Notes on Continuous-time Dynamical Systems

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Abstract

The present manuscript constitutes the lecture notes for the "Mathematics" module of the PhD program on Complexity Science, held at ISCTE-IUL during November and December 2011. The aim of the notes is to provide some auxiliary material for the students to follow this 10 hour length module, devoted to the study of "Dynamical Systems in Continuous-time". A good working knowledge of calculus in several variables and linear algebra is desirable, but not mandatory.

Contents

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1 Introduction

A dynamical system is a rule describing the evolution with time of a point in a given set. This rule might be specified by very different means like ordinary differential equations, iterated maps, partial differential equations or cellular automata. In this notes we will be mostly concerned with dynamical systems defined by differential equations. Indeed, some famous examples of dynamical systems can be written in terms of differential equations equations: the harmonic oscillator, the pendulum and double pendulum, or the N-body problem.

The mathematical theory of dynamical system has its roots in classical mechanics, which started to be developed in the XVI and XVII centuries by Galileo and Newton, respectively. In 1686, with the publication of the Principia, Newton laid down the mathematical principles of classical mechanics with three laws governing the motion of bodies under the presence of external forces and described the universal law of gravity. This inspired the work of mathematicians like Euler, Lagrange, Hamilton and Poincaré that built on Newton's work.

The work of Poincaré was a great influence to the present state of the subject since it led to a change in the motivation from the quantitative to the qualitative and geometrical study of such mechanical systems and more general systems of nonlinear differential equations. This change was a key step for the development of the modern theory of dynamical systems during the XX century. This qualitative way of looking at nonlinear dynamical systems was further developed by Birkhoff in the first half of the XX century. At the same time the subject was flourishing in the Soviet Union with the works of Lyapunov, Andronov, Pontryagin and others. A new wave of development came by around 1960 with the influential works of Smale and Moser in the United States, Kolmogorov, Arnold and Sinai in the Soviet Union and Peixoto in Brazil.

2 Differential equations and Dynamical Systems

A dynamical system is a triple $\left(M, \phi^t, K\right)$ where M is called the *phase space* and is usually a smooth manifold or a subset of $\mathbb{R}^n$, $\phi^t : M \times K \to M$, called the *evolution*, is a smooth action of K in M and K is either a subset of $\mathbb{R}$ in the case of a *continuous time dynamical system* or a subset of $\mathbb{Z}$ in the case of a *discrete time dynamical system*. The smooth action $\phi^t(\boldsymbol{x})$ describes the evolution with time $t \in K$ of a point $\boldsymbol{x}$ in the phase space M.

In this notes we study dynamical systems in continuous time, determined by ordinary differential equations. We will introduce some of the basic concepts of the theory, with a special emphasis on the study of examples to illustrate such concepts. All the concepts, statements and its proofs in this notes can be found in classical references such as [5, 8, 9, 10].

Before proceeding to the rigorous treatment of the subject, we will try to provide some motivation with some very simple examples. First of all, we may say that a differential equation is an equality involving a function, and its derivatives. One of the most simple examples of differential equations is provided by

$$\dot{x} = ax \ , \qquad x \in \mathbb{R} \ ,$$

where a is a fixed parameter. This equation can be used to model many different situations. For instance, in the case where a is a positive real number, this equation may be used to model the growth of a population with unlimited resources or the effect of continuously compounding interest. When a is a negative number, it can model radioactive decay. Due to its very simple form, an explicit expression for the solutions can be obtained:

$$x(t) = x_0 \mathrm{e}^{at} \ , \qquad t \in \mathbb{R}$$

where x_0 is the value of $x(t)$ when $t = 0$. If $a > 0$, the solution tends to infinity as t goes to infinity, while the solution tends to zero if $a < 0$. If $a = 0$ all solutions are constant.

The behavior of the solutions is very simple since the equation is linear. In the example above we are able to obtain the solutions of the differential equation and can then plot such solution as a function of t. However, the solutions to most differential equations can not be expressed explicitly in terms of elementary functions. To deal with such cases, we may introduce phase portraits, exhibiting the solution in the space of all possible values of x. The solution curves may be labeled with arrows to indicate the direction that the variable changes as t increases. This type of phase space analysis can often yield qualitatively important information, even when an explicit representation of the solution can not be obtained.

As already noted above, the differential equation $\dot{x} = ax$ may be seen as a very simple model for population growth. The assumption leading to this differential equation is that the rate of growth of the population is proportional to the size of the population. Indeed, this assumption may be a good description of the reality if the population is small. However, if the population becomes too large, it may exhaust its resources, and the growth rate may turn negative. This case may be modelled by the differential equation:

$$\dot{x} = ax\left(1 - \frac{x}{K}\right) , \qquad x \in \mathbb{R} ,$$

which is usually called the logistic population model. The parameter $a > 0$ is usually called "growth rate" and the parameter $K > 0$ is called the "carrying capacity". We can still obtain an explicit solution for this differential equation by elementary methods. Such solution is given by

$$x(t) = \frac{x_0 \mathrm{e}^{at}}{1 + x_0(\mathrm{e}^{at} - 1)/K} , \qquad t \in \mathbb{R}$$

This solution is valid for any initial population $x(0) = x_0$. When $x(0) = K$ or $x(0) = 0$, the population size remains constant: we call these solutions equilibria.

For a qualitative study of the remaining solutions, we can sketch a phase portrait for this equation. From such analysis, we can see that all solutions for which $x(0)$ is positive tend to the "ideal" population $x(t) = K$. For negative values of $x(0)$, solutions tend to $-\infty$, although these solutions are irrelevant from the point of view of population dynamics. We also note that solutions starting close to $x = 0$, move away from 0, and that solutions starting close to $x = K$, approach K as $t \to +\infty$. This provides us with the stability of these equilibria: $x = 0$ is an unstable equilibria, while $x = K$ is a stable one.

We can modify the logistic model introduced above to take into account harvesting of the population. Suppose the population obeys the logistic model, but is subjected to some form of harvesting at a constant rate $h > 0$. The differential equation becomes

$$\dot{x} = ax\left(1 - \frac{x}{K}\right) - h , \qquad x \in \mathbb{R} ,$$

Instead of solving the differential equation explicitly, we may use the graphs of the family of functions

$$f(x) = ax\left(1 - \frac{x}{K}\right) - h$$

to understand the qualitative behavior of its solutions. It is possible to check that the number of equilibria and its stability change as h is varies, while a and K are kept fixed. Such changes on the number of equilibria and their stability are called bifurcations, and can be visually represented in a bifurcation diagram.

Higher dimensional systems of differential equations may also be considered. A very famous example is provided by the Van der Pol equations:

$$\begin{cases} \dot{x} = y - x^3 + x \\ \dot{y} = -x , \qquad (x, y) \in \mathbb{R} \times \mathbb{R} . \end{cases}$$

These equations were originally introduced to model a "self exciting" electric circuit. They have the following particular feature: if (x_0, y_0) is any point in the plane other than $(0, 0)$, then the solution $(x(t), y(t))$ starting from such point tends to a periodic motion. Thus the system excites itself to this periodic motion, and other than transitory effects, the natural motion of a solution is the single periodic motion. We will in the sequel that this is an attracting periodic solution.

There are many other differential equations modelling population growth, spread of diseases, nonlinear oscillators, and many other physical situations in dimension two. However, chaotic behaviour does not occur for (autonomous) differential equations in one or two dimensions. To obtain such behaviour one must consider either non-autonomous systems with dimension at least two, or autonomous systems with dimension at least three. In what concerns the first case, a well know example is the forced Van der Pol equations:

$$\begin{cases} \dot{x} = y - x^3 + x + g(t) \\ \dot{y} = -x \ , \qquad (x, y) \in \mathbb{R} \times \mathbb{R} \ , \end{cases}$$

where $g(t)$ is a periodic function, such as e.g. $g(t) = \cos(\omega t)$. For an example of three dimensional system of differential equations exhibiting chaotic behaviour, one has the very famous Lorenz equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \ , \qquad (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ , \end{cases}$$

where σ is called the Prandtl number and ρ is called the Rayleigh number. These equations were introduced by Lorenz in 1963 as a model of the fluid flow of the atmosphere (with $\sigma = 10$, $\rho = 28$ and $\beta = 3$. Lorenz studied these equations by means of computer simulations and noticed that they exhibit chaotic behaviour. More precisely, he found that the equations display what is now called a strange attractor. After much investigation over the last decades we understand the features causing the chaos in these equations and have a good geometric model for their behavior, but no one has ever been able to verify analytically that these particular equations satisfy the conditions of the geometric model. However, a computer-assisted proof for the existence of the Lorenz attractor has been published by Warwick Tucker as recently as 2002.

2.1 Existence and uniqueness of solutions

We will study equations of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t; \mu) \ , \tag{1}$$

with $x \in U \subseteq \mathbb{R}^n$, $t \in \mathbb{R}$, and $\mu \in V \subseteq \mathbb{R}^p$ where U and V are open sets in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. The overdot in (1) means $\frac{\mathrm{d}}{\mathrm{d}t}$, and μ is seen as a parameter. The independent variable t is often referred to as time. We refer to (1) as a ordinary differential equation.

A solution of (1) is a map, $\boldsymbol{x}$, from some interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^n$, which we represent as

$$\begin{aligned} \boldsymbol{x} : \ & I \to \mathbb{R}^n \\ & t \mapsto \boldsymbol{x}(t) \end{aligned}$$

such that $\boldsymbol{x}(t)$ satisfies (1), i.e.

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), t; \mu) \ .$$

The map $\boldsymbol{x}(t)$ has the geometrical interpretation of a curve in $\mathbb{R}^n$, and (1) gives the tangent vector at each point of the curve.

We will refer to the space of dependent variables of (1), $U \subseteq \mathbb{R}^n$, as the phase space of (1). The goal of the dynamical systems theory is to understand the geometry of solution curves in phase space. In many applications the structure of the phase space may be more general than $\mathbb{R}^n$, i.e. a manifold, but we will focus our attention on the case where the phase space is an open set of $\mathbb{R}^n$ to keep the exposition simple.

Ordinary differential equations that depend explicitly on time (i.e. $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t; \mu)$) are referred to as non autonomous ordinary differential equations, and ordinary differential equations that do not depend explicitly on time (i.e. $\dot{\boldsymbol{x}} = f(\boldsymbol{x}; \mu)$) are referred to as autonomous. We will focus our attention in the autonomous case here.

It is useful to be able to distinguish a solution curve by a particular point in phase space that it passes through at a specific time, i.e. for a solution $\boldsymbol{x}(t)$ we have $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. In this case, we say that an initial condition has been specified. This information may be included in the expression for a solution by writing $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$. If the initial time is understood to be a specific value, say e.g. $t_0 = 0$, we denote such solution as $\boldsymbol{x}(t, \boldsymbol{x}_0)$. Similarly, it may be useful to explicitly display the parametric dependence of solutions. In this case we may write $x(t, t_0, \boldsymbol{x}_0; \mu)$, or, if we are not interested in the initial condition, just $\boldsymbol{x}(t; \mu)$. If the parameters play no role in our arguments, we omit any specific parameter dependence from the notation.

A solution of $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ may also be referred to as the trajectory or phase curve through the point $\boldsymbol{x}_0$ at $t = t_0$. The graph of $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ over t is referred to as an integral curve. More precisely, an integral curve is a set of the form

$$\operatorname{graph}(\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)) = \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \boldsymbol{x} = \boldsymbol{x}(t, t_0, \boldsymbol{x}_0), t \in I\} \ ,$$

where I is the time interval of existence of the corresponding solution.

Example 2.1.1. *Consider the differential equation*

$$\begin{cases} \dot{u} = v \\ \dot{v} = -u \ , \qquad (u, v) \in \mathbb{R} \times \mathbb{R} \ . \end{cases}$$

The solution passing through the point $(u, v) = (1, 0)$ *at* $t = 0$ *is given by*

$$(u(t), v(t)) = (\cos t, -\sin t) \ .$$

The integral curve passing through $(u, v) = (1, 0)$ *at* $t = 0$ *is the set*

$$\{(u, v, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : (u(t), v(t)) = (cost, -sint), \ t \in \mathbb{R}\}.$$

The existence and uniqueness of solutions of (1) is not obvious, as can be seen from the next example

Example 2.1.2. *For an example of non-uniqueness of solution, take the differential equation*

$$\dot{x} = \sqrt{x} \ , \qquad x \in \mathbb{R} \ .$$

We can check that it has more than one solution satisfying $x(0) = 0$*. Indeed, for any* $a > 0$ *the function*

$$x(t) = \begin{cases} 0 & , \qquad t \leq a \\ (t-a)^2/4 & , \qquad t > a \ . \end{cases}$$

is a solution. The lack of unicity is due to the fact that the map $f(x) = \sqrt{x}$ is not sufficiently regular (in this case, it does not satisfy the Lipschitz condition at $x = 0$).

Some conditions must be placed on the map $f(\boldsymbol{x}, t; \mu)$ in order for solutions to exist and be unique. Properties of the solutions such as differentiability with respect to initial conditions and parameters, are inherited from properties on $f(\boldsymbol{x}, t; \mu)$.

Theorem 2.1.3 (Existence and Uniqueness of solutions of ordinary differential equations)**.** *Consider the differential equation*

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t) \ , \tag{2}$$

where $f(\boldsymbol{x}, t)$ is C^r, $r \geq 1$, on some open set $U \subseteq \mathbb{R}^n \times \mathbb{R}$, and let $(\boldsymbol{x}_0, t_0) \in U$. Then there exists a solution of (2) through the point $\boldsymbol{x}_0$ at $t = t_0$, denoted $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ with $\boldsymbol{x}(t_0, t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0$, for $|t - t_0|$ sufficiently small. This solution is unique in the sense that any other solution of (2) through $\boldsymbol{x}_0$ at $t = t_0$ must be the same as $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ on their common interval of existence. Moreover, $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ is a C^r function of t, t_0, and $\boldsymbol{x}_0$.

It is possible to weaken the assumptions on the map $f(\boldsymbol{x}, t)$ and still obtain existence and uniqueness (see e.g. [4, 6]). The previous theorem only guarantees existence and uniqueness for sufficiently small time intervals. However, it is possible to extend in a unique way the time interval of existence of solutions:

Theorem 2.1.4. *Let $C \subset U \subseteq \mathbb{R}^n \times \mathbb{R}$ be a compact set containing $(\boldsymbol{x}_0, t_0)$. The solution $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)$ of (2) can be uniquely extended backward and forward in t up to the boundary of C.*

What happens is that any solution may leave U after sufficient time. Therefore, the theorem ensuring existence and Uniqueness of solutions of ordinary differential equations is only local. It is easy to provide examples of differential equations whose solutions leave any subset $U \subset \mathbb{R}^n$ in finite time. Another way to look at the previous theorem is by noting that it tells us how solutions may cease to exist. This is illustrated in the following example.

Example 2.1.5. *Consider the equation*

$$\dot{x} = x^2 \ , \qquad x \in \mathbb{R} \ .$$

Its solution through $x_0 \in \mathbb{R}$ at $t = 0$ is given by

$$x(t, 0, x_0) = -\frac{x_0}{x_0 t - 1} \ ,$$

which clearly does not exist for all $t \in \mathbb{R}$, since it becomes infinite as $t \to 1/x_0$. This example also shows that the time interval of existence may depend on the initial condition x_0.

A similar example is provided by the differential equation

$$\dot{x} = 1 + x^2 \ , \qquad x \in \mathbb{R} \ ,$$

with general solution of the form

$$x(t) = \tan(t + C) \ ,$$

where C is a constant determined from the initial condition. As in the previous example, it is clear that $x(t)$ goes to infinity in finite time.

As we have already noted above, an ordinary differential equation may depend on parameters. The following result describes how the solutions of (1) depend on its parameters.

Theorem 2.1.6. *Consider the differential equation* (1) *and assume that* $f(\boldsymbol{x}, t; \mu)$ *is a* C^r *map,* $r \geq 1$*, on some open set* $U \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p$*. For* $(\boldsymbol{x}_0, t_0, \mu) \in U$*, the solution* $\boldsymbol{x}(t, t_0, \boldsymbol{x}_0, \mu)$ *is a* C^r *function of* t*,* t_0*,* $\boldsymbol{x}_0$*, and* μ*.*

This result is illustrated by the following example.

Example 2.1.7. *Consider the system of differential equations*

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \ , \qquad (u, v) \in \mathbb{R} \times \mathbb{R} \ , \omega > 0 \ . \end{cases}$$

The solution passing through the point $(u, v) = (x_0, y_0)$ *at* $t = t_0$ *is given by*

$$(u(t), v(t)) = (x_0 \cos(\omega(t-t_0)) + \frac{y_0}{\omega} \sin(\omega(t-t_0)), y_0 \cos(\omega(t-t_0)) - x_0 \omega \sin(\omega(t-t_0)))$$

and it is differentiable with respect to $(t, t_0, x_0, y_0, \omega)$*.*

From now on we focus our attention on autonomous differential equations

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}; \mu) \ , \qquad \boldsymbol{x} \in \mathbb{R}^n \ , \tag{3}$$

where $f(\boldsymbol{x}; \mu)$ is a C^r map, with $r \geq 1$, on some open set $U \subset \mathbb{R}^n \times \mathbb{R}^p$. For simplicity of exposition, we will assume that the solutions of (3) exist for all $t \in \mathbb{R}$. We summarize below some simple properties of the solutions of (3):

(i) If $\boldsymbol{x}(t)$ is a solution of (3), then so is $\boldsymbol{x}(t + \tau)$ for every $\tau \in \mathbb{R}$.

(ii) For any $\boldsymbol{x}_0 \in \mathbb{R}^n$, there exists only one solution of (3) passing through this point.

Due to property (i) above, it is clear that it is enough to study solutions of (3) with initial conditions of the form $\boldsymbol{x}(0) = \boldsymbol{x}_0$. From now on we will only assume that this is always the case.

Using the solutions $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of (3) we define a map $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by

$$\phi(\boldsymbol{x}_0, t) = \boldsymbol{x}(t, \boldsymbol{x}_0) \ .$$

The map ϕ is called a *flow* and satisfies the following properties:

i) $\frac{\mathrm{d}}{\mathrm{d}t}\phi(\boldsymbol{x}, t) = f(\phi(\boldsymbol{x}, t); \mu)$ for every $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$;

ii) $\phi(\boldsymbol{x}, 0) = \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathbb{R}^n$;

iii) $\phi(\phi(\boldsymbol{x}, s), t) = \phi(\boldsymbol{x}, t + s)$ for every $t, s \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$;

iv) for fixed t, $\phi(\boldsymbol{x}, t)$ is a diffeomorphism of $\mathbb{R}^n$.

We will often use the notation $\phi^t(\boldsymbol{x})$ instead to $\phi(\boldsymbol{x}, t)$ to denote the flow defined by (3).

Example 2.1.8. *The flow defined by the system of differential equations*

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\omega^2 u \ , \qquad (u, v) \in \mathbb{R} \times \mathbb{R} \ , \omega > 0 \ . \end{cases}$$

is the map $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ *given by*

$$\phi(x, y, t) = (x \cos \omega t + \frac{y}{\omega} \sin \omega t, y \cos \omega t - x\omega \sin \omega t) \ .$$

The flow ϕ determines the evolution of the dynamical system defined by (3). It should be remarked that most differential equations solutions can not be written in terms of elementary functions. The dynamical systems theory goal is to study the qualitative and geometric properties of such solutions, without going through the process of determining it explicitly.

2.2 Invariant sets

Invariant sets can be seen as the most basic building blocks for the understanding of the behaviour of a given dynamical system. These sets have the following property: trajectories starting in the invariant set, remain in the invariant set, for all of their future, and all of their past.

Definition 2.2.1. *Let* $S \subset \mathbb{R}^n$ *be a set. Then* S *is said to be invariant under the dynamics of* (3) *if for any* $\boldsymbol{x}_0 \in S$ *we have* $\phi^t(\boldsymbol{x}_0) \in S$ *for all* $t \in \mathbb{R}$. *If we restrict ourselves to positive times t then we refer to* S *as a positively invariant set and, for negative time, as a negatively invariant set.*

We will now discuss some special invariant sets: orbits, equilibria, periodic orbits, and limit sets. We will introduce other invariant sets later in this notes.

Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ be a point in the phase space of (3). The orbit through $\boldsymbol{x}_\mathbf{o}$, which we denote by $\mathcal{O}(\boldsymbol{x}_0)$, is the set of points in phase space that lie on a trajectory of (3) passing through $\boldsymbol{x}_0$:

$$\mathcal{O}(\boldsymbol{x}_0) = \left\{ \phi^t(\boldsymbol{x}_\mathbf{o}) : t \in \mathbb{R} \right\} \ .$$

The positive semiorbit through $\boldsymbol{x}_0$ is the set

$$\mathcal{O}^+(\boldsymbol{x}_0) = \left\{ \phi^t(\boldsymbol{x}_\mathbf{o}) : t \geq 0 \right\} \ .$$

and the negative semiorbit through $\boldsymbol{x}_0$ is the set

$$\mathcal{O}^-(\boldsymbol{x}_0) = \left\{ \phi^t(\boldsymbol{x}_\mathbf{o}) : t \leq 0 \right\} \ .$$

Note that for any $t \in \mathbb{R}$ we have that $\mathcal{O}(\phi^t(\boldsymbol{x}_0)) = \mathcal{O}(\boldsymbol{x}_0)$.

A point $\boldsymbol{p} \in \mathbb{R}^n$ is an *equilibrium* for the flow of (3) if $\phi^t(\boldsymbol{p}) = \boldsymbol{p}$ for all $t \in \mathbb{R}$. Since the flows we consider here are solutions of differential equations, we obtain that an equilibrium $\boldsymbol{p}$ for the flow ϕ^t of the differential equation (3) must satisfy $f(\boldsymbol{p}) = \mathbf{0}$. Equilibria are solutions that do not change in time, thus providing the most simple example of invariant sets.

A point $\boldsymbol{p} \in M$ is a *periodic point of period* T for the flow of (3) if there exists some positive number $T \in \mathbb{R}$ such that $\phi^T(\boldsymbol{p}) = \boldsymbol{p}$ and $\phi^t(\boldsymbol{p}) \neq \boldsymbol{p}$ for every $0 < t < T$. The orbit $\mathcal{O}(\boldsymbol{p})$ of a periodic point is called a *periodic orbit*.

Example 2.2.2. *Consider again the differential equation of example 2.1.1*

$$\begin{cases} \dot{u} = v \\ \dot{v} = -u \ , \qquad (u,v) \in \mathbb{R} \times \mathbb{R} \ . \end{cases} \tag{4}$$

The point $(0,0)$ *is an equilibrium of* (4). *Every other point in* $\mathbb{R}^2$ *is a periodic point for* (4). *The (periodic) orbit passing through* $(u,v) = (1,0)$ *is the unit circle* $u^2 + v^2 = 1$.

A point $\boldsymbol{q}$ is an *ω-limit point of $\boldsymbol{p}$ for the flow ϕ^t* if there exists a sequence of integers $\{t_k\}_{k\in\mathbb{N}}$ going to infinity as k goes to infinity such that

$$\lim_{k\to\infty} \left\| \phi^{t_k}(\boldsymbol{p}) - \boldsymbol{q} \right\| = 0 \ .$$

The set of all ω-limit points of $\boldsymbol{p}$ for ϕ^t is called the *ω-limit set of $\boldsymbol{p}$* and is denoted by $\omega(\boldsymbol{p})$.

A point $\boldsymbol{q}$ is an *α-limit point of $\boldsymbol{p}$ for the flow ϕ^t* if there exists a sequence of integers $\{t_k\}_{k\in\mathbb{N}}$ going to minus infinity as k goes to infinity such that

$$\lim_{k\to\infty} \left\| \phi^{t_k}(\boldsymbol{p}) - \boldsymbol{q} \right\| = 0 \ .$$

The set of all α-limit points of $\boldsymbol{p}$ for ϕ^t is called the *α-limit set of $\boldsymbol{p}$* and is denoted by $\alpha(\boldsymbol{p})$.

Theorem 2.2.3. *The limit sets* $\omega(\boldsymbol{p})$ *and* $\alpha(\boldsymbol{p})$ *are closed and invariant. Moreover, if* $\mathcal{O}^+(\boldsymbol{p})$ *is contained in some compact subset of* $\mathbb{R}^n$, *then* $\omega(\boldsymbol{p})$ *is nonempty, compact, and connected. Similarly, if* $\mathcal{O}^-(\boldsymbol{p})$ *is contained in some compact subset of* $\mathbb{R}^n$, *then* $\alpha(\boldsymbol{p})$ *is nonempty, compact, and connected.*

Example 2.2.4. *Recall the logistic model for the growth of a population:*

$$\dot{x} = ax\left(1 - \frac{x}{K}\right) \ , \qquad x \in \mathbb{R} \ , \ a, K > 0 \ .$$

By studying its phase portrait, we get that

- $\omega(0) = \alpha(0) = \{0\}$ *and* $\omega(K) = \alpha(K) = \{K\}$;
- *if* $x_0 \in (0, K)$, *then* $\alpha(x_0) = \{0\}$ *and* $\omega(x_0) = \{K\}$;
- *if* $x_0 > K$, *then* $\alpha(x_0) = \emptyset$ *and* $\omega(x_0) = \{K\}$;
- *if* $x_0 < 0$, *then* $\alpha(x_0) = \{0\}$ *and* $\omega(x_0) = \emptyset$.

2.3 Stability

We will now discuss the notions of Lyapunov stability and asymptotic stability. Such notions can be intuitively stated as follows: if $\boldsymbol{p}$ is a Lyapunov stable point then for every point $\boldsymbol{q}$ close enough to $\boldsymbol{p}$ its orbit stays close to the orbit of $\boldsymbol{p}$; if $\boldsymbol{p}$ is asymptotically stable if it is Lyapunov stable and for every point $\boldsymbol{q}$ close enough to $\boldsymbol{p}$ the forward orbit of $\boldsymbol{q}$ will converge to the forward orbit of $\boldsymbol{p}$.

Definition 2.3.1 (Lyapunov stability). *The orbit of a point* $\boldsymbol{p} \in \mathbb{R}^n$ *is* Lyapunov stable *by the flow* ϕ^t *of* (3) *if for any* $\epsilon > 0$ *there is* $\delta > 0$ *such that if* $\|\boldsymbol{q} - \boldsymbol{p}\| < \delta$*, then*

$$\left\|\phi^t(\boldsymbol{q}) - \phi^t(\boldsymbol{p})\right\| < \epsilon$$

for all $t \geq 0$*.*

The orbit of a point $\boldsymbol{p} \in \mathbb{R}^n$ which is not stable is said to be unstable.

Definition 2.3.2 (Asymptotic stability). *The orbit of a point* $\boldsymbol{p} \in \mathbb{R}^n$ *is* asymptotically stable *by the flow* ϕ^t *of* (3) *if it is Lyapunov stable and there exists a neighbourhood* V *of* $\boldsymbol{p}$ *such that for every* $\boldsymbol{q} \in V$*,*

$$\left\|\phi^t(\boldsymbol{q}) - \phi^t(\boldsymbol{p})\right\| \to 0 \ ,$$

as t *tends to infinity.*

Example 2.3.3. *Consider again the logistic model for population growth:*

$$\dot{x} = ax\left(1 - \frac{x}{K}\right) \ , \qquad x \in \mathbb{R} \ , \ a, K > 0 \ .$$

From the analysis of its phase portrait, we obtain that $x = 0$ *is an asymptotically stable equilibrium while* $x = K$ *is an unstable equilibrium.*

For an example of a Lyapunov stable equilibrium which is not asymptotically stable, consider the two dimensional system

$$\begin{cases} \dot{u} = v \\ \dot{v} = -u \ , \qquad (u, v) \in \mathbb{R} \times \mathbb{R} \ . \end{cases}$$

The origin $(x, y) = (0, 0)$ *is a Lyapunov stable equilibrium. Moreover, every orbit of this system is Lyapunov stable.*

To see that Lyapunov stability is not a robust property, consider the following second order differential equation:

$$\ddot{x} + \mu\dot{x} + \omega^2 x = 0 \ , \qquad x \in \mathbb{R} \ , \ \mu \in \mathbb{R} \ , \ \omega > 0 \ .$$

This differential equation can be rewritten as a system of two first order differential equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\mu y - \omega^2 x \ , \qquad x \in \mathbb{R} \ , \ \mu \in \mathbb{R} \ , \ \omega > 0 \ . \end{cases}$$

Clearly, for $\mu = 0$ *all the orbits are Lyapunov stable. If* $\mu > 0$ *the origin is an asymptotically stable equilibrium, while in the case where* $\mu < 0$ *the origin is unstable.*

2.4 Linear systems of differential equations

We will now restrict our attention to systems of linear ordinary differential equations, with a special emphasis on those with constant coefficients. We will see later that these may be very helpful to provide a criteria for the stability of equilibria of systems of nonlinear differential equations.

Start by considering the the linear equation

$$\dot{\boldsymbol{x}} = A(t)\boldsymbol{x} \tag{5}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ and $A(t) = (a_{ij}(t))$ an $n \times n$ matrix. We will soon specialize to the where the matrix A doe not depend on t. Before proceeding to that case, note that any linear combination of solutions of (5) is itself a solution of (5):

Proposition 2.4.1. *If $\boldsymbol{x} : I \to \mathbb{R}^n$ and $\boldsymbol{y} : J \to \mathbb{R}^n$ are two solutions of* (5) *and $\alpha, \beta \in \mathbb{R}$, then $\alpha\boldsymbol{x}(t) + \beta\boldsymbol{y}(t)$ is a solution of* (5) *on the interval $I \cap J$.*

Let us now consider the linear differential equation

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \tag{6}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ and $A = (a_{ij})$ is a constant $n \times n$ matrix. This is a generalization of the scalar equation $\dot{x} = ax$ of section 1 with solution $x(t) = x_0 \exp(at)$. By the proposition above, a general solution of (6) can be obtained by linear superposition of n linearly independent solutions $\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)$:

$$\boldsymbol{x}(t) = \sum_{k=1}^{n} c_k \boldsymbol{x}_k(t) \ ,$$

where the n unknown constants $c_1, \ldots, c_n$ are determined by initial conditions. If the matrix A has n linearly independent eigenvectors $\boldsymbol{v}_k$, $k = 1, \ldots, n$, then we may take as a basis for the space of solutions the vector valued functions

$$\boldsymbol{x}_k(t) = \mathrm{e}^{\lambda_k t}\boldsymbol{v}_k \ ,$$

where λ_k is the eigenvalue associated with $\boldsymbol{v}_k$. For complex eigenvalues without multiplicity $\lambda_k = \alpha_k \pm \mathrm{i}\beta_k$, having eigenvectors $\boldsymbol{v}_k = \boldsymbol{v}_k^R \pm \mathrm{i}\boldsymbol{v}_k^I$, we take

$$\begin{aligned} \boldsymbol{x}_k(t) &= e^{\alpha_k t}\left(\boldsymbol{v}_k^R \cos\beta_k t - \boldsymbol{v}_k^I \sin\beta_k t\right) \\ \boldsymbol{x}_{k+1}(t) &= e^{\alpha_k t}\left(\boldsymbol{v}_k^R \sin\beta_k t + \boldsymbol{v}_k^I \cos\beta_k t\right) \end{aligned}$$

as the associated pair of (real) linearly independent solutions. When there are repeated eigenvalues and less than n eigenvectors, one can still obtain what is called *generalized eigenvectors*. We omit such description here and refer the reader to [4, 6]. What should be kept in mind is that it is still possible to obtain a set of n linearly independent solutions.

Let $\{\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)\}$ be a set of n linearly independent solutions of (6). We define the fundamental solution matrix as the matrix having these n solutions for its columns:

$$X(t) = [\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t)] \ .$$

Then, it is possible to prove that the a solution of (6) can be written as

$$\boldsymbol{x}(t) = \mathrm{e}^{At}\boldsymbol{x}_0 \ ,$$

where $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the vector of initial conditions and e^{At} is the $n \times n$ matrix given by

$$\mathrm{e}^{At} = X(t)X(0)^{-1} \ .$$

The matrix e^{At} may be seen as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ for each fixed value of $t \in \mathbb{R}$, i.e. given any point $\boldsymbol{x}_0 \in \mathbb{R}^n$, we have that $e^{At}\boldsymbol{x}_0$ is the point reached by the solution of (6) going through $\boldsymbol{x}_0$ at $t = 0$ after time t. Thus, we obtain that $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by

$$\phi(\boldsymbol{x}, t) = e^{At}\boldsymbol{x} \ ,$$

defines a flow on $\mathbb{R}^n$. As already mentioned before, the flow ϕ can be thought of as the set of all solutions to (6). However, there are some solutions that play a special role: those contained in the linear subspaces spanned by the eigenvectors of the matrix A. Indeed, these subspaces are invariant under the flow ϕ. In particular, if $\boldsymbol{v}_k$ is a (real) eigenvector of A, then a solution based at a point $c_k\boldsymbol{v}_k$ remains in the subspace spanned by $\boldsymbol{v}_k$ for all time. Similarly, if $\boldsymbol{v}_k$ is a (complex) eigenvector of A, then the (two-dimensional) subspace spanned by $\mathrm{Re}(\boldsymbol{v}_k)$ and $\mathrm{Im}(\boldsymbol{v}_k)$ is also invariant under the flow ϕ. Indeed, the general statement is that the eigenspaces of the matrix A are invariant subspaces for the flow ϕ. We define the *stable space* $\mathbb{E}^s$, *unstable space* $\mathbb{E}^u$ and *centre space* $\mathbb{E}^c$ as:

$$\begin{aligned}
\mathbb{E}^s &= \mathrm{span}\{\boldsymbol{v}^s \in \mathbb{R}^n : \ \boldsymbol{v}^s \ \text{is a generalized eigenvector for} \\
&\qquad\qquad \text{an eigenvalue } \lambda_s \text{ of } A \text{ with } \mathrm{Re}(\lambda_s) < 0\} \\
\mathbb{E}^u &= \mathrm{span}\{\boldsymbol{v}^u \in \mathbb{R}^n : \ \boldsymbol{v}^u \ \text{is a generalized eigenvector for} \\
&\qquad\qquad \text{an eigenvalue } \lambda_u \text{ of } A \text{ with } \mathrm{Re}(\lambda_u) > 0\} \\
\mathbb{E}^c &= \mathrm{span}\{\boldsymbol{v}^c \in \mathbb{R}^n : \ \boldsymbol{v}^c \ \text{is a generalized eigenvector for} \\
&\qquad\qquad \text{an eigenvalue } \lambda_c \text{ of } A \text{ with } \mathrm{Re}(\lambda_c) = 0\} \ .
\end{aligned}$$

The names of these invariant spaces are due to their dynamical properties. Solutions lying on $\mathbb{E}^s$ exhibit exponential decay (either monotonic or oscillatory); those lying on $\mathbb{E}^u$ exhibit exponential growth, and those lying on $\mathbb{E}^c$ neither of these. In the absence of multiple eigenvalues, the solutions in $\mathbb{E}^c$ either oscillate at constant amplitude if $\lambda, \bar{\lambda} = \pm \mathrm{i}$, or remain constant $\lambda = 0$. If multiple eigenvalues occur, there solutions lying on $\mathbb{E}^c$ may exhibit polynomial growth or decay.

We now summarize the characterization for the existence and stability of equilibria of (6) obtained above. In what concerns the existence of equilibria of (6), we have that:

Proposition 2.4.2. *The linear system of differential equation* (6) *has*

(i) a unique equilibrium at the origin if $\lambda = 0$ is not an eigenvalue of A;

(ii) a line of equilibria through the origin if $\lambda = 0$ is an eigenvalue of A.

In what concerns stability of the equilibrium point at the origin, we have obtained:

Proposition 2.4.3. *Consider the linear system of differential equation* (6).

(i) If all of the eigenvalues of A have negative real parts, then the origin is an asymptotically stable equilibrium.

(ii) If the matrix A has at least one eigenvalue with positive real part, then the origin is not Lyapunov stable.

We will now discuss the two-dimensional case with some more detail. Let A be a 2×2 matrix. Its eigenvalues are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda \, \mathrm{Id}) \ .$$

This is clearly a degree two polynomial in λ with real coefficients. Thus, it must have either two distinct real roots, one real root with multiplicity two, or a pair of conjugated roots.

Let us consider the case of two distinct real roots λ_1, λ_2 first. Then there is a base of $\mathbb{R}^2$ formed by the eigenvectors of A associated with the eigenvalues λ_1 and λ_2, which we will denote by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. Assuming for the moment that $\lambda_1, \lambda_2 \neq 0$, there are three cases to consider:

(a) $\lambda_1 < 0 < \lambda_2$;

(b) $\lambda_1 < \lambda_2 < 0$;

(c) $0 < \lambda_1 < \lambda_2$.

In what concerns case (a), since $\lambda_1 < 0$ we have that the all solutions lying the line with direction $\boldsymbol{v}_1$ tend to the origin $(0, 0)$ as $t \to \infty$. This line is the stable subspace $\mathbb{E}^s$. Similarly, all solutions lying the line with direction $\boldsymbol{v}_2$ tend away from the origin $(0, 0)$ as $t \to +\infty$. This line is the unstable subspace $\mathbb{E}^u$. All other solutions tend to ∞ in the direction of the unstable subspace, as $t \to +\infty$. In backward time, these solutions tend to ∞ in the direction of the stable subspace. The equilibrium point of a system with eigenvalues satisfying $\lambda_1 < 0 < \lambda_2$ is called a *saddle*.

We now consider case (b). Since both eigenvalues of A are negative, all solutions tend to the origin as $t \to +\infty$. Moreover, since $|\lambda_1| > |\lambda_2|$, we obtain that all solutions lying outside the line spanned by $\boldsymbol{v}_1$ tend to the origin tangentially to the line spanned $\boldsymbol{v}_2$. In this case, the equilibrium point is called a *sink*.

Case (c) is very similar to case (b). Since both eigenvalues of A are now positive, all solutions (except for the equilibrium) tend to ∞ as $t \to +\infty$ and to the origin as $t \to -\infty$. Moreover, since $|\lambda_2| > |\lambda_1|$, we obtain that all solutions lying outside the line spanned by $\boldsymbol{v}_2$ are tangent to the line spanned by $\boldsymbol{v}_1$ at the origin. In this case, the equilibrium point is called a *source*.

Example 2.4.4. *Consider the following* 2×2 *matrices*

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} , \qquad B = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} , \qquad C = \begin{pmatrix} 3 & 1 \\ 1 & 3 & , \end{pmatrix}$$

defining, respectively, the following two-dimensional linear systems:

$$\begin{cases} \dot{x} = x + 3y \\ \dot{y} = 3x + y \end{cases} , \qquad \begin{cases} \dot{x} = -3x - y \\ \dot{y} = -x - 3y \end{cases} , \qquad \begin{cases} \dot{u} = 3x + y \\ \dot{v} = x + 3y \end{cases} .$$

The matrix A has eigenvalues $\lambda_1 = -2$ with eigenvector $\boldsymbol{v}_1 = (-1, 1)$, and $\lambda_2 = 4$ with eigenvector $\boldsymbol{v}_2 = (1, 1)$. The origin is a saddle equilibrium for the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.

The matrix B has eigenvalues $\lambda_1 = -2$ with eigenvector $\boldsymbol{v}_1 = (-1, 1)$, and $\lambda_2 = -4$ with eigenvector $\boldsymbol{v}_2 = (1, 1)$. The origin is a sink equilibrium for the linear system $\dot{\boldsymbol{x}} = B\boldsymbol{x}$.

The matrix C has eigenvalues $\lambda_1 = 2$ with eigenvector $\boldsymbol{v}_1 = (-1, 1)$, and $\lambda_2 = 4$ with eigenvector $\boldsymbol{v}_2 = (1, 1)$. The origin is a source equilibrium for the linear system $\dot{\boldsymbol{x}} = C\boldsymbol{x}$.

We will now discuss the case where the characteristic polynomial of A has a pair of complex conjugated roots $\lambda, \bar{\lambda}$. First note that straight line solutions no longer exist. Instead, solutions oscillate around the origin of $\mathbb{R}^2$. Let $\lambda = \alpha + \mathrm{i}\beta$. Then:

(a) If $\alpha = 0$, all the solutions (apart from the equilibrium) lie inside closed curves centered at the origin. All such solutions are periodic, with period $2\pi/\beta$. The equilibrium point is called a *centre*.

(b) If $\alpha < 0$, all the solutions (apart from the equilibrium) spiral into the origin. The equilibrium point is called a *spiral sink*.

(c) If $\alpha > 0$, all the solutions (apart from the equilibrium) spiral away from. The equilibrium point is called a *spiral source*. cases the equilibrium point is called

Example 2.4.5. *Consider the following 2×2 matrices*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

defining, respectively, the following two-dimensional linear systems:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \qquad \begin{cases} \dot{u} = -x - y \\ \dot{v} = x - y \end{cases}, \qquad \begin{cases} \dot{x} = x + y \\ \dot{y} = -x + y \end{cases}.$$

The matrix A has a pair of complex eigenvalues $\lambda = \pm\mathrm{i}$ with eigenvector $\boldsymbol{v} = (\mp\mathrm{i}, 1)$. The equilibrium at the origin is a centre for the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.

The matrix B has a pair of complex eigenvalues $\lambda = -1 \pm \mathrm{i}$ with eigenvector $\boldsymbol{v} = (\pm\mathrm{i}, 1)$. The equilibrium at the origin is a spiral sink for the linear system $\dot{\boldsymbol{x}} = B\boldsymbol{x}$.

The matrix C has a pair of complex eigenvalues $\lambda = 1 \pm \mathrm{i}$ with eigenvector $\boldsymbol{v} = (\mp\mathrm{i}, 1)$. The equilibrium at the origin is a spiral source for the linear system $\dot{\boldsymbol{x}} = C\boldsymbol{x}$.

We deal now we the case of a repeated eigenvalue, i.e. an eigenvalue with multiplicity two. We need to distinguish between the following two cases:

(a) λ has two linearly independent eigenvectors;

(b) λ has only one eigenvector (up to multiplication by a non-zero constant).

In case (a), every vector in $\mathbb{R}^2$ is an eigenvector of A and thus, all solutions lie on a straight line through the origin and either tend to $(0, 0)$ if $\lambda < 0$, or away from $(0, 0)$ if $\lambda > 0$, as $t \to +\infty$.

In what concerns case (b), we have that all solutions starting on the line through the origin with direction determined by the eigenvector $\boldsymbol{v}$ associated with λ, will remain in that line. All other solutions are tangent to such line at the origin. Furthermore, all solutions tend to the origin if $\lambda < 0$, or away from the origin if $\lambda > 0$, as $t \to +\infty$.

Example 2.4.6. *Consider the following* 2×2 *matrices*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} ,$$

defining, respectively, the following two-dimensional linear systems:

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases} , \qquad \begin{cases} \dot{u} = -x - y \\ \dot{v} = -y \end{cases} .$$

The matrix A *has an eigenvalue* $\lambda_1 = 1$ *with multiplicity two and every vector of* $\mathbb{R}^2$ *is an eigenvector* A*. The equilibrium at the origin is a source for the linear system* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ *and every solution lies on a straight line through the origin.*

The matrix B *has an eigenvalue* $\lambda_1 = -1$ *with multiplicity two, but only one eigenvector* $\boldsymbol{v}_1 = (1, 0)$*. The origin is sink for the linear system* $\dot{\boldsymbol{x}} = B\boldsymbol{x}$*. The solutions are tangent at the origin to the line with direction* $\boldsymbol{v}$*.*

A final remark concerning the degenerate case of zero eigenvalues. If exactly one of the eigenvalues is zero, there exists a line L of equilibria through the origin with direction given by the eigenvector associated with the zero eigenvalue. All other solutions will be contained in lines with direction given by the eigenvector associated with the non-zero eigenvalue. Moreover, these solutions will tend to L in forward time if the second eigenvalue is negative, and tend to L in backward time if the second eigenvalue is positive. If zero is an eigenvalue with multiplicity two and there is only one eigenvector (up to multiplication by constant) associated with it, then all solutions lie in lines with direction given by this eigenvalue. Finally, if zero is an eigenvalue with multiplicity two and there are two linearly independent eigenvectors associated with it, the whole plane is filled with equilibria.

We conclude the discussion regarding the stability of equilibria of two-dimensional linear systems with an alternative characterization in terms of the trace and determinant of the 2×2 matrix A. Note that the characteristic polynomial can be rewritten as

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0 ,$$

where $\operatorname{tr}(A)$ denotes the trace of A and $\det(A)$ denotes the determinant of A. Let λ_1, λ_2 be the (possibly repeated) eigenvalues of the matrix A. Recall that

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 , \qquad \det(A) = \lambda_1 \lambda_2$$

and that

$$\lambda_1 = \frac{-\operatorname{tr}(A) + \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2} , \qquad \lambda_2 = \frac{-\operatorname{tr}(A) - \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2} .$$

We obtain:

(i) if $\det(A) < 0$, then A has two distinct real eigenvalues of opposite signs and the origin is an equilibrium of saddle type.

(ii) if $\det(A) > 0$ and $\operatorname{tr}(A)^2 > 4\det(A)$, then A has two distinct real eigenvalues of the same sign. If $\operatorname{tr}(A) > 0$, the equilibrium at the origin is a source, and if $\operatorname{tr}(A) < 0$ the equilibrium at the origin is a sink.

(iii) if $\det(A) > 0$ and $\mathrm{tr}(A)^2 = 4\det(A)$, then A has two repeated non-zero eigenvalues. If $\mathrm{tr}(A) > 0$, the equilibrium at the origin is a source, and if $\mathrm{tr}(A) < 0$ the equilibrium at the origin is a sink.

(iv) if $\det(A) > 0$ and $\mathrm{tr}(A)^2 < 4\det(A)$, then A has two complex conjugated eigenvalues. If $\mathrm{tr}(A) > 0$, the equilibrium at the origin is a spiral source, if $\mathrm{tr}(A) = 0$ the equilibrium at the origin is a centre, and if $\mathrm{tr}(A) < 0$ the equilibrium at the origin is a spiral sink.

(v) if $\det(A) = 0$ at least one of the eigenvalues of A is zero. If additionally $\mathrm{tr}(A) = 0$ then both eigenvalues are zero.

2.5 Hartman-Grobman Theorem

The stability of equilibria is, under certain conditions, determined by a linear system associated with the differential equation (3). We will now discuss such conditions.

Let $\boldsymbol{p}$ be an equilibrium point for the flow of (3). Since $f(\boldsymbol{p}) = \boldsymbol{0}$, expanding (3) in Taylor series about $\boldsymbol{p}$ we obtain

$$\dot{\boldsymbol{x}} = Df_{\boldsymbol{p}}(\boldsymbol{x} - \boldsymbol{p}) + O\left(|\boldsymbol{x} - \boldsymbol{p}|^2\right) \ ,$$

where $Df_{\boldsymbol{p}}$ denotes the Jacobian matrix of f at $\boldsymbol{p}$. The linearized system at $\boldsymbol{p}$ is then given by

$$\dot{\boldsymbol{y}} = Df_{\boldsymbol{p}}\, \boldsymbol{y} \ , \tag{7}$$

where $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{p}$ measures the displacement between the solution $\boldsymbol{x}(t)$ and the equilibrium $\boldsymbol{p}$.

Similarly to what was done for the linear case, we may define the *stable space* $\mathbb{E}^s$, *unstable space* $\mathbb{E}^u$ and *centre space* $\mathbb{E}^c$ as the subspaces of $\mathbb{R}^n$ given by

$$\begin{aligned}
\mathbb{E}^s = \mathrm{span}\{\boldsymbol{v}^s \in \mathbb{R}^n : \ \boldsymbol{v}^s \ &\text{is a generalized eigenvector for} \\
&\text{an eigenvalue } \lambda_s \text{ of } Df_{\boldsymbol{p}} \text{ with } \mathrm{Re}(\lambda_s) < 0\} \\
\mathbb{E}^u = \mathrm{span}\{\boldsymbol{v}^u \in \mathbb{R}^n : \ \boldsymbol{v}^u \ &\text{is a generalized eigenvector for} \\
&\text{an eigenvalue } \lambda_u \text{ of } Df_{\boldsymbol{p}} \text{ with } \mathrm{Re}(\lambda_u) > 0\} \\
\mathbb{E}^c = \mathrm{span}\{\boldsymbol{v}^c \in \mathbb{R}^n : \ \boldsymbol{v}^c \ &\text{is a generalized eigenvector for} \\
&\text{an eigenvalue } \lambda_c \text{ of } Df_{\boldsymbol{p}} \text{ with } \mathrm{Re}(\lambda_c) = 0\} \ .
\end{aligned}$$

The equilibrium $\boldsymbol{p}$ is said to be *hyperbolic* if $\mathbb{E}^c = \{\boldsymbol{0}\}$. If additionally, $\mathbb{E}^u = \{\boldsymbol{0}\}$ then $\boldsymbol{p}$ is called a *sink*, if $\mathbb{E}^s = \{\boldsymbol{0}\}$ then $\boldsymbol{p}$ is a *source*, and if $\mathbb{E}^s \neq \{\boldsymbol{0}\}$ and $\mathbb{E}^u \neq \{\boldsymbol{0}\}$ then $\boldsymbol{p}$ is called a *saddle*. The equilibrium $\boldsymbol{p}$ is said to be *elliptic* if $\mathbb{E}^s = \mathbb{E}^u = \{\boldsymbol{0}\}$.

The next theorem states that if $\boldsymbol{p}$ is a hyperbolic equilibrium point for the flow of (3) then the linear part of $Df_{\boldsymbol{p}}$ completely determines the stability of $\boldsymbol{p}$. More specifically, the theorem ensures the existence (in a neighbourhood of $\boldsymbol{p}$) of a conjugacy between (3) and its linearization (7).

Theorem 2.5.1 (Hartman–Grobman Theorem)**.** *Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Then, the flow ϕ^t of f is conjugate in a neighbourhood of $\boldsymbol{p}$ to the affine flow $\boldsymbol{p} + \mathrm{e}^{At}(\boldsymbol{y} - \boldsymbol{p})$, where $A = Df_{\boldsymbol{p}}$. More precisely, there exist a neighbourhood U of $\boldsymbol{p}$ and a homeomorphism $h : U \to U$ such that $\phi^t(h(\boldsymbol{x})) = h\left(\boldsymbol{p} + \mathrm{e}^{At}(\boldsymbol{y} - \boldsymbol{p})\right)$ as long as $\boldsymbol{p} + \mathrm{e}^{At}(\boldsymbol{y} - \boldsymbol{p}) \in U$.*

The stability of a hyperbolic equilibrium follows from the Hartman–Grobman theorem.

Corollary 2.5.2. *Let $\boldsymbol{p}$ be a hyperbolic equilibrium for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. If $\boldsymbol{p}$ is a source or a saddle, then $\boldsymbol{p}$ is not Lyapunov stable. If $\boldsymbol{p}$ is a sink, then it is asymptotically stable.*

We illustrate the results in this section in the example below.

Example 2.5.3. *Recall Van der Pol's equations*

$$\begin{cases} \dot{x} = y - x^3 + x \\ \dot{y} = -x \ , \qquad (x, y) \in \mathbb{R} \times \mathbb{R} \ . \end{cases}$$

The origin is the unique equilibrium, and the linearization of Van der Pol's equations around the equilibrium is given by

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = -x \ , \end{cases}$$

with the coefficients matrix being

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

The eigenvalues of the matrix A are $\lambda = \frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$, and thus the origin is a spiral source. Hartman-Grobman theorem implies that in a sufficiently small neighnourhood of the origin, the dynamics of the Van der Pol's equations is conjugated to its linearization, the origin being a spiral source for the nonlinear system too.

For another example, consider the motion of a pendulum, described by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) \ , \qquad x \in [-\pi, \pi] \ , y \in \mathbb{R} \ . \end{cases}$$

As defined above, the pendulum has three equilibria: $(0, 0)$ and $(\pm\pi, 0)$, but it is usually convenient to identify the point $(\pi, 0)$ and $(-\pi, 0)$. The linearized equations around $(0, 0)$ are given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \ , \end{cases} \qquad .$$

The corresponding coefficients matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has eigenvalues $\lambda = \pm\mathrm{i}$. Thus, the origin is a center for the linearized system. Note that in this case Hartman-Grobman theorem provides no extra information regarding the stability of the origin for the nonlinear system. We now consider the remaining equilibria. The linearized equations around the point $(\pi, 0)$ and $(-\pi, 0)$ are given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \ , \end{cases}$$

and the corresponding coefficients matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\lambda = \pm 1$. *Thus, the equilibrium at* $(\pi, 0)$ *is a saddle. Moreover, Hartman-Grobman theorem ensures the existence of a saddle at* $(\pi, 0)$ *for the non-linear system.*

2.6 Invariant manifolds: the stable manifold theorem

Without going into technical details, a manifold is a set which locally has the structure of Euclidean space. In applications, manifolds are usually multidimensional surfaces embedded in $\mathbb{R}^n$, such as e.g. a sphere or a torus. If the surface has no singular points, i.e. the derivative of the function representing the surface has maximal rank, then it can locally be represented as a graph of a given map. The surface is then a C^r manifold if the (local) graphs representing it are C^r functions. An invariant set $S \subset \mathbb{R}^n$ is said to be a C^r, $r \geq 1$, invariant manifold if S is an invariant set with the structure of a C^r differentiable manifold.

Definition 2.6.1. *Let* $\boldsymbol{p}$ *be an equilibrium point for the flow of* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ *and* U *a neighbourhood of* $\boldsymbol{p}$. *The* local stable manifold *for* $\boldsymbol{p}$ *in the neighbourhood* U *is the set*

$$W^s(\boldsymbol{p}, U) = \left\{ \boldsymbol{q} \in U : \phi^t(\boldsymbol{q}) \in U \text{ for } t > 0 \text{ and } d(\phi^t(\boldsymbol{q}), \boldsymbol{p}) \to 0 \text{ as } t \to \infty \right\} \ .$$

The local unstable manifold *for* $\boldsymbol{p}$ *in the neighbourhood* U *is the set*

$$W^u(\boldsymbol{p}, U) = \left\{ \boldsymbol{q} \in U : \phi^{-t}(\boldsymbol{q}) \in U \text{ for } t > 0 \text{ and } d(\phi^{-t}(\boldsymbol{q}), \boldsymbol{p}) \to 0 \text{ as } t \to \infty \right\} \ .$$

To simplify notation we also denote $W^s(\boldsymbol{p}, U)$ and $W^u(\boldsymbol{p}, U)$ by $W^s_{\text{loc}}(\boldsymbol{p})$ and $W^u_{\text{loc}}(\boldsymbol{p})$, respectively, provided this does not causes any ambiguity regarding the point $\boldsymbol{p}$. Furthermore, it is usual to take $U = B(\boldsymbol{p}, \epsilon) = \{\boldsymbol{q} \in M : d(\boldsymbol{q}, \boldsymbol{p}) < \epsilon\}$. In this case we denote the stable and unstable manifolds for $\boldsymbol{p}$ in $B(\boldsymbol{p}, \epsilon)$ by $W^s_\epsilon(\boldsymbol{p})$ and $W^u_\epsilon(\boldsymbol{p})$, respectively.

The invariant manifolds $W^s_{\text{loc}}(\boldsymbol{p})$ and $W^u_{\text{loc}}(\boldsymbol{p})$ are the analogues for the nonlinear case of the stable and unstable spaces $\mathbb{E}^s$ and $\mathbb{E}^u$ in the linear systems. The theorem below states that $W^s_{\text{loc}}(\boldsymbol{p})$ and $W^u_{\text{loc}}(\boldsymbol{p})$ are tangent to $\mathbb{E}^s$ and $\mathbb{E}^u$ at the equilibrium point $\boldsymbol{p}$.

Theorem 2.6.2 (Stable Manifold Theorem for equilibrium points)**.** *Let* f *be a* C^r *map, with* $r \geq 1$ *and let* $\boldsymbol{p}$ *be a hyperbolic equilibrium for the flow of* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. *Then there exist local stable and unstable manifolds* $W^s_{\text{loc}}(\boldsymbol{p})$ *and* $W^u_{\text{loc}}(\boldsymbol{p})$, *of the same dimensions as the stable and unstable spaces* $\mathbb{E}^s$ *and* $\mathbb{E}^u$ *of the corresponding linearized system around* $\boldsymbol{p}$, *and tangent to* $\mathbb{E}^s$ *and* $\mathbb{E}^u$ *at* $\boldsymbol{p}$, *respectively. Moreover,* $W^s_{\text{loc}}(\boldsymbol{p})$ *and* $W^u_{\text{loc}}(\boldsymbol{p})$ *are as smooth as the function* f.

The *global stable and unstable manifolds*, denoted by $W^s(\boldsymbol{p})$ and $W^u(\boldsymbol{p})$ respectively, are obtained from the local stable and unstable manifolds by the relations

$$\begin{aligned} W^s(\boldsymbol{p}) &= \bigcup_{t \leq 0} \phi^t \left(W^s(\boldsymbol{p}, U) \right) \\ W^u(\boldsymbol{p}) &= \bigcup_{t \geq 0} \phi^t \left(W^u(\boldsymbol{p}, U) \right) \ . \end{aligned}$$

Existence and uniqueness of solutions of (3) ensure that two stable or unstable manifolds of distinct equilibria $\boldsymbol{p}$ and $\boldsymbol{q}$ cannot intersect. Moreover, $W^s(\boldsymbol{p})$ and $W^u(\boldsymbol{q})$ can not intersect itself. However, intersections of stable and unstable manifolds of distinct fixed points or the same fixed point can occur. Such intersections are a source for much of the complex behavior in dynamical systems.

Example 2.6.3. *Recall the pendulum equation*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) \ , \qquad x \in [-\pi, \pi] \ , y \in \mathbb{R} \ . \end{cases}$$

We have seen above that $\boldsymbol{p}_\pm = (\pm\pi, 0)$ *are hyperbolic equilibria for the pendulum equation with eigenvalues* $\lambda_1 = -1$ *and* $\lambda_2 = 1$. *The local stable and unstable manifolds* $W^s_{\text{loc}}(\boldsymbol{p})$ *and* $W^u_{\text{loc}}(\boldsymbol{p})$ *are tangent at* $\boldsymbol{p}$ *to the lines through* $\boldsymbol{p}$ *with directions defined by the eigenvectors* $\boldsymbol{v}_1 = (-1, 1)$ *and* $\boldsymbol{v}_2 = (1, 1)$, *respectively. Moreover, it is possible to obtain an explicit description for the invariant manifolds of* $\boldsymbol{p}_\pm$*:*

$$W^s(\boldsymbol{p}_+) = W^u(\boldsymbol{p}_-) = \{(x, y) \in \mathbb{R}^2 : \frac{1}{2}y^2 - \cos(x) = 1 \ , \ y > 0 \ , \ -\pi < x < \pi\}$$
$$W^u(\boldsymbol{p}_+) = W^s(\boldsymbol{p}_-) = \{(x, y) \in \mathbb{R}^2 : \frac{1}{2}y^2 - \cos(x) = 1 \ , \ y < 0 \ , \ -\pi < x < \pi\} \ .$$

Another example is provided by

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y + x^2 \ , (x, y) \in \mathbb{R}^2 \ . \end{cases}$$

Clearly this system has a unique fixed point at the origin. Its linearization at the origin is given by

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \ . \end{cases}$$

The coefficients matrix of the linear system has eigenvalues $\lambda_1 = -1$ *and* $\lambda_2 = 1$ *and eigenvectors* $\boldsymbol{v}_1 = (0, 1)$ *and* $\boldsymbol{v}_2 = (1, 0)$. *The stable and unstable spaces are then given by:*

$$\mathbb{E}^s = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$
$$\mathbb{E}^u = \{(x, y) \in \mathbb{R}^2 : y = 0\} \ .$$

Eliminating the time variable, we obtain from the nonlinear system the following first order differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} + x$$

which can be integrated to obtain the family of solutions:

$$y(x) = \frac{x^3}{2} + \frac{C}{x} \ ,$$

where $C \in \mathbb{R}$ *is a constant to be determined by some initial condition.*

The stable manifold theorem then implies that $W^u_{\text{loc}}(0,0)$ can be represented as a graph of a function $y = h(x)$ with $h(0) = h'(0) = 0$, since $W^u_{\text{loc}}(0,0)$ is tangent to $\mathbb{E}^u$ at $(0,0)$. Hence, we get

$$W^u_{\text{loc}}(0,0) = \left\{(x,y) \in \mathbb{R}^2 : y = \frac{x^3}{2}\right\} ,$$

Finally, noting that if $x(0) = 0$, then $\dot{x} = 0$, and hence $x(t) = 0$, we get that

$$W^s_{\text{loc}}(0,0) = \left\{(x,y) \in \mathbb{R}^2 : x = 0\right\} .$$

2.7 Lyapunov functions

The method of Lyapunov functions can be used to determine the stability of equilibria when the information obtained from linearization is inconclusive, i.e. when the equilibria is not hyperbolic.

The basic idea behind the introduction of Lyapunov functions may be described in a heuristic way for two-dimensional systems of differential equations. Consider a system of two differential equations in the plane with an equilibrium point $\boldsymbol{p}$, the goal being to determine whether or not the equilibrium point $\boldsymbol{p}$ is stable. According to the definitions of Lyapunov stability, it us sufficient to find a neighbourhood U of $\boldsymbol{p}$ for which orbits starting in U remain in U for all positive times. This condition would be satisfied if the two-dimensional differential equation defines a vector field in the plane that is either tangent to the boundary of the neighbourhood U or pointing inwards towards $\boldsymbol{p}$. Moreover, this geometrical description should hold even as the neighbourhood U shrinks down to the point $\boldsymbol{p}$. This method holds for systems of differential equations of arbitrary dimension.

Theorem 2.7.1. *Let $\boldsymbol{p}$ be an equilibrium point for the flow of* (3) *and let $V : U \to \mathbb{R}$ be a C^1 function defined on some neighborhood U of $\boldsymbol{p}$ such that:*

(i) $V(\boldsymbol{p}) = 0$ *and* $V(x) > 0$ *in* $x \in U\backslash\boldsymbol{p}$*;*

(ii) $\dot{V}(\boldsymbol{x}) \leq 0$ *in* $U\backslash\boldsymbol{p}$*.*

Then $\boldsymbol{p}$ is Lyapunov stable and V is called a Lyapunov function.

Moreover, if

(iii) $\dot{V}(\boldsymbol{x}) < 0$ *in* $U\backslash\boldsymbol{p}$*.*

then $\boldsymbol{p}$ is asymptotically stable and V is called a strict Lyapunov function.

The derivative in the statement of the theorem corresponds to the derivative of V along solution curves of (3) and may be written as:

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(\boldsymbol{x}) ,$$

where $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$.

If we can choose $U = \mathbb{R}^n$ in case item (iii), then $\boldsymbol{p}$ is said to be globally asymptotically stable, and all solutions remain bounded and in fact approach $\boldsymbol{p}$ as $t \to +\infty$. Thus, we have one more method for testing stability of equilibria (and boundedness of solutions) without actually having to solve the differential equation. The drawback is that there are no general methods for finding suitable Lyapunov functions, although in mechanical problems the energy is often a good candidate.

Example 2.7.2. *Consider the two-dimensional system of differential equations:*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \epsilon x^2 y \ , \end{cases}$$

with a unique equilibrium $\boldsymbol{p} = (0,0)$*. Computing the corresponding linearized system, it is easy to see that* $\boldsymbol{p}$ *is nonhyperbolic. To decide if this equilibrium point is stable, we consider the following function:*

$$V(x,y) = \frac{1}{2}(x^2 + y^2) \ .$$

Clearly $V(0,0) = 0$ *and* $V(x,y) > 0$ *in any neighborhood of* $(0,0)$*. Moreover, we get that*

$$\dot{V}(x,y) = x\dot{x} + y\dot{y} = \epsilon x^2 y^2 \ .$$

Thus, the equilibrium point $(0,0)$ *is globally Lyapunov stable for* $\epsilon < 0$ *(since* V *is zero on the lines* $x = 0$ *and* $y = 0$*).*

2.8 Periodic orbits and Poincaré maps

We consider again the differential equation (3) and let γ denote a periodic orbit of the flow of (3) with period T and $\boldsymbol{p} \in \gamma$. Then, for some k, the k^{th} coordinate of the map f must be non–zero at $\boldsymbol{p}$, i.e. $f_k(\boldsymbol{p}) \neq 0$. Take the hyperplane given by

$$\Sigma = \{\boldsymbol{x} : x_k = p_k\} \ .$$

The hyperplane Σ is called a *cross section* at $\boldsymbol{p}$. For some $\boldsymbol{x} \in \Sigma$ near $\boldsymbol{p}$, the flow $\phi^t(\boldsymbol{x})$ returns to Σ in time $\tau(\boldsymbol{x})$ close to T. We call $\tau(\boldsymbol{x})$ the *first return time.*

Definition 2.8.1. *Let* $V \subset \Sigma$ *be an open set in* Σ *on which* $\tau(\boldsymbol{x})$ *is a differentiable function. The* Poincaré map*,* $P : V \to \Sigma$*, is defined by*

$$P(\boldsymbol{x}) = \phi^{\tau(\boldsymbol{x})}(\boldsymbol{x}) \ . \tag{8}$$

Thus, the Poincaré map reduces the analysis of a continuous time dynamical system to the analysis of a discrete time dynamical system. This is very useful for the analysis of the behaviour of periodic orbits of flows since such orbits are fixed points of the Poincaré map. We list below some properties of the Poincaré map of a flow near a periodic orbit.

Theorem 2.8.2. *Let* ϕ^t *be the* C^r *flow (*$r \geq 1$*) of* $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$*.*

i) If $\boldsymbol{p}$ *is on a periodic orbit of period* T *and* Σ *is transversal at* $\boldsymbol{p}$*, then the first return time* $\tau(\boldsymbol{x})$ *is defined in a neighbourhood* V *of* $\boldsymbol{p}$ *and* $\tau : V \to \mathbb{R}$ *is* C^r*.*

ii) The Poincaré map (8) *is* C^r*.*

iii) If γ *is a periodic orbit of period* T *and* $\boldsymbol{p} \in \gamma$*, then* $D\phi^T_{\boldsymbol{p}}$ *has 1 as an eigenvalue with eigenvector* $f(\boldsymbol{p})$*.*

iv) If γ *is a periodic orbit of period* T *and* $\boldsymbol{p}, \boldsymbol{q} \in \gamma$*, then the derivatives* $D\phi^T_{\boldsymbol{p}}$ *and* $D\phi^T_{\boldsymbol{q}}$ *are linearly conjugate and so have the same eigenvalues.*

We will now use the Poincaré map to study the stability of periodic orbits of the dynamical system defined by (3). Let γ be a periodic orbit of period T for the flow of (3) with $\boldsymbol{p} \in \gamma$ and let $1, \lambda_1, ..., \lambda_{m-1}$ be the eigenvalues of $D\phi^T_{\boldsymbol{p}}$. The $m-1$ eigenvalues $\lambda_1, ..., \lambda_{m-1}$ are called the *characteristic multipliers* of the periodic orbit γ. We say that

- γ is *hyperbolic* if $|\lambda_j| \neq 1$ for all $j \in \{1, ..., m-1\}$.
- γ is *elliptic* if $|\lambda_j| = 1$ for all $j \in \{1, ..., m-1\}$.

Moreover, in the case where γ is a hyperbolic periodic orbit, we say that

- γ is a *periodic sink* if $|\lambda_j| < 1$ for all $j \in \{1, ..., m-1\}$.
- γ is a *periodic source* if $|\lambda_j| > 1$ for all $j \in \{1, ..., m-1\}$.
- γ is a *saddle periodic orbit* if γ is neither a periodic sink nor a periodic source.

The next result establishes the relation between the characteristic multipliers and the Poincaré map near a periodic orbit.

Theorem 2.8.3. *Let $\boldsymbol{p}$ be a point on a periodic orbit γ of period T for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$. Then, the characteristic multipliers of the periodic orbit are the same as the eigenvalues of the derivative of the Poincaré map at $\boldsymbol{p}$.*

The next theorem regarding the stability of periodic orbits of flows follows the theorem above and the analysis of the stability of fixed points of maps.

Theorem 2.8.4. *Let γ be a periodic orbit of period T for the flow of $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$.*

i) If γ is a periodic sink, then γ is asymptotically stable.

ii) If γ has at least one characteristic multiplier λ_k such that $|\lambda_k| > 1$, then γ is not Lyapunov stable.

We illustrate the results above in the next example.

Example 2.8.5. *Consider the two-dimensional system*

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \ . \end{cases}$$

It is easy to see that the origin is an equilibrium with eigenvalues $\lambda = 1 \pm \mathrm{i}$. Thus, the origin is a spiral source. Introducing polar coordinates $x = r\cos(\theta)$, $y = r\cos(\theta)$, we obtain

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \ . \end{cases} \tag{9}$$

Recall that the variable r represents the distance to the origin of the point (x, y), while θ is equal to the angle formed between the line joining the origin and (x, y), and the positive x-axis. Moreover, note that $\dot{r} = 0$ if and only if $r = 0$ or $r = 1$. Clearly, $r = 0$ corresponds to the the equilibrium at the origin. In what concerns $r = 1$, this corresponds to the periodic orbit γ defined by $x^2 + y^2 = 1$, since on this set the value of r is fixed, while the angle θ evolves at a constant rate 1.

To decide about the stability of the periodic orbit γ, we introduce a Poincaré map. Consider the cross section

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \ , x > 0\} \ ,$$

which in polar coordinates becomes

$$\Sigma = \{(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^1 : \theta = 0\ , r > 0\}\ .$$

Since the two-dimensional system (9) *is given by two independent first order equations, i.e. the system has decoupled when the change to polar coordinates was made, it is possible to solve it explicitly to obtain:*

$$\phi^t(r_0,\theta_0) = \left(\left(1 + \left(\frac{1}{{r_0}^2} - 1 \right) \mathrm{e}^{-2t} \right)^{-1/2}, t + \theta_0 \right)\ .$$

The time taken for any point $\boldsymbol{q} \in \Sigma$ *to return to* Σ *is equal to* 2π*. Thus, the Poincaré map is given by*

$$P(r_0) = \left(1 + \left(\frac{1}{{r_0}^2} - 1 \right) \mathrm{e}^{-4\pi} \right)^{-1/2}\ .$$

Clearly, the map P *has a fixed point at* $r_0 = 1$*, i.e.* $P(1) = 1$*, which corresponds to the periodic orbit* γ*. In this case,* P *is a one-dimensional map and its linearization at* $r_0 = 1$ *is given by*

$$DP(1) = \frac{\mathrm{d}P}{\mathrm{d}r_0}_{|r_0=1} = \mathrm{e}^{-4\pi} < 1\ .$$

Thus, $r_0 = 1$ *is a stable fixed point of* P *and* γ *is a stable or attracting periodic orbit.*

A final remark to point out that $DP(1)$ *could have been computed by considering the linearization of* (9) *near the periodic orbit orbit:*

$$\begin{cases} \dot{s} = -2s \\ \dot{\theta} = 1\ . \end{cases}$$

The linear flow is then given by

$$\phi^t(s_0,\theta_0) = \left(\mathrm{e}^{-2t}s_0, t + \theta_0\right)\ .$$

Hence, $DP(1) = \mathrm{e}^{-4\pi}$*, as expected.*

2.9 Asymptotic behaviour

We will now briefly discuss some of the concepts which are key to study the long term behaviour of orbits of dynamical systems. We have already introduced and discussed the notion of ω and α limit sets of flows. However, these notions do not address the question of stability of those asymptotic motions. In order to do that, we will introduce the notion of attractor. To proceed with this plan, we need to introduce several auxiliary notions before.

Definition 2.9.1 (Attracting set and trapping region)**.** *A closed invariant set* $A \subset \mathbb{R}^n$ *is called an* attracting set *if there is some positively invariant neighbourhood* U *of* A *such that for every* $t \geq 0$ *we have that*

$$\bigcap_{t>0} \phi^t(U) = A\ .$$

The open set U *is called a* trapping region.

It should be clear that finding a Lyapunov function is equivalent to finding a trapping region. Also, it should be noted that all solutions of (3) starting in a trapping region exist for all positive times. This is useful in non-compact phase spaces such as $\mathbb{R}^n$ for proving existence of solutions on semi-infinite time intervals.

To test if a given region is a candidate to be a trapping region, one can evaluate the vector field defined by (3) on the boundary of such region. If, on the boundary of the region, the vector field is pointing towards the interior of the region, or is tangent to the boundary, then such region is a trapping region. Note that the boundary of the region must be at least C^1 in order for this test to be carried out.

Given an attracting set, the natural question to pose is which points in phase space approach the attracting set asymptotically:

Definition 2.9.2 (Basin of attraction)**.** *The basin of attraction of an attracting set A is given by*

$$\bigcup_{t \le 0} \phi^t(U) = A \ .$$

The basin of attraction of an attracting set A is independent of the choice of the open set U being attracted to A, as long as U is as given in definition 2.9.1.

Example 2.9.3. *Consider the two-dimensional system*

$$\begin{cases} \dot{x} = x - x^3 \\ \dot{y} = -y \ , \end{cases}$$

It should be easy to see that it has three equilibria: a saddle at $(0,0)$ and two sinks at $(\pm 1, 0)$. The y-axis is the stable manifold of $(0,0)$. Take a set U containing an ellipse (or a circle) containing the three equilibria. Then U is a trapping region and the set $A = [-1,1] \times \{0\}$ is an attracting set. Its basin of attraction is the whole plane. However, almost all points in the plane will eventually end up near one of the sinks. Hence, the attracting set A "contains two attractors", the sinks $(\pm 1, 0)$.

As the previous example shows, if we are interested in describing where most points in phase space go, we must go beyond the notion of attracting set. This may be achieved by excluding from the definition of an attracting set the case with a collection of distinct attractors. Instead, it should be required that all points in the attracting set eventually come arbitrarily close to every other point in the attracting set under the flow:

Definition 2.9.4 (Topological Transitivity)**.** *A closed invariant set A is said to be* topologically transitive *if, for any two open sets $U, V \subset A$ there exists some $t \in \mathbb{R}$ such that*

$$\phi^t(U) \cap V \neq \emptyset \ .$$

We are now ready to define what we mean by an attractor:

Definition 2.9.5 (Attractor)**.** *An* attractor *is a topologically transitive attracting set.*

It should be noted that the study of attractors and their basin boundaries is still a topic of research in dynamical systems. We will discuss some examples with attractors later in this notes.

2.10 Two dimensional flows

We will now discuss one important result for planar two-dimensional systems. The Poincaré-Bendixon theorem provides a complete determination of the asymptotic behaviour of a large class of flows on the plane under very mild conditions. Systems on two dimensional manifolds other than $\mathbb{R}^2$ are more complicated and can display more subtle behavior.

Theorem 2.10.1 (Poincaré-Bendixon theorem)**.** *Consider the flow ϕ^t of a planar system of differential equations in $\mathbb{R}^2$. Let M be a positively invariant open set by ϕ^t and assume that M contains a finite number of equilibria. Let $\boldsymbol{p} \in M$, and consider its ω-limit set $\omega(\boldsymbol{p})$. Then one of the following possibilities holds:*

(i) $\omega(\boldsymbol{p})$ *is an equilibria;*

(ii) $\omega(\boldsymbol{p})$ *is a closed orbit;*

(ii) $\omega(\boldsymbol{p})$ *consists of a finite number of equilibria $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ and orbits γ such that $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$ for some $i, j \in \{1, \ldots, n\}$.*

The orbits in the third item are referred to as *heteroclinic orbits* when they connect distinct equilibrium points and *homoclinic orbits* when they connect a point to itself. Closed paths formed of heteroclinic orbits are called *homoclinic cycles*. Two-dimensional flows have generally richer dynamics than one-dimensional systems, in which periodic orbits cannot occur and the equilibrium points are ordered and connected to their immediate neighbors and only to them. The existence of heteroclinic connections in higher-dimensional systems depends on the relative dimensions of the stable and unstable manifolds of neighbouring fixed points. These connections are generally very difficult to find, unless the system possesses special symmetries or other properties.

A final note concerns flows on nonplanar two-dimensional manifolds such as the torus. Depending on the geometry of the two-dimensional manifold, it may have limit sets which are not closed orbits, fixed points, homoclinic or heteroclinic orbits cycles. A particularly simple example is provided by the irrational linear flow on the torus, which has dense orbits.

2.11 Gradient systems

A gradient system on $\mathbb{R}^n$ is a system of differential equations of the form

$$\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x}) \,, \tag{10}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a C^r function, $r \geq 1$, and

$$\nabla V(\boldsymbol{x}) = \left(\frac{\partial V}{\partial x_1}(\boldsymbol{x}), \ldots, \frac{\partial V}{\partial x_n}(\boldsymbol{x}) \right) \,.$$

Gradient systems have special properties that make their flows rather simple. One particularly interesting property is the following:

Proposition 2.11.1. *The function V is a Lyapunov function for the gradient system* (10)*. Moreover, $\dot{V}(\boldsymbol{p}) = 0$ if and only if $\boldsymbol{p}$ is an equilibrium point, where $\dot{V}$ denotes the derivative of V along a solution of* (10)*.*

A consequence of the result above is that if $\boldsymbol{p}$ is an isolated minimum of the function V, then $\boldsymbol{p}$ is an asymptotically stable equilibrium of the gradient system (10). We now list some properties for this class of dynamical systems:

Theorem 2.11.2. *The gradient system* (10) *has the following properties:*

(i) If c is a regular value of V, then the vector field defined by $-\nabla V(x)$ is perpendicular to the level set $V^{-1}(c)$.

(ii) The critical points of the function V are the equilibrium points of (10).

(iii) If a critical point is an isolated minimum of V, then this point is an asymptotically stable equilibrium point.

(iv) Every limit point of a solution of (10) *is an equilibrium.*

(v) The linearization of (10) *at any equilibrium point has only real eigenvalues.*

Example 2.11.3. *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be the function given by*

$$V(x, y) = x^2(x-1)^2 + y^2 \ .$$

The gradient system determined by V is

$$\begin{cases} \dot{x} = -2x(x-1)(2x-1) \\ \dot{y} = -2y \ , \end{cases} \tag{11}$$

which has three equilibrium points: $(0, 0)$, $(1/2, 0)$, *and* $(1, 0)$. *Computing the linearization of* (11) *around these equilibria, we get that* $(0, 0)$ *and* $(1, 0)$ *are sinks, while* $(1/2, 0)$ *is a saddle. Moreover, it is clear that* $x = 0$, $x = 1/2$, $x = 1$ *and* $y = 0$ *contain the equilibria invariant manifolds.*

The sets $V^{-1}([0, c])$ are closed, bounded, and positively invariant under the gradient flow (11). *Therefore, each solution entering such a set is defined for all $t \geq 0$, and tends to one of the three equilibria.*

2.12 Hamiltonian systems

Hamiltonian mechanics are one of the possible formulations for classical mechanics. Since its invention in 1833 by William Rowan Hamilton it has been one of the most useful tools for the mathematical analysis of physical systems and it is still a flourishing field as a mathematical theory.

The general framework to deal with Hamiltonian system is evolved from the technical point of view. We provide just the very basic ideas here.

Let U be an open subset of $\mathbb{R}^{2n}$ and $H : U \to \mathbb{R}$ a C^r function, $r \geq 2$. Hamilton's canonical equations are given by

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \ , \qquad (q_1, \ldots, q_n, p_1, \ldots, p_n) \in U \subset \mathbb{R}^{2n} \ . \end{aligned} \tag{12}$$

Introducing the (symplectic) matrix

$$\mathbf{J} = \begin{pmatrix} 0 & \mathrm{Id}_{n\times n} \\ -\mathrm{Id}_{n\times n} & 0 \end{pmatrix} \ ,$$

it is possible to represent (12) in the more compact form

$$\dot{\boldsymbol{x}} = \mathbf{J}\nabla H(\boldsymbol{x}) \ , \qquad x = (q_1, \ldots, q_n, p_1, \ldots, p_n) \in U \subset \mathbb{R}^{2n} \ .$$

Hamiltonian systems conserve the phase space volume measure, and for this reason are included in the class of conservative dynamical systems.

Theorem 2.12.1 (Liouville's Theorem)**.** *Let $\phi^t(\boldsymbol{x})$ denote the flow of Hamilton's equations* (12)*. The flow $\phi^t(\boldsymbol{x})$ preserves the volume in phase space. For any region $D \subset U$ we have*

$$\mathrm{Vol}\left(\psi^t(D)\right) = \mathrm{Vol}\,(D) \ .$$

for every $t \in \mathbb{R}$.

Another key property of Hamiltonian systems is provided in the proposition below:

Proposition 2.12.2. *The Hamiltonian function $H(q,p)$ is constant along orbits of the flow of the Hamiltonian system* (12)*.*

More generally, we refer to any function $F : U \to \mathbb{R}$ which is constant along orbits of the flow of the Hamiltonian system (12) as a first integral or conserved quantity with respect to the dynamics generated (12).

Definition 2.12.3. *Let $\phi^t(\boldsymbol{x})$ denote the flow of Hamilton's equations* (12)*. A* conserved quantity *(or* first integral*) of the Hamiltonian system* (12) *is a map $J : U \to \mathrm{Re}$ such that $J\left(\psi^t(\boldsymbol{x})\right)$ is a constant function of t.*

There is a deep relation between symmetries and conserved quantities of Hamiltonian systems, with very interesting consequences for the analysis of Hamiltonian dynamical systems.

Definition 2.12.4. *We say that the Hamiltonian system* (12) *has (continuous) symmetry if there exists a one–parameter group of transformations of the phase space of the Hamiltonian system, $\phi_\lambda : U \times \mathbb{R} \to U$, $\lambda \in \mathbb{R}$, that preserves both the Hamiltonian function and the form of Hamilton's equations* (12)*.*

The next theorem provides the relation between symmetries and conserved quantities in Hamiltonian systems.

Theorem 2.12.5 (Noether's Theorem)**.** *If the Hamiltonian system* (12) *has a one–parameter group of symmetries ϕ_λ, $\lambda \in \mathbb{R}$, then Hamilton's equations have a conserved quantity $J : U \to \mathbb{R}$.*

Thus, associated to each one–parameter group of symmetries there is one conserved quantity of the Hamiltonian system. The existence of symmetries and conserved quantities in a Hamiltonian system enables the reduction of the dimension of its phase space which might lead to a simplification on the analysis of the dynamical behaviour of Hamilton's equations.

Under certain "mild" conditions on the conserved quantities, for each conserved quantity of a Hamiltonian system we are able to reduce the dimension of its phase space by two dimensions. We will now provide the setting for the particular and fundamental case of a Liouville (or completely) integrable Hamiltonian system: n degrees of freedom with n conserved quantities (independent and in involution).

Definition 2.12.6. *Let $F_1, ..., F_n$ conserved quantities of the Hamiltonian system* (12)*. We say that the conserved quantities are in* involution *if the following set of equalities is satisfied*

$$\{F_i, F_j\} = 0 , \qquad i \neq j ,$$

where $\{F, G\}$ denotes the Poisson bracket of the functions F and G, given by

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i} .$$

Definition 2.12.7. *Let $F_1, ..., F_n$ conserved quantities of the Hamiltonian system* (12) *and consider the following level set:*

$$C_{\boldsymbol{a}} = \left\{\boldsymbol{x} \in \mathbb{R}^{2n} : F_i(\boldsymbol{x}) = a_i \ , \ i \in \{1, ..., n\}\right\} \ , \tag{13}$$

where $\boldsymbol{a} = (a_1, ..., a_n) \in \mathbb{R}^n$. We say that the conserved quantities are independent on the set $C_{\boldsymbol{a}} \subset M$ *if the n gradient vectors ∇F_i, $i \in \{1, ..., n\}$, are linearly independent at each point of $C_{\boldsymbol{a}}$.*

The concepts introduced above lead to the following important result.

Theorem 2.12.8 (Arnold–Liouville Theorem)**.** *Let $((M, \omega), H)$ be a Hamiltonian system with n degrees of freedom and assume that*

i) *$((M, \omega), H)$ has n analytic conserved quantities F_1,..., F_n in involution.*

ii) *the conserved quantities F_1,..., F_n are independent on the level set $C_{\boldsymbol{a}}$ (defined in* (13)*).*

Then the Hamiltonian system (12) *is completely integrable and $C_{\boldsymbol{a}}$ is a smooth manifold invariant under its phase flow.*

Since we are assuming that the flow $\phi^t(\boldsymbol{x})$ is defined for every $t \in \mathbb{R}$ it should be added to the previous theorem that;

i) each connected component of $C_{\boldsymbol{a}}$ is diffeomorphic to the product of a k–dimensional torus $\mathbb{T}^k$ with an $(n-k)$–dimensional Euclidean space $\mathbb{R}^{n-k}$ for some k. If moreover, $C_{\boldsymbol{a}}$ is compact, then $k = n$ and $C_{\boldsymbol{a}}$ is diffeomorphic to a torus $\mathbb{T}^n$.

ii) on $\mathbb{T}^k \times \mathbb{R}^{n-k}$, there exist coordinates $\varphi_1, ..., \varphi_k$ and $z_1, ..., z_{n-k}$ such that Hamilton's equations on $C_{\boldsymbol{a}}$ are

$$\begin{aligned} \dot{\varphi}_i &= \omega_i \ , \qquad 1 \leq i \leq k \\ \dot{z}_j &= c_j \ , \qquad 1 \leq j \leq n-k \ , \end{aligned}$$

where $\omega_i = \omega_i(\boldsymbol{a})$ and $c_j = c_j(\boldsymbol{a})$ are constants.

The coordinates introduced above are called action-angle coordinates.

Example 2.12.9. *Hamiltonian systems are particularly well suited to describe physical systems such as, for instance, the pendulum or the N-body problem.*

The pendulum has Hamiltonian function $H : \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(q, p) = \frac{1}{2}p^2 + 1 - \cos(q) .$$

Hamilton's equations describing the motion or the pendulum are then given by

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\sin(q) \ . \end{cases}$$

The most famous example of a Hamiltonian system is the N-body problem, which has Hamiltonian function $H : \mathbb{R}^{6N} \to \mathbb{R}$ given by

$$H(q,p) = \sum_{i=1}^{N} \frac{1}{2m_i} \|p_i\|^2 + \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|} \ ,$$

where $q = (q_1, \ldots, q_n) \in \mathbb{R}^{3n}$, $p = (p_1, \ldots, p_n) \in \mathbb{R}^{3n}$ are the vectors of positions and momenta, and $m_1, \ldots, m_n$ are the bodies masses.

3 Bifurcations

Consider the differential equation

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \mu) \ , \tag{14}$$

where f is a sufficiently regular function defined on some open set of $\mathbb{R}^n \times \mathbb{R}^p$ and μ is a parameter. In this section we introduce the concept of bifurcation: a change in the qualitative behaviour of the dynamical system (14) when the parameter μ is varied.

3.1 Local bifurcations

We will first address the case of local bifurcations. These occur when the stability of an equilibrium changes as the parameter μ is varied and correspond to the case where the real part of an eigenvalue of that equilibrium passes through zero. We study four distinct types of local bifurcations: saddle-node, transcritical, pitchfork and Hopf bifurcations. For simplicity of exposition, we will restrict our attention to one-dimensional systems in the three first cases and to planar systems during the study of Hopf bifurcations

Theorem 3.1.1 (Saddle-node bifurcation)**.** *Suppose that $\dot{x} = f(x, \mu)$, $x, \mu \in \mathbb{R}$, is a first-order differential equation for which*

(i) $f(x_0, \mu_0) = 0$;

(ii) $\frac{\partial f}{\partial x}(x_0, \mu_0) = 0$;

(iii) $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$;

(iv) $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$.

Then this differential equation undergoes a saddle-node bifurcation at $\mu = \mu_0$.

Example 3.1.2. *The key example of a saddle-node bifurcation is provided by the following differential equation:*

$$\dot{x} = f(x, \mu) = \mu - x^2 \ , \quad x \in \mathbb{R} \ , \ \mu \in \mathbb{R} \ . \tag{15}$$

Clearly, we have that

$$f(0,0) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial x}(0,0) = 0 \ .$$

The set of all equilibria of this differential equation is given by $\mu = x^2$, corresponding to a parabola in the $\mu - x$ plane.

For $\mu < 0$, (15) has no equilibria, and $\dot{x}$ is always negative. For $\mu > 0$, (15) has two equilibria: a stable one, corresponding to one branch of the parabola, and an unstable one, corresponding to the other of the parabola.

Theorem 3.1.3 (Transcritical bifurcation)**.** *Suppose that $\dot{x} = f(x, \mu)$, $x, \mu \in \mathbb{R}$, is a first-order differential equation for which*

(i) $f(x_0, \mu_0) = 0$;

(ii) $\frac{\partial f}{\partial x}(x_0, \mu_0) = 0$;

(iii) $\frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0$.

(iv) $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$;

(v) $\frac{\partial^2 f}{\partial x \partial \mu}(x_0, \mu_0) \neq 0$.

Then this differential equation undergoes a transcritical bifurcation at $\mu = \mu_0$.

Example 3.1.4. *The most simple example of a transcritical bifurcation is provided by the following differential equation:*

$$\dot{x} = f(x, \mu) = \mu x - x^2 \ , \quad x \in \mathbb{R} \ , \ \mu \in \mathbb{R} \ . \tag{16}$$

Clearly, we have that

$$f(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(0,0) = 0 \ .$$

The set of all equilibria of this differential equation is given by $x = 0$ and $\mu = x$, corresponding to two lines in the $\mu - x$ plane.

For $\mu < 0$, (16) has two equilibria: $x = 0$ is stable and $x = \mu$ is unstable. The two equilibria coalesce at $\mu = 0$ and, for $\mu > 0$, $x = 0$ is unstable and $x = \mu$ is stable. An exchange of stability has occurred at $\mu = 0$.

Theorem 3.1.5 (Pitchfork bifurcation)**.** *Suppose that $\dot{x} = f(x, \mu)$, $x, \mu \in \mathbb{R}$, is a first-order differential equation for which*

(i) $f(x_0, \mu_0) = 0$;

(ii) $\frac{\partial f}{\partial x}(x_0, \mu_0) = 0$;

(iii) $\frac{\partial f}{\partial \mu}(x_0, \mu_0) = 0$.

(iv) $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) = 0$;

(v) $\frac{\partial^2 f}{\partial x \partial \mu}(x_0, \mu_0) \neq 0$.

(vi) $\frac{\partial^3 f}{\partial x^3}(x_0, \mu_0) \neq 0$.

Then this differential equation undergoes a pitchfork bifurcation at $\mu = \mu_0$.

Example 3.1.6. *For an example of a differential equation with a pitchfork bifurcation, consider*

$$\dot{x} = f(x, \mu) = \mu x - x^3 \ , \quad x \in \mathbb{R} \ , \ \mu \in \mathbb{R} \ . \tag{17}$$

Clearly, we have that

$$f(0,0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(0,0) = 0 \ .$$

The set of all equilibria of this differential equation is given by $x = 0$ and $\mu = x^2$, corresponding to a line and a parabola in the $\mu - x$ plane.

For $\mu < 0$, (17) has one stable equilibria at $x = 0$. For $\mu > 0$, $x = 0$ is now an unstable equilibria, and two new stable equilibria have been created at $\mu = 0$, one in each branch of the parabola $\mu = x^2$.

Theorem 3.1.7 (Hopf bifurcation)**.** *Suppose that $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \mu)$, $\boldsymbol{x} \in \mathbb{R}^2, \mu \in \mathbb{R}$, is a first-order system of differential equations for which:*

(i) $f(x_0, \mu) = 0$ *for all values of* μ *close to* μ_0*;*

(ii) the eigenvalues of $D_x f(x_0, \mu)$ *are* $\alpha(\mu) \pm \mathrm{i}\beta(\mu)$ *with* $\alpha(\mu_0) = 0$*,* $\beta(\mu_0) \neq 0$ *and* $\alpha'(\mu_0) \neq 0$*.*

Then this differential equation undergoes a Hopf bifurcation at $\mu = \mu_0$*.*

Example 3.1.8. *For an example of a system of differential equations exhibiting a Hopf bifurcation, consider*

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = 1 \ , \end{cases}$$

where (r, θ) *are polar coordinates in the plane.*

Note that the origin is the only equilibrium point for this system, since $\dot{\theta} \neq 0$*. For* $\mu < 0$*, the origin is a sink since* $\mu r - r^3 < 0$ *for all* $r > 0$*. For* $\mu > 0$*, the origin is a source and a stable periodic orbit of radius* $\sqrt{\mu}$ *has been created at* $\mu = 0$*.*

3.2 Global bifurcations

Global bifurcations occur when invariant sets, such as periodic orbits, collide with equilibria, causing the topology of the trajectories in the phase space to change. Such changes are not confined in a small neighbourhood and extend to an arbitrarily large distance in phase space. Global bifurcations can also involve more complicated sets such as chaotic attractors. Since this subject is already quite technical, we provide an illustrative example only.

The simplest global bifurcations occur for planar vector fields when there is a trajectory joining two saddle points or forming a loop containing a single saddle point. Consider the following planar system of differential equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^2 + \mu y \ . \end{cases}$$

When $\mu = 0$ the system is divergence-free and has a first integral

$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} \ .$$

It is possible to see that the origin is a saddle point, and a stable manifold of the origin overlaps an unstable manifold, forming a loop γ. Moreover, the interior of γ is filled by a family of closed orbits. For $\mu \neq 0$, these invariant manifolds do not overlap, radically changing the qualitative behaviour of this dynamical system. In particular, the origin turns into a spiral sink if $\mu < 0$ or a spiral source if $\mu > 0$. No periodic orbit persists when $\mu \neq 0$. Thus, the bifurcation at $\mu = 0$ has had a non-local impact on the behaviour of this dynamical system.

4 Chaotic behaviour

This section is devoted to the study of four continuous-time dynamical systems with chaotic behaviour. Before proceeding with the discussion of these examples, we introduce a very useful tool for detecting chaos.

4.1 A tool for detecting chaos: Lyapunov exponents

Consider again a differential equation of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) \ , \qquad \boldsymbol{x} \in \mathbb{R}^n \ . \tag{18}$$

where f is a C^r map with $r \geq 1$. Let $\boldsymbol{x}(t, \boldsymbol{x}_0)$ be the solution of (18) with initial condition $\boldsymbol{x}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$. To describe the geometry associated with the attraction and/or repulsion of orbits of (18) relative to $\boldsymbol{x}(t, \boldsymbol{x}_0)$. One considers the orbit structure of the linearization of (18) about $\boldsymbol{x}(t, \boldsymbol{x}_0)$, which is given by

$$\dot{\boldsymbol{y}} = Df(\boldsymbol{x}(t, \boldsymbol{x}_0))\boldsymbol{y} \ . \tag{19}$$

Let $X(t, \boldsymbol{x}(t, \boldsymbol{x}_0))$ denote the fundamental solution matrix of (19) and let $\boldsymbol{e} \neq 0$ be a vector in $\mathbb{R}^n$. We define the coefficient of expansion in the direction $\boldsymbol{e}$ along the trajectory through x_0 to be

$$\lambda_t(\boldsymbol{x}_0, \boldsymbol{e}) = \frac{\|X(t, \boldsymbol{x}(t, \boldsymbol{x}_0))\boldsymbol{e}\|}{\|\boldsymbol{e}\|} \ .$$

Note that the coefficient $\lambda_t(\boldsymbol{x}_0, \boldsymbol{e})$ depends on t, on the orbit of (18) through $\boldsymbol{x}_0$ and on $\boldsymbol{e}$.

The *Lyapunov exponent* in the direction $\boldsymbol{e}$ along the trajectory through $\boldsymbol{x}_0$ is defined as

$$\chi(\boldsymbol{x}_0, \boldsymbol{e}) = \overline{\lim}_{t \to +\infty} \frac{1}{t} \lambda_t(\boldsymbol{x}_0, \boldsymbol{e}) \ .$$

For the zero vector it is common to define $\chi(\boldsymbol{x}_0, \boldsymbol{0}) = -\infty$.

Proposition 4.1.1 (Properties of Lyapunov exponents)**.** *The following properties hold:*

(i) For any vectors $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^n$, $\chi(\boldsymbol{x}_0, \boldsymbol{e}_1 + \boldsymbol{e}_2) \leq \max\{\chi(\boldsymbol{x}_0, \boldsymbol{e}_1), \chi(\boldsymbol{x}_0, \boldsymbol{e}_1)\}$.

(ii) For any vector $\boldsymbol{e} \in \mathbb{R}^n$ and constant $c \in \mathbb{R}$, $\chi(\boldsymbol{x}_0, c\boldsymbol{e}) = \chi(\boldsymbol{x}_0, \boldsymbol{e})$.

(iii) The set of numbers $\{\chi(\boldsymbol{x}_0, \boldsymbol{e})\}_{\boldsymbol{e} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}}$ takes at most n values. It is called Lyapunov spectrum.

In practical applications the Lyapunov exponents of a trajectory are typically computed numerically. There has been much rigorous work in recent years involving the development of algorithms to accurately compute Lyapunov exponents

4.2 Chaotic behaviour and strange attractors

Let $\phi^t(\boldsymbol{x})$ denote the flow of (18) and assume $\Lambda \subset \mathbb{R}^n$ is a compact and invariant set $\phi^t(\boldsymbol{x})$, i.e. $\phi^t(\Lambda) \subseteq \Lambda$ for all $t \in \mathbb{R}$.

We say that the flow $\phi^t(\boldsymbol{x})$ has sensitive dependence on initial conditions if for any point $\boldsymbol{x} \in \Lambda$, there is at least one point arbitrarily close to x that diverges from $\boldsymbol{x}$. More precisely:

Definition 4.2.1 (Sensitive dependence on initial conditions)**.** *The flow $\phi^t(\boldsymbol{x})$ is said to have sensitive dependence on initial conditions on Λ if there exists $\epsilon > 0$ such that, for any $\boldsymbol{x} \in \Lambda$ and any neighborhood U of $\boldsymbol{x}$, there exists $\boldsymbol{y} \in U$ and $t > 0$ such that $\left\|\phi^t(\boldsymbol{x}) - \phi^t(\boldsymbol{y})\right\| > \epsilon$.*

Taken just by itself, sensitive dependence on initial conditions is a fairly common property in many dynamical systems. For a set to be chaotic, a couple of other properties need to be added:

Definition 4.2.2 (Chaotic invariant set)**.** *An invariant set Λ is said to be chaotic if*

(i) $\phi^t(\boldsymbol{x})$ has sensitive dependence on initial conditions on Λ.

(ii) $\phi^t(\boldsymbol{x})$ is topologically transitive on Λ.

(iii) The periodic orbits of $\phi^t(\boldsymbol{x})$ are dense in Λ.

We now define what is meant by strange attractor:

Definition 4.2.3 (Strange attractor)**.** *Suppose that $A \subset \mathbb{R}^n$ is an attractor. Then A is called a* strange attractor *if it is chaotic.*

A standard procedure to prove that a dynamical system has a strange attractor is the following:

(1) Find a trapping region U in the phase space.

(2) Show that U contains a chaotic invariant set Λ. This can be achieved by showing that inside the trapping region U there is a homoclinic orbit (or heteroclinic cycle) which has associated with it an invariant Cantor set on which the dynamics are topologically conjugate to a shift.

(3) Consider the attracting set

$$A = \bigcap_{t>0} \phi^t(U) .$$

Noting that $\Lambda \subset A$, to conclude that A is a strange attractor, it is "enough" to prove that the sensitive dependence on initial conditions on Λ extends to A, and that A is topologically transitive.

4.3 Duffing Equation

The periodically forced, damped Duffing Equation is given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - \delta y + \gamma \cos \omega t \ , \end{cases}$$

where δ and γ are positive constants.

Consider first the conservative case $\delta = \gamma = 0$. Then, Duffing equation is a Hamiltonian system with Hamiltonian function given by

$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} \ .$$

It has three equilibria: a hyperbolic saddle at $(0, 0)$ and two centers at $(-1, 0)$ and $(1, 0)$ surrounded by a one-parameter families of periodic orbits. Both families of periodic orbits are bounded by a homoclinic orbit consisting of overlapping pieces of stable and unstable manifolds for the saddle.

Consider now the damped Duffing Equation with no forcing: $\delta > 0$ and $\gamma = 0$. In this case, there are three hyperbolic equilibria: a hyperbolic saddle at the origin $(0, 0)$ with eigenvalues

$$\lambda_\pm = -\frac{\delta}{2} \pm \frac{1}{2}\sqrt{\delta^2 + 4}$$

and two spiral sinks for $0 < \delta < \sqrt{8}$ at $(-1, 0)$ and $(1, 0)$ with eigenvalues

$$\lambda_\pm = -\frac{\delta}{2} \pm \frac{1}{2}\sqrt{\delta^2 - 8} \ .$$

Hence, for $\delta > 0$, $(0, 0)$ is unstable and $(\pm 1, 0)$ are asymptotically stable.

Let us now rewrite the forced, damped Duffing Equation as

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \epsilon(\gamma \cos \omega t - \delta y) \ , \end{cases}$$

where δ, γ and ϵ are positive constants. The new parameter ϵ was introduced to control the size of the forcing and damping terms. We have seen above that for $\epsilon = 0$, Duffing equation has a pair of homoclinic orbits bounding a family of periodic. Then, it is possible to define a cross section Σ to the homoclinic orbit and define a Poincaré map $P : \Sigma \to \Sigma$ with the following property. For small $\epsilon > 0$ and $\delta = 0$, the map P has a hyperbolic fixed point with stable and unstable manifolds with transverse intersections. It is even possible to prove that such intersections persist for all parameter values satisfying the following condition:

$$\delta < \left(\frac{3\pi\omega \mathrm{sech}(\pi\omega/2)}{2\sqrt{2}}\right)\gamma \ .$$

Transverse intersections of stable and unstable manifolds give rise to a horseshoe for the dynamics of the Poincaré map P, and chaotic behaviour for the dynamics of the forced damped Duffing equation.

4.4 The Lorenz Equations

Recall that Lorenz equations were given by

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \ , \qquad (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ , \end{cases}$$

where σ, ρ and β are positive constants.

We start by noting that that the system is invariant under the transformation $(x, y, z) \mapsto (-x, -y, z)$ and that the z axis is an invariant manifold since

$$x(t) = 0 \ , \quad y(t) = 0 \ , \quad z(t) = z_0 \mathrm{e}^{-\beta t}$$

is a solution of Lorenz equations.

If $\rho \leq 1$ there is only equilibrium, located at origin. The linearization around the origin is determined by the matrix

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} ,$$

with eigenvalues given by

$$\lambda_1 = -\beta \ , \qquad \lambda_2 = \frac{1}{2}\left(1 + \sigma \pm \sqrt{(1+\sigma)^2 + 4(\rho-1)\sigma}\right) \ .$$

Hence the origin is asymptotically stable for $\rho < 1$. Moreover, one can check that

$$V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$$

is a Lyapunov function. Indeed, one obtains that

$$\dot{V}(x, y, z) = -\sigma(\rho(x+y)^2 + (1-\rho)y^2 + \beta z^2) < 0 \ , \qquad (x, y, z) \neq (0, 0, 0) \ .$$

In particular, the above means that when $\rho < 1$ the Lorenz equations have only the origin as an equilibrium and all solutions converge to the origin as $t \to +\infty$.

If r grows above 1, there are two new fixed points

$$(x, y, z) = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho - 1) \ ,$$

and the corresponding linearization is determined by the matrix

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{\beta(\rho-1)} \\ \pm\sqrt{\beta(\rho-1)} & \pm\sqrt{\beta(\rho-1)} & -b \end{pmatrix} ,$$

One could compute the eigenvalues of the matrix above, but their analytical expressions turn out to be to long. It should be noted that the eigenvalues must be the same for both points. Moreover, it can be shown that the two new fixed points are asymptotically stable for $1 < r < 470/19$. In what concerns the origin, it is easy to see that one eigenvalue is now positive and the origin is no longer stable.

We will now briefly discuss the dynamics of the Lorenz equations for larger values of ρ. Consider the following modification of the Lyapunov function above

$$V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \ .$$

It is easy to see that

$$\dot{V}(x, y, z) = -2\sigma(\rho x^2 + y^2 + \beta(z-\rho)^2 - \beta\rho^2) \ .$$

Let E be the ellipsoid defined by

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : \dot{V}(x, y, z) \geq 0\right\}$$

and let M be defined by

$$M = \max_{(x,y,z)\in E} V(x, y, z) \ .$$

Define now

$$E_\epsilon = \left\{(x, y, z) \in \mathbb{R}^3 : V(x, y, z) \leq M + \epsilon\right\}$$

for some positive ϵ. Note that any point outside E_ϵ is also outside E and hence $\dot{V}(x, y, z) < 0$. Thus, for $x \in \mathbb{R}^3 \backslash E_\epsilon$ the value of V is strictly decreasing along its trajectory and hence it must enter E_ϵ after some finite time.

The set E_ϵ is a trapping region for the Lorenz equations and there is a corresponding attracting set

$$\Lambda = \bigcap_{t \geq 0} \phi^t(E_0) \; .$$

which is the attractor of the Lorenz equation. Note that all fixed points plus their unstable manifolds must be contained in Λ. Moreover, the dynamics in Λ are chaotic and thus Λ is a strange attractor.

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