(iii) Solve Hamilton's equations to obtain the optimal state trajectory.
(iv) Show that the pair obtained in items (ii) and (iii) is indeed a solution for the optimal control problem.

Exercise 5. Consider the optimal control problem with infinite horizon of example 3 in section 3.1 with $U(c) = c^{1-\nu}/(1-\nu)$, $\nu \in (0, 1)$, and $F(k) = \alpha k$, with $\alpha > 0$. Assume that $k(0) = k_0$.

(i) Use Pontryagin's maximum principle to obtain a necessary condition for optimality.
(ii) Find the optimal consumption $c^*(t)$.
(iii) Solve Hamilton's equations to obtain the optimal state trajectory.
(iv) Show that the pair obtained in items (ii) and (iii) is indeed a solution for the optimal control problem.

4.2 The discrete-time case

In this section we discuss Pontryagin's maximum Principle in the setup of discrete-time dynamical systems, as introduced in section 3.2. Given a dynamical system defined by the difference equations

$$\begin{cases} x(t+1) = f(t, x(t), u(t)) \,, & t \in \Theta \\ x(0) = x_0 \end{cases} \tag{40}$$

maximize the functional

$$J(u(\cdot)) = \sum_{t=0}^{T-1} L(t, x(t), u(t)) + \Phi(x(T)) \,,$$

over all admissible controls $u(\cdot) \in \mathcal{V}_{ad}^{\Theta}$ (see definition 3.3.)

4.2.1 Maximum principle with no state constraints

We will now introduce a formulation of Pontryagin's maximum principle with no state constraints and finite horizon. Consider the following assumptions:

(A1) $S(t) = \mathbb{R}^n$ for every $t \in \Theta$.

(A2) U is a separable metric space (i.e. it contains a countable dense subset).

(A3) The maps $f : \Theta \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $L : \Theta \times \mathbb{R}^n \times U \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.

The following result gives a set of first-order necessary condition for the existence of optimal pairs.

Theorem 4.3 (Discrete-time Pontryagin's maximum principle)**.** *Let (A1)-(A3) hold and let* $(\overline{x}(\cdot), \overline{u}(\cdot))$ *be an optimal pair for the optimal control problem under consideration. Then there exists an adjoint function* $p(\cdot) : \Theta \to \mathbb{R}^n$ *such that* $\overline{x}(\cdot), \overline{u}(\cdot), p(\cdot)$ *satisfy*

$$\begin{cases} \overline{x}(t+1) = H_p(t, \overline{x}(t), \overline{u}(t), p(t+1)) \\ p(t) = H_x(t, \overline{x}(t), \overline{u}(t), p(t+1)) \\ p(T) = \Phi_x(\overline{x}(T)) \\ \overline{x}(0) = x_0 \end{cases} \tag{41}$$

and

$$H_u(t, \overline{x}(t), \overline{u}(t), p(t+1)) = 0 \tag{42}$$

where

$$H(t, x, u, p) = L(t, x, u) + f(t, x, u) \cdot p \ , \qquad (t, x, u, p) \in \Theta \times \mathbb{R}^n \times U \times \mathbb{R}^n. \tag{43}$$

Similarly to the continuous-time case, we call $p(\cdot)$ the *adjoint variable* and (41) the *extended Hamiltonian system.* The identity (42) replaces the maximum condition of the continuous-time case, and is called *variational condition.*

Proof. Left as an exercise. Similar to the continuous-time case. □

If the matrix H_{uu} is non-singular, then the implicit function theorem guarantees that (42) defines the optimal control $\bar{u}(\cdot)$ as a function of the state and adjoint variables:

$$\bar{u}(t) = g\left(t, \overline{x}(t), p(t+1)\right) \ .$$

Substituting the optimal control above in (41), we obtain a system of difference equations of the form

$$\begin{cases} \overline{x}(t+1) = H_p(t, \overline{x}(t), g\left(t, \overline{x}(t), p(t+1)\right), p(t+1)) \\ p(t) = H_x(t, \overline{x}(t), g\left(t, \overline{x}(t), p(t+1)\right), p(t+1)) \\ p(T) = \Phi_x(\overline{x}(T)) \\ \overline{x}(0) = x_0 \end{cases} . \tag{44}$$

4.2.2 Alternative constraints

As done in the continuous-time case, alternative boundary conditions can be considered for the state variable $x(\cdot)$, leading to distinct transversality conditions.

I: Free endpoint. This is the case of theorem 4.3. The transversality condition reduces to:

$$p(T) = \Phi_x(\bar{x}(T)) \ .$$

II: Fixed end point. If the terminal condition is

$$\bar{x}(T) = k \ ,$$

then the transversality condition in theorem 4.3 does not provide any information for $p(T)$, being just a constant to be determined by solving the boundary value problem of difference equations.

III: One-sided constraint. If the state variable has a constraint of the form

$$\bar{x}(T) \geq k \ ,$$

then the transversality conditions is given by

$$p(T) \geq \Phi_x(\bar{x}(T)) \ ,$$

and

$$(p(T) - \Phi_x(\bar{x}(T)))(\bar{x}(T) - k) = 0 \ .$$

IV: One-sided constraint. If the state variable has a constraint of the form

$$\bar{x}(T) \leq k \ ,$$

then the transversality conditions is given by

$$p(T) \leq \Phi_x(\bar{x}(T)) \ ,$$

and

$$(p(T) - \Phi_x(\bar{x}(T)))(\bar{x}(T) - k) = 0 \ .$$

4.2.3 Infinite horizon

Similarly to what was done in the case of the continuous-time problem, we introduce some simplifications to deal with the discrete-time infinite horizon optimal control problem. Suppose that:

(i) the function L depends on the time variable t only through a discounting factor, i.e.

$$L(t, x, u) = \beta^t \ell(x, u) \ , \quad \beta \in (0, 1) \ .$$

(ii) the function f does not dependent on the time variable t.

(iii) the function Φ is identically zero.

Then, the optimal pair must satisfy:

$$\begin{cases} \overline{x}(t+1) = H_p(t, \overline{x}(t), \overline{u}(t), p(t+1)) \\ p(t) = H_x(t, \overline{x}(t), \overline{u}(t), p(t+1)) \\ x(0) = x_0 \\ H_u(t, \overline{x}(t), \overline{u}(t), p(t+1)) = 0 \end{cases}$$

together with some transversality conditions. We deal with the two most useful (and easy) cases now:

I: Free endpoint. In this case the transversality condition is

$$\lim_{T \to +\infty} \beta^T p(T) = 0 \ .$$

II: One-sided constraint. If the ending value of the state variable is of the form

$$\lim_{T \to +\infty} \bar{x}(T) \geq 0 \ ,$$

then the transversality conditions are of the form

$$\lim_{T \to +\infty} \beta^T p(T) = 0 \quad \text{and} \quad \lim_{T \to +\infty} \beta^T p(T) \bar{x}(T) = 0 \ .$$

4.2.4 Example

Consider the example 4 in section 3.2. We will use the maximum principle to find the optimal control and the corresponding state trajectory. Start by defining the Hamiltonian function

$$H(t,k,c,p) = L(t,w,c) + f(t,w,c)\cdot p \ ,$$

where

$$\begin{aligned} L(t,w,c) &= \beta^t U(c) \\ f(t,w,c) &= (1+r)(w-c) \ , \end{aligned}$$

The variational equation is given by

$$\frac{\partial H}{\partial c}(t,w_t,c_t,p_{t+1}) = 0 \ .$$

Computing the derivative above, one gets

$$\beta^t U'(c_t) - (1+r)p_{t+1} = 0 \ ,$$

providing the optimal consumption $c^*(\cdot)$:

$$c_t^* = I((1+r)\beta^{-t}p_{t+1}) \ ,$$

where $I(\cdot)$ denotes the inverse of $U'(\cdot)$. Therefore, the maximized Hamiltonian function is given by

$$H^*(t,w,c^*,p) = \beta^t U\left(I((1+r)\beta^{-t}p)\right) + (1+r)\left(w - I((1+r)\beta^{-t}p)\right)p \ .$$

The difference equations for w and p are then

$$\begin{aligned} w_{t+1} &= \frac{\partial H^*}{\partial p}(t,w_t,c_t^*,p_{t+1}) \\ p_t &= \frac{\partial H^*}{\partial w}(t,w_t,c_t^*,p_{t+1}) \ , \end{aligned}$$

with boundary conditions $w(0) = w_0$ and $w(T) = w_T$. Evaluating the partial derivatives of the Hamiltonian function, one gets

$$\begin{aligned} w_{t+1} &= (1+r)U'\left(I((1+r)\beta^{-t}p_{t+1})\right)I'((1+r)\beta^{-t}p_{t+1}) \\ &\quad (1+r)\left(w_t - I((1+r)\beta^{-t}p_{t+1})\right) - (1+r)^2\beta^{-t}I'((1+r)\beta^{-t}p_{t+1})p_{t+1} \\ p_t &= (1+r)p_{t+1} \ , \end{aligned}$$

Since I is the inverse function of U', the equations above simplify to

$$\begin{aligned} w_{t+1} &= (1+r)\left(w_t - I((1+r)\beta^{-t}p_{t+1})\right) \\ p_t &= (1+r)p_{t+1} \ , \end{aligned}$$

Note that the second difference equation has solutions of the form

$$p_t = p_0(1+r)^{-t} \ ,$$

where p_0 is a constant to be determined from the boundary conditions. Substituting p_{t+1} in the difference equation for w, we obtain

$$w_{t+1} = (1+r)\left(w_t - I\left(((1+r)\beta)^{-t}p_0\right)\right) \ ,$$

which must be solved with boundary conditions w_0 and w_T.

4.2.5 Exercises

Exercise 1. Consider again example 4 in section 3.2 with $U(c) = \ln(c)$.

(i) Use Pontryagin's maximum principle to obtain a necessary condition for optimality.

(ii) Find the optimal consumption $c^*(t)$ as a function of the state and adjoint variables.

Exercise 2. Consider again example 4 in section 3.2 with $U(c) = c^{1-\nu}/(1-\nu)$, $\nu \in (0,1)$.

(i) Use Pontryagin's maximum principle to obtain a necessary condition for optimality.

(ii) Find the optimal consumption $c^*(t)$ as a function of the state and adjoint variables.

5 The dynamic programming principle

We will now study another method which may be used to solve optimal control problems, namely, the method of dynamic programming. Dynamic programming originated from the work of R. Bellman in the early 1950s. It is a mathematical technique that produces a sequence of interrelated decisions, and can be applied to many optimization problems. The basic idea of the dynamic programming method when applied to optimal control problems is to consider a family of optimal control problems with different initial times and states, and then establish relationships among these problems through the Hamilton-Jacobi-Bellman equation (HJB), a first-order partial differential equation in the cases we will be considering in this notes. If the HJB equation is solvable, then the optimal feedback control is obtained by taking the maximizer of the Hamiltonian function involved in the HJB equation.

5.1 The continuous-time case

Recall the formulation of the continuous-time optimal control problem of section 3.1 and note that the initial time $t = 0$ and the initial state $x(0) = x_0$ are fixed. The idea behind the dynamic programming method is to consider a family of optimal control problems with different initial times and states, and then to establish relationships among these problems, so that one finally solves all of them at once.

Let $(s, y) \in [0, T) \times \mathbb{R}^n$ and consider the following control system over the interval $[s, T]$

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [s, T], \\ x(s) = y\ . \end{cases} \tag{45}$$

The control $u(\cdot)$ is a measurable function from the interval $[s, T]$ to the metric space U, i.e. $u(\cdot) \in \mathcal{V}[s, T]$ where $\mathcal{V}[s, T]$ is of the form

$$\mathcal{V}[s, T] = \{u(\cdot) : u : [s, T] \to U \text{ is measurable}\}\ .$$

The functional to be maximized is given by

$$J(s, y; u(\cdot)) = \int_s^T L(t, x(t), u(t))\mathrm{d}t + \Phi(x(T)). \tag{46}$$

The optimal control problem can then be restated as follows. For some given pair $(s, y) \in [0, T) \times \mathbb{R}^n$, maximize the function (46) subject to the dynamical system (45) over the set $\mathcal{V}[s, T]$. Note that the optimal control problem just stated is indeed a family of optimal control problems parameterized by $(s, y) \in [0, T) \times \mathbb{R}^n$. Moreover, the original optimal control problem of section 3.1 coincides with the problem stated above when $s = 0$ and $y = x_0$.

5.1.1 Hamilton-Jacobi-Bellman equation

Consider the following assumptions:

(A1) The horizon $T > 0$ is finite and there are no constraints on the state variable apart from the initial condition.

(A2) U is a separable metric space.

(A3) The maps $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $L : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.

Condition (A3) above can be further relaxed. It is enough to assume that f, L and Φ are continuously differentiable with respect to x and uniformly continuous in t and u.

As in the previous section, under the assumptions (A2) and (A3), for any $(s, y) \in [0, T) \times \mathbb{R}^n$ and $u(\cdot) \in \mathcal{V}[s, T]$, the differential equation (45) admits a a unique solution $x(\cdot) =$

$x(\cdot; s, y, u(\cdot))$, and the functional (46) is well-defined. Furthermore, the *value function* is well-defined by the conditions

$$\begin{cases} V(s,y) = \sup_{u(\cdot)\in\mathcal{V}[s,T]} J(s,y;u(\cdot)) \\ V(T,y) = \Phi(y) \end{cases}, \quad (s,y) \in [0,T) \times \mathbb{R}^n, \tag{47}$$

The value function V plays a very important role in obtaining optimal controls. The following result, called *Bellman's principle of optimality*, characterizes the function V

Theorem 5.1 (Dynamic Programming principle). *Let (A1)-(A3) hold. Then for any $(s,y) \in [0,T) \times R^n$, we have that*

$$V(s,y) = \sup_{u(\cdot)\in\mathcal{V}[s,T]} \Big\{ \int_s^{s'} L(t, x(t; s, y, u(\cdot)), u(t))\mathrm{d}t + V(s', x(s'; s, y, u(\cdot))) \Big\} \tag{48}$$

for every $0 \le s \le s' \le T$.

Proof. Denote the right-hand side of (48) by $\overline{V}(s,y)$. Using (47), we have that

$$V(s,y) \ge J(s,y;u(\cdot)) = \int_s^{s'} L(t, x(t), u(t))\mathrm{d}t + J(s', x(s'); u(\cdot))$$

for every $u(\cdot) \in \mathcal{V}[s,T]$. Taking the supremum over $u(\cdot) \in \mathcal{V}[s,T]$ we get

$$V(s,y) \ge \overline{V}(s,y). \tag{49}$$

Conversely, for any $\epsilon > 0$ there exists $u_\epsilon(\cdot) \in \mathcal{V}[s,T]$ such that

$$\begin{aligned} V(s,y) - \epsilon &\le J(s,y;u_\epsilon(\cdot)) \\ &\le \int_s^{s'} L(t, x_\epsilon(t), u_\epsilon(t))\mathrm{d}t + V(s', x_\epsilon(s')) \\ &\le \overline{V}(s,y) \ , \end{aligned} \tag{50}$$

where $x_\epsilon(\cdot) = x(\cdot; s, y, u_\epsilon(\cdot))$. Combining (49) and (50), we obtain the desired result. □

One remark is now order in concerning identity (48). Suppose $(\overline{x}(\cdot), \overline{u}(\cdot))$ is an optimal pair for the sequence of optimal control problems under consideration, and let $s' \in (s,T)$. Then, we have that

$$\begin{aligned} V(s,y) &= J(s,y;\overline{u}(\cdot)) \\ &= \int_s^{s} L(t, \overline{x}(t), \overline{u}(t))\mathrm{d}t + J(s', \overline{x}(s'), \overline{u}(\cdot)) \\ &\le \int_s^{s'} L(t, \overline{x}(t), \overline{u}(t))\mathrm{d}t + V(s', \overline{x}(s')) \\ &\le V(s,y) \ , \end{aligned} \tag{51}$$

where the last inequality is due to theorem 5.1. Therefore, all the equalities in (51) must hold, and in particular we get that

$$V(s', \overline{x}(s')) = J(s', \overline{x}(s'); \overline{u}(\cdot)) \equiv \int_{s'}^{T} L(t, \overline{x}(t), \overline{u}(t))\mathrm{d}t + \Phi(\overline{x}(T)) \ .$$

The condition above is the essence of Bellman's principle of optimality. In particular, we are able to conclude that $\overline{u}(\cdot)$ is optimal on $[s,T]$ with initial data $(s,y))$, then $\overline{u}|_{[s',T]}(\cdot)$ is optimal on $[s',T]$ with initial data $(s', \overline{x}(s'))$. From an heuristic point of view, the principle of optimality corresponds to the following statement: a globally optimal solution determines a locally optimal solution.

We refer to identity (48) as the dynamic programming equation. It provides a necessary condition for $\overline{u}(\cdot)$ to be an optimal control. However, the dynamic programming equation is too complicated to work with. The following result gives a partial differential equation that a continuously differentiable value function must satisfy.

Theorem 5.2 (The Hamilton-Jacobi-Bellman equation)**.** *Assume that the conditions (A1)-(A3) hold. Suppose that the value function is C^1. Then V is a solution to the following terminal value problem of a first-order partial differential equation*

$$\begin{cases} V_t + \sup_{u\in U} H(t,x,u,V_x) = 0 \\ V(T,x) = \Phi(x) \end{cases} , \qquad (t,x) \in [0,T) \times \mathbb{R}^n , \tag{52}$$

where the Hamiltonian function is given by

$$H(t,x,u,p) = L(t,x,u) + f(t,x,u) \cdot p , \qquad (t,x,u,p) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n. \tag{53}$$

The partial differential equation (52) is called the Hamilton-Jacobi-Bellman equation (HJB).

Proof. Fix an arbitrary point $u \in U$ and let $x(\cdot)$ be the state trajectory corresponding to the control $u(t) \equiv u$. Using the dynamic programming principle 5.1 with $0 < s < s' < T$, we have

$$0 \geq \frac{V(s',x(s')) - V(s,y)}{s'-s} + \frac{1}{s'-s}\int_s^{s'} L(t,x(t),u)\mathrm{d}t .$$

Letting $s' \to s$, we get

$$0 \quad \geq \quad V_t(s,y) + V_x(s,y) \cdot f(s,y,u) + L(s,y,u) ,$$

which results in

$$0 \geq V_t(s,y) + \sup_{u\in U} H(s,y,u,V_x(s,y)) . \tag{54}$$

For any $\epsilon > 0$ and every $0 \leq s \leq s' \leq T$ such that $s' - s > 0$ is small enough, there exists a control $u(\cdot) = u_{\epsilon,s'}(\cdot) \in \mathcal{V}[s,T]$ such that

$$V(s,y) - \epsilon(s'-s) \leq \int_s^{s'} L(t,x(t),u(t))\mathrm{d}t + V(s',x(s')) .$$

Thus, it follows that (noting $V \in C^1([0,T] \times \mathbb{R}^n)$)

$$\begin{aligned} -\epsilon \quad &\leq \quad \frac{V(s',x(s')) - V(s,y)}{s'-s} + \frac{1}{s'-s}\int_s^{s'} L(t,x(t),u(t))\mathrm{d}t \\ &= \quad \frac{1}{s'-s}\int_s^{s'} \Big\{V_t(t,x(t)) + V_x(t,x(t)) \cdot f(t,x(t),u(t)) + L(t,x(t),u(t))\Big\}\mathrm{d}t \\ &= \quad \frac{1}{s'-s}\int_s^{s'} \Big\{V_t(t,x(t)) + H(t,x(t),u(t),V_x(t,x(t)))\Big\}\mathrm{d}t \\ &\leq \quad \frac{1}{s'-s}\int_s^{s'} \Big\{V_t(t,x(t)) + \sup_{u\in U} H(t,x(t),u,V_x(t,x(t)))\Big\}\mathrm{d}t \\ &\to \quad V_t(s,y) + \sup_{u\in U} H(s,y,u,V_x(s,y)) , \quad \text{as } s' \to s. \end{aligned} \tag{55}$$

For the proof of the last limit we use uniform continuity of the functions L and f. The result follows from combining (54) and (55). □

The solutions of the HJB equation provide a strategy to construct the optimal control. Suppose the value function V is obtained by solving the HJB equation (52). Moreover, assume that for any $(t,x) \in [0,T] \times \mathbb{R}^n$, the supremum in (52) is achieved at $u = \overline{u}(t,x)$, i.e. we have that

$$H(t,x,\overline{u}(t,x),V_x(t,x)) = \sup_{u\in U} H(t,x,u,V_x(t,x)) \ . \tag{56}$$

In addition, suppose that for any $(s,y) \in [0,T) \times \mathbb{R}^n$, there exists a solution $\overline{x}(\cdot)$ to the following differential equation

$$\begin{cases} \dot{\overline{x}}(t) = f(t,\overline{x}(t),\overline{u}(t,\overline{x}(t))) \ , & t \in [s,T] \\ \overline{x}(s) = y. \end{cases} \tag{57}$$

Set $\overline{u}(t) = \overline{u}(t,\overline{x}(t))$. Then, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(t,\overline{x}(t)) &= V_t(t,\overline{x}(t)) + V_x(t,\overline{x}(t)) \cdot f(t,\overline{x}(t),\overline{u}(t)) \\ &= -L(t,\overline{x}(t),\overline{u}(t)) \ . \end{aligned}$$

Integrating (58) from s to T, we obtain

$$V(s,y) = \int_s^T L(t,\overline{x}(t),\overline{u}(t))\mathrm{d}t + \Phi(\overline{x}(T)) = J(s,y;\overline{u}(\cdot)) \ ,$$

that is, $(\overline{x}(\cdot),\overline{u}(\cdot))$ is an optimal pair.

The strategy described above is based on the following steps the original optimal control problem of section 3.1:

1) Solve the HJB equation (52) to find the value function $V(t,x)$.
2) Use the maximum condition (56) to find $\overline{u}(t,x)$.
3) Solve (57) with $(s,y) = (0,x_0)$ to get the optimal pair $(\overline{x}(\cdot),\overline{u}(\cdot))$.

5.1.2 Verification theorem

The following result provides a strategy to construct optimal feedback controls. Such a result is called a verification theorem.

Theorem 5.3 (Verification theorem)**.** *Assume that the conditions (A1)-(A3) hold and let* $V \in C^1([0,T] \times \mathbb{R}^n, \mathbb{R})$ *be a solution of the HJB equation* (52)*. Then*

$$V(s,y) \geq J(s,y;u(\cdot)) \tag{58}$$

for all $u(\cdot) \in \mathcal{V}[s,T]$ *and every* $(s,y) \in [0,T) \times \mathbb{R}^n$*. Furthermore, a given admissible pair* $(\overline{x}(\cdot),\overline{u}(\cdot))$ *is optimal if and only if*

$$V_t(t,\overline{x}(t)) + \max_{u\in U} H(t,\overline{x}(\cdot),\overline{u}(\cdot),V_x(t,\overline{x}(t))) = 0 \ .$$

Proof. For any $u(\cdot) \in \mathcal{V}[s,T]$ with the corresponding state trajectory $x(\cdot)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(t,x(t)) &= V_t(t,x(t)) + V_x(t,x(t)) \cdot f(t,x(t),u(t)) \\ &= V_t(t,x(t)) + H(t,x(t),u(t),V_x(t,\overline{x}(t))) - L(t,x(t),u(t)) \\ &\leq -L(t,x(t),u(t)) \ , \end{aligned} \tag{59}$$

where the last inequality in (59) is due to the HJB equation (52). Integrating both sides of the inequality above from s to T gives

$$V(T,\overline{x}(T)) - V(s,y) \leq -\int_s^T L(t,\overline{x}(t),\overline{u}(t))\mathrm{d}t \ ,$$

which reduces to

$$V(s,y) \;\geq\; \int_s^T L(t,\overline{x}(t),\overline{u}(t))\mathrm{d}t + \Phi(\overline{x}(T)) \ ,$$

completing the proof of inequality (58) in the statement of the theorem.

Applying the second equality of (59) to $(\overline{x}(\cdot),\overline{u}(\cdot))$, and integrating from s to T, we have

$$\begin{aligned} V(T,\overline{x}(T)) - V(s,y) \;&=\; -\int_s^T L(t,\overline{x}(t),\overline{u}(t))\mathrm{d}t \\ &\quad + \int_s^T V_t(t,\overline{x}(t)) + H(t,\overline{x}(t),\overline{u}(t),V_x(t,\overline{x}(t)))\mathrm{d}t \ . \end{aligned}$$

or equivalently

$$V(s,y) \;=\; J(s,y;\overline{u}(\cdot)) - \int_s^T V_t(t,\overline{x}(t)) + H(t,\overline{x}(t),\overline{u}(t),V_x(t,\overline{x}(t)))\mathrm{d}t \ .$$

The result now follows from the inequality

$$V_t(t,\overline{x}(t)) + H(t,\overline{x}(t),\overline{u}(t),V_x(t,\overline{x}(t))) \leq 0 \ ,$$

due to the HJB equation (52). □

5.1.3 Infinite horizon

Consider the following additional assumptions:

(i) the function L depends on the time variable t only through a discounting factor, i.e.

$$L(t,x,u) = \exp(-\rho t)\ell(x,u) \ , \quad \rho > 0 \ .$$

(ii) the function f does not dependent on the time variable t.

(iii) the function Φ is identically zero.

Then, the value function $V(t,x)$ must be of the form

$$V(t,x) = e^{-\rho t}V(x) \ ,$$

and, therefore, the HJB equation reduces to the following ordinary differential equation

$$-\rho V(x) + \sup_{u\in U} H(t,x,u,V_x(x)) = 0 \ .$$

5.1.4 Example

Consider the example 2 in section 3.1. Recall that the Hamiltonian function is given by

$$H(t,k,c,p) = L(t,k,c) + f(t,k,c)\cdot p \ ,$$

where

$$\begin{aligned} L(t,k,c) \;&=\; \mathrm{e}^{-\gamma t}U(c) \\ f(t,k,c) \;&=\; w + rk - c \ . \end{aligned}$$

Therefore, the HJB equation is given by

$$V_t(t,k) + \sup_{c\in U}\left[\mathrm{e}^{-\gamma t}U(c) + (w+rk-c)V_k(t,k)\right] = 0$$

with boundary condition

$$V(T,k) = 0 \ .$$

Note that the Hamiltonian H is a concave function of the control variable c and thus, one can obtain the optimal control $c^*(t,k,V(t,k))$ from first order condition

$$\frac{\partial H}{\partial c} = 0 \ .$$

Computing the derivative above, one gets

$$\mathrm{e}^{-\gamma t} U'(c) - V_k(t,k) = 0 \ .$$

Thus, the optimal consumption $c^*(\cdot)$ is given by

$$c^*(t) = I(V_k(t,k)\mathrm{e}^{\gamma t}) \ ,$$

where $I(\cdot)$ denotes the inverse of $U'(\cdot)$. Therefore, the HJB equation reduces to

$$V_t(t,k) + \mathrm{e}^{-\gamma t} U\left(I(V_k(t,k)\mathrm{e}^{\gamma t})\right) + (w + rk - I\left(V_k(t,k)\mathrm{e}^{\gamma t}\right))V_k(t,k) = 0 \ .$$

It should be noted that for general utility functions U, the HJB equation above is nonlinear and usually impossible to solve analytically.

5.1.5 Exercises

Exercise 1. Consider again example 2 in section 3.1 with $U(c) = \ln(c)$.

(i) Find the HJB equation associated with this problem.

(ii) Use the maximum condition to determine the optimal consumption $c^*(t)$ as a function of the solution of the HJB equation.

Exercise 2. Consider again example 2 in section 3.1 with $U(c) = c^{1-\nu}/(1-\nu)$, $\nu \in (0,1)$.

(i) Find the HJB equation associated with this problem.

(ii) Use the maximum condition to determine the optimal consumption $c^*(t)$ as a function of the solution of the HJB equation.

5.2 The discrete-time case

Recall the formulation of the discrete-time optimal control problem of section 3.2 and note that the initial time $t = 0$ and the initial state $x(0) = x_0$ are fixed. As done in the continuous-time case, we will consider a family of optimal control problems with different initial times and states.

Let $(s,y) \in \Theta \times \mathbb{R}^n$ and consider the following control system over the set $\Theta^s = \{s, s+1, \ldots, T-1\}$.

$$\begin{cases} x(t+1) = f(t,x(t),u(t)) \ , & t \in \Theta \\ x(s) = y \end{cases} \tag{60}$$

The control $u(\cdot)$ is a measurable function from the set Θ^s to the metric space U, i.e. $u(\cdot) \in \mathcal{V}^{\Theta^s}$ where $\mathcal{V}^{\Theta^s}$ is the set

$$\mathcal{V}^{\Theta^s} = \{u(\cdot) : u : \Theta^s \to U \ \text{ is measurable}\} \ .$$

The functional to be maximized is given by

$$J(s,y;u(\cdot)) = \sum_{t=s}^{T-1} L(t,x(t),u(t)) + \Phi(x(T)) \ .$$

For some given pair $(s,y) \in [0,T) \times \mathbb{R}^n$, the goal is to maximize the function (5.2) subject to the dynamical system (60) over the set $\mathcal{V}^{\Theta^s}$. The original optimal control problem of section 3.2 coincides with the problem stated above when $s = 0$ and $y = x_0$.

5.2.1 Hamilton-Jacobi-Bellman equation

Consider the following assumptions:

(A1) The horizon $T > 0$ is finite and there are no constraints on the state variable apart from the initial condition.

(A2) U is a separable metric space.

(A3) The maps $f : \Theta \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $L : \Theta \times \mathbb{R}^n \times U \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.

The *value function* is well-defined by the conditions

$$\begin{cases} V(s,y) = \sup_{u(\cdot)\in\mathcal{V}^{\Theta^s}} J(s,y;u(\cdot)) \\ V(T,y) = \Phi(y) \end{cases} , \quad (s,y) \in \Theta \times \mathbb{R}^n . \tag{61}$$

The result below is Bellman's principle of optimality, which characterizes the value function V defined above.

Theorem 5.4. *Let (A1)-(A3) hold. Then for any $(s,y) \in \Theta \times \mathbb{R}^n$, we have that*

$$V(s,y) = \sup_{u(\cdot)\in\mathcal{V}^{\Theta^s}} \Big\{ \sum_{t=s}^{s'-1} L(t, x(t;s,y,u(\cdot)), u(t))\mathrm{d}t + V(s', x(s';s,y,u(\cdot))) \Big\} \tag{62}$$

for every $0 \le s \le s' \le T-1$.

Proof. Left as an exercise. Similar to the continuous-time case. □

Identity (62) is known as the dynamic programming equation. It provides a necessary condition for some control $u(\cdot)$ to be optimal. In the discrete-time case, the HJB equation is an almost immediate consequence of the dynamic programming equation.

Theorem 5.5 (The Hamilton-Jacobi-Bellman equation)**.** *Assume that the conditions (A1)-(A3) hold. The value function V is a solution of the following difference equation:*

$$\begin{cases} V(t,x) = \sup_{u\in U} \{L(t,x,u) + V(t+1, f(t,x,u))\} = 0 \\ V(T,x) = \Phi(x), \end{cases}$$

Proof. Take $s = t$ and $s' = t+1$ in the dynamic programming principle and use (60). □

The HJB equation translates the problem of maximizing the function J into the problem of solving a difference equation.

5.2.2 Infinite horizon

Consider the following additional assumptions:

(i) the function L depends on the time variable t only through a discounting factor, i.e.

$$L(t,x,u) = \beta^t \ell(x,u) , \quad \beta \in (0,1) .$$

(ii) the function f does not dependent on the time variable t.

(iii) the function Φ is identically zero.

Then, the value function $V(t,x)$ must be of the form

$$V(t,x) = \beta^t V(x) ,$$

and, therefore, the HJB equation reduces to the following functional equation

$$V(x) = \sup_{u\in U} \{L(x,u) + \beta V(f(x,u))\} = 0 .$$

5.2.3 Example

Consider again the example 4 in section 3.2. Recall that

$$\begin{aligned} L(t,w,c) &= \beta^t U(c) \\ f(t,w,c) &= (1+r)(w-c) \ . \end{aligned}$$

The HJB equation is given by

$$V(t,w) = \sup_{c \in U} \left\{ \beta^t U(c) + V(t+1,(1+r)(w-c)) \right\} = 0$$

From the maximum condition, one gets the equation

$$\beta^t U'(c) - (1+r) V_w(t+1,(1+r)(w-c)) = 0 \ ,$$

which implicitly defines the optimal control $c^*(t,w)$. Substituting in the HJB, we obtain

$$V(t,w) = \beta^t U(c^*(t,w)) + V(t+1,(1+r)(w-c^*(t,w))) = 0 \ .$$

5.2.4 Exercises

Exercise 1. Consider again example 4 in section 3.2 with $U(c) = \ln(c)$.

(i) Find the HJB equation associated with this problem.

(ii) Use the maximum condition to obtain an implicit definition for the optimal consumption.

Exercise 2. Consider again example 4 in section 3.2 with $U(c) = c^{1-\nu}/(1-\nu)$, $\nu \in (0,1)$.

(i) Find the HJB equation associated with this problem.

(ii) Use the maximum condition to obtain an implicit definition for the optimal consumption.

References

[1] A. C. Chiang and K. Wainwright. *Fundamental Methods of Mathematical Economics*. McGraw Hill, 2005.

[2] A. K. Dixit. *Optimization in Economic Theory*. Oxford University Press, 2nd edition, 1990.

[3] E. A. Ok. *Real Analysis with Economic Applications*. Princeton University Press, 2007.

[4] A. Seierstad and K. Sydsaeter. *Optimal Control Theory with Economic Applications*. North-Holland, 1987.

[5] S. P. Sethi and G. L. Thompson. *Optimal Control Theory: Applications to Management Science and Economics*. North-Holland, 2nd edition, 1987.

[6] R. K. Sundaram. *First Course in Optimization Theory*. Cambridge University Press, 1996.