TWO-PLAYER ZERO-SUM STOCHASTIC DIFFERENTIAL GAMES WITH MARKOV-SWITCHING JUMP-DIFFUSION DYNAMICS

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Abstract. We consider a two-player zero-sum stochastic differential game (SDG) with hybrid state variable dynamics given by a Markov-switching jump-diffusion. We study this game using a combination of dynamic programming and viscosity solution techniques, generalizing the seminal work of Fleming and Souganidis to a broader class of SDGs. Under some mild assumptions, we prove that the value of the game exists and is the unique viscosity solution of a certain nonlinear partial integro-differential equation of Hamilton-Jacobi-Bellman-Isaacs type.

Keywords: Stochastic differential game; Markov-switching jump-diffusion; dynamic programming; viscosity solutions.

1. Introduction

Differential game theory concerns problems modeling interactions among agents whose decisions are continuously evolving. Isaacs [34] first, and Berkovitz and Fleming [15] and Friedman [29, 30] later, studied such problems and provided the first contributions to the field. The goal of the theory is to describe the interaction among agents occurring in the most various situations. One should point out that any action taken by the players both influences and is influenced by the evolution of the state of the system over time. This class of problems is closely related with optimal control theory and, in particular, the Pontryagin maximum principle and Bellman's dynamic programming principle and the corresponding Hamilton-Jacobi-Bellman-Isaacs equation. However, it should be noted that differential games are usually far more complex than optimal control problems, the reason being the fact that, unlike optimal control problems, differential games correspond to the case where more than one player is involved. Besides, there isn't an obvious notion for what a solution is, with multiple proposals put forward instead.

Early definitions of differential game value made use of time discretizations [29] and were later substituted by the Elliott-Kalton notion of differential game value [23]. Using the theory of viscosity solutions introduced by Crandall and Lions [22], Evans and Souganidis [26] characterized the upper and lower Elliott-Kalton value functions as unique viscosity solutions of the corresponding Hamilton-Jacobi-Bellman-Isaacs PDEs and Souganidis [59] showed that the Elliott-Kalton value functions are actually the same as those defined using time discretizations. The notion of value of a differential game extends to the stochastic differential games' setup. Under the assumption that the Isaacs condition holds, Fleming and Souganidis [28] proved the existence of value for two-player zero-sum stochastic differential games (SDGs) with diffusive state variable dynamics. Their approach relies on an asymmetric formulation of the game under consideration whereby two subgames are defined, with one player having an information advantage on one of the subgames and the remaining player having a similar advantage on the other subgame. The stronger player uses Elliot-Kalton strategies while the weaker player resorts to open-loops controls. Katsoulakis [37] and Cardaliaguet and Rainer [21] also rely on asymmetric game formulations while proving a dynamic programming principle. An alternative approach introduced by Sîrbu [55] and building up on previous related work by Bayraktar and Sîrbu [10, 11, 12], relies on the stochastic Perron's method to characterize the value of SDGs formulated symmetrically over appropriately specified elementary feedback strategies. Some recent developments of the theory address SDGs with more general state variable dynamics such as, for instance, jump-diffusion state variable dynamics with no switching variable [16, 17, 18, 38] or diffusive state variable dynamics modulated by a switching process with both finite horizon [9, 32, 43, 53] and infinite horizon [41, 42]. Other extensions include more general payoff functionals [19, 20, 40], alternative control sets [13, 60, 64] and game formulations [21, 27, 51, 64], or more relaxed assumptions regarding boundedness of the problem's data [33].

In this paper, we study a two-player zero-sum stochastic differential game in which the state variable dynamics follow a Markov-switching jump-diffusion process. The game has a hybrid framework, incorporating both continuous and discrete stochastic effects, with state transitions among distinct jump-diffusive dynamics being governed by a continuous-time Markov process modeling the regime switches. The players' payoff functionals depend explicitly on the state of the Markov process governing the switching behavior, adding an extra layer of complexity to the problem under consideration. We employ a combination of dynamic programming and viscosity solution techniques to rigorously establish the existence and uniqueness of the value function for the game. Namely, we prove that the upper and lower value functions associated with this SDG satisfy certain nonlinear partial integro-differential equation of Hamilton-Jacobi-Bellman-Isaacs (HJBI) type which, under appropriate conditions, uniquely determine the value of the game as a viscosity solution. Even though we will regard such HJBI type equation as a single partial differential equation, depending on the finitely many values of the Markov process driving the switching dynamics, we should note it could also be regarded as a system of coupled partial differential equations, one equation for each state of the Markov process. Our choice aims at simplifying technical details and attempting to remain close to the most commonly used definitions of viscosity solutions for partial differential equations. We should stress that the strategy followed here generalizes the seminal work of Fleming and Souganidis in [28] to include the broader class of games under consideration herein. We find this approach to be quite effective, as it allows us to build on established and foundational results, extending only those aspects where the influence of the Markov-switching and jump-diffusion components is significant. We should also note that our analysis differs from other recent works such as [41, 42, 43], which focus on SDGs with Markov-switching diffusive dynamics with no jump component, or [64], which focus on a SDG with state variable dynamics within the class considered herein, but with impulsive controls and a recursive functional.

Incorporating both jump-diffusion processes and Markov-switching into the framework of stochastic differential games results in highly intricate state variable dynamics, leading to additional mathematical challenges. Firstly, the presence of both jump-diffusion processes and regime switching provides the systems under analysis with a sort of hybrid behavior, i.e. with both continuous and discrete sources of uncertainty simultaneously present. Namely, while the Markov-switching component introduces discrete transitions between distinct regimes, the jump-diffusion process models sudden and unpredictable movements occurring continuously. Secondly, the HJBI equation describing the SDG value functions is actually a system of coupled partial integro-differential equations, one for each state of the regime-switching Markov process, each of which including nonlocal integral terms arising from the jump process. Such equations are highly nonlinear, complicating the existence, uniqueness, and regularity analysis of its solutions. Finally, from the point of view of game theory, the presence of multiple interacting stochastic components complicates the design of optimal strategies. Players must account for the effects of both jumps and regime changes in their decision-making, leading to potentially more complicated feedback strategies. In real-world applications such as finance, energy markets, and engineering control, estimating transition probabilities for regime switching and jump intensities adds an extra layer of difficulty. Incomplete or noisy observations of the switching process can further complicate the formulation and solution such games.

We should also point out that, in addition to its relevance from a theoretical point of view, the analysis of SDGs with Markov-switching has potential applications to problems modeled by hydrid dynamic systems originating from areas of knowledge as distinct as finance, ecology or engineering. Indeed, in all of these areas, sudden external variations may lead to significant changes in the evolution of relevant observables. In particular, the importance of such class of systems has contributed to the recent increase of its use in mathematical finance [24, 25, 54, 56, 62, 63] and mathematical biology [5, 14, 65], to name only a few recent works. Moreover, the study of optimal control problems with Markov-switching has also received a lot of attention in recent years: see, for instance, the contributions by Koutsoukos [39], Tang and Hou [60], Mao [46], Azevedo et al. [4], Temoçin and Weber [61], and Song and Zhu [57]. Finally, we believe that such class of systems will find a broader scope of applications in areas germane to game theory [44, 48, 49].

This paper is organized as follows. In Section 2 we describe the problem we propose to address and state our main results, which are then proved in Section 3. We conclude in Section 4.

2. Framework and main results

In this section we formulate the problem under consideration herein and state our main results.

2.1. **Notation and setup.**

Let $T > 0$ be a deterministic finite time horizon and, for every $t \in [0,T]$, let $(\Omega_{t,T}, \mathcal{G}_{t,T}, \mathbb{G}_{t,T}, \mathbb{P}_{t,T})$ be a filtered probability space with filtration $\mathbb{G}_{t,T} = \{\mathcal{G}_{t,s} : s \in [t,T]\}$ satisfying the usual conditions, i.e. $\mathbb{G}_{t,T}$ is an increasing sequence of σ-algebras for any $t \in [0,T]$, it is right-continuous, and $\mathcal{G}_{t,T}$ contains all $\mathbb{P}_{t,T}$-null sets. For each $d \in \mathbb{N}$, let $\mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\}$ and let $\mathcal{B}^d_0$ be the Borel σ-field generated by the open subsets O of $\mathbb{R}^d_0$ whose closure does not contain 0.

We will consider the following stochastic processes throughout this paper:

(i) a M-dimensional Brownian motion $B^t = \{B^t(s) : s \in [t,T]\}$ on the filtered probability space $(\Omega_{t,T}, \mathcal{G}_{t,T}, \mathbb{G}_{t,T}, \mathbb{P}_{t,T})$.

(ii) a K-dimensional Lévy process $\eta^t = \{\eta^t(s) : s \in [t,T]\}$ with Poisson random measure $J^t(s,A)$ with intensity (or Lévy measure) $\nu^t(A)$. Recall that for each $s \in [t,T]$ and $\omega \in \Omega_{t,T}$, $J^t(s,\cdot)(\omega)$ is a counting measure on $\mathcal{B}^K_0$, and that for each $A \in \mathcal{B}^K_0$, $\{J^t(s,A) : s \in [t,T]\}$ is a Poisson process with intensity $\nu^t(A)$. For each $s \in [t,T]$ and $A \in \mathcal{B}^K_0$, define the compensated Poisson random measure of $\eta^t(\cdot)$ by
$$\tilde{J}^t(\mathrm{d}s, \mathrm{d}z) = J^t(\mathrm{d}s, \mathrm{d}z) - \nu^t(\mathrm{d}z)\mathrm{d}s$$
and notice that $\{\tilde{J}^t(s,A) : s \in [t,T]\}$ is a martingale-valued measure [3].

(iii) a homogeneous continuous-time Markov process $\mu^t = \{\mu^t(s) : s \in [t,T]\}$ with a finite state space $S = \{1, \dots, n\}$.

Let us introduce the following technical assumptions:

(A1) U and V are compact metric spaces.

(A2) The transition probability function
$$P_{ij}(s) = \mathbb{P}_{t,T}\{\mu^t(t+s) = j | \mu^t(t) = i\}\ ,\ i,j \in S\ ,\ s \in [0, T-t]$$
of the Markov-process $\mu^t(\cdot)$ is such that both conditions below hold:
 (i) $\lim_{s\to 0^+} P_{ii}(s) = 1$ for all $i \in S$
 (ii) $q_i = \lim_{s \to 0^+} (1 - P_{ii}(s))/t$ is finite for all $i \in S$.

(A3) The maps $f : [0,T] \times \mathbb{R}^N \times S \times U \times V \to \mathbb{R}^N$, $\sigma : [0,T] \times \mathbb{R}^N \times S \times U \times V \to \mathbb{R}^{N\times M}$, $h : [0,T] \times \mathbb{R}^N \times S \times U \times V \times \mathbb{R}^K_0 \to \mathbb{R}^{N\times K}$, $\Psi : \mathbb{R}^N \times S \to \mathbb{R}$ and $L : [0,T] \times \mathbb{R}^N \times S \times U \times V \to \mathbb{R}$ are such that for each fixed $a \in S$, $f(\cdot,\cdot,a,\cdot,\cdot)$, $\sigma(\cdot,\cdot,a,\cdot,\cdot)$, $h(\cdot,\cdot,a,\cdot,\cdot,\cdot)$, $\Psi(\cdot,a)$ and $L(\cdot,\cdot,a,\cdot,\cdot)$ are bounded and uniformly continuous with respect to all the remaining variables, and Lipschitz continuous with respect to the variables $(t,x) \in [0,T] \times \mathbb{R}^N$, uniformly in $(u,v) \in U \times V$.

(A4) The Lévy measure ν^t is a positive Radon measure on $\mathbb{R}^K_0$ that satisfies
$$\int_{\mathbb{R}^K_0} \min(|z|^2, 1)\nu^t(\mathrm{d}z) < \infty\ .$$

(A5) the Brownian motion $B^t(\cdot)$, the Lévy process $\eta^t(\cdot)$ and the Markov process $\mu^t(\cdot)$ are mutually independent and adapted to the filtration $\mathbb{G}_{t,T}$.

Under Assumption (A2), it is known [2] that for each $i \in S$ the rates of change
$$q_{ij} = P'_{ij}(0)$$
exist and are finite for all $j \in S$. Moreover, setting $q_{ii} = -q_i$ for each $i \in S$, we are able to write the generator matrix of the Markov process

μ^t as $Q = (q_{ij})_{i,j=1}^n$. This is related with the transition probability matrix $P(s) = (P_{ij}(s))_{i,j\in S}$ through the identity

$$P(s) = \mathrm{e}^{sQ} \ , \quad s \in [0, T-t] \ .$$

The Markov process $\mu^t(\cdot)$ introduced above admits a representation as a stochastic integral with respect to a Poisson random measure [47, 31], as we will now explain. For $i, j \in S$ such that $i \neq j$, denote by Λ_{ij} the consecutive (with respect to the lexicographical order on $S \times S$), left-closed, right-open intervals of the real line, each with length q_{ij}. Define $\Gamma : S \times \mathbb{R} \to \mathbb{R}$ as

$$\Gamma(i,z) \quad = \quad \begin{cases} j-i & \text{if } z \in \Lambda_{ij} \\ 0 & \text{otherwise} \end{cases} ,$$

where the new state $j \in S$ is determined by the interval Λ_{ij} to which $z \in \mathbb{R}$ belongs to. Roughly, the map Γ determines the size of the switch between states of the Markov process $\mu^t(\cdot)$. Then, the Markov process $\mu^t(\cdot)$ with initial condition $\mu^t(t) = i_0$ admits the representation

$$\mathrm{d}\mu^t(s) \quad = \quad \int_{\mathbb{R}} \Gamma(\mu^t(s_-), z) \ N^t(\mathrm{d}s, \mathrm{d}z) \ , \ s \in [t, T] \ , \tag{1}$$

where $N^t(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure with intensity $\mathrm{d}s \times \ell(\mathrm{d}z)$, in which $\ell(\cdot)$ is the Lebesgue measure on $\mathbb{R}$.

We will use the notation $|\cdot|$ for the Euclidean norm on any finite dimensional space, without indicating the dimension each time for simplicity of exposition. Embedding the state space $S = \{1, \dots, n\}$ of the Markov process $\mu^t(\cdot)$ into $\mathbb{R}^n$ by identifying each element $i \in S$ with the unit vector $\mathbf{e}_i$ of the standard basis of $\mathbb{R}^n$ (i.e. $\mathbf{e}_i$ is the element of $\mathbb{R}^n$ having all components equal to zero except for the ith component, which equals 1), we endow the product space $\mathbb{R}^N \times S$ with the metric

$$d_{N,S}\left((x,a),(y,b)\right) = |(x-y, \mathbf{e}_a - \mathbf{e}_b)| \tag{2}$$

induced by the Euclidean norm on $\mathbb{R}^{N+n}$, i.e. the distance between the points $(x,a) \in \mathbb{R}^N \times S$ and $(y,b) \in \mathbb{R}^N \times S$ equals the distance from the origin of $\mathbb{R}^{N+n}$ to the point $(x-y, \mathbf{e}_a - \mathbf{e}_b) \in \mathbb{R}^{N+n}$ whose first N components are those of $x - y \in \mathbb{R}^N$ and whose last n components are those of $\mathbf{e}_a - \mathbf{e}_b \in \mathbb{R}^n$. For a bounded Lipschitz continuous function g on $\mathbb{R}^N$, denote by $|g|_1$ its Lipschitz norm, defined as

$$|g|_1 = \sup_{x\in\mathbb{R}^N} |g(x)| + \sup_{x,y\in\mathbb{R}^N : x\neq y} \frac{|g(x)-g(y)|}{|x-y|} \, .$$

We denote by $W^{1,\infty}(\mathbb{R}^N)$ the space of all bounded Lipschitz continuous functions g on $\mathbb{R}^N$ with the property that $|g|_1$ is finite. For a fixed $t \in [0,T]$, we will denote by $|g(t,\cdot)|_1$ the Lipschitz norm $|\cdot|_1$ of $g(t,x)$ as a function of x alone. We also denote by $C^{1,2}([0,T) \times \mathbb{R}^N)$ the space of continuous functions on $[0,T) \times \mathbb{R}^N$ with one continuous derivative with respect to its first argument and two continuous derivatives with respect to its second argument, and by $\mathrm{USC}([0,T] \times \mathbb{R}^N)$ and $\mathrm{LSC}([0,T] \times \mathbb{R}^N)$ the set of all upper and lower semicontinuous functions on $[0,T] \times \mathbb{R}^N$, respectively. Finally, we denote by $USC_p([0,T] \times \mathbb{R}^N)$, $LSC_p([0,T] \times \mathbb{R}^N)$ and $C_p^{1,2}([0,T) \times \mathbb{R}^N)$ the sets of functions g, respectively, in $USC([0,T]\times\mathbb{R}^N)$, $LSC([0,T]\times\mathbb{R}^N)$ and $C^{1,2}([0,T)\times\mathbb{R}^N)$ satisfying the growth condition

$$|g(x)| \leq C\,(1+|x|^p) \text{ for all } x \in \mathbb{R}^N$$

for some $p \geq 0$. The condition above should be satisfied uniformly in t if g depends on t.

In what follows, let $s \in [t,T]$. Denote by $\Omega^1_{t,s}$ the set of $\mathbb{R}^M$-valued continuous functions on $[t,s]$ taking the value 0 at t, i.e.

$$\Omega^1_{t,s} = \left\{\omega \in C\left([t,s];\mathbb{R}^M\right) : \omega(t) = 0\right\},$$

and endow $\Omega^1_{t,s}$ with the sup-norm. Let $\mathcal{G}^1_{t,s}$ be the corresponding Borel σ-algebra and note that when endowed with the Wiener measure $\mathbb{P}^1_{t,s}$ on $\mathcal{G}^1_{t,s}$, $\Omega^1_{t,s}$ becomes a classical Wiener space.

Denote by $\mathcal{B}\left([t,s]\times\mathbb{R}^K_0\right)$ the Borel σ-algebra on $[t,s]\times\mathbb{R}^K_0$ and define $\Omega^2_{t,s}$ as the set of all $\mathbb{N}\cup\{\infty\}$-valued measures on $\left([t,s]\times\mathbb{R}^K_0, \mathcal{B}\left([t,s]\times\mathbb{R}^K_0\right)\right)$. Let $\mathcal{G}^2_{t,s}$ be the smallest σ-algebra over $\Omega^2_{t,s}$ such that the maps $q \in \Omega^2_{t,s} \to q(A) \in \mathbb{N}\cup\{\infty\}$ are measurable for all $A \in \mathcal{B}\left([t,s]\times\mathbb{R}^K_0\right)$. Additionally, let the coordinate random measure $J_{t,s}$ be defined as $J_{t,s}(q,A) = q(A)$ for all $q \in \Omega^2_{t,s}, A \in \mathcal{B}\left([t,s]\times\mathbb{R}^K_0\right)$ and denote $\mathbb{P}^2_{t,s}$ as the probability measure on $\left(\Omega^2_{t,s},\mathcal{G}^2_{t,s}\right)$ under which $J_{t,s}$ is a Poisson random measure with Lévy measure ν^t satisfying of Assumption (A4).

Similarly, let $\mathcal{B}\left([t,s]\times\mathbb{R}\right)$ be the Borel σ-algebra on $[t,s]\times\mathbb{R}$ and define $\Omega^3_{t,s}$ as the set of all $\mathbb{N}\cup\{\infty\}$-valued measures on $\left([t,s]\times\mathbb{R}, \mathcal{B}\left([t,s]\times\mathbb{R}\right)\right)$. Let $\mathcal{G}^3_{t,s}$ be the smallest σ-algebra over $\Omega^3_{t,s}$ such that the maps $p \in \Omega^3_{t,s} \to p(A) \in \mathbb{N}\cup\{\infty\}$ are measurable for all $A \in \mathcal{B}\left([t,s]\times\mathbb{R}\right)$. Moreover, let the coordinate random measure $N_{t,s}$ be defined as $N_{t,s}(p,A) = p(A)$ for all $p \in \Omega^3_{t,s}, A \in \mathcal{B}\left([t,s]\times\mathbb{R}\right)$ and denote by $\mathbb{P}^3_{t,s}$ the probability measure on $\left(\Omega^3_{t,s},\mathcal{G}^3_{t,s}\right)$ under which the process $\mu^t(\cdot)$ is a Markov process with respect to its natural filtration.

We define $\Omega_{t,s}$ as the direct product

$$\Omega_{t,s} = \Omega^1_{t,s}\times\Omega^2_{t,s}\times\Omega^3_{t,s}\ ,$$

defining accordingly the probability measure $\mathbb{P}_{t,s}$ as

$$\mathbb{P}_{t,s} = \mathbb{P}^1_{t,s}\otimes\mathbb{P}^2_{t,s}\otimes\mathbb{P}^3_{t,s}\ , \tag{3}$$

and the σ-algebra $\mathcal{G}_{t,s}$, as the completion of $\mathcal{G}^1_{t,s}\otimes\mathcal{G}^2_{t,s}\otimes\mathcal{G}^3_{t,s}$ with respect to the measure $\mathbb{P}_{t,s}$.

We will denote a generic element of $\Omega_{t,T}$ by $\omega = (\omega_1,\omega_2,\omega_3)$ with $\omega_i \in \Omega^i_{t,T}, i \in \{1,2,3\}$. Define the coordinate functions

$$B^t(s,\omega) = \omega_1(s) \text{ and } J^t(A_2,\omega) = \omega_2(A_2) \text{ and } N^t(A_3,\omega) = \omega_3(A_3)$$

for $0 \leq t \leq s \leq T$, $\omega \in \Omega$, $A_2 \in \mathcal{B}([t,T]\times\mathbb{R}^K_0)$, $A_3 \in \mathcal{B}([t,T]\times\mathbb{R})$. For $t \in [0,T]$, define the filtration $\mathbb{G}_{t,T} = \{\mathcal{F}_{t,s} : s \in [t,T]\}$ as follows. Let

$$\hat{\mathcal{F}}_{t,s} = \sigma\{B^t(r), J^t(A_2), N^t(A_3) : A_2 \in \mathcal{B}([t,r]\times\mathbb{R}^K_0), A_3 \in \mathcal{B}([t,r]\times\mathbb{R}), t \leq r \leq s\}\ ,$$

with $t \leq s \leq T$ and make the resulting filtration $\hat{\mathbb{G}}_{t,T} = \{\hat{\mathcal{F}}_{t,s} : s \in [t,T]\}$ right-continuous. Additionally, augment $\hat{\mathbb{G}}^+_{t,T}$ by including all $\mathbb{P}_{t,T}$-null sets. This procedure yields $\mathbb{G}_{t,T}$. If and when needed, we extend the filtration

$\mathbb{G}_{t,T}$ for $s < t$ by taking $\mathcal{F}_{t,s}$ to be the trivial σ- algebra augmented by all the $\mathbb{P}_{t,T}$-null sets.

For $\hat{t} \in (t,T)$ and $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega_{t,T} = \Omega^1_{t,T} \times \Omega^2_{t,T} \times \Omega^3_{t,T}$, let

$$\omega^{t,\hat{t}} = (\omega_1|_{[t,\hat{t}]}, \omega_2|_{[t,\hat{t}]}, \omega_3|_{[t,\hat{t}]}) \in \Omega_{t,\hat{t}} \ ,$$

and

$$\omega^{\hat{t},T} = \Big((\omega_1 - \omega_1(\hat{t}))\, |_{[\hat{t},T]}, \omega_2|_{[\hat{t},T]}, \omega_3|_{[\hat{t},T]} \Big) \in \Omega_{\hat{t},T} \ .$$

Define $\pi : \Omega_{t,T} \to \Omega_{t,\hat{t}} \times \Omega_{\hat{t},T}$ to be the map given by

$$\pi(\omega) = \Big(\omega^{t,\hat{t}}, \omega^{\hat{t},T} \Big) \ . \tag{4}$$

Then, π induces the identification

$$\Omega_{t,T} = \Omega_{t,\hat{t}} \times \Omega_{\hat{t},T}$$

and the inverse of π acts on pairs of paths $(\omega^{t,\hat{t}}, \omega^{\hat{t},T}) \in \Omega_{t,\hat{t}} \times \Omega_{\hat{t},T}$ by concatenation, i.e. $\omega = \pi^{-1}(\omega^{t,\hat{t}}, \omega^{\hat{t},T}) \in \Omega_{t,T}$. Finally, notice that for any $\hat{t} \in (t,T)$ the map π identifies the filtered probability space $(\Omega_{t,T}, \mathcal{G}_{t,T}, \mathbb{G}_{t,T}, \mathbb{P}_{t,T})$ with $\Big(\Omega_{t,\hat{t}} \times \Omega_{\hat{t},T}, \mathcal{G}_{t,\hat{t}} \otimes \mathcal{G}_{\hat{t},T}, \mathbb{G}_{t,\hat{t}} \otimes \mathbb{G}_{\hat{t},T}, \mathbb{P}_{t,\hat{t}} \otimes \mathbb{P}_{\hat{t},T} \Big)$.

2.2. **Differential Game formulation.** The *two-Player zero-sum differential game with Markov-switching jump-diffusion* is defined on the filtered probability space $(\Omega_{t,T}, \mathcal{G}_{t,T}, \mathbb{G}_{t,T}, \mathbb{P}_{t,T})$ and consists of the controlled stochastic differential equation

$$\begin{aligned}
\mathrm{d}X(s) &= f\left(s, \mathcal{X}(s), u(s), v(s)\right) \mathrm{d}s \\
&\quad + \sigma\left(s, \mathcal{X}(s), u(s), v(s)\right) \mathrm{d}B^t(s) \\
&\quad + \int_{\mathbb{R}^K_0} h\left(s_-, \mathcal{X}(s_-), u(s_-), v(s_-), z\right) \tilde{J}^t(\mathrm{d}s, \mathrm{d}z) \quad (s \in [t,T]) \ , \\
\mathrm{d}\mu^t(s) &= \int_{\mathbb{R}} \Gamma\left(\mu^t(s_-), z\right) N^t(\mathrm{d}s, \mathrm{d}z) \ , \\
X(t) &= x, \quad \mu^t(t) = a \ ,
\end{aligned} \tag{5}$$

where $\mathcal{X}(s)$ denotes the pair $(X(s), \mu^t(s))$, and the *payoff* functional

$$\begin{aligned}
J(t,x,a;u(\cdot),v(\cdot)) &= \mathbb{E}_{\mathbb{P}_{t,T}} \Bigg[\int_t^T L\left(s, \mathcal{X}^{u,v}_{t,x,a}(s), u(s), v(s)\right) \mathrm{d}s \\
&\qquad + \Psi\left(\mathcal{X}^{u,v}_{t,x,a}(T)\right)\Bigg] \ ,
\end{aligned} \tag{6}$$

where $\mathcal{X}^{u,v}_{t,x,a}(s) = (X^{u,v}_{t,x,a}(s), \mu_{t,a}(s))$ denotes the solution of the initial value problem (5) associated with a specific choice of controls $u(\cdot)$ and $v(\cdot)$. Note that both the stochastic differential equation in (5) and the payoff functional (6) depend on the Markov process $\mu^t(\cdot)$. Moreover, observe that in light of assumption (A3), equation (5) admits a unique strong solution. We should note, however, that the boundedness assumption on f, σ, h and ψ in (A3) could be replaced by a linear growth condition in x. For instance, Buckdahn et al. [18] implement the use of such weaker condition within the context of controlled stochastic systems with a Brownian motion and a Poisson random measure, and with nonlinear cost functionals defined by controlled backward stochastic differential equations.

We will refer to the functions L and Ψ determining the objective functional J as the *running payoff* and *terminal payoff*, respectively. In what concerns the game under consideration, the payoff functional (6) represents some cost that a first Player is trying to maximize (and thus a second Player seeks to minimize) subject to the state variable dynamics defined by (5) and some constraints of the form $u(t) \in U$ and $v(t) \in V$ for every appropriately defined instant of time $t \geq 0$. Since the gain of one player represents a loss to the other player, this game fits into the class of zero-sum games.

Admissible control process An *admissible control process* $u(\cdot)$ (resp. $v(\cdot)$) for Player I (resp. II) on $[t,T]$ is a $\mathbb{G}_{t,T}$-progressively measurable process taking values in U (resp. V). The set of all admissible controls for Player I (resp. II) on $[t,T]$ is denoted by $\mathcal{U}(t,T)$ (resp. $\mathcal{V}(t,T)$).

We say that two controls $u_1(\cdot), u_2(\cdot) \in \mathcal{U}(t,T)$ are the same on $[t,s]$, for some $s \in (t,T]$, and denote it by $u_1(\cdot) \approx u_2(\cdot)$, if it holds that

$$\mathbb{P}_{t,T}\{u_1(\cdot) = u_2(\cdot) \text{ a.e. } [t,s]\} = 1 \ .$$

A similar convention is used for elements of $\mathcal{V}(t,T)$.

Admissible strategy An *admissible strategy* α (resp. β) for Player I (resp. II) on $[t,T]$ is a mapping $\alpha : \mathcal{V}(t,T) \to \mathcal{U}(t,T)$ (resp. $\beta : \mathcal{U}(t,T) \to \mathcal{V}(t,T)$) such that if $v(\cdot) \approx \tilde{v}(\cdot)$ (resp. $u(\cdot) \approx \tilde{u}(\cdot)$) on $[t,s]$ for every $s \in [t,T]$, then $\alpha[v(\cdot)] \approx \alpha[\tilde{v}(\cdot)]$ (resp. $\beta[u(\cdot)] \approx \beta[\tilde{u}(\cdot)]$). The set of all admissible strategies for Player I (resp. II) on $[t,T]$ is denoted by $\mathcal{A}(t,T)$ (resp. $\mathcal{B}(t,T)$).

Before proceeding, we note that admissible strategies are also referred to as *nonanticipative strategies* in the literature.

Lower and upper value functions The *lower value function* of the SDG (5)-(6) is given by

$$V^-(t,x,a) = \inf_{\beta \in \mathcal{B}(t,T)} \sup_{u(\cdot) \in \mathcal{U}(t,T)} J(t,x,a;u(\cdot),\beta[u(\cdot)]) \tag{7}$$

while the corresponding *upper value function* is

$$V^+(t,x,a) = \sup_{\alpha \in \mathcal{A}(t,T)} \inf_{v(\cdot) \in \mathcal{V}(t,T)} J(t,x,a;\alpha[v(\cdot)],v(\cdot)) \ . \tag{8}$$

We say that the SDG (5)-(6) has a value if

$$V^+(t,x,a) = V^-(t,x,a)$$

and call it the *common value* of the SDG. This definition extends the standard notion of common value of a game, introduced by Elliot and Kalton [23], to SDGs with a Markov-switching jump-diffusion.

Choosing the controls at time t, the Player who moves first (the maximizing Player for the lower game, and the minimizing Player for the upper game) is allowed to use the past of the three stochastic processes driving (5), while the Player with the advantage (Player II for the lower game, Player I for the upper game), is allowed to use both the past of such processes and the other player's control.

2.3. **Statement of main results.** In this section we state our main results. We focus first on a recursive characterization for the value functions of the game, known as dynamic programming principle, before proceeding to an alternative characterization in terms of viscosity solutions of partial integro-differential equations.

Theorem 2.1 (Dynamic programming principle). *Suppose that Assumptions (A1)-(A5) hold and let $t, \hat{t} \in [0,T]$ be such that $t < \hat{t}$. Then, for every $(x,a) \in \mathbb{R}^N \times S$, we have that:*

(i) *the lower value function V^- is determined by the recursive relation*

$$V^-(t,x,a) = \inf_{\beta \in \mathcal{B}(t,T)} \sup_{u \in \mathcal{U}(t,T)} \mathbb{E}_{\mathbb{P}_{t,T}} \Bigg[V^-\left(\hat{t}, \mathcal{X}^{u,v}_{t,x,a}(\hat{t})\right) + \int_t^{\hat{t}} L(s, \mathcal{X}^{u,v}_{t,x,a}(s), u(s), \beta[u(\cdot)](s))\, \mathrm{d}s \Bigg], \tag{9}$$

combined with the boundary condition $V^-(T,x,a) = \Psi(x,a)$, where $\mathcal{X}^{u,v}_{t,x,a}(s)$, $s \in [t,T]$, is the solution of (5) *with $v(\cdot) = \beta[u(\cdot)] \in \mathcal{V}(t,T)$ for $u(\cdot) \in \mathcal{U}(t,T)$.*

(ii) *the upper value function V^+ is determined by the recursive relation*

$$V^+(t,x,a) = \sup_{\alpha \in \mathcal{A}(t,T)} \inf_{v \in \mathcal{V}(t,T)} \mathbb{E}_{\mathbb{P}_{t,T}} \Bigg[V^+\left(\hat{t}, \mathcal{X}^{u,v}_{t,x,a}(\hat{t})\right) + \int_t^{\hat{t}} L(s, \mathcal{X}^{u,v}_{t,x,a}(s), \alpha[v(\cdot)](s), v(s))\, \mathrm{d}s \Bigg], \tag{10}$$

combined with the boundary condition $V^+(T,x,a) = \Psi(x,a)$, where $\mathcal{X}^{u,v}_{t,x,a}(s)$, $s \in [t,T]$, is the solution of (5) *with $u(\cdot) = \alpha[v(\cdot)] \in \mathcal{U}(t,T)$ for $v(\cdot) \in \mathcal{V}(t,T)$.*

It is well known that, in general, the value functions V^- and V^+ determined by the variational identities in the dynamic programming principle above are not smooth. An alternative notion of weak solution, know as viscosity solution, was originally proposed by Crandall and Lions in [22] for the case of first order Hamilton-Jacobi equations. More recently, such notion has been extended to partial integro-differential [1, 6, 7, 8, 35, 36, 50]. Herein we deal with such class of equations, containing additional terms due to the Markov-switching.

Viscosity solution A function $W : [0,T] \times \mathbb{R}^N \times S \to \mathbb{R}$ such that $W(\cdot,\cdot,a) \in USC_p([0,T] \times \mathbb{R}^N)$ for every $a \in S$ (resp. $W(\cdot,\cdot,a) \in LSC_p([0,T] \times \mathbb{R}^N)$) is a *viscosity subsolution* (resp. *supersolution*) of

$$\begin{cases} W_t + \mathcal{H}(t,x,a,W(t,\cdot,\cdot), W_x, W_{xx}) = 0 \\ W(T,x,a) = \Psi(x,a)\ , \end{cases} \tag{11}$$

if for every $(x,a) \in \mathbb{R}^N \times S$ we have that

$$W(T,x,a) \le \Psi(x,a) \qquad (\text{resp. } W(T,x,a) \ge \Psi(x,a)\ )$$

and, additionally,

$$\phi_t(t_0,x_0,a_0) \\ +\mathcal{H}(t_0,x_0,a_0,\phi(t_0,x_0,a_0),\phi_x(t_0,x_0,a_0),\phi_{xx}(t_0,x_0,a_0)) \geq 0$$

$$\big(\text{resp. } \phi_t(t_0,x_0,a_0) \\ +\mathcal{H}(t_0,x_0,a_0,\phi(t_0,x_0,a_0),\phi_x(t_0,x_0,a_0),\phi_{xx}(t_0,x_0,a_0)) \leq 0\big)$$

for every function $\phi : [0,T] \times \mathbb{R}^N \times S \to \mathbb{R}$ such that $\phi(\cdot,\cdot,a) \in C_p^{1,2}([0,T) \times \mathbb{R}^N)$ for every $a \in S$, and any local maximum (resp. minimum) (t_0,x_0,a_0) of $W - \phi$. We say that W is a *viscosity* solution of (11) if it is both a sub- and supersolution.

Before proceeding to the statement of the next result, let us introduce some more notation: we will denote by $\mathbb{S}^N$ the set of symmetric $N \times N$ matrices and by $\mathrm{tr}(A)$ the trace of a matrix $A \in \mathbb{S}^N$.

Theorem 2.2. *Suppose that Assumptions (A1)-(A5) hold. Then the value functions V^- and V^+ are, respectively, the unique viscosity solutions of the HJBI equations:*

$$\begin{cases} W_t + \mathcal{H}^-(t,x,a,W(t,\cdot,\cdot),W_x,W_{xx}) = 0 \\ W(T,x,a) = \Psi(x,a) \end{cases} \tag{12}$$

and

$$\begin{cases} W_t + \mathcal{H}^+(t,x,a,W(t,\cdot,\cdot),W_x,W_{xx}) = 0 \\ W(T,x,a) = \Psi(x,a) \end{cases} \tag{13}$$

where, for $(t,x,a,p,A) \in [0,T] \times \mathbb{R}^N \times S \times \mathbb{R}^N \times \mathbb{S}^N$ and any real valued function w on $[0,T] \times \mathbb{R}^N \times S$ such that $w(\cdot,\cdot,a)$ is smooth for every $a \in S$, we have

$$\begin{aligned} \mathcal{H}^-(t,x,a,w\,(t,\cdot,\cdot)\,,p,A) &= \max_{u\in U}\min_{v\in V} \Big\{ H_1(t,x,a,u,v,p,A) \\ &\qquad + H_2(t,x,a,u,v)[w] \Big\} \\ \mathcal{H}^+(t,x,a,w\,(t,\cdot,\cdot)\,,p,A) &= \min_{v\in V}\max_{u\in U} \Big\{ H_1(t,x,a,u,v,p,A) \\ &\qquad + H_2(t,x,a,u,v)[w] \Big\} \end{aligned}$$

with

$$H_1(t,x,a,u,v,p,A) = L(t,x,a,u,v) + f(t,x,a,u,v)\cdot p \\ + \mathrm{tr}\left(\frac{1}{2} g(t,x,a,u,v) A\right)$$

and

$$\begin{aligned} H_2(t,x,a,u,v)[w] = &\sum_{j\in S: j\neq a} q_{aj}\left(w(t,x,j) - w(t,x,a)\right) \\ &+ \int_{\mathbb{R}_0^K} \Big(w(t,x+h(t,x,a,u,v,z),a) - w(t,x,a) \\ &\qquad - w_x(t,x,a)\cdot h(t,x,a,u,v,z) \Big)\, \nu^t(\mathrm{d}z) \end{aligned}$$

where $g = \sigma\sigma'$, and σ' denotes the transpose of σ.

We say that the *Isaacs condition* holds if for all $(t, x, a, p, A) \in [0, T] \times \mathbb{R}^N \times S \times \mathbb{R}^N \times \mathbb{S}^N$ and any real valued function w on $[0, T] \times \mathbb{R}^N \times S$ such that $w(\cdot, \cdot, a)$ is smooth for every $a \in S$, the following identity holds:

$$\mathcal{H}^+(t, x, a, w(t, \cdot, \cdot), p, A) = \mathcal{H}^-(t, x, a, w(t, \cdot, \cdot), p, A) \ . \tag{14}$$

The next result follows from combining Isaacs condition above with the uniqueness of the viscosity solutions to (12) and (13), guaranteed by Theorem 2.2. In particular, it establishes the existence of a common value for the SDG (5)-(6) in the sense of Elliot and Kalton [23].

Corollary 2.3. *If the Isaacs condition* (14) *holds, then the upper and the lower value functions of the SDG* (5)-(6) *coincide.*

The rest of this paper is devoted to the proof of Theorems 2.1 and 2.2.

3. Dynamic programming principle

The goal of this section is to obtain a dynamic programming principle for the SDG (5)-(6), describing the game value functions. We will resort to the concepts of r-strategies and r-lower and r-upper values introduced by Fleming and Souganidis [28], which combined with an appropriate discretization procedure, yield the existence and uniqueness of viscosity solutions to the HJBI equations (12) and (13) associated with the desired dynamic programming principle.

3.1. **Some preliminary results.**

We will now introduce further notation and terminology that will be useful in the sequel. Let $(t, x, a) \in [0, T) \times \mathbb{R}^N \times S$ be fixed and for any given $u(\cdot) \in \mathcal{U}(t, T)$ and $v(\cdot) \in \mathcal{V}(t, T)$, define

$$\gamma(s, \omega) = (u(s, \omega), v(s, \omega))$$

for every $s \geq t$ and $\omega \in \Omega_{t,T}$. By definition of the control processes $u(\cdot) \in \mathcal{U}(t, T)$ and $v(\cdot) \in \mathcal{V}(t, T)$, it follows that $\gamma(\cdot)$ is $\mathbb{G}_{t,T}$-progressively measurable. By standard results from stochastic differential equations theory (see e.g. [3] for further details), it is known that the SDE (5) admits a unique solution $\mathcal{X}^{u,v}_{t,x,a}(s) \in \mathbb{R}^N \times S$ on the filtered probability space $(\Omega_{t,T}, \mathcal{G}_{t,T}, \mathbb{G}_{t,T}, \mathbb{P}_{t,T})$ for any fixed $u(\cdot) \in \mathcal{U}(t, T)$ and $v(\cdot) \in \mathcal{V}(t, T)$. Moreover, $\mathcal{X}^{u,v}_{t,x,a}(\cdot) = \left(X^{u,v}_{t,x,a}(\cdot), \mu_{t,a}(\cdot)\right)$ satisfies

$$\begin{aligned}
X^{u,v}_{t,x,a}(s) \quad &= \quad X^{u,v}_{t,x,a}(\hat{t}) + \int_{\hat{t}}^{s} f(r, \mathcal{X}^{u,v}_{t,x,a}(r), \gamma(r)) \, \mathrm{d}r \\
&\quad + \int_{\hat{t}}^{s} \sigma(r, \mathcal{X}^{u,v}_{t,x,a}(r), \gamma(r)) \, \mathrm{d}B^t(r) \qquad\qquad (15) \\
&\quad + \int_{\hat{t}}^{s} \int_{\mathbb{R}^K_0} h\left(r_-, \mathcal{X}^{u,v}_{t,x,a}(r_-), u(r_-), v(r_-), z\right) \tilde{J}^t(\mathrm{d}r, \mathrm{d}z) \ , \\
\mu_{t,a}(s) \quad &= \quad \mu_{t,a}(\hat{t}) + \int_{\hat{t}}^{s} \int_{\mathbb{R}} \Gamma(\mu_{t,a}(r_-), z) \, N^t(\mathrm{d}r, \mathrm{d}z) \ ,
\end{aligned}$$

where $t \leq \hat{t} \leq s \leq T$. Additionally, noticing that

$$\begin{aligned} B^t(s,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) - B^{\hat{t}}(s,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) &= \omega_1^{\hat{t},T}(s) \\ J^{\hat{t}}(A_2,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) &= \omega_2^{\hat{t},T}(A_2) \\ N^{\hat{t}}(A_3,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) &= \omega_3^{\hat{t},T}(A_3) \end{aligned} \tag{16}$$

we obtain that for $\mathbb{P}_{t,\hat{t}}$-a.e. $\omega^{t,\hat{t}} \in \Omega_{t,\hat{t}}$ the processes on the left hand side of (16) coincide with the Brownian motion $B^{\hat{t}}(s,\omega^{\hat{t},T})$ and the Poisson random measures $J^{\hat{t}}(A_2,\omega^{\hat{t},T})$ and $N^{\hat{t}}(A_3,\omega^{\hat{t},T})$ on the filtered probability space $(\Omega_{\hat{t},T},\mathcal{G}_{\hat{t},T},\mathbb{G}_{\hat{t},T},\mathbb{P}_{\hat{t},T})$. Define also

$$\begin{aligned} \tilde{\gamma}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T}) &= \gamma(s,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) \\ \tilde{X}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T}) &= X^{u,v}_{t,x,a}(s,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) \\ \tilde{\mu}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T}) &= \mu_{t,a}(s,\pi^{-1}(\omega^{t,\hat{t}},\omega^{\hat{t},T})) \ , \end{aligned}$$

and

$$\tilde{\mathcal{X}}^{u,v}_{t,x,a}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T}) = \left(\tilde{X}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T}),\tilde{\mu}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T})\right) \ .$$

Note that the identity

$$\begin{aligned} \tilde{X}(s,\omega^{t,\hat{t}},\cdot) &= X^{u,v}_{t,x,a}(\hat{t}) + \int_{\hat{t}}^{s} f(r,\tilde{\mathcal{X}}(r,\omega^{t,\hat{t}},\cdot),\tilde{\gamma}(r,\omega^{t,\hat{t}},\cdot))\,\mathrm{d}r \\ &\quad + \int_{\hat{t}}^{s} \sigma(r,\tilde{\mathcal{X}}(r,\omega^{t,\hat{t}},\cdot),\tilde{\gamma}(r,\omega^{t,\hat{t}},\cdot))\,\mathrm{d}B^{\hat{t}}(r) \\ &\quad + \int_{\hat{t}}^{s}\int_{\mathbb{R}_0^K} h\left(r_-,\tilde{\mathcal{X}}(r_-,\omega^{t,\hat{t}},\cdot),u(r_-),v(r_-),z\right)\tilde{J}^t(\mathrm{d}r,\mathrm{d}z) \ , \\ \tilde{\mu}(s,\omega^{t,\hat{t}},\cdot) &= \mu_{t,a}(\hat{t}) + \int_{\hat{t}}^{s}\int_{\mathbb{R}} \Gamma\left(\tilde{\mu}(r_-,\omega^{t,\hat{t}},\cdot),z\right)\ N^t(\mathrm{d}r,\mathrm{d}z) \ , \end{aligned} \tag{17}$$

holds $\mathbb{P}_{t,\hat{t}}$-a.e. $\omega^{t,\hat{t}} \in \Omega_{t,\hat{t}}$ as a consequence of (15) and the comments following it. Moreover, by uniqueness of solutions of (5), we get that the paths of $\tilde{\mathcal{X}}(s,\omega^{t,\hat{t}},\cdot) = \left(\tilde{X}(s,\omega^{t,\hat{t}},\cdot),\tilde{\mu}(s,\omega^{t,\hat{t}},\cdot)\right)$, $s \in [\hat{t},T]$, coincide with those of (5) with initial condition $(\hat{t},X^{u,v}_{t,x,a}(\hat{t}),\mu_{t,a}(\hat{t}))$ and controls $(u(\cdot,\omega^{t,\hat{t}}),v(\cdot,\omega^{t,\hat{t}}))$ for $\mathbb{P}_{t,\hat{t}}$-a.e $\omega^{t,\hat{t}} \in \Omega_{t,\hat{t}}$. From this point onwards we will also use the notation $\mathcal{X}^{u,v}_{t,x,a}(\cdot) = \left(X^{u,v}_{t,x,a}(\cdot),\mu_{t,a}(\cdot)\right)$ to refer to the stochastic process $\tilde{\mathcal{X}}(\cdot) = \left(\tilde{X}(\cdot),\tilde{\mu}(\cdot)\right)$ on the filtered probability space $(\Omega_{\hat{t},T},\mathcal{G}_{\hat{t},T},\mathbb{G}_{\hat{t},T},\mathbb{P}_{\hat{t},T})$.

The observations above, together with the fact that

$$\mathbb{E}_{\mathbb{P}_{t,\hat{t}}\otimes\mathbb{P}_{\hat{t},T}}\left[\phi\left(\omega^{t,\hat{t}},\omega^{\hat{t},T}\right)|\mathcal{G}_{t,\hat{t}}\right] = \mathbb{E}_{\mathbb{P}_{\hat{t},T}}\left[\phi\left(\omega^{t,\hat{t}},\omega^{\hat{t},T}\right)\right] \qquad \mathbb{P}_{t,\hat{t}} - \text{a.s.} \tag{18}$$

for any bounded and measurable function $\phi : \Omega_{t,T} \to \mathbb{R}$, yield the following technical lemma.

Lemma 3.1. *Suppose Assumptions (A1)-(A5) hold and let $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$ denote the solution of* (5) *with initial condition $(t,x,a) \in [0,T) \times \mathbb{R}^N \times S$ and controls $(u(\cdot),v(\cdot)) \in \mathcal{U}(t,T)\times\mathcal{V}(t,T)$. For any bounded continuous function ϕ and any deterministic $s \in [\hat{t},T]$ we have that*

$$\mathbb{E}_{\mathbb{P}_{t,T}}\left[\phi\left(\mathcal{X}^{u,v}_{t,x,a}(s),\gamma(s,\omega)\right)|\mathcal{G}_{t,\hat{t}}\right] = \mathbb{E}_{\mathbb{P}_{\hat{t},T}}\left[\phi\left(\mathcal{X}^{u,v}_{\hat{t},\mathcal{X}^{u,v}_{t,x,a}(\hat{t})}(s),\tilde{\gamma}(s,\omega^{t,\hat{t}},\omega^{\hat{t},T})\right)\right]$$

holds $\mathbb{P}_{t,\hat{t}}$ *almost surely.*

The next lemma concerns regularity of the SDG value functions V^- and V^+ defined in (7) and (8).

Lemma 3.2. *Let Assumptions (A1)-(A5) hold. Then, we have that:*

i) *For every* $u(\cdot) \in \mathcal{U}(t,T)$*,* $v(\cdot) \in \mathcal{V}(t,T)$*,* $\alpha \in \mathcal{A}(t,T)$*, and* $\beta \in \mathcal{B}(t,T)$*, the payoff functionals*

$$(t,x,a) \to J(\cdot,\cdot,\cdot;\alpha[v(\cdot)],v(\cdot)) \quad \text{and} \quad (t,x,a) \to J(\cdot,\cdot,\cdot;u(\cdot),\beta[u(\cdot)]) \tag{19}$$

are bounded and Lipschitz continuous with respect to x*, uniformly in* t*,* a*,* α*,* $v(\cdot)$*,* β*,* $u(\cdot)$*.*

ii) *the SDG value functions* V^- *and* V^+ *in* (7) *and* (8) *are bounded and Lipschitz continuous with respect to* x*, uniformly in* t,a*.*

Proof. Boundedness of the functionals in (19), as well as Lipschitz continuity with respect to x, follow from Assumptions (A1)-(A5) combined with standard results from stochastic differential equations such as moment estimates and Gronwall's inequality (see e.g. [3, 52]). □

The lemma below establishes a regularity property of the solutions of (5) that will be useful in the sequel.

Lemma 3.3. *Suppose Assumptions (A1)-(A5) hold and let* $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$ *be the solution of* (5) *for a pair of admissible controls* $(u(\cdot),v(\cdot)) \in \mathcal{U}(t,T)\times\mathcal{V}(t,T)$*. Then,* $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$ *is stochastically continuous, i.e.*

$$\lim_{r\to s}\mathbb{P}_{t,T}\left(d_{N,S}\left(\mathcal{X}^{u,v}_{t,x,a}(r),\mathcal{X}^{u,v}_{t,x,a}(s)\right) > \epsilon\right) = 0$$

for every $\epsilon > 0$ *and* $s \in [t,T]$*.*

Proof. Let $s^*, s \in [t,T]$. Using (5) we obtain

$$\begin{aligned} X^{u,v}_{t,x,a}(s^*) - X^{u,v}_{t,x,a}(s) = &\int_s^{s^*} f\left(r,\mathcal{X}^{u,v}_{t,x,a}(r),u(r),v(r)\right)\mathrm{d}r \\ &+ \int_s^{s^*} \sigma\left(r,\mathcal{X}^{u,v}_{t,x,a}(r),u(r),v(r)\right)\mathrm{d}B^t(r) \\ &+ \int_s^{s^*}\int_{\mathbb{R}^K_0} h\left(r,\mathcal{X}^{u,v}_{t,x,a}(r),u(r),v(r),z\right)\ J^t(\mathrm{d}r,\mathrm{d}z) \end{aligned}$$

and

$$\mu_{t,a}(s^*) - \mu_{t,a}(s) = \int_s^{s^*}\int_{\mathbb{R}}\Gamma(\mu_{t,a}(r),z)\ N^t(\mathrm{d}r,\mathrm{d}z)$$

By the Itô-Lévy isometry and the boundedness of our data, we have that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{t,T}}\left[|X^{u,v}_{t,x,a}(s^*) - X^{u,v}_{t,x,a}(s)|^2 + |\mu_{t,a}(s^*) - \mu_{t,a}(s)|^2\right] \\
&\le 3\mathbb{E}_{\mathbb{P}_{t,T}}\left[\left|\int_s^{s^*} f(r, \mathcal{X}^{u,v}_{t,x,a}(r), u(r), v(r))\ \mathrm{d}r\right|^2\right] \\
&+3\int_s^{s^*} \mathbb{E}_{\mathbb{P}_{t,T}}\left[\sigma^2(r, \mathcal{X}^{u,v}_{t,x,a}(r), u(r), v(r))\right]\ \mathrm{d}r \\
&+3\int_s^{s^*}\int_{\mathbb{R}^K_0} \mathbb{E}_{\mathbb{P}_{t,T}}\left[h^2(r, \mathcal{X}^{u,v}_{t,x,a}(r), u(r), v(r))\right]\nu(\mathrm{d}z)\ \mathrm{d}r \qquad (20)\\
&+\int_s^{s^*}\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}_{t,T}}\left[\Gamma^2\left(\mu_{t,a}(r), z\right)\right]\ \mathrm{d}z\mathrm{d}r \\
&\le K\left(|s^*-s| + |s^*-s|^2\right)\ .
\end{aligned}$$

Therefore, by Chebyshev's inequality, we get

$$\mathbb{P}_{t,T}\left(d_{N,S}\left(\mathcal{X}^{u,v}_{t,x,a}(s^*), \mathcal{X}^{u,v}_{t,x,a}(s)\right) > \epsilon\right) \le \frac{K}{\epsilon^2}\left(|s-s^*| + |s-s^*|^2\right)\ ,$$

establishing the stochastic continuity of $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$. □

We close this section with a first-moment estimate for Markov-switching jump-diffusions such as considered herein. This result belongs to a class of well-known estimates within the theory of SDEs (see, for instance, [45, Corollary 2.4.6] for difusive SDEs or [47, Theorem 3.3.23] for the case of Markov-switching diffusions, both of which stated and proved under a linear growth condition). Since we are not aware of any reference in the literature with a version of this estimate with the features we require here (i.e., containing both Markov-switching and jump-diffusion components), we state and prove the relevant first-moment estimate below.

Lemma 3.4. *Suppose Assumptions (A1)-(A5) hold and let* $\mathcal{X}^{u,v}_{t,x,a}(\cdot) = \left(X^{u,v}_{t,x,a}(\cdot), \mu_{t,a}(\cdot)\right)$ *be the solution of* (5) *for a pair of admissible controls* $(u(\cdot), v(\cdot)) \in \mathcal{U}(t,T) \times \mathcal{V}(t,T)$*. Then, there exists a positive constant* C *such that*

$$\mathbb{E}_{\mathbb{P}_{t,T}}\left[d_{N,S}\left(\mathcal{X}^{u,v}_{t,x,a}(t^*), (x,a)\right)\right] \le C\left(t^*-t\right)^{1/2}\ .$$

Proof. Without loss of generality we may assume $|t^*-t| \le 1$. Appealing to inequality (20) from the proof of Lemma 3.3 (with $s = t$ and $s^* = t^*$), we obtain that there exists a positive constant K such that

$$\mathbb{E}_{\mathbb{P}_{t,T}}\left[|X^{u,v}_{t,x,a}(t^*) - x|^2 + |\mu_{t,a}(t^*) - a|^2\right] \le K\left(|t^*-t| + |t^*-t|^2\right)\ .$$

Since $|t^*-t| \le 1$, the result then follows from combining the estimate above with the Cauchy-Schwarz inequality. □

In the next section we will prove certain sub- and super-optimal dynamic programming principles, key intermediate steps for the proof of Theorem 2.1. For that purpose, we will resort to a special class of restrictive strategies and the corresponding value functions, following a method originally developed by Fleming and Souganidis [28].

3.2. **Sub-optimal and super-optimal dynamic programming principles.** Fleming and Souganidis, in their seminal paper [28], have introduced the concept of restrictive strategies, r-strategies for short, in order to overcome the measurability issues that hamper a generalization of the method for the proof of the deterministic dynamic programming principle to the stochastic setup.

Before proceeding to the definition of r-strategies, we observe that given a sequence of times $0 \le \bar{t} \le t \le T$ and a Player I admissible control $u(\cdot) \in \mathcal{U}(\bar{t}, T)$, for $\mathbb{P}_{\bar{t},t}$ a.e. path $\omega^{\bar{t},t} \in \Omega_{\bar{t},t}$, we are able to define a Player I admissible control in $\mathcal{U}(t,T)$, which we will denote as $u(\omega_{\bar{t},t})(\cdot)$, through the relation

$$u(\omega^{\bar{t},t})(s, \omega^{t,T}) = u(s, \omega) \ ,$$

where $\omega = \pi^{-1}(\omega^{\bar{t},t}, \omega^{t,T})$ and π is the map defined in (4). Recalling that the preimage $\omega = \pi^{-1}(\omega^{\bar{t},t}, \omega^{t,T}) \in \Omega_{t,T}$ corresponds to the concatenation of $\omega^{\bar{t},t} \in \Omega_{t,\hat{t}}$ with $\omega^{t,T} \in \Omega_{\hat{t},T}$, we note that the process detailed above defines the admissible control $u(\omega_{\bar{t},t})(\cdot) \in \mathcal{U}(t,T)$ over the time interval $[t,T]$ as the restriction of the admissible control $u(\cdot) \in \mathcal{U}(\bar{t},T)$ over the larger time interval $[\bar{t},T]$ when the section of the path $\omega^{\bar{t},t}$ corresponding to the time interval $[\bar{t},t]$ is fixed.

Given the SDG determined by (5) and (6), we say that a *r-strategy* β for Player II on $[t,T]$ is an admissible strategy with the following additional property: for every $\bar{t} < t < \hat{t}$ and $u(\cdot) \in \mathcal{U}(\bar{t},T)$ the map $(s,\omega) \to \beta[u(\omega_{\bar{t},t})(\cdot)](s,\omega^{t,T})$ is $(\mathcal{B}([t,\hat{t}]) \otimes \mathcal{G}_{t,\hat{t}}, \mathcal{B}(U))$-measurable, where $\mathcal{B}(X)$ stands for the Borel σ-algebra of a set X. The set of r-strategies for Player II is denoted by $\mathcal{B}_r(t,T)$. We define r-strategies for Player I in a similar fashion and denote the set of such strategies by $\mathcal{A}_r(t,T)$.

r-lower and r-upper value functions The *r-lower* and *r-upper value functions* of the SDG determined by (5) and (6) with initial data (t,x,a) are given by

$$V_r^-(t,x,a) = \inf_{\beta \in \mathcal{B}_r(t,T)} \sup_{u(\cdot) \in \mathcal{U}(t,T)} J(t,x,a; u(\cdot), \beta[u(\cdot)])$$

and

$$V_r^+(t,x,a) = \sup_{\alpha \in \mathcal{A}_r(t,T)} \inf_{v(\cdot) \in \mathcal{V}(t,T)} J(t,x,a; \alpha[v(\cdot)], v(\cdot)) \ .$$

The result below follows from Lemma 3.2 and the definitions of admissible strategies and r-strategies.

Corollary 3.5. *Suppose Assumptions (A1)-(A5) hold. We have that:*

(a) *the r-value functions V_r^- and V_r^+ of the SDG determined by* (5) *and* (6) *are bounded and Lipschitz continuous in x, uniformly in t,a.*
(b) *for every $(t,x,a) \in [0,T] \times \mathbb{R}^N \times S$,*

$$V^-(t,x,a) \le V_r^-(t,x,a) \ \text{and} \ V_r^+(t,x,a) \le V^+(t,x,a) \ ,$$

where V^- and V^+ are, respectively, the SDG lower and upper value functions defined in (7) *and* (8).

Even though the r-value functions do not satisfy a full dynamic programming principle, it is still possible to obtain sub- and super-optimal dynamic programming principles for such functions.

Proposition 3.6 (Sub-optimal and super-optimal dynamic programming principle)**.** *Assume that conditions (A1)-(A5) hold. For any* $(t,x,a) \in [0,T) \times \mathbb{R}^N \times S$ *and every* $\hat{t} \in [t,T)$*, we have that:*

$$V_r^-(t,x,a) \leq \inf_{\beta \in \mathcal{B}_r(t,T)} \sup_{u(\cdot) \in \mathcal{U}(t,T)} \mathbb{E}_{\mathbb{P}_{t,T}} \left[V_r^- \left(\hat{t}, \mathcal{X}_{t,x,a}^{u,v}(\hat{t})\right) + \int_t^{\hat{t}} L(s, \mathcal{X}_{t,x,a}^{u,v}(s), u(s), \beta[u(\cdot)](s)) \, \mathrm{d}s \right] , \tag{21}$$

where $\mathcal{X}_{t,x,a}^{u,v}(\cdot)$ *is the solution of* (5) *with* $v(\cdot) = \beta[u(\cdot)](\cdot) \in \mathcal{V}(t,T)$ *for* $u(\cdot) \in \mathcal{U}(t,T)$*, and*

$$V_r^+(t,x,a) \geq \sup_{\alpha \in \mathcal{A}_r(t,T)} \inf_{v(\cdot) \in \mathcal{V}(t,T)} \mathbb{E}_{\mathbb{P}_{t,T}} \left[V_r^+(\hat{t}, \mathcal{X}_{t,x,a}^{u,v}(\hat{t})) + \int_t^{\hat{t}} L(s, \mathcal{X}_{t,x,a}^{u,v}(s), \alpha[v(\cdot)](s), v(s)) \, \mathrm{d}s \right] , \tag{22}$$

where $\mathcal{X}_{t,x,a}^{u,v}(\cdot)$ *is the solution of* (5) *with* $u(\cdot) = \alpha[v(\cdot)](\cdot) \in \mathcal{U}(t,T)$ *for* $v(\cdot) \in \mathcal{V}(t,T)$*.*

Proof. We will prove identity (21). The proof of (22) is analogous. In order to simplify notation, the superscripts u, v will be dropped from the notation for the solution $\mathcal{X}_{t,x,a}^{u,v}(\cdot)$.

Let $(t,x,a) \in [0,T) \times \mathbb{R}^N \times S$ be fixed and let $\hat{t} \in [t,T)$ be arbitrary. Denote the right-hand side of (21) by $\overline{V}(t,x,a)$. Observe that for any $\epsilon > 0$ there exists $\beta_\epsilon(\cdot) \in \mathcal{B}_r(t,T)$ such that

$$\overline{V}(t,x,a) \geq \mathbb{E}_{\mathbb{P}_{t,T}} \left[V_r^-(\hat{t}, \mathcal{X}_{t,x,a}(\hat{t})) + \int_t^{\hat{t}} L(s, \mathcal{X}_{t,x,a}(s), \beta_\epsilon[u(\cdot)](s)) \, \mathrm{d}s \right] - \epsilon \tag{23}$$

for every $u(\cdot) \in \mathcal{U}(t,T)$. Moreover, for each $(y,b) \in \mathbb{R}^N \times S$, we have that

$$V_r^-(\hat{t}, y, b) = \inf_{\beta \in \mathcal{B}_r(\hat{t},T)} \sup_{u(\cdot) \in \mathcal{U}(\hat{t},T)} J(\hat{t}, y, b; u(\cdot), \beta[u(\cdot)]) \, . \tag{24}$$

Therefore, there exists $\beta_{(y,b)} \in \mathcal{B}_r(\hat{t},T)$ such that

$$V_r^-(\hat{t}, y, b) \geq \sup_{u(\cdot) \in \mathcal{U}(\hat{t},T)} J(\hat{t}, y, b; u(\cdot), \beta_{(y,b)}[u(\cdot)]) - \epsilon \, . \tag{25}$$

Let $\{D_i\}_{i \in \mathbb{N}}$ be a Borel partition of $\mathbb{R}^N \times S$ with diameter diam $(D_i) < \delta$ (with respect to the metric $d_{N,S}(\cdot,\cdot)$ defined in (2)) and pick $(y_i, b_i) \in D_i$ for each $i \in \mathbb{N}$. Using item a) of Lemma 3.2 and Corollary 3.5, the diameter $\delta > 0$ can be picked sufficiently small so that for any $(y_i, b_i) \in D_i$, we have that

$$\left| J(\hat{t}, y, b; u(\cdot), \beta[u(\cdot)]) - J(\hat{t}, y_i, b_i; u(\cdot), \beta[u(\cdot)]) \right| < \epsilon \tag{26}$$

for every $u(\cdot) \in \mathcal{U}(\hat{t},T)$ and $\beta \in \mathcal{B}(\hat{t},T)$. We also have

$$\left| V_r^-(\hat{t}, y, b) - V_r^-(\hat{t}, y_i, b_i) \right| < \epsilon \, .$$

For each $\hat{t} \in [t,T]$ and $u(\cdot) \in \mathcal{U}(t,T)$, define

$$\tilde{\beta}[u(\cdot)](s,\omega) = \begin{cases} \beta_\epsilon[u(\cdot)](s,\omega) & \text{if } s \in [t,\hat{t}) \\ \sum_{i\in\mathbb{N}} \mathbf{1}_{D_i}(\mathcal{X}_{t,x,a}(\hat{t}))\beta_{(y_i,b_i)}[u(\omega^{t,\hat{t}})(\cdot)](s,\omega^{\bar{t},T}) & \text{if } s \in [\hat{t},T] \end{cases},$$

where $\omega = (\omega^{t,\hat{t}}, \omega^{\bar{t},T}) \in \Omega_{t,\hat{t}} \times \Omega_{\hat{t},T}$ and $u(\omega^{t,\hat{t}})(\cdot) \in \mathcal{U}(\hat{t},T)$ is the admissible control introduced just before the definition of the r-value functions. Note that $\tilde{\beta}$ is, by construction, an r-strategy i.e. $\tilde{\beta} \in \mathcal{B}_r(t,T)$.

Moreover, whenever $\mathcal{X}_{t,x,a}(\hat{t}) \in D_i$ for some $i \in \mathbb{N}$ and $u(\cdot) \in \mathcal{U}(t,T)$, identity (25) and inequality (26) yield

$$\begin{aligned} V_r^-(\hat{t}, y_i, b_i) \quad &\geq \quad J(\hat{t}, y_i, b_i; u(\omega^{t,\hat{t}})(\cdot), \beta_{(y_i,b_i)}[u(\omega^{t,\hat{t}})(\cdot)]) - \epsilon \qquad (27) \\ &\geq \quad J(\hat{t}, \mathcal{X}_{t,x,a}(\hat{t}); u(\omega^{t,\hat{t}})(\cdot), \beta_{(y_i,b_i)}[u(\omega^{t,\hat{t}})(\cdot)]) - 2\epsilon \end{aligned}$$

for all $u(\cdot) \in \mathcal{U}(t,T)$ and $\mathbb{P}_{t,\hat{t}}$-a.e. $\omega^{t,\hat{t}} \in \Omega_{t,\hat{t}}$.

Using the definition of the payoff functional in (6), we get

$$\begin{aligned} & J(t,x,a;u(\cdot),\tilde{\beta}[u(\cdot)]) = \\ &= \mathbb{E}_{\mathbb{P}_{t,T}}\left[\int_t^T L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\tilde{\beta}[u(\cdot)](s)\right)\mathrm{d}s + \Psi\left(\mathcal{X}_{t,x,a}(T)\right)\right] \\ &= \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_t^{\hat{t}} L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\tilde{\beta}[u(\cdot)](s)\right)\mathrm{d}s \\ &\qquad + \sum_{i\in\mathbb{N}} \mathbf{1}_{D_i}(\mathcal{X}_{t,x,a}(\hat{t}))\Bigg(\int_{\hat{t}}^T L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\tilde{\beta}[u(\cdot)](s)\right)\mathrm{d}s \\ &\qquad\qquad + \Psi\left(\mathcal{X}_{t,x,a}(T)\right)\Bigg)\Bigg]. \end{aligned}$$

From the definition of the r-strategy $\tilde{\beta}$, we get

$$\begin{aligned} J(t,x,a;u(\cdot),\tilde{\beta}[u(\cdot)]) &= \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_t^{\hat{t}} L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\beta_\epsilon[u(\cdot)](s)\right)\mathrm{d}s \\ &+ \sum_{i\in\mathbb{N}} \mathbf{1}_{D_i}(\mathcal{X}_{t,x,a}(\hat{t}))\,\mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_{\hat{t}}^T L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\tilde{\beta}[u(\cdot)](s)\right)\mathrm{d}s \\ &\qquad + \Psi\left(\mathcal{X}_{t,x,a}(T)\right)|\mathcal{G}_{t,\hat{t}}\Bigg]. \end{aligned}$$

Combining the previous identity with Lemma 3.1, we obtain

$$\begin{aligned} J(t,x,a;u(\cdot),\tilde{\beta}[u(\cdot)]) &= \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_t^{\hat{t}} L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\beta_\epsilon[u(\cdot)](s)\right)\mathrm{d}s \\ &+ \sum_{i\in\mathbb{N}} \mathbf{1}_{D_i}(\mathcal{X}_{t,x,a}(\hat{t}))J\left(\hat{t},\mathcal{X}_{t,x,a}(\hat{t});u(\omega^{t,\hat{t}})(\cdot),\beta_{(y_i,b_i)}[u(\omega^{t,\hat{t}})(\cdot)]\right)\Bigg]. \end{aligned}$$

Using inequalities (27) and (26), we get

$$\begin{aligned} J(t,x,a;u(\cdot),\tilde{\beta}[u(\cdot)]) &\leq \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_t^{\hat{t}} L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\beta_\epsilon[u(\cdot)](s)\right)\mathrm{d}s \\ &\quad +\sum_{i\in\mathbb{N}}\mathbf{1}_{D_i}(\mathcal{X}_{t,x,a}(\hat{t}))V_r^-(\hat{t},y_i,b_i)\Bigg]+2\epsilon \\ &\leq \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_t^{\hat{t}} L\left(s,\mathcal{X}_{t,x,a}(s),u(s),\beta_\epsilon[u(\cdot)](s)\right)\mathrm{d}s \\ &\quad +V_r^-(\hat{t},\mathcal{X}_{t,x,a}(\hat{t}))\Bigg]+3\epsilon\ . \end{aligned}$$

Finally, from the previous inequality and (23), we get

$$J(t,x,a;u(\cdot),\tilde{\beta}[u(\cdot)]) \leq \overline{V}(t,x,a)+4\epsilon$$

for every $u(\cdot)\in\mathcal{U}(t,T)$. As a consequence, we conclude that

$$V_r^-(t,x,a) \leq \overline{V}(t,x,a)+4\epsilon\ .$$

Letting ϵ go to zero completes the proof. □

The following Hölder continuity estimates for the r-value functions V_r^- and V_r^+ are obtained by combining Proposition 3.6 with the moment estimate of Lemma 3.4.

Corollary 3.7. *Suppose that conditions (A1)-(A5) hold. The r-value functions V_r^- and V_r^+ are $\frac{1}{2}$-Hölder continuous in t, uniformly in x and a, that is, there exist positive constants C_1 and C_2 such that*

$$|V_r^-(t,x,a)-V_r^-(s,x,a)| \leq C_1(t-s)^{1/2}$$

and

$$|V_r^+(t,x,a)-V_r^+(s,x,a)| \leq C_2(t-s)^{1/2}$$

for every $t,s\in[0,T]$, $x\in\mathbb{R}^N$, and $a\in S$.

Proof. We will focus on establishing Hölder continuity of V_r^- with respect to t. The corresponding argument for V_r^+ is similar. To simplify notation, we will drop the superscripts u,v from the solution $\mathcal{X}_{t,x,a}^{u,v}(\cdot)=\left(X_{t,x,a}^{u,v}(\cdot),\mu_{t,a}(\cdot)\right)$, with the specific controls used at each instant being clear from the context.

Without loss of generality, suppose that $t_1, t_2 \in [0, T]$ are such that $t_1 < t_2$ and $|t_2 - t_1| < 1$. Using (21) and rearranging terms, we get

$$\begin{aligned}
&V_r^-(t_1, x, a) - V_r^-(t_2, x, a) \\
&\leq \inf_{\beta \in \mathcal{B}_r(t_1,T)} \sup_{u(\cdot) \in \mathcal{U}(t_1,T)} \mathbb{E}_{\mathbb{P}_{t_1,T}} \Bigg[\int_{t_1}^{t_2} L(s, \mathcal{X}_{t_1,x,a}(s), u(s), \beta[u(\cdot)](s)) \, \mathrm{d}s \\
&\qquad\qquad + V_r^-(t_2, \mathcal{X}_{t_1,x,a}(t_2)) - V_r^-(t_2, x, a) \Bigg] \\
&= \inf_{\beta \in \mathcal{B}_r(t_1,T)} \sup_{u(\cdot) \in \mathcal{U}(t_1,T)} \mathbb{E}_{\mathbb{P}_{t_1,T}} \Bigg[\int_{t_1}^{t_2} L(s, \mathcal{X}_{t_1,x,a}(s), u(s), \beta[u(\cdot)](s)) \, \mathrm{d}s \\
&\qquad + V_r^-(t_2, X_{t_1,x,a}(t_2), \mu_{t_1,a}(t_2)) - V_r^-(t_2, x, \mu_{t_1,a}(t_2)) \\
&\qquad\qquad + V_r^-(t_2, x, \mu_{t_1,a}(t_2)) - V_r^-(t_2, x, a) \Bigg]
\end{aligned} \tag{28}$$

Boundedness of L (from assumption (A3)), uniform Lipschitz continuity of V_r^- in x (from Corollary 3.5), and boundedness of V_r^- (also from Corollary 3.5) combined with the finiteness of the state space S, yield the existence of positive constants C_1, C_2 and C_3 such that the following inequalities hold:

$$\begin{aligned}
&\int_{t_1}^{t_2} L(s, \mathcal{X}_{t_1,x,a}(s), u(s), \beta[u(\cdot)](s)) \, \mathrm{d}s \leq C_1 |t_2 - t_1| \\
&V_r^-(t_2, X_{t_1,x,a}(t_2), \mu_{t_1,a}(t_2)) - V_r^-(t_2, x, \mu_{t_1,a}(t_2)) \leq C_2 |X_{t_1,x,a}(t_2) - x| \\
&V_r^-(t_2, x, \mu_{t_1,a}(t_2)) - V_r^-(t_2, x, a) \leq C_3 d_{N,S}\left((x, \mu_{t_1,a}(t_2)), (x, a)\right) \; .
\end{aligned} \tag{29}$$

Observing that

$$\begin{aligned}
|X_{t_1,x,a}(t_2) - x| &\leq d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2), (x, a)\right) \\
d_{N,S}\left((x, \mu_{t_1,a}(t_2)), (x, a)\right) &\leq d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2), (x, a)\right) \; ,
\end{aligned} \tag{30}$$

and combining the three inequalities in (29) with inequality (28), we obtain that there exists a positive constant C_4 such that

$$V_r^-(t_1, x, a) - V_r^-(t_2, x, a) \leq C_4 \Big(|t_2 - t_1| + \mathbb{E}_{\mathbb{P}_{t_1,T}}\left[d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2), (x, a)\right)\right] \Big) \tag{31}$$

The first-moment estimate of Lemma 3.4 guarantees the existence of a positive constant C_5 such that

$$\mathbb{E}_{\mathbb{P}_{t_1,T}}\left[d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2), (x, a)\right)\right] \leq C_5 |t_2 - t_1|^{1/2} \; . \tag{32}$$

Putting together inequalities (31) and (32), we conclude that

$$V_r^-(t_1, x, a) - V_r^-(t_2, x, a) \leq K_1 |t_2 - t_1|^{1/2} \tag{33}$$

for some positive constant K_1, thus completing the first part of the proof.

Let us now prove the remaining inequality. Given $u(\cdot) \in \mathcal{U}(t_2, T)$, define $u^*(\cdot) \in \mathcal{U}(t_1, T)$ as

$$u^*(s, \omega) = \begin{cases} u(t_2, \omega^{t_1,t_2}) & \text{if } s \in [t_1, t_2] \\ u(s, \omega^{t_1,t_2}) & \text{if } s \in (t_2, T] \end{cases} \; ,$$

and given $\beta^* \in \mathcal{B}_r(t_1,T)$, define $\beta \in \mathcal{B}_r(t_2,T)$ as

$$\beta[u(\cdot)](s,\omega^{t_2,T}) = \beta^*[u^*(\cdot)](s,\pi^{-1}(\omega^{t_1,t_2},\omega^{t_2,T})) \ .$$

We now observe that for $\beta^* \in \mathcal{B}_r(t_1,T)$, we have

$$\begin{aligned}
J(t_1,x,a;u^*(\cdot),\beta^*[u^*(\cdot)]) = \mathbb{E}_{\mathbb{P}_{t_1,T}}\Bigg[\int_{t_1}^{T} L\left(s,\mathcal{X}_{t_1,x,a}(s),u^*(s),\beta^*[u^*(\cdot)](s)\right)\mathrm{d}s \\
+\Psi\left(\mathcal{X}_{t_1,x,a}(T)\right)\Bigg] \\
= \quad \mathbb{E}_{\mathbb{P}_{t_1,T}}\Bigg[\int_{t_1}^{t_2} L\left(s,\mathcal{X}_{t_1,x,a}(s),u^*(s),\beta^*[u^*(\cdot)](s)\right)\mathrm{d}s \\
+J(t_2,\mathcal{X}_{t_1,x,a}(t_2);u(\cdot),\beta[u(\cdot)])\Bigg] \\
= \quad \mathbb{E}_{\mathbb{P}_{t_1,T}}\Bigg[\int_{t_1}^{t_2} L\left(s,\mathcal{X}_{t_1,x,a}(s),u^*(s),\beta^*[u^*(\cdot)](s)\right)\mathrm{d}s \\
+\ J(t_2,X_{t_1,x,a}(t_2),\mu_{t_1,a}(t_2);u(\cdot),\beta[u(\cdot)]) - J(t_2,x,\mu_{t_1,a}(t_2);u(\cdot),\beta[u(\cdot)]) \\
+\ J(t_2,x,\mu_{t_1,a}(t_2);u(\cdot),\beta[u(\cdot)]) - J(t_2,x,a;u(\cdot),\beta[u(\cdot)]) \\
+\ J(t_2,x,a;u(\cdot),\beta[u(\cdot)])\Bigg]
\end{aligned}$$

Boundedness of L (ensured by assumption (A3)), Lipshitz continuity of the map $x \mapsto J(t,x,a;u(\cdot),v(\cdot))$ (guaranteed by Lemma 3.2), and boundedness of $J(t,x,a;u(\cdot),v(\cdot))$ (also from Lemma 3.2) combined with the finiteness of the state space S, yield the existence of positive constants C_6, C_7 and C_8 such that the following inequalities hold:

$$\begin{aligned}
&\int_{t_1}^{t_2} L(s,\mathcal{X}_{t_1,x,a}(s),u^*(s),\beta^*[u^*(\cdot)](s))\,\mathrm{d}s \geq -C_6|t_2-t_1| \\
&J(t_2,X_{t_1,x,a}(t_2),\mu_{t_1,a}(t_2);u(\cdot),\beta[u(\cdot)]) \\
&\qquad - J(t_2,x,\mu_{t_1,a}(t_2);u(\cdot),\beta[u(\cdot)]) \geq -C_7|X_{t_1,x,a}(t_2)-x| \\
&J(t_2,x,\mu_{t_1,a}(t_2);u(\cdot),\beta[u(\cdot)]) \\
&\qquad - J(t_2,x,a;u(\cdot),\beta[u(\cdot)]) \geq -C_8 d_{N,S}\left((x,\mu_{t_1,a}(t_2)),(x,a)\right) \ .
\end{aligned}$$

Combining the three inequalities above with those in (30), we obtain that there exists a positive constant C_9 such that

$$\begin{aligned}
&J(t_1,x,a;u^*(\cdot),\beta^*[u^*(\cdot)]) \\
&\quad \geq -C_9\left(|t_2-t_1| + \mathbb{E}_{\mathbb{P}_{t_1,T}}\left[d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2),(x,a)\right)\right]\right) \\
&\qquad +J(t_2,x,a;u(\cdot),\beta[u(\cdot)]) \ .
\end{aligned}$$

As a consequence, we have that

$$
\begin{aligned}
&\sup_{u(\cdot)\in\mathcal{U}(t_1,T)} J(t_1,x,a;u(\cdot),\beta^*[u(\cdot)])\\
&\quad\geq -C_9\Big(|t_2-t_1| + \mathbb{E}_{\mathbb{P}_{t_1,T}}\left[d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2),(x,a)\right)\right]\Big)\\
&\qquad + \sup_{u(\cdot)\in\mathcal{U}(t_2,T)} J(t_2,x,a;u(\cdot),\beta[u(\cdot)])\\
&\quad\geq -C_9\Big(|t_2-t_1| + \mathbb{E}_{\mathbb{P}_{t_1,T}}\left[d_{N,S}\left(\mathcal{X}_{t_1,x,a}(t_2),(x,a)\right)\right]\Big) + V_r^-(t_2,x,a)\ .
\end{aligned}
$$

Resorting once more to the first-moment estimate (32), and using the previous inequality, we obtain that a positive constant K_2 exists such that

$$
V_r^-(t_1,x,a) - V_r^-(t_2,x,a) \geq -K_2|t_2-t_1|^{1/2}\ . \tag{34}
$$

The result now follows from combining the estimates (33) and (34). □

3.3. **Hamilton-Jacobi-Bellman-Isaacs equation and viscosity solutions.** Before proceeding to prove that the r-value functions V_r^- and V_r^+ are, respectively, viscosity subsolutions and supersolutions of the HJBI equations (12) and (13), we will derive such equations, under a smoothness assumption, from the dynamic programming principle of Theorem 2.1.

Theorem 3.8. *Let Assumptions (A1)-(A5) hold. Suppose that the variational problems* (9) *and* (10) *in the statement of Theorem 2.1 have solutions V^- and V^+ such that $V^\pm(\cdot,\cdot,a)\in C^{1,2}([0,T)\times\mathbb{R}^N)$ for every $a\in S$. Then, V^- and V^+ satisfy the HJBI equations* (12) *and* (13)*, respectively.*

Proof. We only prove the part of the statement concerning the lower value function V^-, the proof for V^+ being similar. Let $(t,x,a)\in[0,T)\times\mathbb{R}^N\times S$ be fixed and denote by $\mathcal{X}^{u,v}_{t,x,a}(\cdot) = (X^{u,v}_{t,x,a}(\cdot),\mu^t_a(\cdot))$ the state variable trajectory associated with the controls $u(\cdot)\in\mathcal{U}(t,T)$ and $v(\cdot)\in\mathcal{V}(t,T)$.

For arbitrary controls $u(\cdot)\in\mathcal{U}(t,T)$ and $v(\cdot)\in\mathcal{V}(t,T)$, taking $\hat{t}\in[t,T)$ and using Itô-Lévy's formula, we obtain

$$
\begin{aligned}
&\frac{\mathbb{E}_{\mathbb{P}_{t,T}}\left[V^-(\hat{t},\mathcal{X}^{u,v}_{t,x,a}(\hat{t})) - V^-(t,x,a)\right]}{\hat{t}-t} =\\
&\quad\frac{1}{\hat{t}-t}\mathbb{E}_{\mathbb{P}_{t,T}}\left[\int_t^{\hat{t}} A_1(s,\mathcal{X}^{u,v}_{t,x,a}(s),u(s),v(s))\right.\\
&\qquad\left. + H_2(s,\mathcal{X}^{u,v}_{t,x,a}(s),u(s),v(s))\left[V^-(t,x,a)\ \right]\mathrm{d}s\right]\ ,
\end{aligned}
\tag{35}
$$

where the function A_1 is given by

$$
\begin{aligned}
A_1(t,x,a,u,v) \quad &= \quad V_t^-(t,x,a) + f(t,x,a,u,v)\cdot V_x^-(t,x,a)\\
&\qquad + \frac{1}{2}\mathrm{tr}\left(\sigma(t,x,a,u,v)V^-_{xx}(t,x,a)\sigma(t,x,a,u,v)\right)
\end{aligned}
$$

and H_2 is as given in the statement of Theorem 2.2.

Under the assumption that V^- satisfies the dynamic programming principle identity (9), for every $\varepsilon>0$ there exists $\beta_\varepsilon[\cdot]\in\mathcal{B}(t,T)$ such that for

$v(\cdot) = \beta_\varepsilon[u(\cdot)](\cdot)$ we have

$$-\frac{\mathbb{E}_{\mathbb{P}_{t,T}}\left[V^-(\hat{t}, \mathcal{X}^{u,v}_{t,x,a}(\hat{t})) - V^-(t,x,a)\right]}{\hat{t}-t} \geq \frac{1}{\hat{t}-t}\mathbb{E}_{\mathbb{P}_{t,T}}\left[\int_t^{\hat{t}} L(s, \mathcal{X}^{u,v}_{t,x,a}(s), u(s), \beta_\varepsilon[u(\cdot)](s))\, \mathrm{d}s\right] - \varepsilon$$

for every $u(\cdot) \in \mathcal{U}(t,T)$. Taking $u(\cdot) \in \mathcal{U}(t,T)$ to be constant and equal to $u \in U$ for every $s \in [t,T]$ and combining identity (35) with the last inequality, we obtain that for every $\varepsilon > 0$ there exists $\beta_\varepsilon[\cdot] \in \mathcal{B}(t,T)$ such that for $v(\cdot) = \beta_\varepsilon[u](\cdot)$ we have

$$\begin{aligned} \frac{1}{\hat{t}-t}\mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\int_t^{\hat{t}} & A_1(s, \mathcal{X}^{u,v}_{t,x,a}(s), u, \beta_\varepsilon[u](s)) \\ & + H_2(s, \mathcal{X}^{u,v}_{t,x,a}(s), u, \beta_\varepsilon[u](s))\left[V^-(t,x,a)\right] \\ & + L(s, \mathcal{X}^{u,v}_{t,x,a}(s), u, \beta_\varepsilon[u](s))\, \mathrm{d}s\Bigg] \leq \varepsilon \end{aligned} \tag{36}$$

for every $u \in U$.

Letting $\hat{t} \downarrow t$ in (36), we obtain that for every $\varepsilon > 0$ there exists $\beta_\varepsilon[\cdot] \in \mathcal{B}(t,T)$ such that

$$\begin{aligned} V^-_t(t,x,a) + H_1 & \left(t,x,a,u,\beta_\varepsilon[u], V^-_x(t,x,a), V^-_{xx}(t,x,a)\right) \\ & + H_2\left(t,x,a,u,\beta_\varepsilon[u]\right)\left[V^-(t,x,a)\right] \leq \varepsilon \end{aligned}$$

holds for every $u \in \mathcal{U}$, where H_1 is as given in the statement of Theorem 2.2. As a consequence, one obtains that

$$\begin{aligned} V^-_t(t,x,a) + \min_{v\in V} \Big\{ & H_1\left(t,x,a,u,v,V^-_x(t,x,a), V^-_{xx}(t,x,a)\right) \\ & H_2\left(t,x,a,u,v\right)\left[V^-(t,x,a)\right] \Big\} \leq \varepsilon \end{aligned}$$

for every $u \in U$. Taking the maximum over $u \in U$, the previous inequality implies that

$$V^-_t(t,x,a) + \mathcal{H}^-\left(t,x,a,V^-(t,\cdot,\cdot), V^-_x(t,x,a), V^-_{xx}(t,x,a)\right) \leq 0\ .$$

On the other hand, for any $\varepsilon > 0$ and any $\hat{t} \in (t,T)$ for which $\hat{t}-t$ is small enough, there exists $u_\varepsilon(\cdot) \in \mathcal{U}(t,T)$ such that

$$\begin{aligned} V^-(t,x,a) - \varepsilon(\hat{t}-t) \leq \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[& V^-\left(\hat{t}, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(\hat{t})\right) \\ & + \int_t^{\hat{t}} L(s, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(s), u_\varepsilon(s), \beta[u_\varepsilon(\cdot)](s))\, \mathrm{d}s\Bigg] \end{aligned}$$

for every $\beta[\cdot] \in \mathcal{B}(t,T)$, where $\tilde{v}$ stands for the control process $\tilde{v}(\cdot) = \beta[u_\epsilon(\cdot)](\cdot)$. Isolating ε in the inequality above, we get

$$-\varepsilon \leq \frac{1}{\hat{t}-t}\mathbb{E}_{\mathbb{P}_{t,T}}\left[V^-\left(\hat{t}, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(\hat{t})\right) - V^-(t,x,a) \right.$$
$$\left. + \int_t^{\hat{t}} L(s, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(s), u_\varepsilon(s), \beta[u_\varepsilon(\cdot)](s))\, \mathrm{d}s\right] ,$$

which, when combined with identity (35), ensures that

$$-\varepsilon \leq \frac{1}{\hat{t}-t}\mathbb{E}_{\mathbb{P}_{t,T}}\left[\int_t^{\hat{t}} A_1(s, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(s), u_\varepsilon(s), \beta[u_\varepsilon(\cdot)](s)) \right.$$
$$+H_2(s, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(s), u_\varepsilon(s), \beta[u_\varepsilon(\cdot)](s))\left[V^-(t,x,a)\right]$$
$$\left. + L(s, \mathcal{X}^{u_\varepsilon,\tilde{v}}_{t,x,a}(s), u_\varepsilon(s), \beta[u_\varepsilon(\cdot)](s))\, \mathrm{d}s\right]$$

holds for every $\beta[\cdot] \in \mathcal{B}(t,T)$.

Letting $\hat{t} \downarrow t$ in (37) once again, we obtain that for every $\varepsilon > 0$ there exists $u_\varepsilon(\cdot) \in \mathcal{U}(t,T)$ such that

$$-\varepsilon \leq V^-_t(t,x,a) + H_1\left(t,x,a,u_\varepsilon(t), \beta[u_\varepsilon(\cdot)](t), V^-_x(t,x,a), V^-_{xx}(t,x,a)\right)$$
$$+H_2\left(t,x,a,u_\varepsilon(t), \beta[u_\varepsilon(\cdot)](t)\right)\left[V^-(t,x,a)\right]$$

for every $\beta[\cdot] \in \mathcal{B}(t,T)$. Hence, we get that

$$-\varepsilon \leq V^-_t(t,x,a) + \min_{v\in V}\left\{H_1\left(t,x,a,u_\varepsilon(t), v, V^-_x(t,x,a), V^-_{xx}(t,x,a)\right)\right.$$
$$\left. +H_2\left(t,x,a,u_\varepsilon(t), v\right)\left[V^-(t,x,a)\right]\right\} .$$

Taking the maximum over $u \in U$, the previous inequality implies that

$$V^-_t(t,x,a) + \mathcal{H}^-\left(t,x,a,V^-(t,\cdot,\cdot), V^-_x(t,x,a), V^-_{xx}(t,x,a)\right) \geq 0 ,$$

completing the proof. □

Corollaries 3.5 and 3.7 guarantee continuity of the r-value functions V^-_r and V^+_r with respect to t and x. The proposition below characterizes V^-_r and V^+_r as, respectively, viscosity subsolution and supersolution of the HJBI equations (12) and (13).

Proposition 3.9. *Suppose that conditions (A1)-(A5) hold. The r-lower value function V^-_r (resp. r-upper value function V^+_r) of the SDG determined by* (5) *and* (6) *is a viscosity subsolution (resp. supersolution) of* (12) *(resp.* (13)*).*

Proof. We restrict our attention to the assertion regarding the r-lower value function V^-_r, with the result concerning V^+_r following in a similar fashion.

Let $\phi(\cdot,\cdot,a)$ be smooth for every $a \in S$ and suppose that $V^-_r - \phi$ attains a local maximum at $(t_0,x_0,a_0) \in [0,T) \times \mathbb{R}^N \times S$. We must prove that

$$\phi_t(t_0,x_0,a_0) + \mathcal{H}^-\left(t_0,x_0,a_0,\phi(t_0,x_0,a_0), \phi_x(t_0,x_0,a_0), \phi_{xx}(t_0,x_0,a_0)\right) \geq 0 .$$

Assume, for a contradiction, that the inequality above fails to hold. Then, there exists $\lambda > 0$ such that

$$\phi_t(t_0, x_0, a_0) \qquad (37)$$
$$+\mathcal{H}^-\left(t_0, x_0, a_0, \phi(t_0, x_0, a_0), \phi_x(t_0, x_0, a_0), \phi_{xx}(t_0, x_0, a_0)\right) \leq -\lambda < 0 \ .$$

Define the map $\Lambda_1 : [0, T] \times \mathbb{R}^N \times S \times U \times V \to \mathbb{R}$ as

$$\begin{aligned} \Lambda_1(t, x, a, u, v) &= \phi_t(t, x, a) + f(t, x, a, u, v)\phi_x(t, x, a) \\ &\quad + \frac{1}{2}\mathrm{tr}\left(g(t, x, a, u, v)\phi_{xx}(t, x, a)\right) + L(t, x, a, u, v) \ , \end{aligned}$$

where $g(t, x, a, u, v)$ is as given in the statement of Theorem 3.8. From inequality (37) one obtains that

$$\max_{u \in U} \min_{v \in V} \left\{\Lambda_1(t_0, x_0, a_0, u, v) + H_2(t_0, x_0, a_0, u, v)[\phi]\right\} \leq -\lambda < 0 \ . \qquad (38)$$

Hence, for each $u \in U$ there exists $v = v(u) \in V$ such that

$$\Lambda_1(t_0, x_0, a_0, u, v(u)) + H_2(t_0, x_0, a_0, u, v(u))[\phi] \leq -\lambda \ .$$

By uniform continuity of $\Lambda_1 + H_2$, we can conclude that

$$\Lambda_1(t_0, x_0, a_0, w, v(u)) + H_2(t_0, x_0, a_0, w, v(u))[\phi] \leq -\frac{3\lambda}{4}$$

for all $w \in B(u, r) \cap U$ and some $r = r(u) > 0$. By compactness of U, there exist finitely many distinct elements $u_1, u_2, \ldots, u_n \in U$, $v_1, v_2, \ldots, v_n \in V$ and numbers $r_1, \ldots, r_n > 0$ such that

$$U \subset \bigcup_{i=1}^{n} B(u_i, r_i)$$

and

$$\Lambda_1(t_0, x_0, a_0, w, v_i) + H_2(t_0, x_0, a_0, w, v_i)[\phi] \leq -\frac{3\lambda}{4} \ ,$$

for all $w \in B(u_i, r_i)$.

Define a map $\psi : U \to V$ by setting $\psi(u) = v_k$ whenever

$$u \in B(u_k, r_k) \setminus \bigcup_{i=1}^{k-1} B(u_i, r_i) \ , \qquad k = 1, \ldots, n \ .$$

Thus, by definition of ψ it follows that

$$\Lambda_1\left(t_0, x_0, a_0, u, \psi(u)\right) + H_2\left(t_0, x_0, a_0, u, \psi(u)\right)[\phi] \leq -\frac{3\lambda}{4} \qquad \text{for all } u \in U \ .$$

Relying once again on the uniform continuity of Λ, we obtain that there exists $R > 0$ such that

$$\Lambda_1\left(t, x, a, u, \psi(u)\right) + H_2\left(t_0, x_0, a_0, u, \psi(u)\right)[\phi] \leq -\frac{\lambda}{2} \qquad (39)$$

for all $u \in U$ and $(t, x, a) \in [0, T) \times \mathbb{R}^N \times S$ such that

$$\max\{|t - t_0|, d_{N,S}((x, a), (x_0, a_0))\} \leq R$$

where $d_{N,S}(\cdot, \cdot)$ denotes the metric on $\mathbb{R}^N \times S$ defined in (2). The map ψ defined above determines a r-strategy β^* for the second Player on $[t_0, T]$ as follows: for any $u(\cdot) \in \mathcal{U}(t_0, T)$ and $(s, \omega) \in [t_0, T] \times \Omega^{\omega}_{t_0,T}$ let

$$\beta^*[u(\cdot)](s) = \psi(u(s)) \ .$$

Since ψ is measurable (with respect to the Borel σ-algebras $\mathcal{B}(U)$ and $\mathcal{B}(V)$), we have $\beta^* \in \mathcal{B}_r(t_0, T)$.

Combining the sub-optimal dynamic programming principle (21) and the choice of $(t_0, x_0, a_0) \in [0,T) \times \mathbb{R}^N \times S$ with the Itô-Lévy formula, for $\hat{t} \in [t, T)$ we have the inequality

$$\sup_{u(\cdot)\in\mathcal{U}(t_0,T)} \mathbb{E}_{\mathbb{P}_{t_0,T}} \left[\int_{t_0}^{\hat{t}} \Lambda_1 \left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s), u(s), \beta^*[u(\cdot)](s)\right) \right.$$
$$\left. + H_2 \left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s), u(s), \beta^*[u(\cdot)](s)\right) \left[\phi\left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s)\right)\right] \mathrm{d}s \right] \geq 0 \ ,$$

where $\tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(\cdot)$ denotes the solution of (5) under the choice of controls $u(\cdot) \in \mathcal{U}(t_0,T)$ and $v(\cdot) = \beta^*[u(\cdot)] \in \mathcal{V}(t_0,T)$, and initial condition (x_0, a_0) at time t_0. Given $\epsilon > 0$ arbitrary, we can always choose $u_\epsilon(\cdot) \in \mathcal{U}(t_0,T)$ such that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}_{t_0,T}} & \left[\int_{t_0}^{\hat{t}} \Lambda_1 \left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s), u_\epsilon(s), \beta^*[u_\epsilon(\cdot)](s)\right) \right. \\ & \left. + H_2 \left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s), u_\epsilon(s), \beta^*[u_\epsilon(\cdot)](s)\right) \left[\phi\left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s)\right)\right] \mathrm{d}s \right] \\ & \geq -\epsilon(\hat{t} - t_0) \ . \end{aligned} \tag{40}$$

In what follows we will use the simplified notation

$$\begin{aligned} \tilde{\Lambda}(s) := \Lambda_1 & \left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s), u_\epsilon(s), \beta^*[u_\epsilon(\cdot)](s)\right) \\ & + H_2 \left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s), u_\epsilon(s), \beta^*[u_\epsilon(\cdot)](s)\right) \left[\phi\left(s, \tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s)\right)\right] \ . \end{aligned}$$

Let us denote by $d_\infty^{t_0,\hat{t}}$ the metric

$$d_\infty^{t_0,\hat{t}}\left(\gamma_1(\cdot), \gamma_2(\cdot)\right) = \sup_{t\in[t_0,\hat{t}]} d_{N,S}\left(\gamma_1(t), \gamma_2(t)\right) \tag{41}$$

on the space of piecewise continuous paths $\gamma_1, \gamma_2 : [t_0, \hat{t}] \mapsto \mathbb{R}^N \times S$. Setting $D^{u,v}_{t_0,x_0,a_0}$ as

$$D^{u,v}_{t_0,x_0,a_0} = d_\infty^{t_0,\hat{t}}\left(\tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(\cdot), (x_0, a_0)\right) \ , \tag{42}$$

where $\tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(\cdot)$ denotes the solution of (5) under the choice of controls $u_\epsilon(\cdot) \in \mathcal{U}(t_0,T)$ and $v(\cdot) = \beta^*[u_\epsilon(\cdot)] \in \mathcal{V}(t_0,T)$, and initial condition (x_0, a_0) at time t_0, we rewrite inequality (40) as

$$\mathbb{E}_{\mathbb{P}_{t_0,T}} \left[\int_{t_0}^{\hat{t}} \tilde{\Lambda}(s) \mathbf{1}_{D^{u,v}_{t_0,x_0,a_0} > R} \, \mathrm{d}s \right.$$
$$\left. + \int_{t_0}^{\hat{t}} \tilde{\Lambda}(s) \mathbf{1}_{D^{u,v}_{t_0,x_0,a_0} \leq R} \, \mathrm{d}s \right] \geq -\epsilon(\hat{t} - t_0) \ .$$

Combining the inequality above with (39), we obtain

$$\mathbb{E}_{\mathbb{P}_{t_0,T}}\left[\int_{t_0}^{\hat{t}} \tilde{\Lambda}(s)\mathbf{1}_{D^{u,v}_{t_0,x_0,a_0}>R}\,\mathrm{d}s\right] \qquad (43)$$
$$-\frac{\lambda}{2}(\hat{t}-t_0)\mathbb{P}_{t_0,T}\left(D^{u,v}_{t_0,x_0,a_0}\le R\right)\ge -\epsilon(\hat{t}-t_0)\ .$$

Note also that applying Cauchy-Schwarz inequality to the space $[t_0,\hat{t}]\times\Omega_{t_0,T}$, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{t_0,T}}\left[\int_{t_0}^{\hat{t}} \tilde{\Lambda}(s)\mathbf{1}_{D^{u,v}_{t_0,x_0,a_0}>R}\,\mathrm{d}s\right]\\
&\le\left(\mathbb{E}_{\mathbb{P}_{t_0,T}}\left[\int_{t_0}^{\hat{t}}\mathbf{1}_{D^{u,v}_{t_0,x_0,a_0}>R}\,\mathrm{d}s\right]\right)^{1/2}\cdot\left(\mathbb{E}_{\mathbb{P}_{t_0,T}}\left[\int_{t_0}^{\hat{t}}(\tilde{\Lambda}(s))^2\,\mathrm{d}s\right]\right)^{1/2}\\
&\le C(\hat{t}-t_0)\left(\mathbb{P}_{t_0,T}\left(D^{u,v}_{t_0,x_0,a_0}>R\right)\right)^{1/2}\ ,
\end{aligned}\qquad (44)$$

where C is a constant depending only on x_0, a_0, t_0, T and the Lipschitz constants from the functions listed in Assumption (A3). Combining (43) and (44) yields

$$-C\left(\mathbb{P}_{t_0,T}\left(D^{u,v}_{t_0,x_0,a_0}>R\right)\right)^{1/2}$$
$$+\frac{\lambda}{2}\,\mathbb{P}_{t_0,T}\left(D^{u,v}_{t_0,x_0,a_0}\le R\right)\le\epsilon\ . \qquad (45)$$

For $\hat{t}-t_0$ sufficiently small, there exists a constant $a>0$ such that

$$\mathbb{P}_{t_0,T}\left(D^{u,v}_{t_0,x_0,a_0}>R\right)\le\mathbb{P}_{t_0,T}\left(d^{t_0,\hat{t}}_{\infty}\left(\left(\zeta_1(\cdot),\zeta_2(\cdot)\right),(0,0)\right)>aR\right)\ , \qquad (46)$$

where $(\zeta_1(\cdot),\zeta_2(\cdot))$ is determined by

$$\begin{aligned}
\zeta_1(t)&=\int_{t_0}^{t}\sigma\left(s,\tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s),u_1(s),\beta^*[u_1(\cdot)](s)\right)\mathrm{d}B^{t_0}(s)\\
&\quad+\int_{t_0}^{t}\int_{\mathbb{R}^K_0}h\left(s_-,\tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(s_-),u_1(s_-),\beta^*[u_1(\cdot)](s_-),z\right)\tilde{J}^{t_0}(\mathrm{d}s,\mathrm{d}z)\ ,\ t\ge t_0\\
\zeta_2(t)&=\int_{t_0}^{t}\int_{\mathbb{R}}\Gamma\left(\tilde{\mu}_{t_0,a_0}(s_-),z\right)N^{t_0}(\mathrm{d}s,\mathrm{d}z)\ ,\ t\ge t_0\ .
\end{aligned}$$

Finally, using Doob's martingale inequality, we obtain that there exists some positive constant K such that

$$\begin{aligned}
\mathbb{P}_{t_0,T}\left(d^{t_0,\hat{t}}_{\infty}\left(\left(\zeta_1(\cdot),\zeta_2(\cdot)\right),(0,0)\right)>aR\right)&\le\frac{\mathbb{E}_{\mathbb{P}_{t_0,T}}\left[|\zeta_1(\hat{t})|^2+|\zeta_2(\hat{t})|^2\right]}{(aR)^2}\\
&\le\frac{K(\hat{t}-t_0)}{(aR)^2}\ ,
\end{aligned}\qquad (47)$$

where $|\zeta_2(\hat{t})|$ should be understood in terms of the embedding of S into $\mathbb{R}^n$. Combining (45) with (46) and (47), we obtain that

$$-C'(\hat{t}-t_0)^{1/2}+\frac{\lambda}{2}\,\mathbb{P}_{t_0,T}\left(D^{u,v}_{t_0,x_0,a_0}\le R\right)\le\epsilon$$

for some positive constant C', contradicting the stochastic continuity of the process $\tilde{\mathcal{X}}^{u,v}_{t_0,x_0,a_0}(\cdot)$ established in Lemma 3.3. $\square$

3.4. **Time-discretization procedure.** This section uses an approximation procedure, based on a discretization of the time variable, to obtain viscosity solutions for (12) and (13). This technique is originally due to Fleming and Souganidis [28, 58, 59].

Let $\pi = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ be a partition of $[0, T]$ and denote by

$$\|\pi\| = \max_{1 \le i \le m} (t_i - t_{i-1})$$

the mesh of the partition π. We specify below the subsets of admissible controls and admissible strategies associated with the partition π and of relevance to the discretization procedure to be pursued here.

π-admissible control A *π-admissible control* $u(\cdot)$ for Player I on $[t, T]$ is an admissible control with the following additional property: If $i_0 \in \{0, \ldots, m-1\}$ is such that $t \in [t_{i_0}, t_{i_0+1})$, then $u(s) = u$ for $s \in [t, t_{i_0+1})$ with $u \in U$ and $u(s) = u_{t_k}$ for $s \in [t_k, t_{k+1})$ for $k = i_0 + 1, \ldots, m-1$ where u_{t_k} is $\mathcal{G}_{t,t_k}$-measurable. The set of π-admissible controls for Player I on $[t, T]$ will be denoted by $\mathcal{U}_\pi(t, T)$. A π-admissible control $v(\cdot)$ for Player II on $[t, T]$ is defined similarly and the set of all such controls will be denoted by $\mathcal{V}_\pi(t, T)$.

π-admissible strategy A *π-admissible strategy* α for Player I on $[t, T]$ is an element of the set of admissible strategies $\mathcal{A}(t, T)$ with the additional properties that $\alpha[\mathcal{V}(t, T)] \subset \mathcal{U}_\pi(t, T)$, if $t \in [t_{i_0}, t_{i_0+1})$ then for every $v(\cdot) \in \mathcal{V}(t, T)$ the resulting control $\alpha[v(\cdot)]|_{[t,t_{i_0+1})}$ does not depend on $v(\cdot)$, and if $v(\cdot) \approx \tilde{v}(\cdot)$ on $[t, t_k]$, then $\alpha[v(\cdot)](t_k) = \alpha[\tilde{v}(\cdot)](t_k)$, $\mathbb{P}_{t,T}$-a.s. for every $k \in \{i_0 + 1, \ldots, m\}$. The set of all π-admissible strategies for Player I on $[t, T)$ will be denoted by $\mathcal{A}_\pi(t, T)$. A π-admissible strategy β for Player II on $[t, T]$ is defined similarly and the set of all such strategies will be denoted by $\mathcal{B}_\pi(t, T)$.

Let $\bar{W}^{1,\infty}(\mathbb{R}^N \times S)$ denote the space of real-valued functions $\phi(x, a)$ such that for every fixed $a \in S$, $\phi(\cdot, a) \in W^{1,\infty}(\mathbb{R}^N)$. For every $t \in [0, T)$ and $\hat{t} \in (t, T]$, define the operator $F^-_{t,\hat{t}} : \bar{W}^{1,\infty}(\mathbb{R}^N \times S) \to \bar{W}^{1,\infty}(\mathbb{R}^N \times S)$ by

$$\begin{aligned} F^-_{t,\hat{t}}\phi(x, a) = \sup_{u \in U} \inf_{v(\cdot) \in \mathcal{V}(t,\hat{t})} \mathbb{E}_{\mathbb{P}_{t,T}} \Bigg[& \phi(\mathcal{X}^{u,v}_{t,x,a}(\hat{t})) \\ & + \int_t^{\hat{t}} L(s, \mathcal{X}^{u,v}_{t,x,a}(s), u, v(s))\, \mathrm{d}s \Bigg], \end{aligned} \tag{48}$$

where $\mathcal{V}(t, \hat{t})$ denotes the set of admissible controls for Player II on $[t, \hat{t})$ and $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$ is the solution of (5) on $[t, \hat{t})$ associated with the choice of admissible controls $u(\cdot) \equiv u$ and $v(\cdot) \in \mathcal{V}(t, \hat{t})$ having initial condition (x, a) at time t.

Similarly, define the operator $F^+_{t,\hat{t}} : \bar{W}^{1,\infty}(\mathbb{R}^N \times S) \to \bar{W}^{1,\infty}(\mathbb{R}^N \times S)$ as

$$\begin{aligned} F^+_{t,\hat{t}}\phi(x, a) = \inf_{v \in V} \sup_{u(\cdot) \in \mathcal{U}(t,\hat{t})} \mathbb{E}_{\mathbb{P}_{t,T}} \Bigg[& \phi(\mathcal{X}^{u,v}_{t,x,a}(\hat{t})) \\ & + \int_t^{\hat{t}} L(s, \mathcal{X}^{u,v}_{t,x,a}(s), u(s), v)\, \mathrm{d}s \Bigg], \end{aligned}$$

where $\mathcal{U}(t,\hat{t})$ denotes the set of admissible controls for Player I on $[t,\hat{t})$ and $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$ is the solution of (5) on $[t,\hat{t})$ associated with the choice of admissible controls $v(\cdot) \equiv v$ and $u(\cdot) \in \mathcal{U}(t,\hat{t})$ having initial condition (x,a) at time t.

Let $w^-_\pi : [0,T] \times \mathbb{R}^N \times S \to \mathbb{R}$ be such that $w^-_\pi(T,x,a) = \Psi(x,a)$ and

$$w^-_\pi(t,x,a) = F^-_{t,t_{i_0+1}} \prod_{k=i_0+2}^{m} F^-_{t_{k-1},t_k} \Psi(x,a) \tag{49}$$

whenever $t \in [t_{i_0}, t_{i_0+1})$, and similarly, let $w^+_\pi : [0,T] \times \mathbb{R}^N \times S \to \mathbb{R}$ be such that $w^+_\pi(T,x,a) = \Psi(x,a)$ and

$$w^+_\pi(t,x,a) = F^+_{t,t_{i_0+1}} \prod_{k=i_0+2}^{m} F^+_{t_{k-1},t_k} \Psi(x,a) \tag{50}$$

whenever $t \in [t_{i_0}, t_{i_0+1})$. Under Assumptions (A1)-(A5), w^-_π and w^+_π are both well defined and admit the stochastic game characterization described below.

Proposition 3.10. *Let Assumptions (A1)-(A5) hold. For every $(t,x,a) \in [0,T] \times \mathbb{R}^N \times S$, we have*

$$w^-_\pi(t,x,a) = \inf_{\beta \in \mathcal{B}(t,T)} \sup_{u(\cdot) \in \mathcal{U}_\pi(t,T)} J(t,x,a;u(\cdot),\beta[u(\cdot)]) \tag{51}$$

and

$$w^+_\pi(t,x,a) = \sup_{\alpha \in \mathcal{A}(t,T)} \inf_{v(\cdot) \in \mathcal{V}_\pi(t,T)} J(t,x,a;\alpha[v(\cdot)],v(\cdot)) \ . \tag{52}$$

Proof. We will only prove identity (51). The proof of (52) is similar and we skip it. In order to prove (51) we will establish that the following two claims hold:

1) For every $(t,x,a) \in [0,T] \times \mathbb{R}^N \times S$ and every $\epsilon > 0$, there exist $\alpha_\epsilon \in \mathcal{A}_\pi(t,T)$ and $\beta_\epsilon \in \mathcal{B}_\pi(t,T)$ such that

$$J\left(t,x,a;u(\cdot),\beta_\epsilon[u(\cdot)]\right) - \epsilon \le w^-_\pi(t,x,a) \le J\left(t,x,a;\alpha_\epsilon[v(\cdot)],v(\cdot)\right) + \epsilon \tag{53}$$

for all $u(\cdot) \in \mathcal{U}_\pi(t,T)$ and $v(\cdot) \in \mathcal{V}_\pi(t,T)$.

2) For any $\beta \in \mathcal{B}(t,T)$, the pair of strategies $\alpha_\epsilon \in \mathcal{A}_\pi(t,T)$ and $\beta \in \mathcal{B}(t,T)$ define controls $u^\epsilon(\cdot) \in \mathcal{U}_\pi(t,T)$ and $v^\epsilon(\cdot) \in \mathcal{V}(t,T)$ for which

$$J(t,x,a;\alpha_\epsilon[v^\epsilon(\cdot)],v^\epsilon(\cdot)) = J(t,x,a;u^\epsilon(\cdot),\beta[u^\epsilon(\cdot)]) \ . \tag{54}$$

The result follows from these two claims by observing that the left hand side of (53) guarantees that

$$w^-_\pi(t,x,a) \ge \inf_{\beta \in \mathcal{B}(t,T)} \sup_{u(\cdot) \in \mathcal{U}_\pi(t,T)} J(t,x,a;u(\cdot),\beta[u(\cdot)]) \ ,$$

while combining the right hand side of (53) with (54) yields the reverse inequality.

We will establish claim 2) first. For that purpose, it is enough that controls $u^\epsilon(\cdot) \in \mathcal{U}_\pi(t,T)$ and $v^\epsilon(\cdot) \in \mathcal{V}(t,T)$ are constructed in such a way that (54) holds. Let $t \in [t_{i_0}, t_{i_0+1}]$ and $v_0 \in V$. For simplicity of presentation, in what follows we will assume that $t = t_{i_0}$, with the case $t \in (t_{i_0}, t_{i_0+1})$ following similarly after appropriate adjustments are performed. Let $u_{i_0} = \alpha_\epsilon[v_0]$ and

$v_{i_0} = \beta[u_{i_0}]$ and define $v_k \in \mathcal{V}(t,T)$ and $u_k \in \mathcal{U}_\pi(t,T)$, for $k = i_0+1,\dots,m$, through the recursive relation

$$v_k = \beta[u_k] \qquad \text{and} \qquad u_k = \alpha_\epsilon[v_{k-1}] \ .$$

It is enough to check that $u_{k+1} \approx u_k$ and $v_{k+1} \approx v_k$ on $[t_{i_0}, t_k]$ for $k = i_0+1,\dots,m-1$. We resort to induction here. Start by noting that the case $k = i_0+1$ follows from the fact that α_ϵ is independent of the V-valued control on $[t_{i_0}, t_{i_0+1})$. Assume that $u_k \approx u_{k-1}$ and $v_k \approx v_{k-1}$ on $[t_{i_0}, t_{k-1}]$. Since $u_{k+1} = \alpha_\epsilon[v_k]$ and $u_k = \alpha_\epsilon[v_{k-1}]$, using the definition of π-admissible control, $u_{k+1}(t_{k-1}) = u_k(t_{k-1})$, which combined with the fact that u_{k+1} and u_k are constant on $[t_{k-1}, t_k)$, yields $u_{k+1} \approx u_k$ on $[t_{i_0}, t_k]$. Similarly, since $v_k = \beta[u_k]$ and $v_{k+1} = \beta[u_{k+1}]$, we get that $v_{k+1} \approx v_k$ on $[t_{i_0}, t_k]$.

We will now prove claim 1). For $\varphi \in \bar{W}^{1,\infty}(\mathbb{R}^N \times S)$, $(x,a) \in \mathbb{R}^N \times S$, $u \in U$, $t \in [0,T]$ and $\hat{t} \in (t,T]$, define

$$\psi(x,a,u,t,\hat{t},\varphi) = \inf_{v(\cdot)\in\mathcal{V}(t,\hat{t})} \mathbb{E}_{\mathbb{P}_{t,T}} \left[\varphi(\mathcal{X}^{u,v}_{t,x,a}(\hat{t})) + \int_t^{\hat{t}} L(s, \mathcal{X}^{u,v}_{t,x,a}(s), u, v(s))\, \mathrm{d}s \right] ,$$

where $\mathcal{X}^{u,v}_{t,x,a}(\cdot)$ is the solution of (5) under the choice of the admissible controls $u(s) \equiv u$ and $v(\cdot) \in \mathcal{V}(t,\hat{t})$ and initial condition (x,a) at time t. Using Assumptions (A1)-(A5), we obtain that $\psi(\cdot,a,\cdot,t,\hat{t},\varphi) \in W^{1,\infty}(\mathbb{R}^N \times U)$ for every $a \in S$, and

$$F^-_{t,\hat{t}}\varphi(x,a) = \sup_{u\in U} \psi(x,a,u,t,\hat{t},\varphi) \ ,$$

where $F_{t,\hat{t}}$ is the operator defined in (48).

If $t \in [t_{i_0}, t_{i_0+1})$ for $i_0 \in \{0,1,\dots,m-1\}$, let

$$\begin{aligned} \varphi_m &= \Psi(\cdot,\cdot) \\ \varphi_j &= F^-_{t_j,t_{j+1}}\varphi_{j+1} \ , \qquad j = i_0+1,\dots,m-1 \\ \varphi_{i_0} &= F^-_{t,t_{i_0+1}}\varphi_{i_0+1} \ , \end{aligned}$$

to obtain that

$$\varphi_{i_0}(x,a) = w^-_\pi(t,x,a) \ .$$

We partition $\mathbb{R}^N \times S$ and U into Borel sets of diameter less than some positive constant δ, to be determined below. Denote such partitions by $\{A_k : k = 1,2,\dots\}$ and $\{B_\ell : \ell = 1,2,\dots,L\}$, respectively, and pick $(x_k,a_k) \in A_k$ and $u_\ell \in B_\ell$ for each $k = 1,2,\dots$ and $\ell = 1,2,\dots,L$. For any $\gamma > 0$ there exists δ small enough and $u^*_{kj} = u_{\ell(k,j)} \in U$, $k = 1,2,\dots$ and $j = i_0+1,\dots,m$, such that

$$\psi(x_k,a_k,u^*_{kj},t_{j-1},t_j,\varphi_j) > F^-_{t_{j-1},t_j}\varphi_j(x_k,a_k) - \gamma \ . \tag{55}$$

In addition, we choose $v_{kj}^{\ell}(\cdot) \in V(t_{j-1}, t_j)$ such that for $u(\cdot)$ identically equal to $u_\ell \in U$ on the interval $[t_{j-1}, t_j)$ we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{t_{j-1},T}}\Bigg[\varphi(\mathcal{X}^{\ell}_{t_{j-1},x_k,a_k}(t_j)) \\
&\quad + \int_{t_{j-1}}^{t_j} L(s, \mathcal{X}^{\ell}_{t_{j-1},x_k,a_k}(s), u_\ell, v_{kj}^{\ell}(s))\,\mathrm{d}s\Bigg] < \psi\left(x_k, a_k, u_\ell, t_{j-1}, t_j, \varphi_j\right) + \gamma \ ,
\end{aligned}$$

with $t_{i_0} = t$ whenever $j = i_0 + 1$. The notation $\mathcal{X}^{\ell}_{t_{j-1},x_k,a_k}(\cdot)$ stands for the solution of (5) with initial condition (x_k, a_k) at time t_{j-1} subject to the admissible controls $u(\cdot) \equiv u_\ell$ and $v_{kj}^{\ell}(\cdot)$.

We will now exhibit the strategies α_ϵ and β_ϵ in (53). Fix $(t, x, a) \in [0, T) \times \mathbb{R}^N \times S$. For $v(\cdot) \in \mathcal{V}(t, T)$, we define

$$\begin{aligned}
\alpha_\epsilon[v(\cdot)](s) \quad &= \quad I_{[t,t_{i_0+1})}(s) \sum_k u^*_{ki_0} I_{A_k}(x, a) \\
&\quad + \sum_{j=i_0+1}^{m-1} I_{[t_j,t_{j+1})}(s) \sum_k u^*_{kj} I_{A_k}(\mathcal{X}(t_j)) \ ,
\end{aligned}$$

where $\mathcal{X}(\cdot)$ is defined on each of the intervals $[t, t_{i_0+1}]$ and $[t_j, t_{j+1}]$, $j = i_0+1, \ldots, m-1$, as the solution of (5) with $u(\cdot) = \alpha_\epsilon[v(\cdot)]$. For $u(\cdot) \in \mathcal{U}(t, T)$, we define

$$\begin{aligned}
\beta_\epsilon[u(\cdot)](s) \quad &= \quad I_{[t,t_0+1)}(s) \sum_{k,\ell} \hat{v}^{\ell}_{ki_0}(s) I_{A_k}(x, a) I_{B_\ell}(u(s)) \\
&\quad + \sum_{j=i_0+1}^{m-1} \sum_{k,\ell} I_{[t_j,t_{j+1})}(s) \hat{v}^{\ell}_{kj}(s) I_{A_k}(\mathcal{X}(t_j)) I_{B_\ell}(u(s)) \ ,
\end{aligned}$$

where $\mathcal{X}(\cdot)$ is now defined on each of the intervals $[t, t_{i_0+1}]$ and $[t_j, t_{j+1}]$, $j = i_0+1, \ldots, m-1$, as the solution of (5) with $v(\cdot) = \beta_\epsilon[u(\cdot)]$, and $\hat{v}^{\ell}_{kj}(\cdot, \omega) = v^{\ell}_{kj}(\cdot, \omega^{t_j,T})$ using the identification of $\Omega_{t,T}$ with $\Omega_{t,t_j} \times \Omega_{t_j,T}$ provided by $\pi(\omega) = (\omega^{t,t_j}, \omega^{t_j,T})$ discussed in Section 2.1.

Let J stand either for $J(t, x, a; \alpha_\epsilon[v(\cdot)], v(\cdot))$ or $J(t, x, a; u(\cdot), \beta_\epsilon[u(\cdot)])$. For any $v(\cdot) \in \mathcal{V}(t, T)$ and $u(\cdot) = \alpha_\epsilon[v(\cdot)]$ or $u(\cdot) \in \mathcal{U}_\pi(t, T)$ and $v(\cdot) = \beta_\epsilon[u(\cdot)]$, we have

$$
\begin{aligned}
&w_\pi^-(t,x,a) - J \\
&\quad = \varphi_{i_0}(x,a) - \mathbb{E}_{\mathbb{P}_{t,T}}\left[\int_t^T L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s + \varphi_m(\mathcal{X}(T))\right] \\
&\quad = \sum_{j=i_0+1}^{m}\left\{\mathbb{E}_{\mathbb{P}_{t,T}}\left[\varphi_{j-1}\left(\mathcal{X}(t_{j-1})\right)\right] - \mathbb{E}_{\mathbb{P}_{t,T}}\left[\varphi_j(\mathcal{X}(t_j))\right]\right\} \\
&\qquad\qquad - \mathbb{E}_{\mathbb{P}_{t,T}}\left[\int_t^T L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s\right] \\
&\quad = \sum_{j=i_0+1}^{m}\Bigg\{\mathbb{E}_{\mathbb{P}_{t,T}}\left[\varphi_{j-1}(\mathcal{X}(t_{j-1}))\right] - \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\varphi_j(\mathcal{X}(t_j)) \\
&\qquad\qquad + \int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s\Bigg]\Bigg\} \\
&\quad = \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\sum_{j=i_0+1}^{m}\Bigg\{\varphi_{j-1}(\mathcal{X}(t_{j-1})) - \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\varphi_j(\mathcal{X}(t_j)) \qquad (56)\\
&\qquad\qquad + \int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s\,\Big|\mathcal{F}_{t,t_{j-1}}\Bigg]\Bigg\}\Bigg] .
\end{aligned}
$$

Using the identity above, we obtain (53) by checking that the following two statements hold $\mathbb{P}_{t,T}$-a.s.:
(A) For any $v(\cdot)\in\mathcal{V}(t,T)$ and $u(\cdot)=\alpha_\epsilon[v(\cdot)]$, we have that

$$
\begin{aligned}
&\varphi_{j-1}(\mathcal{X}(t_{j-1})) \le \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\varphi_j(\mathcal{X}(t_j)) \\
&\qquad + \int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s\,\Big|\mathcal{F}_{t,t_{j-1}}\Bigg] + \epsilon(t_j - t_{j-1}) .
\end{aligned}
$$

(B) For any $u(\cdot)\in\mathcal{U}_\pi(t,T)$ and $v(\cdot)=\beta_\epsilon[u(\cdot)]$, we have that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{t,T}}\left[\varphi_j(\mathcal{X}(t_j)) + \int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s\,\Big|\mathcal{F}_{t,t_{j-1}}\right] \\
&\qquad \le \varphi_{j-1}(\mathcal{X}(t_{j-1})) + \epsilon(t_j - t_{j-1}) .
\end{aligned}
$$

Using identity (18), the conditional expectations in the two statements (A) and (B) above can be replaced by expectations with respect to $\mathbb{P}_{t_{j-1},T}$. In addition, taking $\hat{t}=t_{j-1}$ in (17) we have that $\mathcal{X}(\omega^{t,t_{j-1}},\cdot)(s)$, $s\in[t_{j-1},T]$, is a solution of the stochastic differential equation (17) for $\mathbb{P}_{t,t_{j-1}}$-a.e. $\omega^{t,t_{j-1}}\in\Omega_{t,t_{j-1}}$. Finally, for $\mathcal{X}(t_{j-1})\in A_k$ and $u(t_{j-1})\in B_\ell$, there exists a positive constant C (depending on the constants of assumption (A3)) such that the following inequalities hold simultaneously:

$$
\begin{aligned}
& d_{N,S}\left(\mathcal{X}(t_{j-1}),(x_k,a_k)\right) < C\delta\ , \\
& \mathbb{E}_{\mathbb{P}_{t_{j-1},T}}\left[d_{N,S}\left(\mathcal{X}(t_j),\mathcal{X}^{\ell}_{t_{j-1},x_k,a_k}(t_j)\right)\right] < C\delta\ , \\
& |\varphi_{j-1}(\mathcal{X}(t_{j-1}))-\varphi_{j-1}(x_k,a_k)| < C\delta\ , \quad \text{and} \\
& \left|\mathbb{E}_{\mathbb{P}_{t_{j-1},T}}[\varphi_j\left(\mathcal{X}(t_j)\right)]-\mathbb{E}_{\mathbb{P}_{t_{j-1},T}}[\varphi_j\left(\mathcal{X}^{\ell}_{t_{j-1},x_k,a_k}(t_j)\right)]\right| < C\delta\ .
\end{aligned}
\tag{57}
$$

Using (55), we obtain that for each positive integer k and each $v(\cdot)\in\mathcal{V}(t_{j-1},T)$, we have

$$
\begin{aligned}
\varphi_{j-1}(x_k,a_k) < \mathbb{E}_{\mathbb{P}_{t_{j-1},T}}\Bigg[&\varphi_j\left(\mathcal{X}^{u^*_{kj},v(\cdot)}_{t_{j-1},x_k,a_k}(t_j)\right) \\
&+\int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}^{u^*_{kj},v(\cdot)}_{t_{j-1},x_k,a_k}(s),u^*_{kj},v(s)\right)\mathrm{d}s\ \Bigg|\mathcal{F}_{t,t_{j-1}}\Bigg]+\gamma\ .
\end{aligned}
\tag{58}
$$

Recalling that any $v(\cdot)\in\mathcal{V}(t,T)$ determines an element of $\mathcal{V}(t_{j-1},T)$ of the form $v(\omega^{t,t_{j-1}},\cdot)|_{[t_{j-1},T]}$, and combining the estimates in (57) and (58) with assumption (A3), we obtain that

$$
\begin{aligned}
&\varphi_{j-1}\left(\mathcal{X}(t_{j-1})\right) \le \varphi_{j-1}(x_k,a_k)+C\delta \\
&\le\ \mathbb{E}_{\mathbb{P}_{t_{j-1},T}}\Bigg[\varphi_j\left(\mathcal{X}^{u^*_{kj},v(\cdot)}_{t_{j-1},x_k,a_k}(t_j)\right) \\
&\qquad +\int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}^{u^*_{kj},v(\cdot)}_{t_{j-1},x_k}(s),u^*_{kj},v(s)\right)\mathrm{d}s\ \Bigg|\mathcal{F}_{t,t_{j-1}}\Bigg]+\gamma+C\delta \\
&\le\ \mathbb{E}_{\mathbb{P}_{t,T}}\Bigg[\varphi_j\left(\mathcal{X}(t_j)\right) \\
&\qquad +\int_{t_{j-1}}^{t_j} L\left(s,\mathcal{X}(s),u(s),v(s)\right)\mathrm{d}s\ \Bigg|\mathcal{F}_{t,t_{j-1}}\Bigg] \\
&\qquad +\gamma+(2C+C'(t_j-t_{j-1}))\delta\ ,
\end{aligned}
$$

where C' depends on the constants of assumption (A3). We conclude that statement (A) holds as long as $\gamma+(2C+C'(t_j-t_{j-1}))\delta<\epsilon(t_j-t_{j-1})$.

Statement (B) follows from a similar argument, completing the proof. □

The lemma below follows as a consequence of Assumptions (A1)-(A5) and the characterizations of w^-_π and w^+_π obtained above.

Lemma 3.11. *There exists a positive constant C, depending solely on Assumptions (A1)-(A5), such that the inequalities*

$$
\begin{aligned}
& |w^{\pm}_\pi(t,x,a)| \le C\ , \quad \textit{and} \\
& |w^{\pm}_\pi(t,x,a)-w^{\pm}_\pi(\hat{t},\hat{x},\hat{a})| \le C\left(d_{N,S}\left((x,a),(\hat{x},\hat{a})\right)+|t-\hat{t}|^{1/2}\right)
\end{aligned}
$$

hold for all $(x,a),(\hat{x},\hat{a})\in\mathbb{R}^N\times S$ and $t,\hat{t}\in[0,T]$.

Combining Lemma 3.11 with the Arzela-Ascoli Theorem, we obtain that the families of functions $\{w^-_\pi\}$ and $\{w^+_\pi\}$ converge uniformly as $\|\pi\|\to 0$ along subsequences to bounded uniformly continuous functions. The result below guarantees that such uniform limits are viscosity solutions of (12) and (13).

Proposition 3.12. *Assume that (A1)-(A5) hold and let w_π^- and w_π^+ be given by* (49) *and* (50)*, respectively. Then the limits*

$$w^- = \lim_{\|\pi\|\to 0} w_\pi^- \qquad \text{and} \qquad w^+ = \lim_{\|\pi\|\to 0} w_\pi^+$$

exist locally uniformly and are the unique viscosity solution of (12) *and* (13)*, respectively.*

Proof. We only prove the statement concerning w^-, with the proof of the corresponding statement for w^+ being similar.

Existence of w^- follows from a comparison theorem (Theorem A.1 in the Appendix) together with Lemma 3.11, provided we guarantee that any subsequential limit of the family $\{w_\pi^-\}$ as $\|\pi\| \to 0$ is a viscosity solution of (12). To see that this is indeed the case, let w^- be a locally uniform limit of a subsequence of the family $\{w_\pi^-\}$. We will only argue that w^- is a viscosity subsolution of (12), with the proof that w^- is also a viscosity supersolution being similar. To achieve such goal, we will use what is now a classical argument [58, 59].

Let ϕ be a smooth test function such that $w^- - \phi$ has a strict local maximum at (t_0, x_0, a_0). We need to show that

$$\phi_t(t_0, x_0, a_0) + \mathcal{H}^-\left(t_0, x_0, a_0, \phi(t_0, \cdot, \cdot), \phi_x(t_0, x_0, a_0), \phi_{xx}(t_0, x_0, a_0)\right) \geq 0 \ . \tag{59}$$

Since $w_\pi^- \to w^-$ locally uniformly as $\|\pi\| \to 0$, there exists a family $\{(t_\pi, x_\pi, a_\pi)\}_\pi$ with the property that $(t_\pi, x_\pi, a_\pi) \to (t_0, x_0, a_0)$ as $\|\pi\| \to 0$ and $w_\pi^- - \phi$ attains a local maximum at (t_π, x_π, a_π) which, without loss of generality, may be assumed to be global. Using the definition of w_π^- in (49), if $t_\pi \in [t_{i_0}^\pi, t_{i_0+1}^\pi)$, we have that

$$w_\pi^-(t_\pi, x_\pi, a_\pi) = F^-_{t_\pi, t^\pi_{i_0+1}}\, w_\pi^-(t^\pi_{i_0+1}, \cdot, \cdot)(x_\pi, a_\pi) \ .$$

Hence, we conclude that

$$\phi(t_\pi, x_\pi, a_\pi) \leq F^-_{t_\pi, t^\pi_{i_0+1}}\, \phi(t^\pi_{i_0+1}, \cdot, \cdot)(x_\pi, a_\pi) \ . \tag{60}$$

Using Itô-Lévy's formula, for any smooth test function ϕ we have that

$$\lim_{s\downarrow t} \frac{F^-_{t,s}\phi(\cdot,\cdot) - \phi(\cdot,\cdot)}{s-t} = \mathcal{H}^-(t, \cdot, \cdot, \phi(t, \cdot, \cdot), \phi_x, \phi_{xx}) \ . \tag{61}$$

The proof is completed by noticing that (59) follows as a consequence of (60) and (61). □

3.5. **Completing the proofs of Theorems 2.1 and 2.2.** We will now combine the results obtained so far to complete the characterization of the value functions V^- and V^+ as the unique viscosity solutions of (12) and (13), respectively, leading us, also, to the proof of the Dynamic Programming Principle of Theorem 2.1.

The first step is based on the observation that since the limit functions w^- and w^+ of Proposition 3.12 are, respectively, the unique viscosity solutions of (12) and (13), combining the Comparison theorem A.1 (in the Appendix) with Proposition 3.9 yields the inequalities in following lemma.

Lemma 3.13. *Assume that (A1)-(A5) hold. For every* $(t, x, a) \in [0, T] \times \mathbb{R}^N \times S$ *we have that*

$$V_r^-(t, x, a) \leq w^-(t, x, a) \qquad \text{and} \qquad V_r^+(t, x, a) \geq w^+(t, x, a) \ .$$

The next step consists in showing that $V^-(t, x, a) \geq w^-(t, x, a)$ and that $V^+(t, x, a) \leq w^+(t, x, a)$ for every $(t, x, a) \in [0, T] \times \mathbb{R}^N \times S$. As a consequence, we complete the proof of Theorem 2.2, which states that the lower and upper value functions V^- and V^+ are the unique viscosity solutions of (12) and (13), respectively.

Proof of Theorem 2.2. We will focus on proving the part of the statement concerning V^-, with a similar proof holding for the corresponding statement concerning V^+. Putting together Corollary 3.5 and Lemma 3.13, we obtain that $V^- \leq V_r^- \leq w^-$ on $[0, T] \times \mathbb{R}^N \times S$. On the other hand, combining identities (9) and (51), we obtain that for every partition π of $[0, T]$, the inequality $w_\pi^- \leq V^-$ holds on $[0, T] \times \mathbb{R}^N \times S$. In addition, Proposition 3.12 implies that $w^- \leq V^-$ on $[0, T] \times \mathbb{R}^N \times S$, guaranteeing that $w^- = V^-$ on $[0, T] \times \mathbb{R}^N \times S$. □

Proof of Corollary 2.3. If the Isaacs condition holds, then the HJBI equations (12) and (13) coincide. Hence, uniqueness of the viscosity solutions, as guaranteed by the Comparison theorem A.1, ensures that V^- and V^+ are identical. □

We are finally ready to complete the proof of the dynamic programming principle of Theorem 2.1.

Proof of Theorem 2.1. We will prove that the lower value function of the SDG determined by (5) and (6) satisfies identity (9). The corresponding proof for the upper value function is similar and we omit it here.

Let $\hat{t} \in (0, T]$ be fixed and let $\overline{V}(t, x, a)$ denote the right hand side of (9). It is enough to consider in (9) controls $u(\cdot)$ and strategies β defined in $[t, \hat{t}]$. By Theorem 2.2, we have that $\overline{V}$ is the viscosity solution of (12) on $[0, \hat{t}] \times \mathbb{R}^N \times S$ with $\overline{V}(\hat{t}, x, a) = V^-(\hat{t}, x, a)$. Uniqueness of viscosity solutions yields that $\overline{V} = V^-$. □

4. Conclusions

In this paper, we have analyzed a two-player zero-sum stochastic differential game with Markov-switching jump-diffusion state variable dynamics. Employing dynamic programming and viscosity solution techniques, we have rigorously established the existence of value for the problem under consideration and demonstrated that it satisfies a nonlinear partial integro-differential HJBI equation as its unique viscosity solution. These results contribute to the theory of stochastic differential games by extending more classical formulations to a broader setting, thereby allowing for the modeling of a wider range of real-world phenomena.

Combining Markov-switching dynamics and jump-diffusion processes into a single framework is particularly relevant in applications whose system behavior is influenced by both abrupt random events and underlying structural changes. Such models arise naturally in areas such as financial markets, where asset prices are subject to sudden jumps and macroeconomic regime shifts; in energy markets, where electricity prices and demand fluctuate based on external shocks and policy shifts; and in engineering and control systems, where operational modes switch due to external interventions or failures.

Our results provide a theoretical foundation for analyzing optimal strategies within such elaborate stochastic settings, offering a framework for decision-making under uncertainty when multiple sources of randomness are present. Future research directions could explore extensions to non-zero-sum games, learning-based approaches for estimating transition dynamics, and numerical methods for solving the associated HJBI equations in high-dimensional settings.

Appendix A. Comparison theorem for viscosity solutions

We will now state the comparison theorem for non-local equations that plays a key role in establishing the proof of Proposition 3.12 and Lemma 3.13, both of which lead to the proof of the dynamic programming principle in Theorem 2.1. The statement provided below can be obtained from the comparison principle due to Pham [50, Theorem 4.1] by taking into account the stochastic integral representation for the Markov process $\mu^t(\cdot)$, described in Section 2.1 and given in (1), and the associated embedding of its state space S into $\mathbb{R}^n$.

Before proceeding to the statement of the comparison theorem, we need to introduce additional notation. Namely, we will denote by $UC_x\left([0,T]\times\mathbb{R}^N\times S\right)$ the set of continuous functions in $[0,T]\times\mathbb{R}^N\times S$, uniformly continuous in x, uniformly in $(t,a)\in[0,T]\times S$.

Theorem A.1 (Comparison Theorem)**.** *Assume that (A1)-(A5) hold. If $v,\tilde{v}\in UC_x\left([0,T]\times\mathbb{R}^N\times S\right)$ (resp. $u,\tilde{u}\in UC_x\left([0,T]\times\mathbb{R}^N\times S\right)$) are bounded viscosity sub- and super-solutions of* (12) *(resp.* (13)*) with boundary conditions Ψ and $\tilde{\Psi}$ and if $\Psi\le\tilde{\Psi}$ on $\mathbb{R}^N\times S$, then $v\le\tilde{v}$ (resp. $u\le\tilde{u}$) on $[0,T]\times\mathbb{R}^N\times S$.*

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