OPTIMAL CONSUMPTION, INVESTMENT AND LIFE INSURANCE SELECTION UNDER ROBUST UTILITIES

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ABSTRACT. We study the problem faced by a wage earner with an uncertain lifetime who has access to a Black-Scholes type financial market consisting of one risk-free security and one risky asset. His preferences relative to consumption, investment and life insurance purchase are described by a robust expected utility. We rewrite this problem in terms of a two-player zero-sum stochastic differential game and we derive the wage earner optimal strategies for a general class of utility functions, studying the case of discounted constant relative risk aversion utility functions with more detail.

Keywords: Optimal consumption and investment; Life insurance; Uncertain lifetime; Stochastic differential games

1. INTRODUCTION

This paper extends the contributions of Yaari (1965), who introduced an optimal consumption problem for an individual with uncertain time of death within a deterministic investment environment setup, and Hakansson (1969, 1971), who studied Yaari's model in the discrete-time case. Merton (1969, 1971) studied a closely related problem: a continuous-time optimal consumption and investment problem without any life insurance component, with Richard (1975) combining these earlier approaches to obtain a continuous-time model for optimal consumption, investment and life insurance purchase.

This class of problems has been the subject of intense research activity in recent years. For instance, Pliska and Ye (2007) studied a continuous-time model combining the more realistic features of all of those present in the contemporary literature. Duarte et al. (2014) extended their approach by considering a financial market comprised of one risk-free security and an arbitrary number of risky securities, Guambe and Kufakunesu (2015) broadened it to the case of financial markets determined by geometric Itô–Lévy jump processes, while Mousa et al. (2016) added to this problem an insurance market with a fixed number of life insurance providers. Liang and Guo (2016) looked at this class of problems with an underlying incomplete market, for which the stock price has a mean-reverting drift, while Pirvu and Zhang (2012) and Shen and Wei (2016) considered a complete financial market with parameters given by random processes adapted to the Brownian motion filtration. Huang and Milevsky (2008) and Huang et al. (2008) analyzed a portfolio choice problem for the case of mortality contingent claims using general HARA utilities. Kraft and Steffensen (2008) contributed with extensions through a continuous-time multi-state

Markovian framework, Bruhn and Steffensen (2011) considered the case of a two-person household, and Kronborg and Steffensen (2013) studied the problem faced by a wage earner endowed with deterministic labor income and the possibility to invest in a Black Scholes market and to buy life insurance or annuities. Finally, Kwak et al. (2009) studied this problem for a family with two adults with uncertain lifetime receiving deterministic labor income.

The most common optimality criteria are based on functionals of von Neumann-Morgenstern form. Such choice corresponds to an implicit assumption that a financial market investor has complete knowledge regarding the probability measure describing the dynamics of the financial market risky assets. However, such information is seldom available and even sophisticated investors have some degree of uncertainty regarding such probability measure and need to consider several probability measures while planning their investment strategies. To address problems with model uncertainty such as these, Gilboa and Schmeidler (1989) proposed the use of robust utility functionals of the form

$$X \mapsto \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[U(X)] ,$$

where $\mathcal{Q}$ is a set of prior probability measures and X is a random variable on an appropriate probability space. Maccheroni et al. (2006) introduced robust utility functionals of the form

$$X \mapsto \inf_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[U(X)] + \gamma[\mathbb{Q}]\} , \tag{1}$$

where $\gamma[\cdot]$ is a penalty function defined on the set $\mathcal{Q}$ of prior probability measures. For robust utility functionals of this type, the most popular choice for penalty functional has been the *Kuhlback-Leibler* entropy between two measures, given by

$$\gamma[\mathbb{Q}] = \frac{1}{\beta}\mathcal{H}[\mathbb{Q}|\mathbb{P}] \tag{2}$$

for a certain positive penalization parameter β and reference probability measure $\mathbb{P}$. See, for instance, Hansen and Sargent (2001) and Bordigoni et al. (2007) for further details in this topic. See also the results by Baltas and Yannacopoulos (2016) and Baltas et al. (2018) concerning portfolio optimization under uncertainty. More closely connected with life insurance purchase, Liang and Young (2020) consider the problem of determining the optimal robust strategy of an individual seeking to maximize a penalized probability of reaching a bequest goal under uncertainty regarding the drift of the risky asset and the individual hazard rate of mortality. Shen and Su (2019) consider a life-cycle planning problem, sharing some similarities to ours, for an agent seeking to determine robust strategies regarding consumption, investment and life-insurance purchase. Their model contains uncertainty regarding the economic conditions, in both the risky asset and the agent income, as well as the individual hazard rate of mortality. Unlike ours, their model includes a single life-insurance company. Their objective functional also differs from ours on the modeling of all three components (life-time consumption, terminal wealth, and bequest). Moreover, under the setup under consideration herein, we obtain and study explicit solutions for the robust optimal control problem under consideration. Wang et al. (2021) study optimal decisions on consumption, investment and life-insurance purchase for a household over two consecutive generations, both of which are supposed to be ambiguity-averse expected utility maximizers.

Before proceeding any further, we mention alternative formulations and extensions for the problem under consideration herein, broadening its scope and range of applicability. Such formulations may include alternative forms of coefficient randomness such as, for instance, Markov-switching state variable dynamics as in Azevedo et al. (2014) and Temoçin and Weber (2014) or semi-Markov modulated state variable dynamics as in Azevedo et al. (2022), as well as the presence of terms with delay as considered by Savku (2017), Savku et al. (2017) and Savku and Weber (2018). Additional extensions may also concern alternative forms of objective functionals such as in Kara et al. (2019) or Korn and Müller (2022), as well as potential applications to pension funds such as in Arık et al. (2023) and Baltas et al. (2022).

In this paper, we consider a robust optimal consumption, investment and life insurance selection-and-purchase problem subject to an underlying financial market whose assets prices evolve according to a linear stochastic differential equation (SDE). The relevance of our analysis lies on the use of a more realistic model describing the preferences of a wage-earner interacting with an insurance market composed of several insurance companies. Specifically, we assume that the wage-earner is a fully rational agent whose ultimate aim is to maximize a robust expected utility describing his preferences towards consumption, life insurance selection and purchase, and wealth at retirement time. To address such problem, we employ a stochastic differential games (SDGs) approach to the maximization of such robust utility functionals. This leads to a characterization of the value of the game and the corresponding optimal strategies through a Hamilton-Jacobi-Bellman-Isaacs (HJBI) type partial differential equation (PDE). One of our key conclusions is that a typical wage-earner would, at any instant of time, buy life insurance from the company offering the cheaper premium for the same level of protection. Additionally, in the case of discounted constant relative risk aversion utilities, we prove that the optimal consumption rate is an increasing function of both the wage earner wealth and the present value of his cumulative future earnings with respect to an appropriately chosen discount rate. Furthermore, we provide a detailed characterization of some qualitative features of the optimal portfolio. We also characterize the least-favorable martingale measure $\mathbb{Q} \in \mathcal{Q}$ in the sense of Föllmer and Gundel (2006) as corresponding to absorbing eventual profits available to the wage earner when investing in the financial market.

This paper is organized as follows. In Section 2 we formulate the problem under consideration, describing the underlying financial and life insurance markets, as well as the problem faced by a wage earner with an uncertain lifetime when choosing his optimal strategies for consumption, investment, and life insurance purchase. Then, in Section 3, we rewrite the problem using the specific language of SDGs. In Section 4 we find the optimal robust strategies for the class of power utility functions. Finally, in Section 5, we summarize our conclusions.

2. Problem Formulation

Let $T > 0$ be a deterministic finite time horizon and, for every $t \in [0, T]$ and $s \in [t, T]$, let $\Omega^\omega_{t,s}$ be the set of $\mathbb{R}^M$-valued continuous functions on $[t, s]$ taking the value 0 at t, i.e.

$$\Omega^\omega_{t,s} = \left\{\omega \in C\left([t, s]; \mathbb{R}^M\right) : \omega(t) = 0\right\} .$$

Let $\mathcal{F}^\omega_{t,u}$ be the σ-algebra generated by paths $\omega \in \Omega^\omega_{t,s}$ up to some time $u \in [t, s]$ with $\mathbb{F}^\omega_{t,s} = \left\{\mathcal{F}^\omega_{t,u} : u \in [t, s]\right\}$ being the corresponding filtration. When endowed with the Wiener measure $\mathbb{P}^\omega_{t,s}$ on $\mathcal{F}^\omega_{t,s}$, $\Omega^\omega_{t,s}$ becomes a classical Wiener space. Let $B^t = \left\{B^t(s) : s \in [t, T], B^t(t) = 0\right\}$ be a Brownian motion on the filtered probability space $(\Omega^\omega_{t,T}, \mathcal{F}^\omega_{t,T}, \mathbb{F}^\omega_{t,T}, \mathbb{P}^\omega_{t,T})$.

We also consider an absolutely continuous (with respect to the Lebesgue measure) random variable τ defined on the probability space $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P}^\tau)$ and taking values on $\mathbb{R}^+ = (0, +\infty)$ representing the wage-earner random life time.

For each $t \in [0, T]$, the probability measure $\mathbb{P}^\tau$ of the random variable τ induces a conditional probability measure on $\Omega^\tau_t = (t, \infty)$ determined by

$$\mathbb{P}^\tau_t(\tau \in A) = \mathbb{P}^\tau(\tau \in A | \tau > t) , \qquad A \in \mathcal{F}^\tau_t ,$$

where $\mathcal{F}^\tau_t = \mathcal{B}(\Omega^\tau_t)$ denotes the Borel σ-algebra of Ω^τ_t. Additionally, for each $t \in [0, T]$ the random variable τ is assumed to be independent of the filtration $\mathbb{F}^\omega_{t,T}$ generated by the Brownian motion B^t. Using independence between the Brownian motion B^t and the random variable τ, we define the sample space Ω_t as the direct product

$$\Omega_t = \Omega^\omega_{t,T} \times \Omega^\tau_t ,$$

defining accordingly the probability measure

$$\mathbb{P}_t = \mathbb{P}^\omega_{t,T} \otimes \mathbb{P}^\tau_t$$

and the σ-algebra $\mathcal{F}_t$ on Ω_t as the completion of $\mathcal{F}^\omega_{t,T} \otimes \mathcal{F}^\tau_t$ with respect to $\mathbb{P}_t$.

2.1. **Financial market model.** We define a continuous-time financial market consisting of one risk-free asset and one risky-asset. More precisely, we assume that the prices of the risk-free asset $\{S_0(s) : s \in [t, T]\}$ and the risky asset $\{S_1(s) : s \in [t, T]\}$ evolve according to the SDEs

$$\begin{aligned} \mathrm{d}S_0(s) &= r(s)S_0(s)\mathrm{d}s, \\ \mathrm{d}S_1(s) &= \mu(s)S_1(s)\mathrm{d}s + \sigma(s)S_1(s)\mathrm{d}B^t(s) , \qquad s \geq t \end{aligned}$$

with positive initial conditions $S_0(t) = s_0$ and $S_1(t) = s_1$.

Assumption 2.1. *For each $t \in [0, T]$, the riskless interest rate $r(\cdot)$, the risky-asset appreciation rate $\mu(\cdot)$ and the risky-asset volatility $\sigma(\cdot)$, are deterministic continuous functions on the interval $[t, T]$. Additionally, we assume that:*

(i) *the risk-free interest rate $r(\cdot)$ is positive on the interval $[t, T]$;*
(ii) *there exist positive real numbers σ^- and σ^+ such that $\sigma^- < \sigma(s) < \sigma^+$ for all $s \in [t, T]$;*

(iii) *there exists a real valued* $\mathbb{F}^{\omega}_{t,T}$*-progressively measurable process* $\zeta(\cdot) \in \mathbb{R}$ *such that for all* $s \in [t,T]$*, it holds that*

$$\mu(s) - r(s) = \sigma(s)\zeta(s) \qquad a.s.$$

and the following two conditions hold

$$\int_t^T \zeta^2(s)\ \mathrm{d}s < \infty \qquad a.s.$$

$$\mathbb{E}_{\mathbb{P}^{\omega}_{t,T}}\left[\exp\left(-\int_t^T \zeta(s)\ \mathrm{d}B^t(s) - \frac{1}{2}\int_t^T \zeta^2(s)\ \mathrm{d}s\right)\right] = 1\ .$$

The existence of the market price of risk $\zeta(\cdot)$ ensures the absence of arbitrage opportunities in the financial market defined above. See the excelent monograph by Karatzas and Shreve (1998) for further details on market viability.

2.2. **Life insurance market model.** We assume that the wage-earner is alive at time $t = 0$ and his lifetime is the nonnegative continuous random variable τ defined earlier.

Assumption 2.2. *The random variable* τ *has distribution function* $G^- : [0,\infty) \to [0,1]$ *with bounded and Lipschitz continuous density* $g : [0,\infty) \to \mathbb{R}^+$ *such that*

$$G^-(t) = \mathbb{P}^{\tau}(\tau \le t) = \int_0^t g(s)\ \mathrm{d}s\ .$$

Recall that the *survival function* of τ, $G^+ : [0,\infty) \to [0,1]$, is defined as the probability that the random variable τ exceeds time t, that is

$$G^+(t) = \mathbb{P}^{\tau}(\tau > t) = 1 - G^-(t)\ .$$

For all $0 \le t \le s$, denote by $G^+(s;t)$ and $G^-(s;t)$ the conditional probabilities

$$\begin{aligned} G^+(s;t) &= \mathbb{P}^{\tau}_t(\tau > s) = \mathbb{P}^{\tau}(\tau > s|\tau > t) \\ G^-(s;t) &= \mathbb{P}^{\tau}_t(\tau \le s) = \mathbb{P}^{\tau}(\tau \le s|\tau > t)\ . \end{aligned} \tag{3}$$

Moreover, notice that for each fixed $t \in [0,T]$, $G^-(s;t)$ is the probability distribution function of a continuous random variable and let $g^-(s;t)$ denote the conditional density function associated with it, that is

$$g^-(s;t) = \frac{\mathrm{d}}{\mathrm{d}s} G^-(s;t)\ .$$

Finally, observe that $g^-(t;t)$ is precisely the *hazard rate function* associated with the random variable τ.

The life insurance market under consideration herein is composed by K insurance companies, with each insurance company continuously offering life insurance contracts. The wage-earner buys life insurance from the insurance company k by paying a *premium insurance rate* $p_k(t)$ for each $k = 1,2,\ldots,K$. The insurance contracts are like term insurance, with an infinitesimally small term. If the wage-earner dies at time $\tau \le T$ while buying insurance at the rate $p_k(t)$ from the k^{th} insurance company, then that insurance company pays an amount

$$Z_k(\tau) = \frac{p_k(\tau)}{\eta_k(\tau)}$$

to his estate, where $\eta_k : [0,T] \to \mathbb{R}^+$ is the k^{th} *insurance company premium-payout ratio.*

Assumption 2.3. *For every $k \in \{1,\ldots,K\}$, the k^{th} insurance company premium-payout ratio $\eta_k(t)$ is a continuous and deterministic function. Additionally, we will assume that the K insurance companies under consideration here offer pairwise distinct contracts in the sense that $\eta_{k_1}(t) \neq \eta_{k_2}(t)$ for every $k_1 \neq k_2$ and Lebesgue-almost-every $t \in [0,T]$.*

As a consequence of Assumption 2.3, we have that the $K \times K$ symmetric matrix $\eta(t)^T \eta(t)$, where $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_K(t))^T \in (\mathbb{R}^+)^K$, is non-singular for Lebesgue almost-every $t \in [0,T]$.

The life insurance contract ends when the wage-earner dies or achieves retirement age, whichever happens first. Therefore, the wage-earner's total legacy to his estate in the event of a premature death at time $\tau \leq T$ is given by

$$Z(\tau) = X(\tau) + \sum_{k=1}^{K} \frac{p_k(\tau)}{\eta_k(\tau)} \ , \tag{4}$$

where $X(t)$ denotes the wage-earner's wealth at time $t \in [0,T]$.

We represent the wage-earner life insurance purchase rate as a vector

$$p(t) = (p_1(t), p_2(t), \ldots, p_K(t))^T \in (\mathbb{R}_0^+)^K \ ,$$

where for each $k \in \{1,2,\ldots,K\}$, the quantity $p_k(t)$ denotes the life insurance purchase rate from the k^{th} insurance company at time $t \in [0, \min\{\tau, T\}]$. Note that a zero component in $p(t)$ represents the absence of any life insurance contract between the wage-earner and a certain insurance company.

2.3. **The wealth process.**

Let us define the *random horizon* ξ as

$$\xi = \min\{\tau, T\}$$

and notice that ξ takes values on the interval $[0,T]$.

Given an initial time $t \in [0,T]$, the wage-earner receives income $i(s)$ at a continuous rate during the period $[t, \xi]$, i.e. the income will be terminated either by his death or his retirement, whichever happens first.

Assumption 2.4. *The income function $i : [0,T] \to \mathbb{R}_0^+$ is a deterministic Borel-measurable function satisfying the integrability condition:*

$$\int_0^T i(s) \, \mathrm{d}s < \infty \ .$$

Let $c(s)$ denote the consumption rate adopted by the wage-earner at time $s \in [t,T]$ and let $\theta(s)$ denote the fraction of the wage-earner's wealth allocated to the risky asset S_1 at time $s \in [t,T]$. Clearly, the wage-earner invests $1 - \theta(s)$ of her wealth on the risk-free asset S_0.

Assumption 2.5. *For every $t \in [0,T]$, the control variables satisfy the following:*

(i) *the* consumption process $\{c(s) : s \in [t,T]\}$ *is a* $\mathbb{F}^{\omega}_{t,T}$*-progressively measurable nonnegative process satisfying:*

$$\int_t^T c(s)\ \mathrm{d}s < \infty \qquad a.s..$$

(ii) *the* portfolio process $\{\theta(s) : s \in [t,T]\}$ *is* $\mathbb{F}^{\omega}_{t,T}$*-progressively measurable and such that*

$$\int_t^T \theta^2(s)\ \mathrm{d}s < \infty \qquad \text{a.s..}$$

(iii) *for all* $k = 1, 2, \dots, K$*, the* k^{th} company premium insurance rate $\{p_k(s) : s \in [t,T]\}$ *is nonnegative a* $\mathbb{F}^{\omega}_{t,T}$*-predictable process, i.e.* $p_k(t)$ *is measurable with respect to the smallest* σ*-algebra on* $\mathbb{R}^+_0 \times \Omega$ *such that all left-continuous and adapted processes are measurable.*

The *wealth process* $X(s)$, for $s \in [t,T]$, is defined through the SDE

$$\begin{aligned} \mathrm{d}X(s) \quad &= \quad \left(i(s) - c(s) - \sum_{k=1}^{K} p_k(s) + \Big((1-\theta(s))r(s) + \theta(s)\mu(s) \Big) X(s) \right) \mathrm{d}s \\ &\quad + \theta(s)X(s)\sigma(s)\mathrm{d}B^t(s)\ , \end{aligned} \tag{5}$$

with initial condition $X(t) = x$.

2.4. **The robust consumption, investment and life insurance purchase problem.** Define $\mathcal{Y}_t$ as the set of $\mathbb{R}$-valued $\mathbb{F}^{\omega}_{t,T}$-progressively measurable stochastic processes $\{y(s) : s \in [t,T]\}$ satisfying

$$\mathbb{P}^{\omega}_{t,T}\left[\int_t^T y^2(s)\ \mathrm{d}s < \infty \right] = 1$$

and the Novikov condition

$$\mathbb{E}_{\mathbb{P}^{\omega}_{t,T}}\left[\exp\left(\frac{1}{2} \int_t^T y^2(s)\ \mathrm{d}s \right) \right] < \infty\ . \tag{6}$$

Note that the Novikov condition (6) ensures that the stochastic process

$$M(s) = \exp\left\{ \int_t^s y(u)\ \mathrm{d}B^t(u) - \frac{1}{2}\int_t^s y^2(u)\ \mathrm{d}u \right\},\ s \in [t,T] \tag{7}$$

is a $(\mathbb{F}^{\omega}_{t,T}, \mathbb{P}^{\omega}_{t,T})$ martingale. Additionally, each element $y(\cdot) \in \mathcal{Y}_t$ uniquely determines a probability measure $\mathbb{Q}_t(y)$ on $\mathcal{F}^{\omega}_{t,T}$ with the property that

$$\mathrm{d}\mathbb{Q}_t(y) = M(T)\ \mathrm{d}\mathbb{P}^{\omega}_{t,T}\ , \tag{8}$$

where $M(\cdot)$ is as given in (7). In what follows, we denote by $\mathcal{Q}$ the set of all probability measures $\mathbb{Q}_t(y)$ on $\mathcal{F}^{\omega}_{t,T}$ for which property (8) holds. Finally, we denote by $\tilde{\mathcal{Q}}$ the set of probability measures

$$\tilde{\mathcal{Q}} = \{\mathbb{Q}_t(y) \otimes \mathbb{P}^{\tau}_t : \mathbb{Q}_t(y) \in \mathcal{Q}\}\ .$$

Let $U(t,c,x)$ be the wage-earner's utility derived from a consumption level $c \in [0,+\infty)$ at time t while holding wealth x, let $L(t,z)$ be the utility function for the size of the wage-earners's legacy z, as given in (4), in case death occurs at time $t \in [0,T]$, and let $\Psi(x)$ be the utility obtained from holding wealth x at retirement time T.

Assumption 2.6. *The following conditions hold for the utility functions U, L and Ψ:*

a) $U : D(U) \subseteq [0,T] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ *is such that for every* $(t,x) \in [0,T] \times \mathbb{R}$ *the function* $U(t,\cdot,x)$ *is twice differentiable, strictly increasing, strictly concave, and its first derivative maps* $\mathbb{R}^+$ *onto* $\mathbb{R}^+$*. Additionally, both* U *and the partial derivative of* U *with respect to its second variable are continuous functions of* (t,x)*.*

b) $L : D(L) \subseteq [0,T] \times \mathbb{R} \to \mathbb{R}$ *is such that for every* $t \in [0,T]$ *there exists an interval* $I(t)$ *of the form* $(a(t),+\infty)$ *on which* $L(t,\cdot)$ *is twice differentiable, strictly increasing, strictly concave, and its first derivative maps* $I(t)$ *onto* $\mathbb{R}^+$*. Additionally, both* L *and the partial derivative of* L *with respect to its second variable are continuous functions of* t*.*

c) $\Psi : \mathbb{R} \to \mathbb{R}$ *is a twice differentiable, strictly increasing and strictly concave function.*

We now define the set of admissible controls. The formulation used here is standard in optimal control (see, e.g. Øksendal and Sulem (2005) or Yong and Zhou (1999)).

Definition Denote by $\mathcal{A}(t,x)$ the set of admissible control processes on $[t,T]$, i.e. triples (c,θ,p) satisfying Assumption 2.6 for which:

i) the SDE (5) subject to the boundary condition $X(t) = x$ has a unique solution $X_{t,x}^{c,\theta,p}(\cdot)$ under the choice of control $(c,\theta,p) \in \mathcal{A}(t,x)$;

ii) the following integrability conditions hold:

$$\mathbb{E}_{\mathbb{P}_{t,T}^{\omega}}\left[\int_t^T \left|U(s,c(s),X_{t,x}^{c,\theta,p}(s))\right| \, \mathrm{d}s\right] < \infty$$

$$\mathbb{E}_{\mathbb{P}_{t,T}^{\omega}}\left[\int_t^T \left|L\left(s,Z_{t,x}^{c,\theta,p}(s)\right)\right| \, \mathrm{d}s\right] < \infty$$

$$\mathbb{E}_{\mathbb{P}_{t,T}^{\omega}}\left[\left|\Psi(X_{t,x}^{c,\theta,p}(T))\right|\right] < \infty \ ,$$

where $Z_{t,x}^{c,\theta,p}(\cdot)$ is as given in (4) under the choice of control $(c,\theta,p) \in \mathcal{A}(t,x)$.

Let us define an entropy functional of Kuhlback-Leibler type consistent with the random horizon ξ as

$$\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}_{t,T}^{\omega}] \quad = \quad \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}_t^{\tau}}\Big[\log M(\xi)\Big] \ , \tag{9}$$

where $M(\cdot)$ is as given in (7). Given $(c,\theta,p) \in \mathcal{A}(t,x)$, we consider a payoff functional of the same form as the robust utility functional (1)-(2), namely

$$\begin{aligned} J(t,x;c(\cdot),\theta(\cdot),p(\cdot)) = \inf_{\tilde{\mathbb{Q}}_t(y)\in\tilde{\mathcal{Q}}} \Bigg\{ & \mathbb{E}_{\tilde{\mathbb{Q}}_t(y)}\Bigg[\int_t^\xi U\left(s,c(s),X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)\right) \mathrm{d}s \\ & +\Psi(X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(T))I_{(T,+\infty)}(\tau) + L(\tau,Z(\tau))I_{[0,T]}(\tau)\Bigg] \\ & +\frac{1}{\beta}\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}_{t,T}^{\omega}]\Bigg\} \ , \end{aligned} \tag{10}$$

where $X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)$ is now the solution of equation (5) associated with the admissible strategies $(c,\theta,p) \in \mathcal{A}(t,x)$, the initial condition x when $s=t$, and the least favorable probability measure $\tilde{\mathbb{Q}}_t(y) = \mathbb{Q}_t(y) \otimes \mathbb{P}_t^{\mathcal{T}} \in \tilde{\mathcal{Q}}$.

The wage-earner's goal is to solve the robust consumption, investment and life insurance purchase problem

$$\sup_{(c,\theta,p)\in\mathcal{A}(t,x)} J(t,x;c(\cdot),\theta(\cdot),p(\cdot)) \; . \tag{11}$$

In the next section we will rephrase the robust optimization problem above from the point of view of SDGs.

3. An equivalent stochastic differential game

In this section we formulate the robust consumption, investment and life insurance purchase problem (11) in the language of SDGs. We start by obtaining a convenient alternative representation for the payoff function defined in (10) before proceeding to formulate the robust utility problem under consideration as a SDG.

3.1. An equivalent representation for the payoff functional.

Recall the definition of the set $\mathcal{Y}_t$ given in Section 2.4. Girsanov Theorem yields that the stochastic process $\tilde{B}^t = \{\tilde{B}^t(s) : s \in [t,T]\}$ with decomposition given by

$$\tilde{B}^t(s) = B^t(s) - \int_t^s y(u)\,\mathrm{d}u \; , \; t \le s < T \tag{12}$$

is a one-dimensional $(\mathbb{F}^{\omega}_{t,T}, \mathbb{Q}_t(y))$ Brownian motion, where $\mathbb{Q}_t(y)$ is the element of $\mathcal{Q}$ determined by the process $y(\cdot) \in \mathcal{Y}_t$.

The change of measure from $\mathbb{P}^{\omega}_{t,T}$ to $\mathbb{Q}_t(y)$ will produce a change in the drift part of the SDE (5) as detailed in the following proposition.

Lemma 3.1. *Let $\mathbb{Q}_t(y)$ denote the probability measure on $\mathcal{Q}$ associated with the stochastic process $y(\cdot) \in \mathcal{Y}_t$. The wealth process $X(\cdot)$ under the equivalent probability measure $\mathbb{Q}_t(y) \in \mathcal{Q}$ is determined by*

$$\begin{aligned}
\mathrm{d}X(s) &= \Bigg(i(s) - c(s) - \sum_{k=1}^{K} p_k(s) \\
&\quad + \Big((1-\theta(s))r(s) + \theta(s)\left(\mu(s) + \sigma(s)y(s)\right) \Big) X(s) \Bigg) \mathrm{d}s \\
&\quad + \theta(s)X(s)\sigma(s)\mathrm{d}\tilde{B}^t(s) \\
X(t) &= x \; .
\end{aligned} \tag{13}$$

Proof. Substituting equation (12) into (5), yields

$$\begin{aligned}\mathrm{d}X(s) &= \left(i(s) - c(s) - \sum_{k=1}^{K} p_k(s) + \Big((1-\theta(s))r(s) + \theta(s)\mu(s)\Big)X(s)\right)\mathrm{d}s \\ &\quad +\theta(s)X(s)\sigma(s)\mathrm{d}\tilde{B}^t(s) + \theta(s)X(s)\sigma(s)y(s)\mathrm{d}s \\ &= \left(i(s) - c(s) - \sum_{k=1}^{K} p_k(s)\right. \\ &\quad \left.+\Big((1-\theta(s))r(s) + \theta(s)\left(\mu(s) + \sigma(s)y(s)\right)\Big)X(s)\right)\mathrm{d}s \\ &\quad +\theta(s)X(s)\sigma(s)\mathrm{d}\tilde{B}^t(s)\ ,\end{aligned}$$

as required. □

The result below provides a characterization of the entropy functional (9) in terms of the stochastic processes $y(\cdot) \in \mathcal{Y}_t$.

Lemma 3.2. *Let $\mathbb{Q}_t(y)$ denote the probability measure on $\mathcal{Q}$ associated with the stochastic process $y(\cdot) \in \mathcal{Y}_t$. The entropy functional $\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}^\omega_{t,T}]$ is given by*

$$\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}^\omega_{t,T}] = \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^\tau_t}\left[\frac{1}{2}\int_t^\xi y^2(s)\ \mathrm{d}s\right]\ , \tag{14}$$

where $y(\cdot)$ is the element of $\mathcal{Y}_t$ associated with $\mathbb{Q}_t(y) \in \mathcal{Q}$.

Proof. By definition, we have that

$$\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}^\omega_{t,T}] \ = \ \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^\tau_t}\Big[\log M(\xi)\Big]\ .$$

Given the representation (7) for $M(\cdot)$, we obtain

$$\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}^\omega_{t,T}] \ = \ \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^\tau_t}\Big[\int_t^\xi y(s)\ \mathrm{d}B^t(s) - \frac{1}{2}\int_t^\xi y^2(s)\ \mathrm{d}s\Big]\ .$$

Combining the equality above with the decomposition (12), we obtain

$$\mathcal{H}_\xi[\mathbb{Q}_t(y)|\mathbb{P}^\omega_{t,T}] \ = \ \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^\tau_t}\Big[\int_t^\xi y(s)\ \mathrm{d}\tilde{B}^t(s) + \frac{1}{2}\int_t^\xi y^2(s)\ \mathrm{d}s\Big]\ .$$

Noticing that $\tilde{B}^t$ is an $(\mathbb{F}^\omega_{t,T}, \mathbb{Q}_t(y))$ one-dimensional Brownian motion, the expected value of the Itô integral is zero and we arrive at (14). □

Proposition 3.3. *Let $\mathbb{Q}_t(y)$ denote the probability measure on $\mathcal{Q}$ associated with the stochastic process $y(\cdot) \in \mathcal{Y}_t$. The payoff functional J defined in (10) admits the representation*

$$\begin{aligned}&J(t, x; c(\cdot), \theta(\cdot), p(\cdot)) = \\ &\quad \inf_{y(\cdot)\in\mathcal{Y}_t} \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^\tau_t}\Big[\int_t^\xi U\left(t, c(s), X^{c,\theta,p,\mathbb{Q}_t(y)}_{t,x}(s)\right) + \frac{1}{2\beta}y^2(s)\ \mathrm{d}s \\ &\qquad +\Psi(X^{c,\theta,p,\mathbb{Q}_t(y)}_{t,x}(T))I_{(T,+\infty)}(\tau) + L(\tau, Z(\tau))I_{[0,T]}(\tau)\Big]\ .\end{aligned} \tag{15}$$

Proof. Given the robust utility functional (10), the characterization of the set $\mathcal{Q}$ in terms of $\mathcal{Y}_t$ and that of $\tilde{\mathcal{Q}}$ in terms of $\mathcal{Q}$, we obtain that J is given by

$$\begin{aligned}
J(t,x;c(\cdot),\theta(\cdot),p(\cdot)) &= \inf_{\tilde{\mathbb{Q}}_t(y)\in\tilde{\mathcal{Q}}} \mathbb{E}_{\tilde{\mathbb{Q}}_t}\Bigg[\int_t^{\xi} U\left(t,c(s),X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)\right)\,\mathrm{d}s \\
&\quad +\Psi(X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(T))I_{(T,+\infty)}(\tau)+L(\tau,Z(\tau))I_{[0,T]}(\tau) \\
&\quad +\frac{1}{\beta}\mathcal{H}_{\xi}[\mathbb{Q}_t(y)|\mathbb{P}^{\omega}_{t,T}]\Bigg] \\
&= \inf_{\mathbb{Q}_t(y)\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^{\tau}_t}\Bigg[\int_t^{\xi} U\left(t,c(s),X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)\right)\,\mathrm{d}s \\
&\quad +\Psi(X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(T))I_{(T,+\infty)}(\tau)+L(\tau,Z(\tau))I_{[0,T]}(\tau) \\
&\quad +\frac{1}{\beta}\mathcal{H}_{\xi}[\mathbb{Q}_t(y)|\mathbb{P}^{\omega}_{t,T}]\Bigg] \\
&= \inf_{y(\cdot)\in\mathcal{Y}_t} \mathbb{E}_{\mathbb{Q}_t(y)\otimes\mathbb{P}^{\tau}_t}\Bigg[\int_t^{\xi} U\left(t,c(s),X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)\right)\,\mathrm{d}s \\
&\quad +\Psi(X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(T))I_{(T,+\infty)}(\tau)+L(\tau,Z(\tau))I_{[0,T]}(\tau) \\
&\quad +\frac{1}{\beta}\mathcal{H}_{\xi}[\mathbb{Q}_t(y)|\mathbb{P}^{\omega}_{t,T}]\Bigg].
\end{aligned}$$

Combining the identity above with Lemma 3.2 and rearranging terms, we arrive at (15). □

The following Lemma enables the transformation of the optimal control problem described above to an equivalent one with a fixed planning horizon. The statement presented below extends the analogous result by Ye (2006) to the case of robust expected utilities.

Lemma 3.4. *Suppose that Assumptions 2.1–2.6 hold and let* $(c,\theta,p)\in\mathcal{A}(t,x)$*. If the random variable* τ *is independent of the filtration* $\mathbb{F}^{\omega}_{t,T}$*, then*

$$\begin{aligned}
&J(t,x;c(\cdot),\theta(\cdot),p(\cdot)) \\
&\quad= \inf_{y(\cdot)\in\mathcal{Y}_t}\mathbb{E}_{\mathbb{Q}_t(y)}\Bigg[\int_t^T\Bigg\{G^+(s;t)\Bigg(U\left(t,c(s),X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)\right)+\frac{1}{2\beta}y^2(s)\Bigg) \\
&\qquad\qquad + g^-(s;t)L\left(s,Z(s)\right)\Bigg\}\mathrm{d}s+G^+(T;t)\ \Psi(X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(T))\Bigg],
\end{aligned}$$

where the conditional probabilities $G^+(s;t)$ *and* $g^-(s;t)$ *are as given* (3)*.*

3.2. **SDG Formulation.** In what follows we characterize the optimal consumption, investment and life insurance purchase strategies under a robust expected utility in terms of the equilibria of an appropriate two-player zero-sum SDG.

Relying on Proposition 3.3 and Lemma 3.4, we define a payoff functional, which we will denote as $\mathbb{J}$, as follows

$$\begin{aligned}
&\mathbb{J}(t,x;c(\cdot),\theta(\cdot),p(\cdot),y(\cdot)) \\
&= \mathbb{E}_{\mathbb{Q}_t(y)}\Bigg[\int_t^T \bigg\{G^+(s;t)\bigg(U\left(t,c(s),X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(s)\right)+\frac{1}{2\beta}y^2(s)\bigg) \\
&\qquad + g^-(s;t)L\left(s,Z(s)\right)\bigg\}\mathrm{d}s + G^+(T;t)\ \Psi(X_{t,x}^{c,\theta,p,\mathbb{Q}_t(y)}(T))\Bigg] ,
\end{aligned} \tag{16}$$

where $\mathbb{Q}_t(y)$ denotes the probability measure on $\mathcal{Q}$ associated with the stochastic process $y(\cdot) \in \mathcal{Y}_t$.

The *lower value function* of the SDG with state variable dynamics given by (13) and payoff functional (16), is defined as

$$V^-(t,x) = \inf_{y\in\mathcal{Y}_t} \sup_{(c,\theta,p)\in\mathcal{A}(t,x)} \mathbb{J}(t,x;c(\cdot),\theta(\cdot),p(\cdot),y(\cdot)) \ . \tag{17}$$

The corresponding *upper value function* is defined as

$$V^+(t,x) = \sup_{(c,\theta,p)\in\mathcal{A}(t,x)} \inf_{y\in\mathcal{Y}_t} \mathbb{J}(t,x;c(\cdot),\theta(\cdot),p(\cdot),y(\cdot)) \ . \tag{18}$$

The next result provides the HJBI equations for the value functions (17) and (18). We skip its proof and refer the interested reader to Ferreira et al. (2019).

Theorem 3.5 (HJBI equation)**.** *Suppose that Assumptions 2.1–2.6 hold and that the value functions V^- and V^+ are $C^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$. Then, V^- and V^+ are the solution to the HJBI equations*

$$\begin{cases} W_t(t,x) - g^-(t;t)W(t,x) + \mathbb{H}^-(t,x,W_x,W_{xx}) = 0 \\ W(T,x) = \Psi(x) \end{cases} \tag{19}$$

and

$$\begin{cases} W_t(t,x) - g^-(t;t)W(t,x) + \mathbb{H}^+(t,x,W_x,W_{xx}) = 0 \\ W(T,x) = \Psi(x) \ , \end{cases} \tag{20}$$

where, for $A, q \in \mathbb{R}$, $x \in \mathbb{R}$ and $t \in [0,T]$, we have

$$\begin{aligned}
\mathbb{H}^-(t,x,q,A) &= \max_{c\geq 0,\theta\in\mathbb{R},p\in(\mathbb{R}_0^+)^K} \min_{y\in\mathbb{R}} H(t,x,c,\theta,p,y,q,A) \\
\mathbb{H}^+(t,x,q,A) &= \min_{y\in\mathbb{R}} \max_{c\geq 0,\theta\in\mathbb{R},p\in(\mathbb{R}_0^+)^K} H(t,x,c,\theta,p,y,q,A) \ ,
\end{aligned}$$

and

$$\begin{aligned}
&H(t,x,c,\theta,p,y,q,A) \\
&= \left(i(t) - c - \sum_{k=1}^K p_k + \Big(r(t) + \theta\left(\mu(t) - r(t) + \sigma(t)y\right)\Big)x\right)q \\
&\quad + \frac{1}{2}\left(\theta x\sigma(t)\right)^2 A + \frac{1}{2\beta}y^2 + U(t,c,x) + g^-(t;t)L\left(t, x + \sum_{k=1}^K \frac{p_k}{\eta_k(t)}\right) \ .
\end{aligned} \tag{21}$$

For each $(t,x) \in [0,T]\times\mathbb{R}$, let $U_c(t,\cdot,x)$ and $L_z(t,\cdot)$ denote, respectively, the derivatives of the utility functions $U(t,\cdot,x)$ and $L(t,\cdot)$ with respect to their second argument. Since, by Assumption 2.6, $U(t,\cdot,x)$ and $L(t,\cdot)$ are twice differentiable and strictly concave with respect to its second argument, the corresponding derivatives are invertible. Hence, we define $I_1 : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ and $I_2 : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ to be the (unique) functions such that

$$\begin{aligned} I_1(t, U_c(t,y,x), x) = y \quad &\text{and} \quad U_c(t, I_1(t,y,x), x) = y \\ I_2(t, L_z(t,y)) = y \quad &\text{and} \quad L_z(t, I_2(t,y)) = y \end{aligned}$$

for every $(t,x) \in [0,T] \times \mathbb{R}$ and $y \in \mathbb{R}$.

Proposition 3.6. *Suppose that each of the HJBI equations* (19) *and* (20) *admits a unique smooth solution. For the SDG with state variable dynamics given by* (13) *and payoff functional* (16)*, it holds that*

$$V^-(t,x) = V^+(t,x)$$

for every $(t,x) \in [0,T] \times \mathbb{R}$*. Moreover, the optimal strategies in feedback form are given by*

$$\begin{aligned} \hat{y}(t,x) &= \frac{\mu(t) - r(t)}{\sigma(t)} \frac{\beta V_x^2}{V_{xx} - \beta V_x^2} \\ \hat{c}(t,x) &= I_1(t, V_x, x) \\ \hat{\theta}(t,x) &= -\frac{\mu(t) - r(t)}{x\sigma^2(t)} \frac{V_x}{V_{xx} - \beta V_x^2} \end{aligned}$$

and, for each $k \in \{1, 2, \ldots, K\}$*, we have that*

$$\hat{p}_k(t,x) = \begin{cases} \max\left\{0, \left[I_2\left(t, \eta_k(t)\left(g^-(t;t)\right)^{-1} V_x\right) - x\right]\eta_k(t)\right\} & , \text{ if } k = k^*(t) \\ 0 & , \text{ otherwise} \end{cases} ,$$

where

$$k^*(t) = \underset{k \in \{1,2,\ldots,K\}}{\arg\min} \; \{\eta_k(t)\} \; , \tag{22}$$

and $V_x \equiv V_x(t,x)$ *and* $V_{xx} \equiv V_{xx}(t,x)$ *denote, respectively, the first and second derivatives of* V *with respect to* x*.*

Proof. It is enough to check that the Isaacs condition holds, i.e.

$$\mathbb{H}^-(t,x,q,A) = \mathbb{H}^+(t,x,q,A) \; ,$$

for every $t \in [0,T]$, $x \in \mathbb{R}$ and $q, A \in \mathbb{R}$.

We evalute the function $\mathbb{H}^+$ first. Differentiating the Hamiltonian function H given in (21) with respect to c, θ and p we get

$$\begin{aligned} H_c(t,x,c,\theta,p,y,q,A) &= -q + U_c(t,c,x) \\ H_\theta(t,x,c,\theta,p,y,q,A) &= (\mu(t) - r(t) + \sigma(t)y)\, xq + \theta x^2 \sigma^2(t) A \\ H_{p_k}(t,x,c,\theta,p,y,q,A) &= -q + \frac{g^-(t;t)}{\eta_k(t)} L_z\Big(t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)}\Big) \; . \end{aligned}$$

Setting the first two derivatives equal to zero, yields

$$\begin{aligned} c(t,x,y,q,A) &= I_1(t,q,x) \\ \theta(t,x,y,q,A) &= -\frac{(\mu(t) - r(t) + \sigma(t)y)\, q}{x\sigma^2(t)A} \; . \end{aligned} \tag{23}$$

To solve the constrained optimization problem associated with the variable $p \in (\mathbb{R}_0^+)^K$, we resort to the Kuhn-Tucker conditions. Namely, we look for a solution $(p_1, \ldots, p_K, \mu_1, \ldots, \mu_K)$ to the following set of equalities and inequalities with all the previous functions being functions of (t, x, y, q, A):

$$\begin{array}{rcll} \dfrac{g^-(t;t)}{\eta_k(t)} L_z\left(t, x + \displaystyle\sum_{k=1}^{K} \dfrac{p_k}{\eta_k(t)}\right) - q & = & -\mu_k \ , & \\ p_k & \geq & 0 \ , & \\ \mu_k & \geq & 0 \ , & k = 1, 2, \ldots, K \\ p_k \mu_k & = & 0 \ . & \end{array} \tag{24}$$

We note that for $k_1 \neq k_2$, if we have $\mu_{k_1}(t, x, y, q, A) = \mu_{k_2}(t, x, y, q, A)$ for some $(t, x, y, q, A) \in [0, T] \times \mathbb{R}^4$, one must have that $\eta_{k_1}(t) = \eta_{k_2}(t)$. Thus, relying on the assumption that all insurance companies offer pairwise distinct contracts, we obtain that for every $k_1, k_2 \in \{1, 2, \ldots, K\}$ such that $k_1 \neq k_2$ and every $x \in \mathbb{R}$, $\mu_{k_1}(t, x, y, q, A) \neq \mu_{k_2}(t, x, y, q, A)$ for Lebesgue a.e. $t \in [0, T]$. In particular, we obtain that for every $(x, y, q, A) \in \mathbb{R}^4$ and Lebesgue a.e. $t \in [0, T]$, there is at most one $k \in \{1, 2, \ldots, K\}$ such that $\mu_k(t, x, y, q, A) = 0$. Therefore, for Lebesgue a.e. $t \in [0, T]$, there is at most one $k \in \{1, 2, \ldots, K\}$ such that $p_k(t, x, y, q, A) \neq 0$.

Using once again the first identity in (24), we get that

$$\eta_{k_1}(t)\,(q - \mu_{k_1}) = \eta_{k_2}(t)\,(q - \mu_{k_2}) \,,$$

where the dependence of μ_{k_1} and μ_{k_2} on (t, x, y, q, A) has been dropped to simplify the notation. As a consequence of the identity above, we conclude that if $\mu_{k_1}(t, x, y, q, A) > \mu_{k_2}(t, x, y, q, A)$ for $(t, x, y, q, A) \in [0, T] \times \mathbb{R}^4$, then $\eta_{k_1}(t) > \eta_{k_2}(t)$. Furthermore, if for some $t \in [0, T]$ we have $\mu_{k_1}(t, x, y, q, A) = 0$, then $\eta_{k_1}(t) < \eta_{k_2}(t)$ for every $k_2 \in \{1, 2, \ldots, K\}$ such that $k_1 \neq k_2$.

From this point onward, let $k^*(t)$ be as given in (22). Then, either we have $p_k(t, x, y, q, A) = 0$ for every $k \in \{1, 2, \ldots, K\}$ or else $p_{k^*(t)}(t, x, y, q, A) > 0$ is a solution to

$$\frac{g^-(t;t)}{\eta_{k^*(t)}(t)} L_z\left(t, x + \frac{p_{k^*(t)}}{\eta_{k^*(t)}(t)}\right) = q \ ,$$

yielding

$$p_k(t, x, q, A) = \begin{cases} \max\left\{0, \left[I_2\left(t, \frac{\eta_k(t)\ q}{g^-(t;t)}\right) - x\right] \eta_k(t)\right\} & , \text{ if } \ k = k^*(t) \\ 0 & , \text{ otherwise} \end{cases} . \tag{25}$$

Substituting the expressions of c, θ and p_k given in (23) and (25) back into the Hamiltonian function H, we get

$$\begin{aligned} \tilde{H}(t, x, y, q, A) \ = \ & i(t) - I_1(t, q, x)q - p_{k^*(t)}(t, x, q, A)q + r(t)xq \\ & -\frac{1}{2} \frac{(\mu(t) - r(t) + \sigma(t)y)^2}{\sigma^2(t)} \frac{q^2}{A} + \frac{1}{2\beta} y^2 + U(t, I_1(t, q, x), x) \\ & + g^-(t;t) L\left(t, x + \frac{p_{k^*(t)}(t, x, q, A)}{\eta_{k^*(t)}(t)}\right) \ . \end{aligned}$$

Differentiating the function $\tilde{H}$ given above with respect to y, we obtain

$$\tilde{H}_y(t,x,y,q,A) = \left(\frac{1}{\beta} - \frac{q^2}{A}\right) y - \frac{(\mu(t)-r(t))}{\sigma(t)} \frac{q^2}{A} \ .$$

Setting the derivative above equal to zero, we get

$$\hat{y}(t,x,q,A) = \frac{\mu(t)-r(t)}{\sigma(t)} \frac{\beta q^2}{A-\beta q^2} \ . \tag{26}$$

Finally, substituting the expression in (26) into (23), we get the saddle point control functions

$$\begin{aligned} \hat{c}(t,x,q,A) &= I_1(t,q,x) \\ \hat{\theta}(t,x,q,A) &= -\frac{\mu(t)-r(t)}{x\sigma^2(t)} \frac{q}{A-\beta q^2} \\ \hat{p}_k(t,x,q,A) &= \begin{cases} \max\left\{0, \left[I_2\left(t, \frac{\eta_k(t)\ q}{g^-(t;t)}\right) - x\right] \eta_k(t)\right\} & ,\ \text{if}\ \ k = k^*(t) \\ 0 & ,\ \text{otherwise} \end{cases} \ . \end{aligned} \tag{27}$$

Substituting c, θ, p, y by $\hat{c}, \hat{\theta}, \hat{p}, \hat{y}$ given in (26) and (27) in H, yields

$$\begin{aligned} \mathbb{H}^+(t,x,q,A) &= i(t) + \big(r(t)x - I_1(t,q,x) - \hat{p}_{k^*(t)}(t,x,q,A)\big)\, q \\ &\quad + \frac{(\mu(t)-r(t))^2}{2\sigma^2(t)} \frac{\beta q^4 - q^2 A}{(A-\beta q^2)^2} + U(t, I_1(t,q,x), x) \\ &\quad + g^-(t;t) L\left(t, x + \frac{\hat{p}_{k^*(t)}(t,x,q,A)}{\eta_{k^*(t)}(t)}\right) \ . \end{aligned}$$

We now evaluate the function $\mathbb{H}^-$. Differentiating the Hamiltonian function H given in (21) with respect to y, we get

$$H_y(t,x,c,\theta,y,q,A) = \theta\sigma(t)xq + \frac{1}{\beta} y \ ,$$

and setting the derivative equal to zero, yields

$$y(t,x,c,\theta,q,A) = -\beta\sigma(t)xq\theta \ . \tag{28}$$

Substituting the expression in (28) back into the Hamiltonian function H, gives

$$\begin{aligned} &\hat{H}(t,x,c,\theta,q,A) \\ &= \left(i(t) - c - \sum_{k=1}^{K} p_k + \left(r(t) + \theta\left(\mu(t) - r(t) - \sigma^2(t)\beta x q\theta\right)\right) x\right) q \\ &\quad + \frac{1}{2}\left(\theta x\sigma(t)\right)^2 A + \frac{1}{2\beta}(\beta\sigma(t)xq\theta)^2 + U(t,c,x) \\ &\quad + g^-(t;t) L\left(t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)}\right) \ . \end{aligned}$$

Again, differentiating the function above with respect to c, θ and p, we get

$$\begin{aligned}
\hat{H}_c(t,x,c,\theta,q,A) &= -q + U_c(t,c,x) \\
\hat{H}_\theta(t,x,c,\theta,q,A) &= (\mu(t)-r(t))xq + \left(x^2\sigma^2(t)A - \beta\sigma^2(t)x^2q^2\right)\theta + \theta x^2\sigma^2(t)A \\
\hat{H}_{p_k}(t,x,c,\theta,q,A) &= -q + \frac{g^-(t;t)}{\eta_k(t)} L_z\Big(t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)}\Big) \ .
\end{aligned}$$

Setting the three derivatives above equal to zero, yields

$$\begin{aligned}
\hat{c}(t,x,q,A) &= I_1(t,q,x) \\
\hat{\theta}(t,x,q,A) &= -\frac{\mu(t)-r(t)}{x\sigma^2(t)} \frac{q}{A-\beta q^2} \ .
\end{aligned} \tag{29}$$

Using a similar argument as before, resorting to the Kuhn-Tucker conditions, we get

$$\hat{p}_k(t,x,q,A) = \begin{cases} \max\left\{0, \left[I_2\left(t, \frac{\eta_k(t)}{g^-(t;t)}\frac{q}{}\right) - x\right]\eta_k(t)\right\} & \text{, if } \ k = k^*(t) \\ 0 & \text{, otherwise} \end{cases} \ . \tag{30}$$

As before, substituting the second expression in (29) into (28), we obtain

$$\hat{y}(t,x,q,A) = \frac{\mu(t)-r(t)}{\sigma(t)} \frac{\beta q^2}{A-\beta q^2} \ . \tag{31}$$

Substituting c, θ, p, y by $\hat{c}, \hat{\theta}, \hat{p_k}, \hat{y}$ given in (29) and (31) in H, yields

$$\begin{aligned}
\mathbb{H}^-(t,x,q,A) &= i(t) + \left(r(t)x - I_1(t,q,x) - \hat{p}_{k^*(t)}(t,x,q,A)\right) q \\
&\quad + \frac{(\mu(t)-r(t))^2}{2\sigma^2(t)} \frac{\beta q^4 - q^2 A}{(A-\beta q^2)^2} + U(t, I_1(t,q,x), x) \\
&\quad + g^-(t;t) L\left(t, x + \frac{\hat{p}_{k^*(t)}(t,x,q,A)}{\eta_{k^*(t)}(t)}\right) \ .
\end{aligned}$$

ensuring that $\mathbb{H}^+(t,x,q,A) = \mathbb{H}^-(t,x,q,A)$ for every $t \in [0,T]$, $x \in \mathbb{R}$ and $q, A \in \mathbb{R}$. □

Under the assumption that each of the HJBI equations (19) and (20) admits a unique smooth solution, Proposition 3.6 guarantees that the SDG lower and upper value functions given in (17) and (18) are identical and satisfy the PDE

$$\begin{aligned}
W_t - g^-(t;t)W + \left(i(t) + r(t)x - I_1(t,W_x,x) - \hat{p}_{k^*(t)}(t,x,W_x,W_{xx})\right) W_x \\
+ \frac{(\mu(t)-r(t))^2}{2\sigma^2(t)} \frac{\beta W_x^4 - W_x^2 W_{xx}}{(W_{xx} - \beta W_x^2)^2} + U(t, I_1(t,W_x,x), x) \\
+ g^-(t;t) L\left(t, x + \frac{\hat{p}_{k^*(t)}(t,x,W_x,W_{xx})}{\eta_{k^*(t)}(t)}\right) = 0
\end{aligned} \tag{32}$$

with boundary condition

$$W(T,x) = \Psi(x) \ . \tag{33}$$

It should be stressed that the optimal life insurance selection and purchase strategy given in Theorem 3.6 directs the wage-earner to focus all of his purchases on the life insurance company offering the highest premium for the same price, i.e. the insurance company with the lowest premium payout ratio $\eta_{k^*(t)}(t)$. Typically, investments on financial markets are of a diversified nature. This unusual behavior arises under the assumptions used herein where the life insurance contracts carry no default risk.

4. A special case with Closed Form Solutions

In this section, we will find closed form representations for both the solution of a boundary value problem of the type (32)-(33) and the corresponding optimal robust strategies. For that purpose, we will restrict our attention to the case where:

i) the life insurance market has one representative company, trading life-insurance contracts with (continuous) premium payout ratio $\eta(t) = \eta_{k^*(t)}(t)$, where $k^*(t)$ is as given in (22). Moreover, the life insurance purchase rate is allowed to be any real value.

ii) the wage preferences are described by the constant relative risk aversion utility functions

$$U(t,c,x) = \mathrm{e}^{-\rho t}\frac{c^\gamma}{\gamma} \quad , \quad L(t,z) = \mathrm{e}^{-\rho t}\frac{z^\gamma}{\gamma} \quad , \quad \text{and} \quad \Psi(x) = \mathrm{e}^{-\rho T}\frac{x^\gamma}{\gamma}$$

with the parameter $\gamma \neq 0$ is such that $1-\gamma > 0$ and the discount rate ρ is positive.

iii) the penalization parameter β of (2) is a function of both t and x, namely

$$\beta(t,x) = \frac{\beta_0}{\gamma W(t,x)} \ ,$$

where β_0 is a positive constant.

Remark 4.1. *A few comments are now in order regarding the assumptions listed immediately above:*

- *In what concerns item i), we recall that Proposition 3.6 implies that the wage-earner will buy life insurance only from the company with the lowest premium payout ratio, thus calling for a better understanding of the representative life insurance company market case described above. We drop the nonnegativity constraint to pursue a more complete understanding of the rational wage-earner preferences towards life insurance purchase simplify the analysis that follows.*
- *As for item iii), we observe that such model for the preference for robustness was initially proposed by Maenhout (2004) and later adopted by many other authors (e.g., Anderson et al. (2009), Branger et al. (2013), Flor and Larsen (2014) and Liu (2010), among others). Maenhout (2004) proposed replacing the constant preference for robustness parameter, by a nonconstant, state dependent* $\beta(t,x) > 0$. *The overall intuition is that larger values of* $\beta(t,x)$ *correspond to less faith in the model.*

Before proceeding to the statement of our next result, we introduce additional notation. Let $\xi(t)$ denote the market price of risk or Sharpe ratio, i.e. the average return per unit of volatility $\sigma(t)$, of the financial market under consideration:

$$\xi(t) = \frac{\mu(t) - r(t)}{\sigma(t)} \ .$$

Proposition 4.2. *Suppose that Assumptions 2.1-2.5 hold. Then, under the setup described in itens i), ii) and iii) above, the HJBI boundary value problem* (32)-(33) *has a solution of the form*

$$W(t,x) = \frac{a(t)}{\gamma} \left(x + b(t)\right)^{\gamma} \ ,$$

where

$$\begin{aligned}
a(t) &= \mathrm{e}^{-\rho t} \left(\mathrm{e}^{-\int_t^T P_2(s)\ \mathrm{d}s} + \int_t^T \mathrm{e}^{-\int_t^s P_2(u)\ \mathrm{d}u}\ P_1(s)\mathrm{d}s \right)^{1-\gamma} \\
P_1(t) &= 1 + \eta(t)^{-\frac{\gamma}{1-\gamma}} \left(g^-(t;t)\right)^{\frac{1}{1-\gamma}} \\
P_2(t) &= \frac{1}{1-\gamma} \left(\rho + g^-(t;t) - \gamma \left(\eta(t) + r(t) + \frac{\xi^2(t)}{2(\beta_0 + 1 - \gamma)} \right) \right) \\
b(t) &= \int_t^T i(s) \mathrm{e}^{-\int_t^s r(v) + \eta(v) \mathrm{d}v} \mathrm{d}s \ .
\end{aligned} \tag{34}$$

Moreover, the corresponding optimal strategies are given by

$$\begin{aligned}
\hat{c}(t,x) &= \left(\mathrm{e}^{\rho t} a(t)\right)^{-\frac{1}{1-\gamma}} (x + b(t)) \\
\hat{\theta}(t,x) &= \frac{(x + b(t))}{x\sigma(t)(\beta_0 + 1 - \gamma)} \xi(t) \\
\hat{y}(t,x) &= -\frac{\beta_0}{\beta_0 + 1 - \gamma}\, \xi(t) \\
\hat{p}(t,x) &= \left[\left(\frac{\mathrm{e}^{\rho t} a(t) \eta(t)}{g^-(t;t)} \right)^{-\frac{1}{1-\gamma}} (x + b(t)) - x \right] \eta(t)
\end{aligned}$$

Proof. If a solution to (32)-(33) is to exist, then using the proof of Proposition 3.6, we obtain that:

$$\begin{aligned}
\hat{c}(t,x) &= I_1(t, W_x, x) \\
\hat{\theta}(t,x) &= -\frac{\mu(t) - r(t)}{x\sigma^2(t)} \frac{W_x}{W_{xx} - \beta W_x^2} \\
\hat{y}(t,x) &= \frac{\mu(t) - r(t)}{\sigma(t)} \frac{\beta W_x^2}{W_{xx} - \beta W_x^2} \\
\hat{p}(t,x) &= \left[I_2 \left(t, \frac{\eta(t)\ W_x}{g^-(t;t)} \right) - x \right] \eta(t) \ .
\end{aligned}$$

Substituting in (32)-(33) an ansatz of the form

$$W(t,x) = \frac{a(t)}{\gamma} \left(x + b(t)\right)^{\gamma} \ ,$$

we get

$$\frac{1}{\gamma}\frac{\mathrm{d}a}{\mathrm{d}t}(t)+\frac{a(t)}{x+b(t)}\frac{\mathrm{d}b}{\mathrm{d}t}(t)-\frac{g^{-}(t;t)}{\gamma}a(t)$$
$$+\frac{1-\gamma}{\gamma}\left(1+(\eta(t))^{-\frac{\gamma}{1-\gamma}}\left(g^{-}(t;t)\right)^{\frac{1}{1-\gamma}}\right)\mathrm{e}^{-\frac{\rho t}{1-\gamma}}a(t)^{-\frac{\gamma}{1-\gamma}}$$
$$+\frac{i(t)+(\eta(t)+r(t))\,x}{x+b(t)}a(t)+\frac{(\mu(t)-r(t))^2}{2\sigma^2(t)\,(\beta_0+1-\gamma)}=0\ .$$

We observe that this problem decouples into two independent boundary value problems for $a(t)$ and $b(t)$ which are given, respectively, by

$$\frac{1}{\gamma}\frac{\mathrm{d}a}{\mathrm{d}t}(t)+\left(\eta(t)+r(t)-\frac{g^{-}(t;t)}{\gamma}+\frac{(\mu(t)-r(t))^2}{2\sigma^2(t)\,(\beta_0+1-\gamma)}\right)a(t)$$
$$+\frac{1-\gamma}{\gamma}\left(1+(\eta(t))^{-\frac{\gamma}{1-\gamma}}\left(g^{-}(t;t)\right)^{\frac{1}{1-\gamma}}\right)\mathrm{e}^{-\frac{\rho t}{1-\gamma}}a(t)^{-\frac{\gamma}{1-\gamma}}=0 \quad (35)$$
$$a(T)=\mathrm{e}^{-\rho t}\ ,$$

and

$$\frac{\mathrm{d}b}{\mathrm{d}t}(t)-(\eta(t)+r(t))\,b(t)+i(t)=0$$
$$b(T)=0\ . \quad (36)$$

To find a solution to the boundary value problem (35), we write $a(t)$ in the form

$$a(t)=\mathrm{e}^{-\rho t}A(t)^{1-\gamma}\ ,$$

obtaining a new boundary value problem for $A(t)$, that is given by

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t}-P_2(t)A(t)+P_1(t)=0$$
$$A(T)=1\ , \quad (37)$$

where $P_1(t)$ and $P_2(t)$ are as in (34). The solution to (37) is given by

$$A(t)=\mathrm{e}^{-\int_t^T P_2(s)\ \mathrm{d}s}+\int_t^T \mathrm{e}^{-\int_t^s P_2(u)\ \mathrm{d}u}\ P_1(s)\mathrm{d}s\ ,$$

and the solution to (35) is as in (34). Finally, the solution to (36) is as in (34), completing the proof. □

The next result provides a qualitative characterization for the optimal purchase strategy, $\hat{p}(t,x)$. Before proceeding to its statement, we introduce some notation: Let $D(t)$ be the quantity given by

$$D(t)=\left(\frac{\mathrm{e}^{\rho t}a(t)\eta(t)}{g^{-}(t;t)}\right)^{-\frac{1}{1-\gamma}}. \quad (38)$$

Corollary 4.3. *The optimal insurance purchase strategy $\hat{p}(t,x)$ has the following properties:*

a) *it is an increasing function of the wealth x if $D(t)\geq 1$ for all $t\in[0,T]$, and a decreasing function of the wealth x if $D(t)\leq 1$ for all $t\in[0,T]$;*
b) *it is an increasing function of the wage earner's human capital $b(t)$;*

c) *it is negative for suitable pairs of wealth x and time t;*
d) *with all other parameters constant, including t and x, the optimal life insurance purchase rate $\hat{p}(t,x)$ is an increasing function of the discount rate ρ.*

Proof. Using (38) and Proposition 4.2, we write the optimal insurance purchase strategy $\hat{p}(t,x)$ as

$$\hat{p}(t,x) = \eta(t)\left(\left(D(t)-1\right)x + D(t)b(t)\right).$$

Items a) and b) follow immediately from the representation above. For the proof of item c), note that $\hat{p}(t,x)$ is negative for all $(t,x) \in [0,T] \times \mathbb{R}^+$ such that

$$\begin{aligned} x &> \frac{D(t)}{1-D(t)} b(t) \\ &= \frac{\left(\mathrm{e}^{\rho t} a(t)\eta(t)\right)^{-\frac{1}{1-\gamma}}}{\left(g^-(t;t)\right)^{-\frac{1}{1-\gamma}} - \left(\mathrm{e}^{\rho t} a(t)\eta(t)\right)^{-\frac{1}{1-\gamma}}} b(t) > 0 \ , \end{aligned}$$

and positive otherwise. To prove d), note that $P_1(t)$ is independent of the discount factor ρ. Moreover, $P_2(t)$ is a decreasing function of ρ. As a consequence, $\mathrm{e}^{\rho t} a(t)$ is decreasing with ρ. This leads to the conclusion that $\hat{p}(t,x)$ is increasing with ρ. □

We will now discuss the qualitative properties of the optimal portfolio process, $\hat{\theta}(t,x)$.

Corollary 4.4. *The optimal portfolio process $\hat{\theta}(t,x)$ is such that:*
a) *$\hat{\theta}(t,x)$ has the same sign as $\xi(t)$;*
b) *$\hat{\theta}(t,x)$ is a decreasing function of the total wealth x if $\xi(t) > 0$ and an increasing function of x if $\xi(t) < 0$ for all $t \in [0,T]$;*
c) *$\hat{\theta}(t,x)$ is an increasing function of the wage earner's human capital $b(t)$ if $\xi(t) > 0$ for all $t \in [0,T]$ and a decreasing function of $b(t)$ if $\xi(t) < 0$ for all $t \in [0,T]$.*

Furthermore, the following equalities hold

$$\begin{aligned} \lim_{x\to 0^+} \hat{\theta}(t,x) &= +\infty, \\ \lim_{x\to\infty} \hat{\theta}(t,x) &= \frac{\xi(t)}{\sigma(t)\left(\beta_0 + 1 - \gamma\right)}, \\ \lim_{t\to T} \hat{\theta}(t,x) &= \frac{\xi(T)}{\sigma(T)\left(\beta_0 + 1 - \gamma\right)} \ . \end{aligned}$$

Proof. Using the expression of $\hat{\theta}(t,x)$ in Proposition 4.2 and the positivity of $b(t)$, items a), b) and c) follow immediately. The limiting behaviors presented on the second part of the corollary also follow from the form of $\hat{\theta}(t,x)$ in Proposition 4.2. □

Finally, we list the properties for the optimal consumption rate, $\hat{c}(t,x)$. The proof follows directly from the closed-form expression given in Proposition 4.2.

Corollary 4.5. *The optimal consumption rate* $\hat{c}(t,x)$ *is an increasing function of both the wealth* x *and the human capital* $b(t)$*.*

Finally, we observe that the stochastic process $\hat{y}(t,x)$ determining the least favorable martingale measure is proportional to the market price of risk $\xi(t)$. Intuitively, such choice may be regarded as an absorption of eventual profits available to the wage earner when investing in the financial market by the choice of such least favorable probability model.

5. Conclusions

We have studied the problem faced by a wage earner whose preferences relative to consumption, investment and life insurance purchase are described by a robust expected utility. We interpreted such problem in terms of a two-player zero-sum stochastic differential game and derived the wage earner optimal strategies for a general class of utility functions, studying the case of discounted constant relative risk aversion utility functions with more detail. Future extensions may include more general asset dynamics, alternative forms of objective functions, as well as applications to pension planning.

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