OPTIMAL CONSUMPTION, INVESTMENT AND LIFE-INSURANCE PURCHASE UNDER A STOCHASTICALLY FLUCTUATING ECONOMY

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Abstract. We study the optimal consumption, investment and life-insurance purchase and selection strategies for a wage-earner with an uncertain lifetime with access to a financial market comprised of one risk-free security and one risky-asset whose prices evolve according to linear diffusions modulated by a continuous-time stochastic process determined by an additional diffusive nonlinear stochastic differential equation. The process modulating the linear diffusions may be regarded as an indicator describing the state of the economy in a given instant of time. Additionally, we allow the Brownian motions driving each of these equations to be correlated. The life-insurance market under consideration herein consists of a fixed number of providers offering pairwise distinct contracts. We use dynamic programming techniques to characterize the solutions to the problem described above for a general family of utility functions, studying the case of discounted constant relative risk aversion utilities with more detail.

Keywords: Uncertain lifetime; Life-insurance purchase and selection; Stochastic optimal control.
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1. Introduction

We consider the problem of optimal consumption, investment and life-insurance selection and purchase subject to an underlying financial market whose asset prices evolve according to a linear SDE with coefficients depending on an additional real-valued continuous-time stochastic process reflecting the state of the overall economy. Such process may be regarded as an index aggregating economic information such as the GDP growth rate, interest rates, inflation rates, unemployment rate, government debt to GDP, trade balance, prices of natural resources and commodities, among many other pieces of relevant information. Given the regular flow of such economic information available to financial market observers nowadays – multiple pieces of information published daily – as well as the market agents' expectations towards the future evolution of such pieces of information, it seems only natural to model the time-evolution of such a process by a stochastic differential equation of diffusive type. Our goal is to better understand the choices of a rational wage-earner, who is simultaneously a saver and investor, with access to both a financial market such as described above and an insurance market composed by multiple competing insurance companies. The wage-earner aims to maximize a given expected utility which encodes his preferences regarding consumption, wealth and protection against untimely death as provided by life-insurance.

The class of problems under consideration here has received substantial attention by the scientific community. Such attention dates back to the initial work of Yarri [44], who introduced an optimal consumption problem for an individual with uncertain time of death within the setup of a pure deterministic investment environment, Hakansson [14, 15], who extended Yaari's model to a discrete-time setup with stochastic risky-assets, Merton [27, 28], who studied a continuous-time optimal consumption and investment problem with no life-insurance purchase component, and Richard [33], who combined the earlier approaches to obtain a continuous-time model for optimal consumption, investment and life-insurance purchase. More recently, Pliska and Ye [32, 45] considered the problem faced by a wage-earner with an uncertain lifetime having to reach decisions concerning consumption, investment and life-insurance purchase. Blanchet-Scalliet et al. [7], on the other hand, considered the problem of optimal portfolio selection with an uncertain exit time without considering any kind of life-insurance purchase. Other recent and interesting studies on the subject of optimal life-insurance purchase include [8, 9, 13, 16, 17, 20, 21, 22, 29, 31].

Financial markets with random coefficients such as the one considered herein have also received plentiful attention recently. Liu [24] considers the problem of dynamic portfolio selection in stochastic enviroments when the asset returns are quadratic and the agent has a constant relative risk aversion. Therein, Liu defines asset returns to be *quadratic* when all four characteristics of their dynamics (the short rate, the maximal squared-Sharpe ratio, the hedging covariance vector, and the unspanned covariance matrix) are quadratic functions of what Liu refers to as a quadratic process – a Markovian diffusion process whose drift and diffusion coefficients are quadratic functions of such process themselves. We also mention Souza and Zubelli [40], who used the hypothesis of fast mean-reversion in an investment decision problem to show that stochastic volatility can alter the optimal time investment curve, and Fouque et al. [12], who studied the Merton portfolio optimization problem with stochastic volatility relying on asymptotic approximations when the volatility process is characterized by its timescales of fluctuation. Bichuch and Sircar [6, 5] considered Merton problems with stochastic volatility and transaction costs, while Lorig and Sircar [25] considered a finite horizon Merton portfolio optimization problem in the presence of stochastic volatility and derived series approximations for both the value function and the optimal investment strategy. Finally, back within the setup of optimal life-insurance purchase, Liang and Guo [23] considered the optimal life insurance problem with an underlying incomplete market, for which the stock price has mean-reverting drift, while Shen and Wei [38] treated such problem with an underlying complete financial market for which parameters governing the market model and the wage earner income are random processes adapted to the Brownian motion filtration.

Before proceeding, we should mention alternative interesting formulations for the problem under consideration herein. These may include different forms of coefficient randomness such as, for instance, Markov-switching state variable dynamics as in Azevedo et al. [2] and Temoçin and Weber [41] or semi-Markov modulated state variable dynamics as in Azevedo et al. [1], as well as the presence of terms with delay as considered by Savku and coauthors [35, 36, 37]. In what concerns the form of the objective functional, one could consider the case of robust utilities as done by Baltas and coauthors [4, 3] and by Uğurlu [42, 43] in lieu of the expected utilities treated here.

At this point, we should stress some of the innovative aspects introduced in this paper:

(i) To the best of our knowledge, this is the first time the dependence of life-insurance purchase on an external factor, which may be regarded as the overall state of the economy, is considered and comprehensively analyzed. We perform such analysis by a variety of means. Firstly, we derive feedback controls depending on the optimal control problem value function and its derivatives for general families of utility functions. Secondly, we find explicit closed-form formulas for both the optimal controls and the value function in the case of discounted constant relative risk aversion utilities and "frozen economy dynamics". This allow us to perform a rather detailed analysis of the parametric dependence of both the value function and the optimal controls with respect to the variable representing the state of the economy. Finally, considering again discounted constant relative risk aversion utilities, we perform an illustrative numerical experiment under the assumption that the economy dynamics are given by an Ornstein-Uhlenbeck process, well known to be a mean-reverting Gaussian and Markov process.

(ii) We consider a more general class of utility functions than used in the current literature to describe the wage-earner preferences concerning the eventual life-insurance payment to his family in case he dies before retirement. The common choice in the literature is to evaluate the wage-earner's utility regarding the bequest to his estate at a value equal to his full wealth at time of death plus any payment arising from life-insurance contracts. However, we feel that such choice might have a drastic influence on what concerns decisions regarding life-insurance purchase, mainly due to the debasing of the life-insurance payment caused by the wealth term in the full bequest size. Instead, we allow for the utility describing the life-insurance payment preferences to be evaluated at a quantity of the form $z-\chi(t,x)$, where z denotes the total life-insurance payment to the wage-earner estate in case of premature death, and $\chi(t,x)$ denotes some relevant benchmark value. As noted above, the current choice in the literature corresponds to taking $\chi(t,x)=-x$, i.e. the wage-earner chooses how much life-insurance his family would need in addition to the accumulated wealth at the time of his eventual death. On one hand, we feel that this choice somewhat debases life-insurance in comparison with consumption. On the other hand, it is widely accepted that a large portion of the population reaches retirement age with an inadequately small amount of savings. Among such individuals, the life-insurance payout is much more of a replacement to the loss of income due to a premature death of the wage-earner, than it is a complement to the accumulated wealth at the time of the eventual death. Henceforth, we believe it is highly relevant to consider and study also the case where $\chi(t, x) = 0$ for all time t and wealth value x. Such choice places life-insurance purchase in a role akin to consumption, a point of view we believe might be appropriate to model the decision-making process of wage-earners with low to moderate levels of wealth. Other reasonable choices are also included in our analysis. For instance, one could also take $\chi(t, x)$ to be the time t present value of the wage-earner future income up until retirement. In such case, the utility of the amount $z - \chi(t, x)$ would be measuring how large the life-insurance payment z should be to compensate the family for the loss of this future income. More generally, the benchmark $\chi(t, x)$ may be regarded as a value of wealth that, ideally, ought to be exceeded by the life-insurance payment if the wage-earner happens to die during the term of the corresponding contract. Such modeling point of view should yield positive optimal life-insurance purchase rates for non-negative $\chi(t, x)$, unlike the standard benchmark $\chi(t, x) = -x$, known to yield negative optimal life-insurance purchase rates for certain choices of time t and wealth x. Finally, it should be added that, for increasing and concave utility functions evaluated at $z - \chi(t, x)$, a larger benchmark $\chi(t, x)$ decreases the wage-earner risk-aversion regarding life-insurance purchase, thus increasing the corresponding purchase rate.

This article is organized as follows. In Section 2 we introduce the underlying financial and insurance markets and formulate the problem we wish to address within the framework of a stochastic optimal control problem with a random horizon. We then proceed to restate such problem as one with a fixed planning horizon, providing the corresponding dynamic programming principle and Hamilton-Jacobi-Bellman (HJB) equation. Section 3 is devoted to a characterization of the optimal strategies associated with general utility functions. Section 4 contains a discussion of the optimal strategies in the special case of discounted constant relative risk aversion utility functions. We conclude in Section 5.

2. Problem formulation

In this section, we will introduce the underlying financial market available to the wage-earner, as well as the setup describing the insurance contracts under consideration herein. We will then formulate the problem faced by a wage-earner with an uncertain lifetime seeking to optimize his decisions regarding consumption, investment and life-insurance purchase and selection.

2.1. **The financial market model.**

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, where each sub-$\sigma$-algebra $\mathcal{F}_t$ represents the information available to any given agent observing the financial market during the time interval $[0, t]$. On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, define two one-dimensional standard Brownian motions, $B_1(\cdot)$ and $B_2(\cdot)$, with correlation $\rho \in (-1,1)$.

Consider a financial market consisting of one risk-free asset and one risky-asset, where the latter may be regarded as an index representative of the overall risky-assets performance. Their respective prices, $(S_0(t))_{t\in[0,T]}$ and $(S_1(t))_{t\in[0,T]}$, evolve according to the following stochastic differential equations:

$$\begin{aligned}
\mathrm{d}S_0(t) &= r(t,Y(t))S_0(t)\mathrm{d}t\ , & S_0(0) &= s_0 \\
\mathrm{d}S_1(t) &= S_1(t)\Big(\mu(t,Y(t))\mathrm{d}t + \sigma(t,Y(t))\mathrm{d}B_1(t)\Big), & S_1(0) &= s_1 > 0\ , \qquad (1)
\end{aligned}$$

where $r(t,y)$ is the riskless interest rate, $\mu(t,y)$ is the risky-asset appreciation rate, and $\sigma(t,y)$ is the risky-asset volatility, all of which are assumed to depend on the time t and on the state of some auxiliary stochastic process $Y(\cdot)$. Such process $Y(\cdot)$ can be regarded as an economic indicator, with evolution determined by

$$\mathrm{d}Y(t) = \alpha(t,Y(t))\mathrm{d}t + \beta(t,Y(t))\mathrm{d}B_2(t)\ , \qquad Y(0) = y_0\ . \tag{2}$$

The following assumption ensures the existence and uniqueness of solutions to the SDE (2), and thus, well-posedeness of the process $Y(\cdot)$. Please see any of the monographs [19, 26] for further information concerning the theory of SDEs, as well as [46] for its connections with the theory of stochastic optimal control.

Assumption 1. *The functions $\alpha, \beta : [0,T]\times\mathbb{R} \to \mathbb{R}$ are uniformly continuous with respect to both its variables, Lipschitz continuous with respect to the variable $y \in \mathbb{R}$, and bounded with respect to $t \in [0,T]$ when restricted to $y = 0$.*

It is well known that the two correlated Brownian motions $B_1(\cdot)$ and $B_2(\cdot)$ can be written in terms of two independent standard Brownian motions. For that purpose, let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent one-dimensional Brownian motions and set

$$\begin{aligned}
B_1 &= \rho W_1 + \sqrt{1-\rho^2}W_2\ , \\
B_2 &= W_2\ .
\end{aligned}$$

Then, the system of SDEs (1)-(2) may be written as

$$\begin{aligned}
\mathrm{d}S_0(t) &= r(t,Y(t))S_0(t)\mathrm{d}t \\
\mathrm{d}S_1(t) &= S_1(t)\left(\mu(t,Y(t))\mathrm{d}t + \sigma(t,Y(t))\left(\rho\mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)\right)\right) \\
\mathrm{d}Y(t) &= \alpha(t,Y(t))\mathrm{d}t + \beta(t,Y(t))\mathrm{d}W_2(t)\ ,
\end{aligned}$$

where $\rho \in (-1,1)$ denotes the correlation of the original Brownian motions $B_1(\cdot)$ and $B_2(\cdot)$.

The following assumption ensures market viability.

Assumption 2. *The functions $r(t,y)$, $\mu(t,y)$ and $\sigma(t,y)$ are deterministic continuous real-valued functions on $[0,T]\times\mathbb{R}$. Additionally, the following conditions hold:*

(i) *the interest rate $r(t,y)$ is positive for all $(t,y) \in [0,T]\times\mathbb{R}$;*

(ii) *there exist positive real numbers* $\underline{\sigma} < \bar{\sigma}$ *such that* $\underline{\sigma} < \sigma(t, y) < \bar{\sigma}$ *for all* $(t, y) \in [0, T] \times \mathbb{R}$*;*

(iii) *there exist* $\mathbb{F}$*-progressively measurable real-valued processes* $\xi_1(\cdot), \xi_2(\cdot)$ *such that*

$$\mu(t, Y(t)) - r(t, Y(t)) = \sigma(t, Y(t)) \left(\rho\xi_1(t) + \sqrt{1 - \rho^2}\ \xi_2(t)\right) , \quad 0 \leq t \leq T \ \textit{a.s.}$$

and the following two conditions hold

$$\int_0^T \xi_1^2(t)\ \mathrm{d}t < \infty \quad \textit{a.s.} \qquad \textit{and} \qquad \int_0^T \xi_2^2(t)\ \mathrm{d}t < \infty \quad \textit{a.s.}\ .$$

The existence of the processes $\xi_1(t)$ and $\xi_2(t)$, regarded as market price of risk for the sources of uncertainty $W_1(\cdot)$ and $W_2(\cdot)$, ensures the absence of arbitrage opportunities in the financial market defined above. Please see [18] for further details on market viability and [34] and references therein for the specific case of financial markets with stochastic volatility.

2.2. **The life-insurance market model.** The wage-earner is assumed to be alive at time $t = 0$ and to have a lifetime determined by a non-negative continuous random variable τ defined on the probability space $(\Omega, \mathcal{F}, P)$.

Assumption 3. *The random variable* τ *is independent of the filtration* $\mathbb{F}$ *and has distribution function* $G^- : [0, \infty) \to [0, 1]$ *with a bounded Lipschitz continuous density* $g : [0, \infty) \to \mathbb{R}^+$ *such that*

$$G^-(t) = P(\tau \leq t) = \int_0^t g(s)\ \mathrm{d}s\ .$$

The *survival function* $G^+ : [0, \infty) \to [0, 1]$ is defined as the probability for the wage-earner to survive past time t, i.e.

$$G^+(t) = P(\tau > t) = 1 - G^-(t)\ .$$

We will also use the *hazard rate function* $\lambda : [0, \infty) \to \mathbb{R}^+$, that is, the conditional, instantaneous death rate for the wage-earner surviving past time t, given by

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(t < \tau \leq t + \delta t \mid \tau > t)}{\delta t} = \frac{g(t)}{G^+(t)}\ .$$

For every $0 \leq t \leq s$, let $G^+(s; t)$ denote the conditional probability for the wage-earner to be alive at time s conditional upon being alive at time $t \leq s$, given by

$$G^+(s; t) = P\Big(\{\tau > s\} \mid \{\tau > t\}\Big) \tag{3}$$

and let $G^-(s; t)$ denote the conditional probability for the wage-earner time of death to occur before time s conditional upon being alive at time $t \leq s$, given by

$$G^-(s; t) = P\Big(\{\tau \leq s\} \mid \{\tau > t\}\Big).$$

Finally, let $g^-(s; t)$ denote the density function associated with the conditional distribution function $G^-(s; t)$, given by

$$g^-(s; t) = \frac{\mathrm{d}}{\mathrm{d}s}\ G^-(s; t)\ . \tag{4}$$

We remark that

$$g^{-}(t;t) = \lambda(t) \ .$$

The life-insurance market under consideration here is composed by K insurance companies, with each insurance company continuously offering life-insurance contracts. The wage-earner buys life-insurance from insurance company k by paying a *premium insurance rate* $p_k(t)$ for each $k = 1, 2, \ldots, K$. The insurance contracts are like term insurance, but with an infinitesimally small term. If the wage-earner dies at time $\tau \leq T$ while buying insurance at the rate $p_k(t)$ from the k^{th} insurance company, then that insurance company pays an amount

$$Z_k(\tau) = \frac{p_k(\tau)}{\eta_k(\tau)}$$

to his estate, where $\eta_k : [0, T] \to \mathbb{R}^+$ is the k^{th} *insurance company premium-payout ratio.*

We shall represent the wage-earner life-insurance purchase rate as a vector

$$p(t) = (p_1(t), p_2(t), \ldots, p_K(t))^T \in (\mathbb{R}_0^+)^K \ ,$$

where, for $k \in \{1, 2, \ldots, K\}$, the quantity $p_k(t)$ denotes the life-insurance purchase rate from the k^{th} insurance company at time $t \in [0, \min\{\tau, T\}]$. A zero component in $p(t)$ represents the absence of any life-insurance contract between the wage-earner and a certain insurance company.

Assumption 4. *For every $k \in \{1, \ldots, K\}$, the k^{th} insurance company premium-payout ratio $\eta_k(t)$ is a continuous and deterministic function. Additionally, we will assume that the K insurance companies under consideration here offer pairwise distinct contracts in the sense that $\eta_{k_1}(t) \neq \eta_{k_2}(t)$ for every $k_1 \neq k_2$ and Lebesgue-almost-every $t \in [0, T]$.*

As a consequence of Assumption 4 above, we have that the $K \times K$ symmetric matrix $\eta(t)^T \eta(t)$, where $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_K(t))^T \in (\mathbb{R}^+)^K$, is non-singular for Lebesgue almost-every $t \in [0, T]$.

The contract ends when the wage-earner dies or achieves retirement age, whichever happens first. Therefore, in the event of premature death at time $\tau \leq T$, the wage-earner's estate receives an amount $Z(\tau)$ given by

$$Z(\tau) = \sum_{k=1}^{K} Z_k(\tau) = \sum_{k=1}^{K} \frac{p_k(\tau)}{\eta_k(\tau)} \ . \tag{5}$$

2.3. **The wealth process.** The wage-earner receives income $i(t, y)$ at a continuous rate during the period $[0, \min\{\tau, T\}]$, i.e. the income will be terminated either by his death or his retirement, whichever happens first.

Assumption 5. *The income function $i : [0, T] \times \mathbb{R} \to \mathbb{R}_0^+$ is a deterministic Borel-measurable function satisfying the integrability condition*

$$\int_0^T i(t, y) \ \mathrm{d}t < \infty$$

for every $y \in \mathbb{R}$.

The *consumption process* $(c(t))_{0\leq t\leq T}$ is a $\mathbb{F}$-progressively measurable non-negative process satisfying the following integrability condition for the investment horizon $T > 0$:

$$\int_0^T c(t)\ \mathrm{d}t < \infty \qquad \text{a.s. .}$$

We assume also that for all $k = 1, 2, \ldots, K$, the k^{th} *company premium insurance rate* $(p_k(t))_{0\leq t\leq T}$ is a non-negative $\mathbb{F}$-predictable process, i.e. $p_k(t)$ is measurable with respect to the smallest σ-algebra on $\mathbb{R}_0^+ \times \Omega$ such that all left-continuous and adapted processes are measurable.

For each $t \in [0, T]$, let $\theta(t)$ denote the fraction of the wage-earner's wealth allocated to the risky-asset S_1 at time t. The wage-earner *portfolio process* is then given by $\Theta(t) = (1-\theta(t), \theta(t))^T$. We assume that the portfolio process is $\mathbb{F}$-progressively measurable and that for the fixed investment horizon $T > 0$ we have that

$$\int_0^T \|\Theta(t)\|^2\ \mathrm{d}t < \infty \qquad \text{a.s. ,}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

The *wealth process* $X(t)$, $t \in [0, \min\{\tau, T\}]$, is then defined by

$$\begin{aligned} X(t) &= x + \int_0^t \left(i(s, Y(s)) - c(s) - \sum_{k=1}^K p_k(s) \right) \mathrm{d}s \\ &\quad + \int_0^t \frac{(1-\theta(s))\, X(s)}{S_0(s)}\ \mathrm{d}S_0\,(s) + \int_0^t \frac{\theta(s) X(s)}{S_1(s)}\ \mathrm{d}S_1\,(s)\ , \end{aligned}$$

where x is the wage-earner's initial wealth and one should keep in mind the additional dependence of the wealth process $X(\cdot)$ on the economic indicator $Y(\cdot)$ via the financial asset prices $S_0(\cdot)$ and $S_1(\cdot)$. This last equation, when combined with (2), may be rewritten in differential form as

$$\begin{aligned} \mathrm{d}X(t) &= \Bigg(i(t, Y(t)) - c(t) - \sum_{k=1}^K p_k(t) \\ &\qquad + \left(r(t, Y(t)) + \theta(t) \left(\mu(t, Y(t)) - r(t, Y(t))\right)\right) X(t) \Bigg) \mathrm{d}t \\ &\quad + \theta(t)\sigma(t, Y(t))X(t) \left(\rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t) \right), \qquad (6) \\ \mathrm{d}Y(t) &= \alpha(t, Y(t))\mathrm{d}t + \beta(t, Y(t))\mathrm{d}W_2(t)\ , \end{aligned}$$

where $0 \leq t \leq \min\{\tau, T\}$.

Finally, we describe the class of constraints on the portfolio process $\Theta(\cdot)$. For that purpose, we introduce an interval-valued function, which we denote as I_θ, assigning to the wage-earner level of wealth x an interval of real numbers $I_\theta(x) \subseteq \mathbb{R}$ to which the fraction θ of the wage-earner's wealth allocated to the risky-asset must belong. Hence, given the wage-earner wealth $X(t)$ at time t, the interval-valued function I_θ determines the constraint $\theta(t) \in I_\theta(X(t))$ as well as the respective constraint on the portfolio process $\Theta(t) = (1-\theta(t), \theta(t))^T$ at time t. To be more concrete, we mention some possible simple portfolio constraints (or lack thereof) within the setup just described. First, the absence of any sort of constraint corresponds to the choice where $I_\theta(x) = \mathbb{R}$ for every $x \in \mathbb{R}$, that is, the fraction of the wage-earner's wealth allocated to the risky-asset can take any real value (where we are abusing language while referring to θ as a fraction). A second simple choice, corresponding to the case where the wage-earner is not allowed to take short positions on any of the two underlying financial assets, corresponds to setting $I_\theta(x) = [0, 1]$ for every $x \in \mathbb{R}$. Finally, the requirement that θ is constrained to be zero whenever the wage-earner's wealth is negative and unconstrained whenever the wage-earner's wealth is nonnegative, corresponds to the choice

$$I_\theta(x) = \begin{cases} \mathbb{R} & x \geq 0 \\ \{0\} & x < 0 \end{cases},$$

which models the case where a wage-earner in debt does not take any long position on the risky asset, paying the risk-free interest rate on the accumulated debt.

2.4. **Utility functions and admissible strategies.** The wage-earner is faced with the problem of finding strategies that jointly maximize the expected utility obtained from: his family consumption for all $t \leq \min\{\tau, T\}$; his wealth at retirement date T if he lives that long; and the life-insurance payment to his estate in the event of premature death.

This problem can be formulated by means of optimal control theory. The wage-earner goal is to maximize some cost functional subject to the (stochastic) dynamics of the state variables, i.e. the dynamics of the wealth process $X(t)$ and the economic indicator $Y(t)$ given by (6); constraints on the control variables, i.e. the consumption process $c(t)$, the premium insurance rates $p(t)$ and the portfolio process $\theta(t)$; and boundary conditions on the state variables.

We will assume that the utility functions U, B and W describing the wage-earner's preferences regarding, respectively, his family consumption level, the life-insurance payout in the event of premature death, and his wealth at retirement time, satisfy the following conditions.

Assumption 6. *The following conditions hold for the utility functions U, B and W:*

a) $U : D(U) \subseteq [0, T] \times \mathbb{R}_0^+ \to \mathbb{R}$ *is such that for every $t \in [0, T]$ the function $U(t, \cdot)$ is twice differentiable, strictly increasing, strictly concave, and its first derivative maps $\mathbb{R}^+$ onto $\mathbb{R}^+$. Additionally, both U and the partial derivative of U with respect to its second variable are continuous functions of $t \in [0, T]$.*
b) $B : D(B) \subseteq [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is such that for every $(t, x) \in [0, T] \times \mathbb{R}$ there exists an interval $I(t, x)$ of the form $(a(t, x), +\infty)$ on which $B(t, \cdot, x)$ is twice differentiable, strictly increasing, strictly concave, and its first derivative maps $I(t, x)$ onto $\mathbb{R}^+$. Additionally, both B and the partial derivative of B with respect to its second variable are continuous functions of $(t, x) \in [0, T] \times \mathbb{R}$.*
c) $W : \mathbb{R} \to \mathbb{R}$ *is a twice differentiable, strictly increasing and strictly concave function.*

In Section 4 we will specialize our analysis to the case where the wage-earner's preferences are described by discounted CRRA utility functions.

We now recall the notion of admissible control in use here. Please see the monographs [30, 46] for further information.

Admissible control Given $t \in [0,T]$, we say that the three-tuple $\nu = (c(s), \theta(s), p(s))_{s\in[t,T]}$ is an admissible control, if:

i) ν is $(\mathcal{F}_s)_{s\in[t,T]}$-adapted and the following integrability conditions hold:

$$\int_t^T c(s)\, \mathrm{d}s < \infty \ , \quad \int_t^T \theta^2(s)\, \mathrm{d}s < \infty \ , \quad \int_t^T p_k(s)\, \mathrm{d}s < \infty \ , \quad k = 1, \dots K \ ;$$

ii) the SDE (6) subject to the boundary conditions $X(t) = x$ and $Y(t) = y$ has a unique solution $\left(X^\nu_{t,x,y}(\cdot), Y_{t,y}(\cdot)\right)$;

iii) the following integrability conditions hold:

$$\mathbb{E}\left[\int_t^T |U(s, c(s))|\ \mathrm{d}s\right] < \infty \ , \qquad \mathbb{E}\left[\left|W(X^\nu_{t,x,y}(T))\right|\right] < \infty \ ,$$
$$\mathbb{E}\left[\int_t^T \left|B\left(s, Z^\nu_{t,x,y}(s), X^\nu_{t,x,y}(s)\right)\right|\ \mathrm{d}s\right] < \infty \ ,$$

where $Z^\nu_{t,x,y}(\cdot)$ is as given in (5) under the choice of controls $\nu = (c(s), \theta(s), p(s))_{s\in[t,T]}$ subject to the boundary conditions $X(t) = x$ and $Y(t) = y$.

We will denote by $\mathcal{A}[t,T]$ the set of admissible controls defined on $[t,T]$.

2.5. **Main optimal control problem.** The wage-earner's problem can now be stated as follows: find $\nu \in \mathcal{A}[0,T]$ which maximizes the expected utility

$$\mathbb{E}\Bigg[\int_0^{\tau\wedge T} U(s, c(s))\ \mathrm{d}s + B(\tau, Z^\nu_{0,x,y}(\tau), X^\nu_{0,x,y}(\tau)) I_{[0,T]}(\tau) \\ + W(X^\nu_{0,x,y}(T)) I_{(T,+\infty)}(\tau)\Bigg] \ , \quad (7)$$

where the random time horizon is of the form $\tau \wedge T = \min\{\tau, T\}$, $I_A(\cdot)$ denotes the indicator function of the set A, $U(t, \cdot)$ is the utility function describing the wage-earner's family preferences regarding the consumption level c at a certain time $t \in [0,T]$, $W(\cdot)$ is the utility function for the terminal wealth $X(T)$ at retirement time T, and $B(\tau, \cdot, \cdot)$ is the utility function associated with the size of the life-insurance payout $Z(\tau)$ in the event of premature death, at time $\tau \leq T$, of an agent with wealth $X(\tau)$.

2.6. **The dynamic programming principle.** We use the techniques introduced in [32, 45] to restate the stochastic optimal control problem formulated above as one with a fixed planning horizon. We then state a dynamic programming principle and the corresponding HJB equation for the resulting optimal control problem with deterministic horizon.

Recall the definition of the set of admissible controls $\mathcal{A}[t,T]$ given above. For any $\nu \in \mathcal{A}[t,T]$, we define the objective functional

$$\begin{aligned} J(t,x,y;\nu) \quad = \quad & \mathbb{E}\bigg[\int_t^{\tau\wedge T} U\left(s,c(s)\right)\mathrm{d}s + B(\tau, Z^{\nu}_{t,x,y}(\tau), X^{\nu}_{t,x,y}(\tau))I_{[0,T]}(\tau) \\ & \qquad\qquad + W(X^{\nu}_{t,x,y}(T))I_{(T,+\infty)}(\tau) \;\Big|\; \mathcal{F}_t\bigg] \end{aligned} \tag{8}$$

and observe that the objective functional above reduces to the expected utility (7) when $t=0$.

The following lemma enables the transformation of the optimal control problem described above to an equivalent one with a fixed planning horizon. The statement presented below extends the analogous result in [45] by removing an assumption concerning the sign of the utility function U.

Lemma 2.1. *Suppose that Assumptions 1–6 hold and let $\nu \in \mathcal{A}[t,T]$. We have that*

$$\begin{aligned} J(t,x,y;\nu) \quad = \quad & \mathbb{E}\bigg[\int_t^{T} \left(G^{+}(s;t)\; U(s,c(s)) + \; g^{-}(s;t)B\left(s, Z^{\nu}_{t,x,y}(s), X^{\nu}_{t,x,y}(s)\right)\right)\mathrm{d}s \\ & \qquad\qquad + G^{+}(T;t)\; W(X^{\nu}_{t,x,y}(T)) \;\Big|\; \mathcal{F}_t\bigg] \,, \end{aligned}$$

where the conditional probabilities $G^{+}(s;t)$ and $g^{-}(s;t)$ are as given in (3) *and* (4)*, respectively.*

Proof. Let $t \in [0,T]$ and take $\nu \in \mathcal{A}[t,T]$. Start by rewriting the functional J in (8) as

$$\begin{aligned} J(t,x,y;\nu) \quad = \quad & \mathbb{E}\bigg[I_{(T,+\infty)}(\tau)\left(\int_t^{T} U\left(s,c(s)\right)\mathrm{d}s + W(X^{\nu}_{t,x,y}(T))\right) \\ & + I_{[0,T]}(\tau)\left(\int_t^{\tau} U(s,c(s))\;\mathrm{d}s + B(\tau, Z^{\nu}_{t,x,y}(\tau), X^{\nu}_{t,x,y}(\tau))\right) \;\Big|\; \mathcal{F}_t\bigg] \,. \end{aligned}$$

Resorting to the probability density function $g(\cdot)$ of the random variable τ and noting that τ is independent of the filtration $\mathbb{F}$, it follows that

$$\begin{aligned} J(t,x,y;\nu) \quad = \quad & \mathbb{E}\bigg[G^{+}(T;t)\left(\int_t^{T} U(s,c(s))\;\mathrm{d}s + W(X^{\nu}_{t,x,y}(T))\right) \\ & + \int_t^T g^{-}(u;t)\int_t^u U(s,c(s))\;\mathrm{d}s\mathrm{d}u \\ & + \int_t^T g^{-}(s;t)B(s, Z^{\nu}_{t,x,y}(s), X^{\nu}_{t,x,y}(s))\,\mathrm{d}s \;\Big|\; \mathcal{F}_t\bigg] \,. \end{aligned} \tag{9}$$

We will now show that the order of integration in the second term of (9) can be interchanged. By Assumption 3 the density function $g(\cdot)$ is continuous. As a consequence, for each fixed $t \in [0,T]$, the conditional density $g^{-}(\cdot;t)$ is also continuous, and thus bounded on $[t,T]$. Denoting by $K(t)$

the upper bound of $|g^-(\cdot;t)|$ on $[t,T]$, we obtain that

$$\begin{aligned}\mathbb{E}\left[\int_t^T\int_s^T \left|g^-(u;t)U(s,c(s))\right| \,\mathrm{d}u\mathrm{d}s\right] &\le K(t)\mathbb{E}\left[\int_t^T\int_s^T |U(s,c(s))| \,\mathrm{d}u\mathrm{d}s\right]\\ &\le K(t)(T-t)\mathbb{E}\left[\int_t^T |U(s,c(s))| \,\mathrm{d}s\right] < \infty ,\end{aligned}$$

where finiteness of the third term in the sequence of inequalities above follows from admissibility of the control $\nu \in \mathcal{A}[t,T]$. Therefore, *Fubini-Tonelli theorem* can be applied to interchange the order of integration and yield

$$\mathbb{E}\left[\int_t^T\int_t^u g^-(u;t)U(s,c(s))\,\mathrm{d}s\mathrm{d}u\right] = \mathbb{E}\left[\int_t^T\int_s^T g^-(u;t)U(s,c(s))\,\mathrm{d}u\mathrm{d}s\right] . \tag{10}$$

Observing that

$$G^+(s;t) - G^+(T;t) = \int_s^T g^-\,(u;t)\,\mathrm{d}u \ ,$$

we rewrite (10) as

$$\int_t^T\int_t^u g^-(u;t)U(s,c(s))\,\mathrm{d}s\mathrm{d}u \;=\; \int_t^T \big(G^+(s;t) - G^+(T;t)\big)\,U(s,c(s))\,\mathrm{d}s \ .$$

Combining the identity above with (9), we obtain

$$\begin{aligned}J(t,x,y;\nu) \;=\; \mathbb{E}\Big[&G^+(T;t)\left(\int_t^T U(s,c(s))\,\mathrm{d}s + W(X^\nu_{t,x,y}(T))\right)\\ &+\int_t^T \big(G^+(s;t) - G^+(T;t)\big)\,U(s,c(s))\,\mathrm{d}s\\ &+\int_t^T g^-(s;t)B(s,Z^\nu_{t,x,y}(s),X^\nu_{t,x,y}(s))\,\mathrm{d}s\;\Big|\;\mathcal{F}_t\Big] .\end{aligned}$$

We complete the proof by rearranging the terms in the identity above. □

The transformation to a fixed planning horizon provided above can be given the following interpretation: a wage-earner facing unpredictable death acts as if he will live until time T, but with a subjective rate of time preferences equal to his "force of mortality", that is, the conditional, instantaneous death rate for the wage-earner surviving past time t, for the consumption of his family and his terminal wealth.

Note that the optimal control problem (7) can be restated in dynamic programming form as

$$\begin{cases}V(t,x,y) = \sup\limits_{\nu\in\mathcal{A}[t,T]} J(t,x,y;\nu)\\ V(T,x,y) = W(x)\end{cases} .$$

Using the previous lemma, one can state a dynamic programming principle, obtaining a recursive relationship for the value function $V(t,x,y)$ defined above. Its proof is similar to the corresponding result in [45] and we omit it here.

Lemma 2.2 (Dynamic programming principle)**.** *Suppose that Assumptions 1–6 hold. For $0 \leq t < s \leq T$, the maximum expected utility $V(t,x,y)$ satisfies the recursive relation*

$$V(t,x,y) = \sup_{\nu \in \mathcal{A}[t,T]} \mathbb{E}\Bigg[G^{+}(s;t) V(s, X^{\nu}_{t,x,y}(s), Y_{t,y}(s)) \\ + \int_t^s \left(G^{+}(u;t)\ U(u,c(u)) + g^{-}(u;t) B\left(u, Z^{\nu}_{t,x,y}(u), X^{\nu}_{t,x,y}(u)\right)\right) \mathrm{d}u \,\Bigg|\, \mathcal{F}_t \Bigg] .$$

The dynamic programming principle above can be used to derive the following HJB equation

$$\begin{cases} V_t - \lambda(t) V + \sup\limits_{(c,\theta,p) \in \mathbb{R}_0^{+} \times I_\theta(x) \times (\mathbb{R}_0^{+})^K} \mathcal{H}(t,x,y;c,\theta,p) = 0 \\ V(T,x,y) = W(x) \end{cases}, \tag{11}$$

where the Hamiltonian function $\mathcal{H}$ is given by

$$\mathcal{H}(t,x,y;c,\theta,p) = U(t,c) + \lambda(t) B\left(t, \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)}, x\right) \\ + \left(i(t,y) - c(t) - \sum_{k=1}^{K} p_k + \left(r(t,y) + \theta(\mu(t,y) - r(t,y))\right) x \right) V_x(t,x,y) \\ + \alpha(t,y)\ V_y(t,x,y) + \frac{1}{2}\beta^2(t,y)\ V_{yy}(t,x,y) + \frac{1}{2}\left(x\theta\sigma(t,y)\right)^2\ V_{xx}(t,x,y) \\ + x\theta\sigma(t,y)\beta(t,y)\sqrt{1-\rho^2}\ V_{xy}(t,x,y)\ . \tag{12}$$

The techniques used in the derivation of the HJB equation above and the proof of the next theorem follow closely those in [10, 45, 46].

Theorem 2.3 (Verification Theorem)**.** *Suppose that Assumptions 1–6 hold and that V is of class $C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then V satisfies the Hamilton-Jacobi-Bellman equation* (11) *and, moreover, the inequality*

$$V(t,x,y) \geq J(t,x,y;\nu)$$

holds for every $\nu \in \mathcal{A}[t,T]$ and $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. Furthermore, an admissible strategy $\nu^* = (c^*(\cdot), \theta^*(\cdot), p^*(\cdot)) \in \mathcal{A}[t,T]$ associated with the state values (X^*, Y^*) is optimal if and only if for a.e. $s \in [t,T]$ and P-a.s. we have

$$V_t(s, X^*(s), Y^*(s)) - \lambda(s) V(s, X^*(s), Y^*(s)) + \mathcal{H}(s, X^*(s), Y^*(s); \nu^*) = 0\ . \tag{13}$$

3. The optimal strategies

We will now employ Theorem 2.3 to compute the optimal life-insurance selection and purchase, portfolio and consumption strategies for the wage-earner with uncertain lifetime under consideration herein.

For each $(t,x) \in [0,T] \times \mathbb{R}$, let $U_c(t,\cdot)$ and $B_z(t,\cdot,x)$ denote, respectively, the derivatives of the utility functions $U(t,\cdot)$ and $B(t,\cdot,x)$. By Assumption 6, both $U(t,\cdot)$ and $B(t,\cdot,x)$ are strictly concave, and the corresponding derivatives are invertible on appropriate intervals. Hence, we define I_1 and I_2 to be the (unique) functions such that

$$I_1(t, U_c(t,a)) = a \quad \text{and} \quad U_c(t, I_1(t,a)) = a\ , \quad \text{for } a \in \mathbb{R}^+ \tag{14}$$

for every $t \in [0,T]$, and

$$\begin{aligned} I_2(t, B_z(t,a,x), x) &= a \quad \text{for } a \in I(t,x) \\ B_z(t, I_2(t,a,x), x) &= a \quad \text{for } a \in \mathbb{R}^+ \end{aligned} \tag{15}$$

for every $(t,x) \in [0,T] \times \mathbb{R}$.

Before proceeding, let us define the function $\hat{\theta} : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by

$$\hat{\theta}(t,x,y) = -\frac{(\mu(t,y) - r(t,y))\, V_x(t,x,y) + \sigma(t,y)\beta(t,y)\sqrt{1-\rho^2} V_{xy}(t,x,y)}{x\sigma^2(t,y)V_{xx}(t,x,y)} . \tag{16}$$

The next result provides a characterization for the optimal strategies in terms of the value function and its derivatives.

Theorem 3.1. *Suppose that Assumptions 1–6 hold and that the value function V is of class $C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then the Hamiltonian function $\mathcal{H}$ given in* (12) *has a unique maximum. Moreover, the optimal strategies are given by*

$$\begin{aligned} c^*(t,x,y) &= I_1\Big(t, V_x(t,x,y)\Big) \\ \theta^*(t,x,y) &= \begin{cases} \sup I_\theta(x) \ , & \text{if} \quad \hat{\theta}(t,x,y) \geq \sup I_\theta(x) \\ \hat{\theta}(t,x,y) \ , & \text{if} \quad \hat{\theta}(t,x,y) \in I_\theta(x) \\ \inf I_\theta(x) \ , & \text{if} \quad \hat{\theta}(t,x,y) \leq \inf I_\theta(x) \end{cases} , \end{aligned}$$

and, for each $k \in \{1,2,\ldots,K\}$, we have that

$$p_k^*(t,x,y) = \begin{cases} \max\Big\{0, \eta_k(t) I_2\Big(t, \eta_k(t)\,(\lambda(t))^{-1} V_x(t,x,y), x\Big)\Big\}, & \text{if } k = k^*(t) \\ 0 \ , & \text{otherwise} \end{cases} ,$$

where

$$k^*(t) = \underset{k \in \{1,2,\ldots,K\}}{\arg\min} \ \{\eta_k(t)\} \ . \tag{17}$$

Proof. Using the second part of Theorem 2.3, an optimal admissible strategy $\nu^* = (c^*, \theta^*, p^*) \in \mathcal{A}[t,T]$ with wealth process X^* and economic indicator Y^* must satisfy (13). Therefore, ν^* must be such that $\mathcal{H}$ given in (12) attains its supremum. We start by remarking that the optimality condition on $\mathcal{H}$ decouples into three independent conditions:

$$\begin{aligned} \sup_{(c,\theta,p) \in \mathbb{R}_0^+ \times I_\theta(x) \times (\mathbb{R}_0^+)^K} \mathcal{H}(t,x,y;\nu) &= \sup_{c \in \mathbb{R}_0^+} \{U(t,c) - cV_x(t,x,y)\} + r(t,y)xV_x(t,x,y) \\ &+ \sup_{p \in (\mathbb{R}_0^+)^K} \left\{ \lambda(t) B\left(t, \sum_{k=1}^K \frac{p_k}{\eta_k(t)}, x\right) - V_x(t,x,y) \sum_{k=1}^K p_k \right\} + i(t,y)V_x(t,x,y) \\ &+ \sup_{\theta \in I_\theta(x)} \left\{ \frac{1}{2}\left(x\theta\sigma(t,y)\right)^2 V_{xx}(t,x,y) + \theta\left(\mu(t,y) - r(t,y)\right) xV_x(t,x,y) \right. \\ &\left. \qquad + x\theta\sigma(t,y)\beta(t,y)\sqrt{1-\rho^2} V_{xy}(t,x,y) \right\} \\ &+ \alpha(t,y)V_y(t,x,y) + \frac{1}{2}\beta^2(t,y)V_{yy}(t,x,y) \ . \end{aligned}$$

Therefore, it is enough to study the variation of $\mathcal{H}$ with respect to each one of the variables c, θ and p independently. We deal with the optimization problem associated with c first. Computing the first-order conditions with respect to c we obtain

$$-V_x(t,x,y) + U_x(t,c^*) = 0 \ . \tag{18}$$

Resorting to the inverse functions introduced before the statement of Theorem 3.1, we can solve equation (18) for the control variable c^* and get

$$c^*(t,x,y) = I_1\left(t, V_x(t,x,y)\right) .$$

Given the smoothness assumption on V and the monotonicity and concavity properties of the utility functions under consideration, we obtain that $V_x(t,x,y)$ is positive, and thus, given that I_1 is a bijection of $\mathbb{R}^+$ onto $\mathbb{R}^+$, so is the optimal consumption c^*.

We now consider the optimization problem associated with the variable θ. The unconstrained case associated with $I_\theta(x) = \mathbb{R}$ is straightforward. Let us instead focus on the constrained case where the constraint is given by a bounded interval. Recalling that we seek to determine the supremum of a function (which is quadratic on the relevant variable θ) over a bounded interval, the solution technique in the case where the constraint is determined by a closed interval applies also to the cases where the constraint is an interval that is not necessarily closed. Thus, and without loss of generality, in what follows we restrict our attention to the case where $I_\theta(x) = [\theta^-(x), \theta^+(x)]$, where $\theta^-(x), \theta^+(x) \in \mathbb{R}$ are such that $\theta^-(x) < \theta^+(x)$ for every $x \in \mathbb{R}$, with the cases where $I_\theta(x)$ is open at one or both of its endpoints following in a similar fashion. Resorting to the Kuhn-Tucker conditions, we seek to find a solution $(\theta(t,x,y), \mu_1(t,x,y), \mu_2(t,x,y))$ to the following set of equalities and inequalities:

$$\begin{aligned}
(\mu(t,y) - r(t,y))\, xV_x(t,x,y) + x\sigma(t,y)\beta(t,y)\sqrt{1-\rho^2}V_{xy}(t,x,y) \quad & \\
+\ (x\sigma(t,y))^2 V_{xx}(t,x,y)\ \theta \ &= \ \mu_1 - \mu_2 \\
\theta - \theta^+(x) \ &\leq \ 0 \\
-\theta + \theta^-(x) \ &\leq \ 0 \\
\mu_1 \ &\geq \ 0 \\
\mu_2 \ &\geq \ 0 \\
\mu_1\left(\theta - \theta^+(x)\right) \ &= \ 0 \\
\mu_2\left(-\theta + \theta^-(x)\right) \ &= \ 0 \ .
\end{aligned}$$

We start by noting that μ_1 and μ_2 can not both be positive simultaneously. The remaining three cases lead us to the desired description for $\theta^*(t,x,y)$ given in the statement. Namely, if $\mu_1 = 0$ and μ_2 is positive, we get

$$\theta^*(t,x,y) = \theta^-(x) = \inf I_\theta(x) \ .$$

On the other hand, if $\mu_2 = 0$ and μ_1 is positive, we obtain

$$\theta^*(t,x,y) = \theta^+(x) = \sup I_\theta(x) \ .$$

Finally, whenever μ_1 and μ_2 are both zero, we obtain that $\theta^*(t,x,y)$ is given by the expression $\hat{\theta}(t,x,y)$ given in (16). The cases where $I_\theta(x)$ is only bounded above or only bounded below may be dealt with in an analogous fashion. We skip their discussion for the sake of brevity.

To solve the constrained optimization problem associated with the variable $p \in (\mathbb{R}_0^+)^K$, we resort to the Kuhn-Tucker conditions once more. Namely, we look for a solution $(p_1(t,x,y),\ldots,p_K(t,x,y),\mu_1(t,x,y),\ldots,\mu_K(t,x,y))$ to the following set of equalities and inequalities:

$$\begin{aligned}\frac{\lambda(t)}{\eta_k(t)}B_z\left(t,\sum_{k=1}^{K}\frac{p_k}{\eta_k(t)},x\right)-V_x(t,x,y) &= -\mu_k \\ p_k &\geq 0 \\ \mu_k &\geq 0\ , \qquad k=1,2,\ldots,K \\ p_k\mu_k &= 0\ .\end{aligned} \tag{19}$$

We start by noting that for $k_1 \neq k_2$, if we have $\mu_{k_1}(t,x,y) = \mu_{k_2}(t,x,y)$ for some $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, one must have that $\eta_{k_1}(t) = \eta_{k_2}(t)$. Thus, relying on the assumption that all insurance companies offer pairwise distinct contracts, we obtain that for every $k_1, k_2 \in \{1,2,\ldots,K\}$ such that $k_1 \neq k_2$ and every $x \in \mathbb{R}$, $\mu_{k_1}(t,x,y) \neq \mu_{k_2}(t,x,y)$ for Lebesgue a.e. $t \in [0,T]$. In particular, we obtain that for every $x \in \mathbb{R}$ and Lebesgue a.e. $t \in [0,T]$, there is at most one $k \in \{1,2,\ldots,K\}$ such that $\mu_k(t,x,y) = 0$. Therefore, for Lebesgue a.e. $t \in [0,T]$, there is at most one $k \in \{1,2,\ldots,K\}$ such that $p_k(t,x,y) \neq 0$.

Using once again the first identity in (19), we get that

$$\eta_{k_1}(t)\left(V_x(t,x,y)-\mu_{k_1}\right) = \eta_{k_2}(t)\left(V_x(t,x,y)-\mu_{k_2}\right).$$

As a consequence of the identity above, we conclude that if $\mu_{k_1}(t,x,y) > \mu_{k_2}(t,x,y)$ for $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}$, then $\eta_{k_1}(t) > \eta_{k_2}(t)$. Furthermore, if for some $t \in [0,T]$ we have $\mu_{k_1}(t,x,y) = 0$, then $\eta_{k_1}(t) < \eta_{k_2}(t)$ for every $k_2 \in \{1,2,\ldots,K\}$ such that $k_1 \neq k_2$.

From this point onwards, let $k^*(t)$ be as given in (17). Then, either $p_k(t,x,y) = 0$ for every $k \in \{1,2,\ldots,K\}$ or else $p_{k^*(t)}(t,x,y) > 0$ is a solution to

$$\frac{\lambda(t)}{\eta_{k^*(t)}(t)}B_z\left(t,\frac{p_{k^*(t)}}{\eta_{k^*(t)}(t)},x\right) = V_x(t,x,y)\ ,$$

yielding

$$p_k^*(t,x,y) = \begin{cases} \max\left\{0,\eta_k(t)I_2\left(t,\frac{\eta_k(t)\ V_x(t,x,y)}{\lambda(t)},x\right)\right\}, & \text{if } k = k^*(t) \\ 0\ , & \text{otherwise}\ .\end{cases}$$

Computing the second derivative of the Hamiltonian (12) with respect to each control variable, we obtain

$$\begin{aligned}\mathcal{H}_{cc}(t,x,y;\nu^*) &= U_{cc}(t,c^*) \\ \mathcal{H}_{p_{k_1}p_{k_2}}(t,x,y;\nu^*) &= \frac{\lambda(t)}{\eta_{k_1}(t)\eta_{k_2}(t)}B_{zz}\left(t,\frac{p^*_{k^*(t)}}{\eta_{k^*(t)}(t)},x\right),\quad k_1,k_2=1,\ldots,K \\ \mathcal{H}_{\theta\theta}(t,x,y;\nu^*) &= (x\sigma(t,y))^2V_{xx}(t,x,y)\ .\end{aligned} \tag{20}$$

Optimality of c^* and p^* follows from strict concavity of the functions U and B with respect to their second variables, which makes the first-order conditions and the Kuhn-Tucker conditions not only necessary, but also sufficient.

Optimality of θ^* follows from the smoothness assumption on the value function V, as we will now describe. First, we observe that the value function V is concave in the variable x, a consequence of the strict concavity of the utility functions described in Assumption 6. The proof of such fact follows standard methods (see e.g. [10, Section IV.10]) and we omit it. Combining concavity of V with respect to x with the smoothness assumption on V, we obtain that the second derivative V_{xx} is nonpositive. At this point, we break our analysis into the following two cases: (i) the unconstrained case associated with $I_\theta(x) = \mathbb{R}$; and (ii) the case where the constraint is a closed and bounded interval of real numbers, that is, $I_\theta(x) = [\theta^-(x), \theta^+(x)]$, where $\theta^-(x), \theta^+(x) \in \mathbb{R}$ are such that $\theta^-(x) < \theta^+(x)$.

Let us deal with the unconstrained case first, for which it is enough to guarantee that the second derivative V_{xx} must be negative. Suppose this is not the case. We have already established that V_{xx} is nonpositive. If V_{xx} is zero, then the Hamiltonian $\mathcal{H}$, being a linear function in θ, would not be bounded above. As a consequence of the HJB equation, we would then conclude that either V_t or V would be unbounded, contradicting the smoothness assumption on the value function V. Therefore, V_{xx} must be strictly negative in this case, ensuring that $\mathcal{H}_{\theta\theta}$, as given in (20), is negative and $\theta^* = \hat{\theta}(t, x, y)$ actually yields an interior maximum of $\mathcal{H}$.

We now consider the case where the constraint is of the form $I_\theta(x) = [\theta^-(x), \theta^+(x)]$, where $\theta^-(x), \theta^+(x) \in \mathbb{R}$ are such that $\theta^-(x) < \theta^+(x)$. This is simpler than the unconstrained case, as we are now looking for the supremum of a quadratic function over a closed and bounded interval. Indeed, if V_{xx} is zero such function is linear in θ and the supremum is attained at either $\theta^-(x)$ or $\theta^+(x)$ as determined by the Kuhn-Tucker conditions discussed earlier in the proof. In the case where V_{xx} is negative, it may also happen that the supremum (maximum, really, in such case) is attained in an interior point of the interval besides its endpoints $\theta^-(x)$ or $\theta^+(x)$. Which of these three possibilities yields the desired supremum is again determined by the Kuhn-Tucker conditions for this problem.

Finally, we observe that the cases where $I_\theta(x)$ is bounded but not closed, and the cases where $I_\theta(x)$ is only bounded above or only bounded below may be addressed in a way similar to what was done above. □

We should remark that the optimal life-insurance selection and purchase strategy given in Theorem 3.1 calls for the wage-earner to concentrate all of his purchases on the life-insurance company offering the highest premium for the same price, i.e. the insurance company with the lowest premium-payout ratio $\eta_k(t)$, $k = 1, 2, \ldots, K$. This is unlike the usual diversification associated with investment on financial markets, but completely natural once one notes that, under the assumptions used herein, the life-insurance contracts carry no default risk.

4. The family of discounted CRRA utilities

We will now consider the special case where the wage-earner has the same discounted CRRA-type utility functions for the consumption of his family, the life-insurance payment to his estate in the event of premature death, and his terminal wealth. Henceforth, we assume that the utility functions are given by

$$U(t,c)=\mathrm{e}^{-\varrho t}\frac{c^{\gamma_1}}{\gamma_1} \qquad W(x)=\mathrm{e}^{-\varrho T}\frac{x^{\gamma_2}}{\gamma_2} \qquad B(t,z,x)=\frac{\mathrm{e}^{-\varrho t}}{\gamma_3}\left(z-\chi(t,x)\right)^{\gamma_3}\ , \tag{21}$$

where the risk aversion parameters γ_i are such that $\gamma_i<1$, with $\gamma_i\neq 0$ for all $i=1,2,3$, the discount rate ϱ is positive, and the function $\chi(t,x)$ reflects a benchmark with respect to which the wage-earner evaluates his family needs concerning life-insurance protection in the event of premature death. The case where $\chi(t,x)=0$ corresponds to an absolute evaluation of life-insurance needs, while $\chi(t,x)=-x$ reflects the family life-insurance needs in addition to the wage-earner wealth x at the eventual premature time of death. The latter is the most common choice in the literature. A third possible interesting choice would be to set $\chi(t,x)$ equal to the time t value of the wage-earner accumulated future income.

We will also assume that the value function V is of class $C^{1,2,2}\left([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R}\right)$, that no short-selling constraints are in place (i.e., $I_\theta(x)=\mathbb{R}$ for every x), that the life-insurance market has only one company with premium-payout ratio $\eta(t)$, and that the nonnegativity constraint for the life-insurance purchase is disregarded. According to Theorem 3.1 and its proof, the optimal strategies are given in terms of the value function V as

$$\begin{aligned} c^*(t,x,y) &= \left(\mathrm{e}^{\varrho t}V_x(t,x,y)\right)^{-1/(1-\gamma_1)} \\ \theta^*(t,x,y) &= \hat\theta(t,x,y) \\ p^*(t,x,y) &= \left(\left(\frac{\eta(t)\mathrm{e}^{\varrho t}V_x(t,x,y)}{\lambda(t)}\right)^{-1/(1-\gamma_3)}+\chi(t,x)\right)\eta(t)\ , \end{aligned} \tag{22}$$

where $\hat\theta(t,x,y)$ is as given in (16). Indeed, under the assumptions listed above, the optimal strategies can be readily obtained from the statement of Theorem 3.1 after observing that, for the family of discounted CRRA utilities (21), the functions I_1 and I_2 defined in (14) and (15), respectively, are given by

$$I_1(t,c)=\left(\mathrm{e}^{\varrho t}c\right)^{-1/(1-\gamma_1)} \quad \text{and} \quad I_2(t,z,x)=\left(\mathrm{e}^{\varrho t}z\right)^{-1/(1-\gamma_3)}+\chi(t,x)\ .$$

Substituting c, θ and p_k in the HJB equation (11) by the optimal strategies given in (22) and combining similar terms, we arrive at the following nonlinear second order partial differential equation

$$\begin{aligned} &V_t(t,x,y)-\lambda(t)V(t,x,y)+\big(i(t,y)+r(t,y)x-\eta(t)\chi(t,x)\big)V_x(t,x,y) \\ &\quad+\frac{1-\gamma_3}{\gamma_3}\mathrm{e}^{-\varrho t/(1-\gamma_3)}(V_x(t,x,y))^{-\gamma_3/(1-\gamma_3)}\left(\frac{\lambda(t)}{(\eta(t))^{\gamma_3}}\right)^{1/(1-\gamma_3)} \\ &\quad+\frac{1-\gamma_1}{\gamma_1}\mathrm{e}^{-\varrho t/(1-\gamma_1)}(V_x(t,x,y))^{-\gamma_1/(1-\gamma_1)}+\ \frac{1}{2}\beta^2(t,y)V_{yy}(t,x,y) \\ &\quad+\alpha(t,y)V_y(t,x,y)-\frac{\left(\Sigma(t,y)V_x(t,x,y)+\beta(t,y)\sqrt{1-\rho^2}V_{xy}(t,x,y)\right)^2}{2V_{xx}(t,x,y)}=0\ , \end{aligned} \tag{23}$$

where $\Sigma(t, y)$ is given by

$$\Sigma(t, y) = \frac{\mu(t, y) - r(t, y)}{\sigma(t, y)} ,$$

and the boundary condition is

$$V(T, x, y) = W(x) .$$

4.1. **A limit case with closed-form solutions.** In this subsection we consider the case where:

a) the evolution of the economic indicator is "frozen" but the financial market still depends on the state of the economy in a parametric fashion;
b) investment in the financial market is unconstrained, i.e. $I_\theta(x) = \mathbb{R}$ for every x;
c) the life-insurance market has only one reference company, trading life-insurance contracts with premium-payout ratio given by

$$\eta(t) = \eta_{k^*(t)}(t) ,$$

and the life-insurance purchase rate is allowed to be any real value.

To proceed with such analysis, we set the functions $\alpha(t, y)$ and $\beta(t, y)$ constant and equal to zero and regard $y \in \mathbb{R}$ as a fixed parameter representative of the economy strength. This will enable us to derive an explicit solution to the optimal control problem under consideration herein and to characterize the corresponding optimal strategies in the case where the benchmark function $\chi(t, x)$ is of the form

$$\chi(t, x) = f_1(t)x + f_2(t) , \tag{24}$$

where $f_1(t)$ and $f_2(t)$ are (eventually nonlinear) continuous real-valued functions on $[0, T]$.

Proposition 4.1. *Suppose that Assumptions 1–6 hold and $\chi(t, x)$ is as given in (24). Then, under the setup described in items a), b) and c) above, the optimal strategies associated with utility functions of the form (21) with identical risk aversion parameters $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ are given by*

$$\begin{aligned}
c^*(t, x, y) &= \frac{1}{E(t, y)} (x + b(t, y)) , \\
\theta^*(t, x, y) &= \frac{1}{1 - \gamma} \frac{x + b(t, y)}{x} \frac{\Sigma(t, y)}{\sigma(t, y)}, \\
p^*(t, x, y) &= \eta(t)\Big((D(t, y) + f_1(t)) x + D(t, y)b(t, y) + f_2(t)\Big)
\end{aligned}$$

where

$$
\begin{aligned}
b(t,y) &= \int_t^T (i(s,y)-f_2(s))\ \exp\left(-\int_t^s (r(v,y)-\eta(v)f_1(v))\,\mathrm{d}v\right)\mathrm{d}s \\
D(t,y) &= \frac{1}{E(t,y)}\left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \\
E(t,y) &= \exp\left(-\int_t^T \bar{H}(v,y)\,\mathrm{d}v\right)+\int_t^T \exp\left(-\int_t^s \bar{H}(v,y)\,\mathrm{d}v\right)L(s)\,\mathrm{d}s \\
\bar{H}(t,y) &= \frac{\lambda(t)+\varrho}{1-\gamma}-\frac{\gamma}{2}\left(\frac{\Sigma(t,y)}{1-\gamma}\right)^2-\frac{\gamma}{1-\gamma}\left(r(t,y)-\eta(t)f_1(t)\right) \\
L(t) &= 1+\left(\frac{\lambda(t)}{(\eta(t))^\gamma}\right)^{1/(1-\gamma)} \\
\Sigma(t,y) &= \frac{\mu(t,y)-r(t,y)}{\sigma(t,y)}\ .
\end{aligned}
$$

Proof. Assume that the utility functions U, B and W are as given in (21). Combining the optimal strategies given in (22) with the representation (24) for $\chi(t,x)$ and the "frozen" dynamics assumption of item a), which corresponds to setting $\alpha(t,y)$ and $\beta(t,y)$ constant and equal to zero, we obtain that the optimal strategies, in terms of the value function V, are given by

$$
\begin{aligned}
c^*(t,x,y) &= \left(\mathrm{e}^{\varrho t}V_x(t,x,y)\right)^{-1/(1-\gamma)} \\
\theta^*(t,x,y) &= -\frac{\left(\mu(t,y)-r(t,y)\right)V_x(t,x,y)}{x\sigma^2(t,y)V_{xx}(t,x,y)} \\
p^*(t,x,y) &= \left(\left(\frac{\eta(t)\mathrm{e}^{\varrho t}V_x(t,x,y)}{\lambda(t)}\right)^{-1/(1-\gamma_3)}+f_1(t)x+f_2(t)\right)\eta(t)\ .
\end{aligned}
\tag{25}
$$

We are now going to find an explicit solution for the HJB equation (11). We substitute c, θ and p_k in the HJB equation by the optimal strategies in (25) and combine similar terms to arrive at the following second order nonlinear partial differential equation

$$
\begin{aligned}
&V_t(t,x,y)-\lambda(t)V(t,x,y)+\big(i(t,y)+r(t,y)x-\eta(t)f_1(t)x-f_2(t)\big)V_x(t,x,y) \\
&-\Sigma^2(t,y)\frac{(V_x(t,x,y))^2}{2V_{xx}(t,x,y)}+\frac{1-\gamma}{\gamma}\mathrm{e}^{-\varrho t/(1-\gamma)}L(t)(V_x(t,x,y))^{-\gamma/(1-\gamma)}=0\ ,
\end{aligned}
\tag{26}
$$

where $\Sigma(t,y)$ and $L(t)$ are as given in the statement of this proposition, and the terminal condition is given by

$$
V(T,x,y)=\mathrm{e}^{-\varrho T}\frac{x^\gamma}{\gamma}\ . \tag{27}
$$

We consider an ansatz of the form

$$
V(t,x,y)=\frac{a(t,y)}{\gamma}(x+b(t,y))^\gamma\ , \tag{28}
$$

and substitute it in (26) so that $a(t,y)$ and $b(t,y)$ are determined by the differential equation

$$\frac{1}{\gamma}\frac{\mathrm{d}a}{\mathrm{d}t}(t,y)+\frac{a(t,y)}{x+b(t,y)}\frac{\mathrm{d}b}{\mathrm{d}t}(t,y)+\frac{i(t,y)+r(t,y)x-\eta(t)f_1(t)x-f_2(t)}{x+b(t,y)}a(t,y)$$
$$-\frac{\lambda(t)}{\gamma}\,a(t,y)-\frac{\Sigma^2(t,y)}{2(\gamma-1)}a(t,y)+\frac{1-\gamma}{\gamma}\mathrm{e}^{-\varrho t/(1-\gamma)}L(t)(a(t,y))^{-\gamma/(1-\gamma)}=0\ . \tag{29}$$

Rewrite the third term in (29) as

$$\begin{aligned}\frac{i(t,y)+r(t,y)x-\eta(t)f_1(t)x-f_2(t)}{x+b(t,y)}&a(t,y)\\ &=\left(r(t,y)-\eta(t)f_1(t)\right)a(t,y)\\ &\quad+\frac{i(t,y)-\left(r(t,y)-\eta(t)f_1(t)\right)b(t,y)-f_2(t)}{x+b(t,y)}a(t,y)\ ,\end{aligned} \tag{30}$$

and combine (29) and (30) to get

$$\begin{aligned}&\frac{1}{\gamma}\frac{\mathrm{d}a}{\mathrm{d}t}(t,y)+\left(r(t,y)-\eta(t)f_1(t)-\frac{\lambda(t)}{\gamma}+\frac{\Sigma^2(t,y)}{2(1-\gamma)}\right)a(t,y)\\ &\quad+\frac{1-\gamma}{\gamma}\mathrm{e}^{-\varrho t/(1-\gamma)}L(t)(a(t,y))^{-\gamma/(1-\gamma)}\\ &\quad+\frac{a(t,y)}{x+b(t,y)}\left(\frac{\mathrm{d}b}{\mathrm{d}t}(t,y)-\left(r(t,y)-\eta(t)f_1(t)\right)b(t,y)+i(t,y)-f_2(t)\right)=0\ .\end{aligned}$$

Note that the previous differential equation and the terminal condition (27) decouple into two independent boundary value problems for $a(t,y)$ and $b(t,y)$ which are given, respectively, by

$$\begin{aligned}&\frac{1}{\gamma}\frac{\mathrm{d}a}{\mathrm{d}t}(t,y)+\left(r(t,y)-\eta(t)f_1(t)-\frac{\lambda(t)}{\gamma}+\frac{\Sigma^2(t,y)}{2(1-\gamma)}\right)a(t,y)\\ &\qquad\qquad+\frac{1-\gamma}{\gamma}\mathrm{e}^{-\varrho t/(1-\gamma)}L(t)(a(t,y))^{-\gamma/(1-\gamma)}=0\\ &a(T,y)=\mathrm{e}^{-\varrho T}\ ,\end{aligned} \tag{31}$$

and

$$\begin{aligned}&\frac{\mathrm{d}b}{\mathrm{d}t}(t,y)-\left(r(t,y)-\eta(t)f_1(t)\right)b(t,y)+i(t,y)-f_2(t)=0\\ &b(T,y)=0\ .\end{aligned} \tag{32}$$

To find a solution to the boundary value problem (31), we write $a(t,y)$ in the form

$$a(t,y)=\mathrm{e}^{-\varrho t}(E(t,y))^{1-\gamma}\ ,$$

obtaining a new boundary value problem for the function $E(t,y)$ of the form

$$\begin{aligned}&\frac{\mathrm{d}E}{\mathrm{d}t}(t,y)-\bar{H}(t,y)E(t,y)+L(t)=0\\ &E(T,y)=1\ ,\end{aligned} \tag{33}$$

where $L(t)$ and $\bar{H}(t,y)$ are as given in the statement of this Proposition. Since equation (33) is a linear, non-autonomous, first order ordinary differential equation with respect to the independent variable t, we can find an explicit solution of the form

$$E(t,y) = \exp\left(-\int_t^T \bar{H}(v,y)\ \mathrm{d}v\right) + \int_t^T \exp\left(-\int_t^s \bar{H}(v,y)\ \mathrm{d}v\right) L(s)\ \mathrm{d}s\ .$$

Therefore, we obtain that the solution of (31) is given by

$$\begin{aligned} a(t,y) \quad &= \quad \mathrm{e}^{-\varrho t}\Bigg[\exp\left(-\int_t^T \bar{H}(v,y)\ \mathrm{d}v\right) \\ &\qquad + \int_t^T \exp\left(-\int_t^s \bar{H}(v,y)\ \mathrm{d}v\right)\ L(s)\ \mathrm{d}s\Bigg]^{1-\gamma} \end{aligned} \tag{34}$$

To find a solution for the boundary value problem (32), we just note that this is again a linear, non-autonomous, first order differential equation and its solution is given by

$$b(t,y) = \int_t^T (i(s,y) - f_2(s))\ \exp\left(-\int_t^s (r(v,y) - \eta(v)f_1(v))\,\mathrm{d}v\right)\mathrm{d}s\ . \tag{35}$$

Combining (25) with (28), (34) and (35), we obtain the optimal strategies

$$\begin{aligned} c^*(t,x,y) \quad &= \quad \frac{1}{E(t,y)}\left(x + b(t,y)\right), \\ \theta^*(t,x,y) \quad &= \quad \frac{1}{1-\gamma}\ \frac{x+b(t,y)}{x}\ \frac{\Sigma(t,y)}{\sigma(t,y)}, \\ p^*(t,x,y) \quad &= \quad \eta(t)\Big(\ (D(t,y)+f_1(t))\,x + D(t,y)b(t,y) + f_2(t)\Big) \end{aligned}$$

where $E(t,y)$ and $D(t,y)$ are as given in the statement of this Proposition.

□

The quantities $b(t,y)$ and $x+b(t,y)$ play a central role in the interpretation of the optimal strategies obtained in Proposition 4.1. The quantity $b(t,y)$ can be seen as the time t value of the wage earner future income from time t to time T with respect to the adjusted discount factor $r(t,y) - \eta(t)f_1(t)$, while the quantity $x + b(t,y)$ might be regarded as the wage-earner full potential wealth (present wealth plus future income).

The next Corollary is a consequence of Proposition 4.1. It states that the optimal consumption rate increases with wealth, while the proportion of wealth allocated to the risky asset decreases with increasing wealth. More importantly, it provides conditions determining the monotonicity of the optimal life-insurance purchase with respect to wealth.

Corollary 4.2. *Assume that the conditions of Proposition 4.1 are satisfied. Then, the following also hold:*

(i) *the maximum expected utility $V(t,x,y)$ is increasing with x;*
(ii) *the optimal consumption rate $c^*(t,x,y)$ is increasing with x;*
(iii) *the optimal risky-asset allocation $\theta^*(t,x,y)$ is decreasing with x; and*
(iv) *the optimal life-insurance purchase $p^*(t,x,y)$ is increasing with x whenever $D(t,y)+f_1(t)$ is non-negative and decreasing whenever $D(t,y)+f_1(t)$ is non-positive.*

We will now consider the dependence of the value function and the corresponding optimal strategies with respect to the economic indicator. Namely, we provide condition under which the optimal consumption rate, the optimal life-insurance purchase and the optimal risky-asset allocation all increase with an improving state of the economy.

Corollary 4.3. *Suppose that the hypotheses of Proposition 4.1 hold. Moreover, assume that the following additional conditions are satisfied:*

(i) *the discounted CRRA utilities risk aversion parameters* $\gamma_1 = \gamma_2 = \gamma_3$ *are negative;*
(ii) $r(t,y)$, $\mu(t,y)$ *and* $(\mu - r)(t,y)$ *are all increasing functions of* $y \in \mathbb{R}$ *for every fixed* $t \in [0,T]$*;*
(iii) $\sigma(t,y)$ *is a decreasing function of* $y \in \mathbb{R}$ *for every fixed* $t \in [0,T]$*;*
(iv) *for every* $0 \le t \le s \le T$*, the function*

$$\ln\left(i(s,y) - f_2(s)\right) - \int_t^s \left(r(v,y) - \eta_{k^*(v)}(v)\right) \mathrm{d}v$$

is increasing with $y \in \mathbb{R}$*.*

Then, the following hold:

(i) *the maximum expected utility* $V(t,x,y)$ *is increasing with* y*; and*
(ii) *the optimal consumption rate* $c^*(t,x,y)$*, the optimal life-insurance purchase* $p^*(t,x,y)$ *and the optimal risky-asset allocation* $\theta^*(t,x,y)$ *are all increasing with* y*.*

Proof. The result follows from the closed-form expressions provided in Proposition 4.1 after noting that, under the monotonicity assumptions on r, μ and σ, for $\gamma_1 = \gamma_2 = \gamma_3 < 0$, we have that:

(a) $\Sigma(t,y)$, $H(t,y)$ and $b(t,y)$ as given in the statement of Proposition 4.1 are increasing functions of y; and
(b) $E(t,y)$ as given in the statement of Proposition 4.1 is a decreasing functions of y.

□

4.2. **Numerical study.** An explicit solution for the nonlinear second order partial differential equation (23) is, in general, not available in closed form. Hence, we resort to a numerical solution of (23) in order to study how the optimal strategies of the problem under consideration herein depend upon the indicator $Y(\cdot)$ modeling the state of the overall economy.

We will consider a financial market as described in Section 2.1, with coefficient functions which are independent of time and affine functions of the state y of the economic indicator. More concretely, we assume that the financial market coefficients have the form

$$
\begin{aligned}
r(y) &= r_0 + r_1 y \\
\mu(y) &= \mu_0 + \mu_1 y \\
\sigma(y) &= \begin{cases} \sigma_0 + \sigma_1 y^* & \text{if } y \leq -y^* \\ \sigma_0 - \sigma_1 y & \text{if } -y^* < y < y^* \\ \sigma_0 - \sigma_1 y^* & \text{if } y \geq y^* \end{cases} \\
\alpha(y) &= \alpha_0 - \alpha_1 y \\
\beta(y) &= \beta_0 \ ,
\end{aligned}
$$

where the parameters r_0, r_1, μ_0, μ_1, σ_0, σ_1, α_0, α_1, β_0, and y^*, are all positive and, additionally, σ_0, σ_1 and y^* are such that $\sigma_0 - \sigma_1 y^* > 0$. The choices listed above for the coefficient functions aim at modeling the following practical feature: with an improving state of the economy, as reflected by a larger value of the economic indicator, we have that

a) the riskless interest rate increases;
b) the risky-asset mean appreciation rate increases;
c) the volatility decreases, but remains bounded away from zero.

In what concerns the dynamics of the economic indicator $Y(\cdot)$, we note that the coefficient functions α and β have been chosen in such a way that $Y(\cdot)$ is an Ornstein–Uhlenbeck process. Given the well-known features of such process – it is mean-reverting, Gaussian, and Markov – we believe this constitutes a very simple, yet reasonably realistic, model for the evolution of the indicator describing the state of the economy. As for the wage-earner preferences, these are assumed to be described by utility functions such as those given in (21). For concreteness of exposition, we consider here the case where $\chi(t,x)$ is identically zero. We note that positive values of $\chi(t,x)$ increase the utility that the wage-earner derives from buying larger increments of life-insurance, thus contributing to a larger allocation of his savings to the purchase of such protection, with a corresponding decrease in the amounts allocated to consumption and investment. Clearly, negative values of $\chi(t,x)$ have the opposite effect, lowering the wage-earner eagerness to purchase life-insurance, and increasing his consumption rate and wealth invested in the financial market. We also work under the assumption that the wage-earner is not allowed to hold any leveraged or short position in the risky-asset and that, if the wage-earner wealth becomes negative, he is obliged to pay an interest rate equal to the risk-free rate on his debt. Such investment constraints translate into a restriction of the form $\theta \in I_\theta(x)$, where $I_\theta(x)$ is given by

$$
I_\theta(x) = \begin{cases} [0,1] & \text{if } x \geq 0 \\ \{0\} & \text{if } x < 0 \end{cases} .
$$

The parameters used in the numerical experiment below were selected to fit the modeling assumptions described above concerning the dependence of the market coefficients dependence on the economic indicator $Y(\cdot)$. Nevertheless, we should stress that the results reported here are robust with respect to reasonable changes in the choice of such parameter values.

We should note that the numerical scheme producing the approximate solutions to both the HJB equation (23) and the corresponding (feedback form) optimal controls is based on a discretization procedure inspired in the techniques developed by Souganidis in [39] and Fleming and Souganidis [11] (see Appendix A for further details), while the solution to the SDEs defining the financial market and the wage-earner wealth process were obtained using an Euler-Maruyama numerical integrator combined with the aforementioned optimal controls numerical approximations. The discretization procedure leading to the numerical solution of the HJB equation (23) is based on a three-dimensional grid within the (t, x, y)-space $[0, T] \times [2, 152] \times [-2, 2]$, where T was taken to be either 5 or 20 depending on whether the focus lied on a short horizon scenario (as in Subsection 4.2.1) or a long horizon scenario (as in Subsection 4.2.2). Such grid has vertices (t_i, x_j, y_k), where

$$x_j = x_0 + j\Delta_x \text{ , with } x_0 = 2,\ \Delta_x = 0.25 \text{ and } j = 0, \ldots, 600$$
$$y_k = y_0 + k\Delta_y \text{ , with } y_0 = -2,\ \Delta_y = 0.05 \text{ and } k = 0, \ldots, 80$$

and

$$t_i = T - i\Delta_t \text{ , with } T = 5,\ \Delta_t = 0.0025 \text{ and } i = 0, \ldots, 2000$$

for a short horizon scenario, or

$$t_i = T - i\Delta_t \text{ , with } T = 20,\ \Delta_t = 0.005 \text{ and } i = 0, \ldots, 4000$$

for a long horizon scenario. Approximations for the corresponding optimal controls $c^*(t, x, y)$, $p^*(t, x, y)$ and $\theta^*(t, x, y)$ in feedback form are obtained at each point of the grid described above as the solution of a steepest descent algorithm associated with a finite-dimensional optimization problem. No further discretization besides the one described above is required to determine the optimal controls. Finally, the Euler-Maruyama numerical integrator employed to obtain numerical solutions to the SDEs defining the financial market and the wage-earner wealth process uses the same step-size Δ_t as the numerical approximation to the HJB equation (23). Whenever values of the optimal controls at points not in the grid described above are needed in the course of the Euler-Maruyama method implementation, a bilinear interpolation in the x and y variables is used to numerically approximate such values.

4.2.1. *Short time horizon scenario.* In what follows we will fix the maximum planning horizon as $T = 5$ (years). Our goal is to analyze the optimal strategies of wage-earners close to retirement age and still considering the acquisition of life-insurance, allowing also a comparison with the long time horizon scenario treated in Section 4.2.2. From an Economics point of view, the key factor distinguishing the two scenarios – short and long time horizons – is the wage-earner's "human capital", i.e. the present value of his cumulative future income with respect to an appropriately chosen discount rate. Indeed, as we will see below, the planning horizon has a key influence in the wage-earner choices, as it significantly limits the available "human capital" which, when all else remains equal, is a decreasing function of the time left until retirement age. The substantial influence of such quantity can be appreciated both by an inspection of the "frozen dynamics" results of Section 4.1, as well as the numerical results presented below.

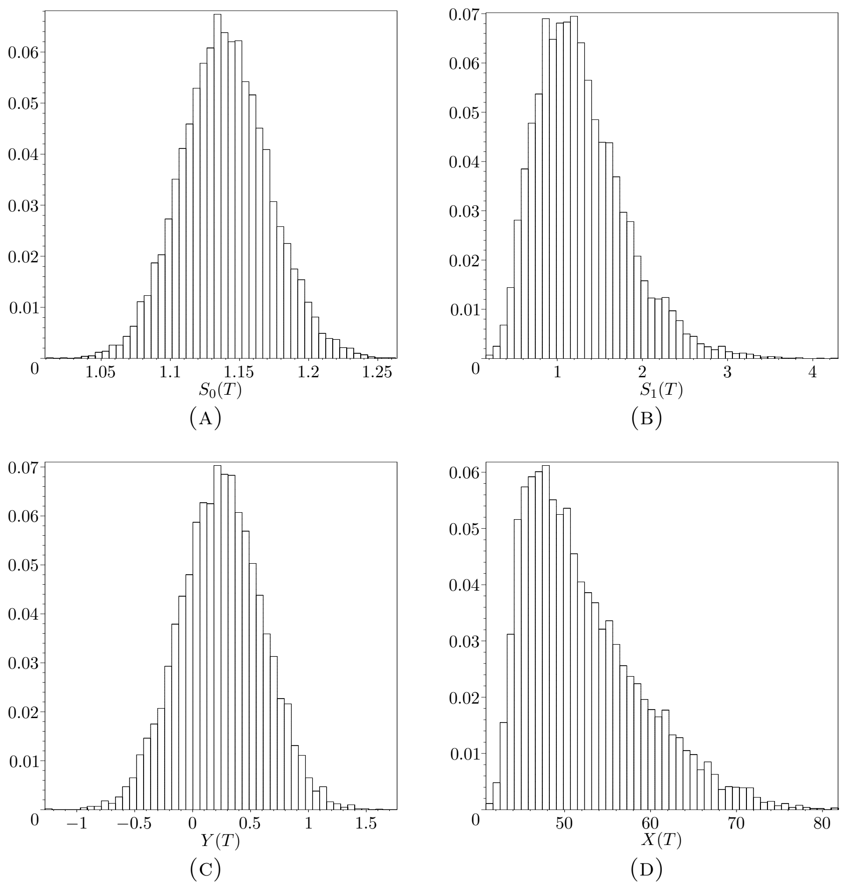

FIGURE 1. Histograms for the time $T = 5$ values of the risk-free asset $S_0(T)$ (Fig. 1a); the risky-asset price $S_1(T)$ (Fig. 1b); the economic indicator $Y(T)$ (Fig. 1c); and the wage-earner wealth $X(T)$ (Fig. 1d). Each histogram was built using 10,000 realizations of the corresponding process. The horizontal axes represent the processes time $T = 5$ values while the vertical axes represent the corresponding relative frequencies. The initial values used were $S_0(0) = 1$, $S_1(0) = 1$, $Y(0) = 0$, and $X(0) = 20$. The remaining model's parameters are as follows: $r(y) = 0.02 + 0.03y$, $\mu(y) = 0.03 + 0.07y$, $\sigma(y) = 0.17 - 0.06y$ for $y \in (-2, 2)$, $\alpha(y) = 0.25 - y$, $\beta(y) = 0.5$, $\rho = 0.5$, $\gamma_1 = -0.6$, $\gamma_2 = -0.5$, $\gamma_3 = -0.4$, $\varrho = 0.01$, $i(t, y) = 50\mathrm{e}^{(0.03+0.03y)t}$, $\lambda(t) = 0.001\mathrm{e}^{-9.5+0.1t}$, $\eta(t) = 1.05\lambda(t)$.

In Figure 1 we present the time $T = 5$ empirical probability distributions of the following stochastic processes: the risk-free asset price $S_0(T)$ (Fig.

1a); the risky-asset price $S_1(T)$ (Fig. 1b); the economic indicator $Y(T)$ (Fig. 1c); and the wage-earner wealth $X(T)$ in the event he survives until retirement time T (Fig. 1d). These histograms are the result of a Monte-Carlo simulation generating $10,000$ trajectories of the stochastic processes under consideration herein. Note that the histogram corresponding to the economic indicator $Y(\cdot)$ is very close to fitting a Gaussian curve, as it should, since $Y(\cdot)$ is assumed to be an Ornstein–Uhlenbeck process for the purposes of this experiment. The histogram for $S_0(T)$ inherits all of its randomness through the dependence of $S_0(\cdot)$ on $Y(\cdot)$, as expressed in (1). Indeed, since $Y(\cdot)$ is Gaussian, $S_0(T)$ has a log-normal distribution. Note also that the histograms for $S_1(T)$ and $X(T)$ share some similarities, such as their skewness to the right and a fatter right tail, reflecting the strong dependence that the final wealth $X(T)$ has on the financial market performance, as expressed through the time evolution of $S_1(\cdot)$.

In Figure 2 we provide a single realization of the time evolution of some of the stochastic processes of interest to our analysis: in figure 2a paths are shown for the risk-free asset S_0 (in blue), the risky-asset S_1 (in red), and the economic indicator Y (in green), while paths for the optimal controls θ^*, c^* and p^* are provided in figures 2b, 2c and 2d, respectively. As should be expected from observing the form of (1) in connection with the modeling assumptions listed above, the rate of growth of the risk-free asset S_0 is higher for correspondingly larger values of the economic indicator, while the value of the risky-asset trends up when the economic indicator shows high positive values, and trends down when the economic indicator shows negative values. In what concerns the optimal strategies, we observe the following: extreme positive values of the economic indicator translate into an optimal investment strategy whereby the wage-earner invests all of his savings in the risky-asset, while extreme negative values of the economic indicator correspond to an optimal investment strategy whereby the wage-earner allocates all of his savings to the risk-free asset. Intermediate values of the economic indicator correspond to investment decisions leading to a mixed portfolio. As for the optimal consumption and life-insurance purchase strategies, we remark that, up to a factor of scale, their time-evolution looks strikingly similar. This is a reasonably natural outcome once one notes that life-insurance purchase might be regarded as a very specific form of consumption. Otherwise, both quantities increase with an increasing economic factor, reflecting a healthier economy and larger disposable income. All of these observations are in agreement with intuition and our closed-form solutions of Section 4.1.

Figure 3 shows a swarm of 50 realizations of the wealth process over the time interval $[0, 5]$. Note that all paths shown are mostly trending up, due to a conjugation of gains from earned income and investment in the financial market. We remark that some of the paths in Figure 3 show arc segments with no volatility, corresponding to intervals of time where the economic indicator took extreme negative values and the corresponding optimal portfolio choice was to allocate all savings to the risk-free asset.

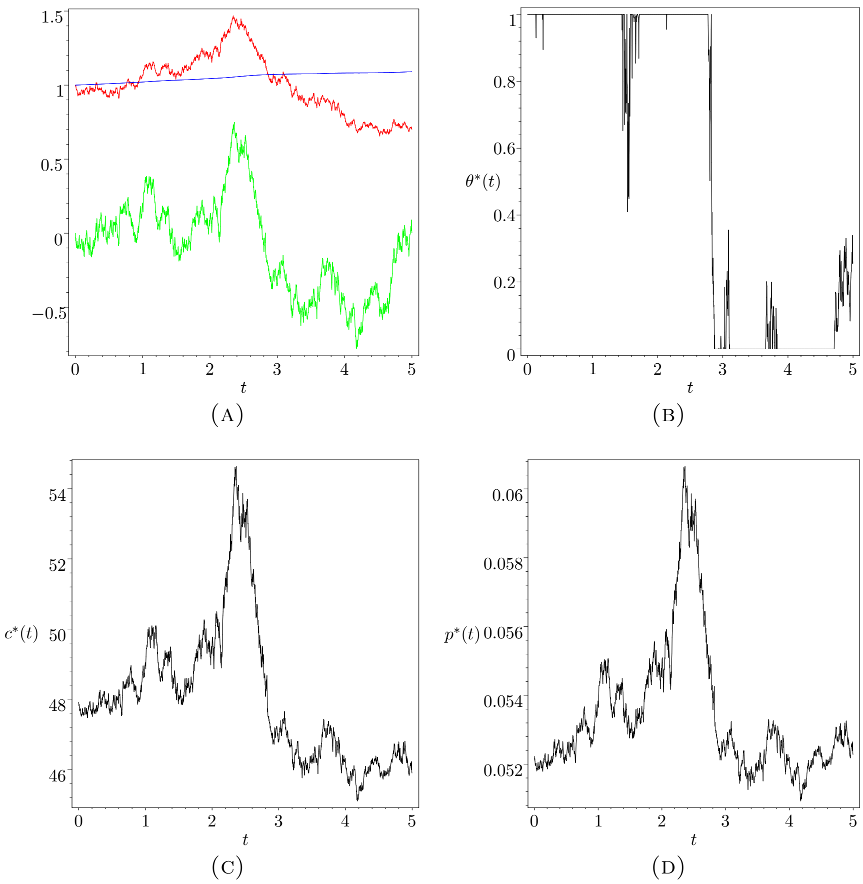

FIGURE 2. One realization yielding the time evolution of some relevant stochastic processes: risk-free asset S_0 (blue), risky-asset S_1 (in red) and economic indicator Y (in green) in Fig. 2a, optimal portfolio θ^* in Fig. 2b, optimal consumption c^* in Fig. 2c, and optimal life-insurance purchase p^* in Fig. 2d. The horizontal axes represent time $t \in [0, 5]$ and the vertical axes represent the stochastic processes values. The initial values and parameters are as given in Figure 1.

Figure 4 shows plots of the value function V, as well as of the optimal controls θ^*, c^* and p^* in feedback form, in terms of the wealth x and the economic indicator y for fixed time $t = 2.5$. As expected, in Figure 4a one can observe that the value function is strictly increasing and strictly concave with respect to the wealth level x, and strictly increasing with respect to the economic indicator y. In what concerns the optimal portfolio θ^*, we note that this is an increasing function of the economic indicator y, switching rather abruptly from full investment on the risk-free asset to full investment on the risky asset. As for its dependence on the wealth level x, one can see that θ^* is decreasing (note that the steep increase from 0 to 1 occurs at larger values of y with increasing values of x). Finally, the optimal consumption

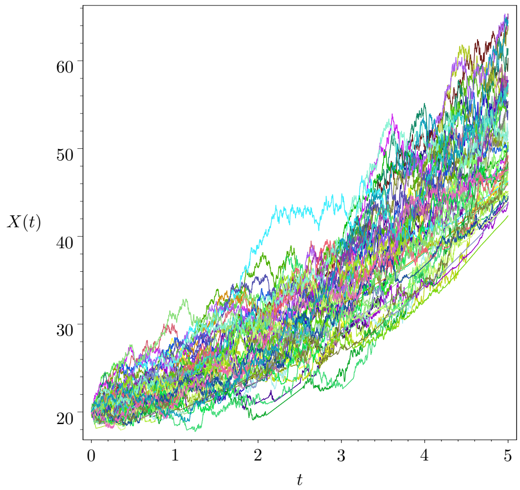

FIGURE 3. Swarm with 50 realizations of the wealth process $X(\cdot)$. The horizontal axis represents time $t \in [0,5]$ and the vertical axis represents wealth. The initial values and parameters are as given in Figure 1.

c^* and the optimal life-insurance purchase p^* are both stricly increasing and strictly convex functions of the economic indicator y and increasing functions of the wealth x. As before, all of these observations are in agreement the qualitative properties of the closed-form solutions of the special limit case considered in Section 4.1.

4.2.2. *Long time horizon scenario.* We will now fix the maximum planning horizon as being $T = 20$ (years) and discuss the optimal strategies of a wage-earner with a larger amount of time until retirement. The "human capital" of such individuals is very often large when compared to their wealth leading to interesting, yet very realistic, outcomes.

Figure 5 is the counterpart to Figure 1 when $T = 20$: it gives histograms approximating the probability distributions of the risk-free asset price $S_0(T)$ (Fig. 5a), the risky-asset price $S_1(T)$ (Fig. 5b), the economic indicator $Y(T)$ (Fig. 5c), and the wage-earner wealth $X(T)$ in the event he survives until retirement time (Fig. 5d). As in the short-term horizon case, all these histograms are based on a Monte-Carlo simulation generating $10,000$ realizations for the corresponding stochastic processes. We remark that the histogram of $Y(20)$ in figure 5c is almost identical to the corresponding one for $Y(5)$ in figure 1c, an indication that both of these histograms are already a very good approximation for the probability density function of the stationary distribution of the process $Y(\cdot)$. On the other hand, the histogram associated with $S_0(20)$ is slightly more asymmetric than that of $S_0(5)$, showing more clearly its non-Gaussian nature. As for the histograms describing $S_1(20)$ and $X(20)$, it should be noted that the longer planning horizon allowed for the appearance of more extreme values in both processes, exacerbating the skewness to the right of both distributions.

Finally, Figure 6 shows a swarm of 50 realizations of the wealth process over the time interval $[0, 20]$. Note that for negative values of the wealth

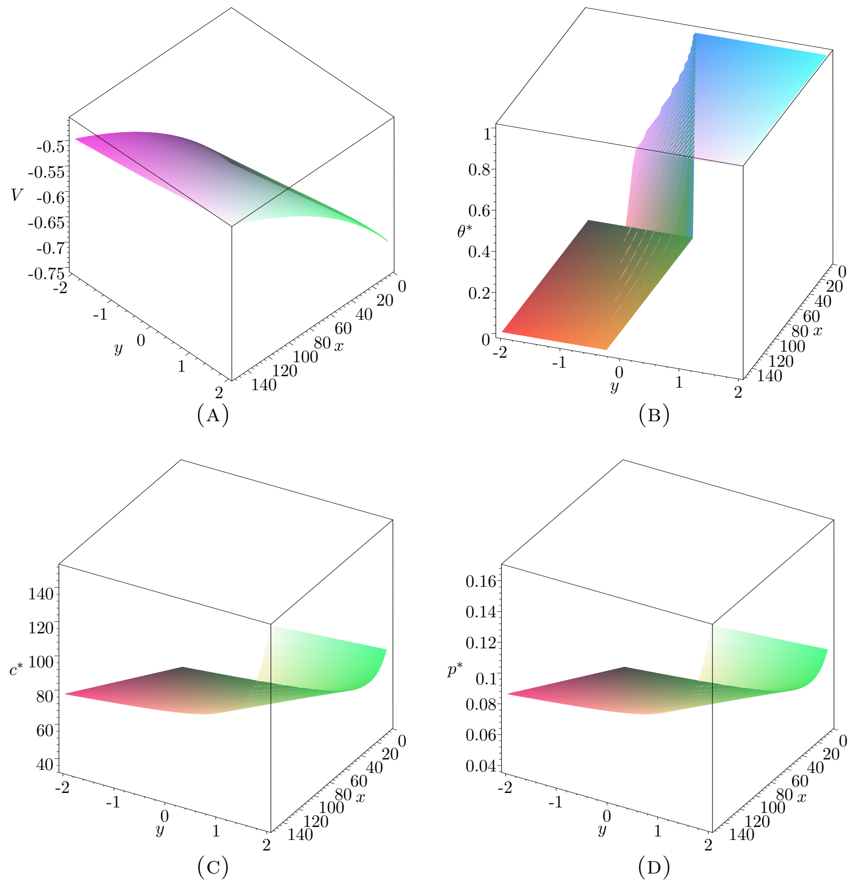

FIGURE 4. Plots of the value function V (Fig. 4a), optimal portfolio θ^* (Fig. 4b), optimal consumption c^* (Fig. 4c), and optimal life-insurance purchase p^* (Fig. 4d) in terms of the wealth x and the economic indicator y for fixed time $t = 2.5$. The parameters are as given in Figure 1.

process the paths seem to exhibit little to no volatility. This is due to the requirement that the wage earner pays the risk-free interest rate on any debts he might have, eliminating from the wealth process dynamics the risky-asset volatility. When the wealth is positive most paths show the same volatility as those from Figure 3 (the scale at which Figure 3 is displayed makes it hard to see the volatility – zoom in to make such detail more noticeable). More importantly, we remark that, unlike Figure 3, the paths displayed in Figure 6 are not trending up over the whole time period under consideration. Indeed, for long planning horizons, the "human capital" is large and the optimal strategy turns out to be to borrow against future earnings to increase the utility derived from consumption and life-insurance purchase, paying off the debt and accumulating savings as the wage-earner income progressively rises and his retirement time nears. Such behavior is even more notorious with larger planning horizons and increasing values of the utility functions discount rate ϱ. Even though the model under consideration herein is a

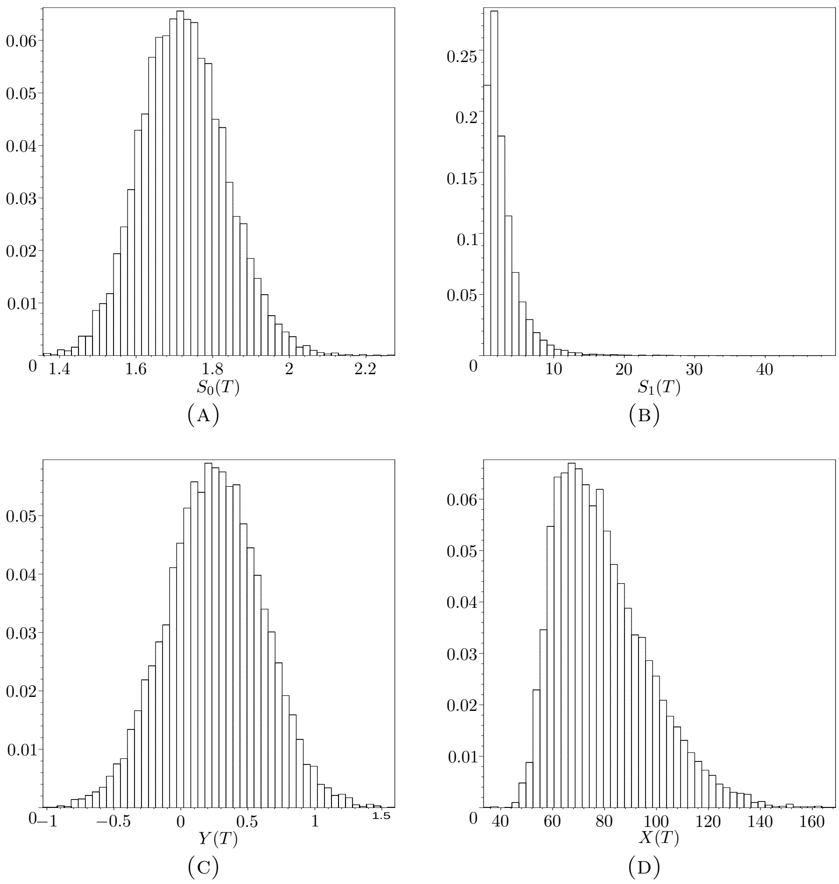

FIGURE 5. Histograms for the time $T = 20$ values of the risk-free asset $S_0(T)$ (Fig. 5a); the risky-asset price $S_1(T)$ (Fig. 5b); the economic indicator $Y(T)$ (Fig. 5c); and the wage-earner wealth $X(T)$ (Fig. 5d). Each histogram was built using 10,000 realizations of the corresponding process. The horizontal axes represent the processes time $T = 20$ values while the vertical axes represent the corresponding relative frequencies. The initial values and parameters are as given in Figure 1.

simple mathematical idealization of very complex personal and economic decisions, the conclusions one can draw from it seem very reasonable, and even natural. While observing Figure 6, one can idealize an agent planning for the long term, taking loans while young to finance consumption (e.g. the purchase of a car or a house) and the purchase of life-insurance to guarantee his family some financial stability, and then slowly but steadily paying down the corresponding mortgages (corresponding to negative value of wealth) while progressing in his career, accumulating also enough wealth to retire later in life.

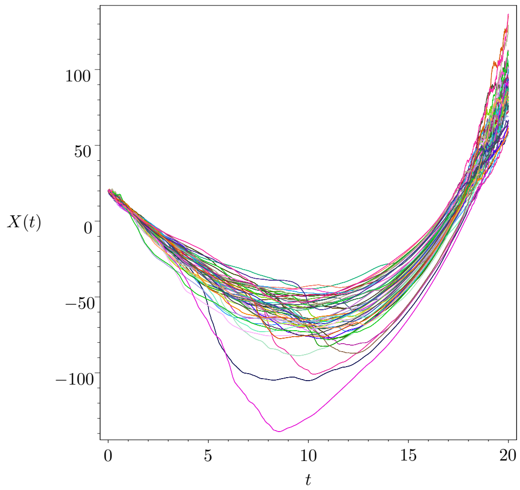

FIGURE 6. Swarm with 50 realizations of the wealth process $X(\cdot)$. The horizontal axis represents time $t \in [0, 20]$ and the vertical axis represents wealth. The initial values and parameters are as given in Figure 1.

5. CONCLUSIONS

We have studied an optimal consumption, investment and life-insurance selection and purchase problem for a wage-earner whose lifetime is uncertain. The problem considered here has the special feature of including a financial market whose assets prices follow a linear SDE with stochastic coefficients depending on the evolution of an underlying economic indicator. We formulated this problem in terms of a stochastic optimal control problem and used dynamic programming techniques to characterize the wage-earner decision-making process, paying particular attention to how the wage-earner decisions are influence by the economic indicator. In the special case of discounted CRRA utility functions we were able to provide a more detailed characterization, combining analytical techniques yielding closed form solution for certain special limiting cases, with an illustrative numerical experiment proving validation to our theoretical results and further insights into the wage-earner decision-making process.

Indeed, based on the "frozen dynamics" limit case for which closed-form solutions are available, we conclude that the wage-earner optimal consumption rate is an increasing function of his wealth while the fraction of wealth invested in the risky-asset is a decreasing function of his wealth. In what concerns the wager-earner life-insurance purchase rate, it may be either increasing with wealth or decreasing with wealth, depending on the sign of a quantity reflecting the agent preferences regarding life-insurance purchase as well as the relative size of the wage-earner hazard rate function to the minimum life insurance contract premium-payout ratio. Under appropriate conditions on the model parameters, we have also seen that the wage-earner optimal consumption rate, optimal life-insurance purchase rate and optimal risky-asset allocation all increase with an improving state of the economy. In addition to validating these conclusions, the numerical experiments of

Sections 4.2.1 and 4.2.2 also stress the importance of the wage-earner "human capital": when the retirement time is sufficiently far into the future this quantity is large and the wage-earner is able to borrow against his future earnings to finance spending in the form of consumption and life-insurance purchase. However, as the wage-earner retirement age starts to approach and the planning horizon shortens correspondingly, it is no longer optimal to borrow against future earnings. Instead, the wage-earner accumulates savings to optimize also the utility of his wealth at the time of retirement, which plays a more relevant role as the retirement age approaches, in addition to the utility derived from his family consumption and from the life-insurance payout in the event of premature death.

Appendix A. Numerical Method

The numerical method described here is inspired in the techniques developed by Souganidis [39] and Fleming and Souganidis [11]. Let $f : [0,T] \times \mathbb{R}^N \times U \to \mathbb{R}^N$, $g : [0,T] \times \mathbb{R}^N \times U \to \mathbb{R}^{N\times M}$, $L : [0,T] \times \mathbb{R}^N \times U \to \mathbb{R}$ and $\psi : \mathbb{R}^N \to \mathbb{R}$ be sufficiently regular functions and let $W(\cdot)$ be an M-dimensional standard Brownian motion defined on an appropriate probability space. Consider a stochastic optimal control problem with state variable dynamics given by a SDE of the form

$$\mathrm{d}X(t) = f(t, X(t), u(t))\mathrm{d}t + g(t, X(t), u(t))\mathrm{d}W(t) \ , \tag{36}$$

and objective functional of the form

$$J(t, x; u) = \mathbb{E}\left[\int_t^T L(s, X(s), u(s))\mathrm{d}s + \psi(X(T)) \,\Big|\, \mathcal{F}_t \right] . \tag{37}$$

Let $\pi = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ be a partition of $[0,T]$. Perform a discretization of (36) to arrive at the following (random) recursive relation

$$\begin{aligned} X_{k+1} &= X_k + f(t_k, X_k, u_k)\Delta_k + g(t_k, X_k, u_k)(\Delta_k)^{1/2}\eta_k \ , \quad k = i_0, \ldots, m-1 \\ X_t &= x \ , \end{aligned}$$

where $i_0 \in \{0, \ldots, m-1\}$ is such that $t \in [t_{i_0}, t_{i_0+1})$, $\Delta_{i_0} = t_{i_0+1} - t$, $\Delta_k = t_{k+1} - t_k$ for $k = i_0 + 1, \ldots, m-1$, and $\eta_0, \ldots, \eta_{m-1}$ are independent and identically distributed random vectors of dimension M with $\mathbb{E}[\eta_k] = 0$ and $\mathbb{E}[\eta_k \eta_k^T] =$ identity. To avoid any measurability issues and simplify the numerical method, let the random variables η_k, $k = 0, \ldots, m-1$, take only finitely many values.

Denote by $C_b^{0,1}(\mathbb{R}^N)$ the set of bounded, Lipschitz continuous functions on $\mathbb{R}^N$. For any $\varphi \in C_b^{0,1}(\mathbb{R}^N)$, $\Delta > 0$, and random variable η with the same distribution as the random variables $\eta_0, \ldots, \eta_{m-1}$, define the operator

$$F(t,\Delta)[\varphi(x)] = \max_{u\in U} \left\{ \mathbb{E}\left[\varphi\left(x + f(t,x,u)\Delta + g(t,x,u)\Delta^{1/2}\eta \right) \right] + L(t,x,u)\Delta \right\}$$

and approximate the value function associated with the stochastic optimal control problem (36)-(37) by the function $V_\pi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$, given by

$$V_\pi(t,x) = F(t, t_{i_0+1} - t) \prod_{k=i_0+1}^{m-1} F(t_k, t_{k+1} - t_k)[\psi(x)] \ ,$$

where $\psi \in C_b^{0,1}(\mathbb{R}^N)$ is the boundary condition associated with the stochastic optimal control problem (36)-(37), i.e. $J(T, x; u) = \psi(x)$. We remark that the definition of the approximated value function V_π given above mimics a time-discretization of the dynamic programing principle associated with the optimal control problem (36)-(37).

Suppose that U is a compact metric space, the functions f, g and L are bounded, uniformly continuous, and Lipschitz continuous with respect to (t, x) uniformly in $u \in U$, and the function ψ is bounded and Lipschitz continuous. Then, it can be proved that the limit $V = \lim_{|\pi|\to 0} V_\pi$ exists locally uniformly and is the unique viscosity solution of HJB equation associated with (36)-(37). The proof of such fact falls outside of the scope of this work, but we believe that the argument of [11, Thm 3.1 and Prop. 2.5] can be adjusted to obtain a complete proof.

The numerical solutions discussed in Section 4.2 were obtained using a Fortran implementation of the method described above to approximate both the value function and the optimal controls of the problem under consideration herein, coupled with Fortran implementations of a steepest descent algorithm to solve the finite-dimensional optimization problem defining the corresponding operator $F(t, \Delta)$, and of an Euler–Maruyama method to numerically approximate the solutions of the SDE describing the wealth process time evolution.

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