DYNAMIC PROGRAMMING FOR SEMI-MARKOV MODULATED SDES

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Abstract. We consider a stochastic optimal control problem with state variable dynamics described by a stochastic differential equation of diffusive type modulated by a semi-Markov process with a finite state space. The time horizon is both deterministic and finite. Within such setup, we provide a detailed proof of the dynamic programming principle and use it to characterize the value function as a viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation. We illustrate our results with an application to Mathematical Finance: the generalization of Merton's optimal consumption-investment problem to financial markets with semi-Markov switching.

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AMS classification: 49L20; 60K15; 90C39; 91G80

1. Introduction

The research effort leading to the development of optimal control theory is partially due to the study of differential games from the 1940s onwards by a group based in the United States including, among others, Bellman, LaSalle, Blackwell, Isaacs, Fleming, and Berkovitz. Another group, organized around Pontryagin and based in the former Soviet Union, independently developed an alternative approach to solve optimal control problems. From the work of these groups, resulted the two central techniques in the analysis of optimal control problems: the dynamic programming principle [4, 5, 6, 7] and the Pontryagin maximum principle [8, 9].

The development of dynamic programming was initiated in the 1950s with the pioneering work of Bellman, followed by Florentin [16, 17] and Kushner [26], among many others. Indeed, given its mathematical relevance and important real-life applications, optimal control and the dynamic programming technique have been the focus of much attention from the scientific community. The excellent monographs by Fleming and Soner [15], Yong and Zhou [36] and Oksendal and Sulem [27] provide a rather detailed account of the current knowledge concerning optimal control problems, including the dynamic programming principle.

After the initial development of the theory with a strong focus on state variable dynamics described by either ordinary differential equations or Itô diffusions, more recent research has tended to focus on systems with more

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general state variable dynamics. A necessary maximum principle for jump-diffusions was first obtained by Tang and Li [32], with a sufficient maximum principle for jump-diffusions being due to Framstad et al. [18]. A dynamic programming principle for jump-diffusions with Markov-switching was obtained by Azevedo et al. [3], with related computational methods due to Temoçin and Weber [33]. A sufficient maximum principle for jump-diffusions with a semi-Markovian switching was derived by Deshpande [11]. This paper contributes to the current literature by extending the dynamic programming principle to include stochastic optimal control problems with state variable dynamics determined by a diffusive stochastic differential equation (SDE) whose coefficients depend on a semi-Markov process with a finite state space. Unlike continuous-time Markov processes, which have exponentially distributed sojourn times, the distributions of the sojourn times for semi-Markov processes are given by general distributions (see e.g. [35] and references therein for further details on semi-Markov processes). This creates a technical obstruction to the extension of the dynamic programming principle proposed herein. Indeed, the argument leading to the proof of the dynamical programming principle relies heavily on the Markov property – and it is known that semi-Markov processes only satisfy such property at jump times. Nevertheless, resorting to some results in the theory of semi-Markov and renewal processes [21], we are able to overcome such difficulties by pairing the semi-Markov process with certain suitable processes. We then use the dynamic programming principle to obtain an appropriate generalization to the classical Hamilton-Jacobi-Bellman equation (HJB equation) and characterize the value function of the stochastic optimal control problem under consideration as a viscosity solution of such equation (see [10] and references therein for further details on the theory of viscosity solutions of Hamilton-Jacobi equations). Alternative forms of switching have been explored in [1, 19], while systems with delay were considered by Savku and coauthors [29, 30, 31]. Future developments in the area may be related with extending the later to encompass also semi-Markovian switching.

We also need to stress the broad applicability of optimal control theory for stochastic systems with switching to a diverse number of fields of knowledge. These include, for instance, Economics [23, 22], Finance [37, 39, 38, 12, 20, 13], and Neuroscience [14, 25, 34]. To illustrate our results herein, we will also discuss an optimal consumption-investment problem within the setup of a diffusive financial market model whose coefficients are driven by a semi-Markov process. It is our belief that semi-Markov processes are rather well-suited to capture some of the complexities arising from financial markets data. First of all, these processes allow the dynamics to switch between different states of the market – think of "bull", "bear" and "sideways" market modes – allowing the modeler plenty of freedom to carefully calibrate the transition probabilities between these states. Perhaps as importantly, the transition probabilities are not necessarily exponential, providing even greater flexibility. On top of that, one can use the semi-Markov process time component, i.e. the time elapsed since the last state switch, to model interesting phenomena such as a "bull" market gaining strength at its beginning or loosing it when it becomes nearly exhausted.

In a similar fashion, such process – the time component – can be used to model the evolution of the market agents' preferences during the period of time spent in each state of the market, a feature with impact on the utility functions describing the agents' preferences.

This paper is organized as follows. In Section 2 we provide background material and formulate the problem under consideration. Section 3 is devoted to the proof of the dynamic programming principle for stochastic optimal control problems with semi-Markov modulated diffusive state variable dynamics. We deal with the characterization of the value function as a viscosity solution of the HJB equation in Section 4. We illustrate our results in Section 5 by looking at Merton's optimal consumption-investment problem for a financial market with semi-Markov modulated coefficients.

2. Setup and problem formulation

Let $T > 0$ be a deterministic finite horizon and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t : t \in [0, T])$ satisfying the usual conditions, i.e. $\mathbb{F}$ is an increasing, right-continuous filtration and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

We will consider the following stochastic processes throughout this paper:

(i) a standard M-dimensional Brownian motion $W(t) = (W(t) : t \in [0, T])$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

(ii) a continuous time semi-Markov process $(\alpha(t) : t \in [0, T])$ with a finite state space $S = \{a_1, \ldots, a_n\}$, transition probabilities p_{ij} from state $a_i \in S$ to state $a_j \in S$ and conditional holding time distributions $G(\cdot|i)$ (for each $a_i \in S$, $G(\cdot|i)$ is a distribution function), i.e. if $0 < t_0 < t_1 < t_2 < \ldots$ are jump times, then

$$\mathbb{P}\left(\alpha(t_{n+1}) = j, t_{n+1} - t_n \leq t | \alpha(t_n) = i\right) = p_{ij} G(t|i) \ .$$

Additionally, we will assume that the transition matrix $[p_{ij}]_{\{a_i, a_j \in S\}}$ is irreducible, i.e. every state of α is accessible from every other state, and that for each $a_i \in S$, $G(\cdot|i)$ is continuously differentiable with a positive, bounded and Lipschitz continuous density function $g(\cdot|i)$.

We will now describe a representation for the semi-Markov process $\alpha(t)$ as a stochastic integral with respect to a certain Poisson random measure. Embed S into $\mathbb{R}^n$ by identifying the element $a_i \in S$ with the i^{th} element $e_i \in \mathbb{R}^n$ of the standard basis of $\mathbb{R}^n$. For $\tau \in \mathbb{R}^+$ and $a_i, a_j \in S$, set

$$\begin{aligned} \lambda_{ij}(\tau) &= p_{ij} \frac{g(\tau|i)}{1 - G(\tau|i)} \geq 0 \ , \quad \text{for } i \neq j \ , \\ \lambda_{ii}(\tau) &= - \sum_{j \in S,\, j \neq i} \lambda_{ij}(\tau) \ , \quad \text{for } i \in S \ , \end{aligned}$$

and, for $i \neq j$, denote by $\Lambda_{ij}(t)$ the consecutive (with respect to the lexicographical order on $S \times S$), left-closed, right-open intervals of the real line, each with length $\lambda_{ij}(t)$. Define functions $\Gamma_1 : S \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ and

$\Gamma_2 : S \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ as

$$\begin{aligned}\Gamma_1(i,t,z) &= \begin{cases} e_j - e_i & \text{if } z \in \Lambda_{ij}(t) \\ 0 & \text{otherwise} \end{cases} \\ \Gamma_2(i,t,z) &= \begin{cases} t & \text{if } z \in \Lambda_{ij}(t),\ i \neq j \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and note that Γ_1 determines the jump from the state a_i to state a_j on S and Γ_2 determines the time elapsed since the last state change of $\alpha(\cdot)$. Denote by $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R})$ the set of all nonnegative integer valued σ-finite measures on the Borel σ-field of $\mathbb{R}^+ \times \mathbb{R}$ and let $N(\mathrm{d}t, \mathrm{d}z)$ be a $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R})$-valued Poisson random measure with intensity $\mathrm{d}t\mathrm{d}z$, independent of $\alpha(0)$, and $(\mathcal{T}(t) : t \in [0,T])$ be the process representing the time elapsed since the semi-Markov process $\alpha(\cdot)$ last state switch. According to [20, Thm 2.1], the process $(\alpha(t), \mathcal{T}(t))$ is determined by the SDEs:

$$\begin{aligned}\alpha(t) &= \alpha(0) + \int_0^t \int_{\mathbb{R}} \Gamma_1(\alpha(s_-), \mathcal{T}(s_-), z)\, N(\mathrm{d}s, \mathrm{d}z) \\ \mathcal{T}(t) &= \mathcal{T}(0) + t - \int_0^t \int_{\mathbb{R}} \Gamma_2(\alpha(s_-), \mathcal{T}(s_-), z)\, N(\mathrm{d}s, \mathrm{d}z)\ , \end{aligned} \tag{1}$$

where the integrations are over the interval $(0, t]$. Finally, we remark that the process determined by the pair $(\alpha(t), \mathcal{T}(t))_{t \geq 0}$ is Markov [21, Ch. III, Sec. 3].

Throughout this paper, by abuse of notation, we will denote by $|\cdot|$ the norm in the Euclidean space $\mathbb{R}^d$, regardless of the specific dimension d, which will be apparent from the context. We introduce the following technical assumptions:

- **(A1)** (U, d_U) is a Polish metric space.
- **(A2)** The state space $S = \{a_1, \ldots, a_2\}$ is endowed with the discrete topology and the corresponding discrete metric $d_S(\cdot, \cdot)$.
- **(A3)** The maps $f : [0,T] \times \mathbb{R}^N \times S \times \mathbb{R}_0^+ \times U \to \mathbb{R}^N$, $\sigma : [0,T] \times \mathbb{R}^N \times S \times \mathbb{R}_0^+ \times U \to \mathbb{R}^{N \times M}$, $\Psi : \mathbb{R}^N \times S \times \mathbb{R}_0^+ \to \mathbb{R}$ and $L : [0,T] \times \mathbb{R}^N \times S \times \mathbb{R}_0^+ \times U \to \mathbb{R}$, are such that:
  (i) for each fixed $a \in S$, $f(\cdot,\cdot,a,\cdot,\cdot)$, $\sigma(\cdot,\cdot,a,\cdot,\cdot)$, $\Psi(\cdot,a,\cdot)$, $L(\cdot,\cdot,a,\cdot,\cdot)$ are uniformly continuous with respect to all its variables;
  (ii) for each fixed $a \in S$ there exists $C > 0$ such that for $\varphi(t,x,\tau,u) = f(t,x,a,\tau,u), g(t,x,a,\tau,u), \Psi(x,a,\tau), L(t,x,a,\tau,u)$, we have

$$\begin{aligned}&|\varphi(t,x,\tau,u) - \varphi(t,y,\tau,u)|^2 < C|x-y|^2 \\ &|\varphi(t,0,\tau,u)|^2 < C\end{aligned}$$

  for every $(t,\tau) \in [0,T] \times \mathbb{R}_0^+$ uniformly in u.
- **(A4)** The Brownian motion $W(\cdot)$ and the semi-Markov process $\alpha(\cdot)$ are independent and adapted to the filtration $\mathbb{F}$.

The state space under consideration herein is the product space

$$\mathcal{N} = \mathbb{R}^N \times S \times \mathbb{R}_0^+\ ,$$

which we endow with the metric

$$d_{\mathcal{N}}(\bar{x}_1, \bar{x}_2) = \max\{|x_1 - x_2|, d_S(a_1, a_2), |\tau_1 - \tau_2|\}$$

for $\bar{x}_1 = (x_1, a_1, \tau_1) \in \mathcal{N}$ and $\bar{x}_2 = (x_2, a_2, \tau_2) \in \mathcal{N}$. We define on $\mathcal{N}$ a stochastic controlled system with state variable dynamics given by

$$\begin{aligned} X(t) &= x_0 + \int_0^t f\left(r, X(r), \alpha(r_-), \mathcal{T}(r_-), u(r_-)\right)\, \mathrm{d}r \\ &\quad + \int_0^t \sigma\left(r, X(r), \alpha(r_-), \mathcal{T}(r_-), u(r_-)\right)\, \mathrm{d}W(r) \\ \alpha(t) &= a_0 + \int_0^t \int_{\mathbb{R}} \Gamma_1(\alpha(r_-), \mathcal{T}(r_-), z)\, N(\mathrm{d}r, \mathrm{d}z) \\ \mathcal{T}(t) &= \tau_0 + t - \int_0^t \int_{\mathbb{R}} \Gamma_2(\alpha(r_-), \mathcal{T}(r_-), z)\, N(\mathrm{d}r, \mathrm{d}z)\ , \end{aligned} \tag{2}$$

together with an objective functional of the form

$$J(\bar{x}_0; u(\cdot)) = \mathbb{E}\left[\int_0^T L\left(t, \mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(t), u(t)\right)\, \mathrm{d}t + \Psi\left(\mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(T)\right)\right]\ , \tag{3}$$

where $\bar{x}_0 = (x_0, a_0, \tau_0) \in \mathcal{N}$ and the ordered triple

$$\mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(t) = (X_{0,\bar{x}_0}(t; u(\cdot)), \alpha_{0,\bar{x}_0}(t), \mathcal{T}_{0,\bar{x}_0}(t)) \in \mathcal{N}$$

denotes the state trajectory determined by the SDE (2) associated with the control $u(\cdot)$ and initial condition $\mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(0) = \bar{x}_0 \in \mathcal{N}$. Before proceeding, we remark that the triple $\mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(t)$ is a jointly Markov process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ [21, Ch. III, Sec. 3].

We say that $u : [0, T] \times \Omega \to U$ is a *strong admissible control* if $u(\cdot)$ is measurable, $\mathbb{F}$-adapted, and the system of SDEs (2) has a unique strong solution $\mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(t)$ satisfying the integrability conditions

$$\mathbb{E}\left[\int_0^T \left|L\left(t, \mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(t), u(t)\right)\right|\, \mathrm{d}t\right] < \infty\ ,$$

$$\mathbb{E}\left[\left|\Psi\left(\mathcal{X}^{u(\cdot)}_{0,\bar{x}_0}(T)\right)\right|\right] < \infty\ .$$

We denote by $\mathcal{U}^s[0, T]$ the set of all strong admissible controls .

Our main aim is to find a control $u(\cdot) \in \mathcal{U}^s[0, T]$ which maximizes the objective functional $J(\bar{x}_0; u(\cdot))$ given in (3) subject to the state variable dynamics (2) over the set of admissible controls $\mathcal{U}^s[0, T]$. Assumptions (A1) and (A3) above generalize the standard set of assumptions from the Stochastic Differential Equations and Optimal Control Theory literature to the setup under consideration here (see [2, 27, 28]) and guarantee existence and uniqueness of strong solutions of (2) via an interlacing argument with a finite number of diffusive SDEs.

We will now introduce the weak formulation of the stochastic control problem under consideration. For any $s \in [0, T)$ and $\bar{x} = (x, a, \tau) \in \mathcal{N}$, consider the state equations:

$$\begin{aligned} X(t) &= x + \int_s^t f\left(r, X(r), \alpha(r_-), \mathcal{T}(r_-), u(r_-)\right)\, \mathrm{d}r \\ &\quad + \int_s^t \sigma\left(r, X(r), \alpha(r_-), \mathcal{T}(r_-), u(r_-)\right)\, \mathrm{d}W(r) \\ \alpha(t) &= a + \int_s^t \int_{\mathbb{R}} \Gamma_1(\alpha(r_-), \mathcal{T}(r_-), z)\; N(\mathrm{d}r, \mathrm{d}z) \\ \mathcal{T}(t) &= \tau + t - s - \int_s^t \int_{\mathbb{R}} \Gamma_2(\alpha(r_-), \mathcal{T}(r_-), z)\; N(\mathrm{d}r, \mathrm{d}z) \end{aligned} \tag{4}$$

along with the objective functional

$$J(s, \bar{x}; u(\cdot)) = \mathbb{E}\left[\int_s^T L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + \Psi\left(\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(T)\right) \right], \tag{5}$$

where

$$\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t) = (X_{s,\bar{x}}(t; u(\cdot)), \alpha_{s,\bar{x}}(t), \mathcal{T}_{s,\bar{x}}(t))$$

is the solution of (4) associated with the control $u(\cdot)$ and the initial condition $\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(s) = \bar{x}$.

For each $s \in [0, T)$ we denote by $\mathcal{U}^w[s, T]$ the set of *weak admissible controls*, composed of 7-tuples

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W(\cdot), \alpha(\cdot), u(\cdot))$$

for which the following conditions hold:

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space;
- (ii) $\mathbb{F} = (\mathcal{F}_t^s : t \in [s, T])$ is a right-continuous filtration;
- (iii) $(W(t) : t \in [s, T])$ is a M-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[s, T]$ and adapted to the filtration $\mathbb{F}$;
- (iv) $(\alpha(t) : t \in [s, T])$ is a continuous-time semi-Markov process on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite state space S and adapted to the filtration $\mathbb{F}$;
- (vi) $u : [s, T] \times \Omega \to U$ is an $\mathbb{F}$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$;
- (vii) under $u(\cdot)$, for any $\bar{x} = (x, a, \tau) \in \mathcal{N}$, the SDE (4) admits a unique solution $\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t)$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

For simplicity of exposition, and as long as no confusion arises, we will use only $u(\cdot) \in \mathcal{U}^w[s, T]$ to denote the 7-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W(\cdot), \alpha(\cdot), u(\cdot)) \in \mathcal{U}^w[s, T]$.

We write the optimal control problem under consideration in dynamic programming form as follows. For any $(s, \bar{x}) \in [0, T) \times \mathcal{N}$, find $\bar{u}(\cdot) \in \mathcal{U}^w[s, T]$ such that

$$J(s, \bar{x}; \bar{u}(\cdot)) = \sup_{u(\cdot) \in \mathcal{U}^w[s,T]} J(s, \bar{x}; u(\cdot)) \,. \tag{6}$$

For any $(s, \bar{x}) \in [0, T) \times \mathcal{N}$ and $u(\cdot) \in \mathcal{U}^w[s, T]$, assumptions (A1) and (A3) guarantee that the SDE (4) admits a unique solution $\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\cdot)$ and the objective functional in (5) is well-defined. Therefore, the optimal control problem *value function* is also well-defined by

$$\begin{cases} V(s,\bar{x}) = \sup_{u(\cdot)\in\mathcal{U}^w[s,T]} J\left(s,\bar{x};u(\cdot)\right) \ , \\ V(T,\bar{x}) = \Psi(\bar{x})\,, \quad ((s,\bar{x})\in[0,T)\times\mathcal{N}) \ . \end{cases} \tag{7}$$

In Section 3 we prove a dynamic programming principle for the value function V. Section 4 is devoted to the corresponding Hamilton-Jacobi-Bellman equation and the characterization of the value function in terms of its viscosity solutions.

3. Dynamic programming principle and HJB equation

The main goal of this section is to characterize the value function (7) by means of a dynamic programming principle. Before moving on, we state a property of the value function which will turn out to be helpful in the proof of the dynamic programming principle. Well-known results from SDEs theory (see, e.g. [2, 28]) ensure that for any $s_1,s_2\in[0,T]$ such that $s_1\le s_2$, any $\bar{x}_1=(x_1,a_1,\tau_1)\in\mathcal{N}$ and $\bar{x}_2=(x_2,a_2,\tau_2)\in\mathcal{N}$, and any weak admissible control $u(\cdot)\in\mathcal{U}^w[s_1,T]$, there exists $C_1>0$ such that

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[s_2,T]} d_{\mathcal{N}}\left(\mathcal{X}^{u(\cdot)}_{s_1,\bar{x}_1}(t),\mathcal{X}^{u(\cdot)}_{s_2,\bar{x}_2}(t)\right)\right] \\ &\qquad\le C_1\left\{d_{\mathcal{N}}(\bar{x}_1,\bar{x}_2)+(1+\max\{|(x_1,\tau_1)|,|(x_2,\tau_2)|\})|s_1-s_2|^{1/2}\right\} \ . \end{aligned} \tag{8}$$

Combining assumptions (A2)–(A3) and property (8), we get that for any $s_1,s_2\in[0,T]$ such that $s_1\le s_2$, any $\bar{x}_1=(x_1,a_1,\tau_1)\in\mathcal{N}$ and $\bar{x}_2=(x_2,a_2,\tau_2)\in\mathcal{N}$, and any weak admissible control $u(\cdot)\in\mathcal{U}^w[s_1,T]$, there is a positive constant C_2 such that

$$\begin{aligned} &|J(s_1,\bar{x}_1;u(\cdot))-J(s_2,\bar{x}_2;u(\cdot))| \\ &\qquad\le C_2\left\{d_{\mathcal{N}}(\bar{x}_1,\bar{x}_2)+(1+\max\{|(x_1,\tau_1)|,|(x_2,\tau_2)|\})|s_1-s_2|^{1/2}\right\} \ . \end{aligned}$$

Finally, taking the supremum over $u(\cdot)\in\mathcal{U}^w[s_1,T]$, we get the following property for the value function V defined in (7).

Lemma 3.1. *Let assumptions (A1)–(A4) hold. Then there exists a positive constant C such that*

$$|V(s_1,\bar{x}_1)-V(s_2,\bar{x}_2)|\le C\left\{d_{\mathcal{N}}(\bar{x}_1,\bar{x}_2)+(1+\max\{|(x_1,\tau_1)|,|(x_2,\tau_2)|\})|s_1-s_2|^{1/2}\right\}$$

for every $s_1,s_2\in[0,T]$ and $\bar{x}_1,\bar{x}_2\in\mathcal{N}$.

Let $\mathbb{F}^s=(\mathcal{F}^s_t:t\in[s,T])$, where $\mathcal{F}^s_t=\sigma\left(W(r),\alpha(r):s\le r\le t\right)$. Since for every $\hat{s}\in[s,T)$ the triple $\mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})$ is $\mathcal{F}^s_{\hat{s}}$-measurable, the solutions $\mathcal{X}^{u(\cdot)}_{s,\bar{x}}(t)$ and $\mathcal{X}^{u(\cdot)}_{\hat{s},\mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})}(t)$ agree a.s. for every $t\in[\hat{s},T]$.

Lemma 3.2. *Let $(s,\bar{x})\in[0,T)\times\mathcal{N}$ and $u(\cdot)\in\mathcal{U}^w[s,T]$. For any $\hat{s}\in[s,T)$, the following equality holds $\mathbb{P}$-a.s. $\omega\in\Omega$:*

$$\begin{aligned} &J(\hat{s},\mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s});u(\cdot)) \\ &\quad= \mathbb{E}\left[\int_{\hat{s}}^T L\left(t,\mathcal{X}^{u(\cdot)}_{\hat{s},\mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})}(t),u(t)\right)\,\mathrm{d}t+\Psi\left(\mathcal{X}^{u(\cdot)}_{\hat{s},\mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})}(T)\right)\;\Big|\;\mathcal{F}^s_{\hat{s}}\right](\omega) \ . \end{aligned}$$

Using the two previous lemmas we prove the following dynamic programming principle for the value function (7).

Theorem 3.3 (Dynamic programming principle)**.** *Assume that conditions (A1)–(A4) hold and let $s, \hat{s} \in [0,T]$ be such that $s < \hat{s}$. Then, for every $\bar{x} \in \mathcal{N}$ we have that*

$$V(s,\bar{x}) = \sup_{u(\cdot)\in\mathcal{U}^w[s,T]} \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + V\left(\hat{s}, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s})\right)\right] \tag{9}$$

for all $\hat{s} \in [s,T]$.

Proof. Start by denoting the right-hand side of (9) by $\overline{V}(s,\bar{x})$. Note that for any $\epsilon > 0$ there exists $u(\cdot) \in \mathcal{U}^w[s,T]$ such that

$$V(s,\bar{x}) - \epsilon < J(s,\bar{x};u(\cdot))\ .$$

Letting $\hat{s} \in [s,T)$ and taking into account the definition of the objective functional given in (5), we get

$$V(s,\bar{x}) - \epsilon < \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + \mathbb{E}\left[\int_{\hat{s}}^T L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t +\ \Psi\left(\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(T)\right)\,\Big|\, \mathcal{F}_{\hat{s}}^s\right]\right]\ .$$

Recalling that the triple $\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\cdot)$ determined by (4) is a Markov process, we obtain

$$V(s,\bar{x}) - \epsilon < \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + \mathbb{E}\left[\int_{\hat{s}}^T L\left(t, \mathcal{X}_{\hat{s},\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s})}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + \Psi\left(\mathcal{X}_{\hat{s},\mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s})}^{u(\cdot)}(T)\right)\,\Big|\, \mathcal{F}_{\hat{s}}^s\right]\right]\ .$$

Resorting to the representation provided by Lemma 3.2, we get from the inequality above that

$$V(s,\bar{x}) - \epsilon < \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + J\left(\hat{s}, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s})\right)\right]\ .$$

Therefore, from the definition of the value function (7) and the inequality above, we obtain

$$\begin{aligned} V(s,\bar{x}) - \epsilon &< \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(t), u(t)\right)\, \mathrm{d}t + V\left(\hat{s}, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s})\right)\right] \\ &\le \overline{V}(s,\bar{x})\ . \end{aligned}$$

We now prove the converse statement. Let $(s,\bar{x}) \in [0,T) \times \mathcal{N}$ and fix an arbitrary control $u(\cdot) \in \mathcal{U}^w[s,T]$. From Lemma 3.1 it follows that for any $\hat{s} \in [s,T]$ and any $\epsilon > 0$ there is $\delta = \delta(\epsilon)$ such that for every $\bar{y} \in \mathcal{N}$ within distance δ of $\bar{x} \in \mathcal{N}$ we have

$$\begin{aligned} \left|J(\hat{s}, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s}); u(\cdot)) - J(\hat{s}, \mathcal{X}_{s,\bar{y}}^{u(\cdot)}(\hat{s}); u(\cdot))\right| &+ \\ \left|V(\hat{s}, \mathcal{X}_{s,\bar{x}}^{u(\cdot)}(\hat{s})) - V(\hat{s}, \mathcal{X}_{s,\bar{y}}^{u(\cdot)}(\hat{s}))\right| &\le \epsilon \end{aligned} \tag{10}$$

for every $u(\cdot) \in \mathcal{U}^w[\hat{s}, T]$. Let $\{D_j\}_{j\in\mathbb{N}}$ be a Borel partition for $\mathcal{N}$ with diameter $\operatorname{diam}(D_j) < \delta$ and take $\bar{x}_j = (x_j, a_j, \tau_j) \in D_j$. For each $j \in \mathbb{N}$ there is $(\Omega_j, \mathcal{F}_j, \mathbb{F}_j, \mathbb{P}_j, W_j(\cdot), \alpha_j(\cdot), u_j(\cdot)) \in \mathcal{U}^w[\hat{s}, T]$ such that

$$V(\hat{s}, \bar{x}_j) - \epsilon \leq J(\hat{s}, \bar{x}_j; u_j(\cdot)) \ . \tag{11}$$

Hence for any $\bar{x} \in D_j$, using inequalities (10) and (11), we obtain

$$J(\hat{s}, \bar{x}; u_j(\cdot)) \geq J(\hat{s}, \bar{x}_j; u_j(\cdot)) - \epsilon \geq V(\hat{s}, \bar{x}_j) - 2\epsilon \geq V(\hat{s}, \bar{x}) - 3\epsilon \ . \tag{12}$$

By the definition of weak admissible control $(\Omega_j, \mathcal{F}_j, \mathbb{F}_j, \mathbb{P}_j, W_j(\cdot), \alpha_j(\cdot), u_j(\cdot))$, there exists a progressively measurable process $v_j : [0, T] \times \Omega_j \to U$ such that

$$u_j(t, \omega) = v_j(t, W_j(\cdot \wedge t, \omega), \alpha_j(\cdot \wedge t, \omega)) \qquad \mathbb{P}_j - \text{a.s. } \omega \in \Omega_j$$

for every $t \in [\hat{s}, T]$. For any control $u(\cdot) \in \mathcal{U}^w[s, T]$, let $\mathcal{X}(\cdot) := \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\cdot)$ denote the corresponding state trajectory. Consider the process

$$\tilde{u}(t, \omega) = \begin{cases} u(t, \omega) & \text{if } t \in [s, \hat{s}) \\ v_j(t, W(\cdot \wedge t, \omega) - W(\hat{s}, \omega), \alpha(\cdot \wedge t, \omega)) & \text{if } t \in [\hat{s}, T] \text{ and } \mathcal{X}(t, \omega) \in D_j \end{cases}$$

and observe that $\tilde{u}(\cdot) \in \mathcal{U}^w[s, T]$. Then, we have that the inequality

$$\begin{aligned} V(s, \bar{x}) \geq J(s, \bar{x}; \tilde{u}(\cdot)) = \mathbb{E}\Bigg[&\int_s^{\hat{s}} L\left(t, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(t), u(t)\right) \, \mathrm{d}t \\ + \mathbb{E}\left[\int_{\hat{s}}^T L\left(t, \mathcal{X}^{u(\cdot)}_{\hat{s}, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})}(t), u(t)\right) \, \mathrm{d}t + \Psi\left(\mathcal{X}^{u(\cdot)}_{\hat{s}, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})}(T)\right) \,\Big|\, \mathcal{F}^s_{\hat{s}} \right] \Bigg] \end{aligned}$$

holds. Using Lemma 3.2 and inequality (12), we obtain

$$V(s, \bar{x}) \geq \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(t), u(t)\right) \, \mathrm{d}t + J(\hat{s}, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s}); u_\epsilon(\cdot)) \right] \ .$$

As a consequence, inequality (11) implies that

$$V(s, \bar{x}) \geq \mathbb{E}\left[\int_s^{\hat{s}} L\left(t, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(t), u(t)\right) \, \mathrm{d}t + V(\hat{s}, \mathcal{X}^{u(\cdot)}_{s,\bar{x}}(\hat{s})) - 3\epsilon \right] \ .$$

We conclude the proof by taking the supremum over all $u(\cdot) \in \mathcal{U}^w[s, T]$. □

For the statement of the next result, we will denote by $\bar{u}(\cdot)$ the optimal control for problem (4)-(5) and by $\bar{\mathcal{X}}(\cdot) = (\bar{X}(\cdot), \bar{\alpha}(\cdot), \bar{\tau}(\cdot))$ the corresponding state variable optimal path. Its proof may be adapted from arguments that are now standard in the literature [36, 3].

Proposition 3.4. *Assume that conditions (A1)–(A4) hold. If the pair* $\left(\bar{\mathcal{X}}(\cdot), \bar{u}(\cdot)\right)$ *is optimal for* (6), *then*

$$V(t, \bar{\mathcal{X}}(t)) = \mathbb{E}\left[\int_t^T L\left(r, \bar{\mathcal{X}}(r), \bar{u}(r)\right) \, \mathrm{d}r + \Psi\left(\bar{\mathcal{X}}(T)\right) \,\Big|\, \mathcal{F}^s_t \right] \qquad \mathbb{P} - \text{a.s.}$$

for every $t \in [s, T]$.

4. The value function as a viscosity solution of the HJB equation

We will now use the dynamic programming principle to derive its corresponding HJB equation and to characterize the value function (7) of the stochastic optimal control under consideration as a viscosity solution of such equation.

Let $I \subseteq \mathbb{R}$ be an interval with interior $\mathrm{int}(I)$ and denote by $C^{1,2,1}\left(I \times \mathbb{R}^N \times \mathbb{R}_0^+;\mathbb{R}\right)$ the set of all continuous functions $F : I \times \mathbb{R}^N \times \mathbb{R}_0^+ \to \mathbb{R}$ such that F_t, F_x, F_τ and F_{xx} are all continuous functions of $(t,x,\tau) \in \mathrm{int}(I) \times \mathbb{R}^N \times \mathbb{R}^+$. Additionally, let $\mathrm{Sym}_{n\times n}(\mathbb{R})$ denote the set of symmetric $N \times N$ real matrices, $\mathrm{tr}(A)$ the trace of $A \in \mathrm{Sym}_{n\times n}(\mathbb{R})$, and $\langle \cdot,\cdot \rangle$ the standard inner product on $\mathbb{R}^N$.

Proposition 4.1 (Hamilton-Jacobi-Bellman equation)**.** *Suppose that conditions (A1)–(A4) hold and that the value function V is such that $V(\cdot,\cdot,a,\cdot) \in C^{1,2,1}\left([0,T] \times \mathbb{R}^N \times \mathbb{R}_0^+;\mathbb{R}\right)$ for every state $a \in S$. Then, the value function satisfies*

$$\begin{cases} V_t + \sup\limits_{u\in U} \mathcal{H}(t,x,a,\tau,u,V,V_x,V_{xx},V_\tau) = 0, \\ V(T,x,a,\tau) = \Psi(x,a,\tau) \ , \end{cases} \tag{13}$$

where for $(t,x,a,\tau,p,A,q) \in [0,T] \times \mathbb{R}^N \times S \times \mathbb{R}_0^+ \times \mathbb{R}^N \times \mathrm{Sym}_{n\times n}(\mathbb{R}) \times \mathbb{R}$ and any real valued function ω on $[0,T] \times \mathbb{R}^{\mathbb{N}} \times S \times \mathbb{R}_0^+$ such that $\omega(\cdot,\cdot,a,\cdot)$ is smooth for every $a \in S$, we have

$$\mathcal{H}(t,x,a,\tau,\omega,p,A,q) = H_1(t,x,a,\tau,u,p,A,q) + H_2(t,x,a,\tau)[\omega] \ ,$$

with

$$\begin{aligned} H_1(t,x,a,\tau,u,p,A,q) \ &= \ L(t,x,a,\tau,u) + q + \langle p, f(t,x,a,\tau,u)\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sigma^T(t,x,a,\tau,u)A\sigma(t,x,a,\tau,u)\right) \ , \end{aligned}$$

and

$$H_2(t,x,a,\tau)[\omega] = \sum_{j\in S: j\neq a} \lambda_{\alpha j}(\tau)\left(\omega(t,x,j,0) - \omega(t,x,a,\tau)\right) \ .$$

Proof. Let $\bar{x} = (x,a,\tau) \in \mathcal{N}$, $u \in U$, and $\mathcal{X}(\cdot)$ be the state trajectory determined by (4) with initial condition $(s,\bar{x}) \in [0,T) \times \mathcal{N}$ and the constant control path $u(\cdot) \equiv u \in \mathcal{U}^w[s,T]$. Letting $\hat{s} \in [s,T]$ and using Itô's formula for semi-Markov modulated diffusive processes (see Lemma A.1 in Appendix A), we get

$$V(\hat{s},\mathcal{X}(\hat{s})) - V(s,\bar{x}) = \int_s^{\hat{s}} a\left(t,\mathcal{X}(t),u(t)\right)\,\mathrm{d}t + M(\hat{s}) \ , \tag{14}$$

where the integrand function is given by

$$\begin{aligned} a(t,x,a,\tau,u) &= V_t(t,x,a,\tau)+V_\tau(t,x,a,\tau)\\ &\quad+\langle V_x(t,x,a,\tau), f(t,x,a,\tau,u)\rangle \qquad (15)\\ &\quad+\frac{1}{2}\mathrm{tr}\left(\sigma^T(t,x,a,\tau,u)V_{xx}(t,x,a,\tau)\sigma(t,x,a,\tau,u)\right)\\ &\quad+\sum_{j\in S:j\neq a}\lambda_{aj}(\tau)\left(V(t,x,j,0)-V(t,x,a,\tau)\right) \end{aligned}$$

and $M(\cdot)$ is the martingale

$$M(r) = \int_s^r b(t,\mathcal{X}(t),u(t))\,\mathrm{d}W(t)+\int_s^r\int_{\mathbb{R}} c\left(t,\mathcal{X}(t),z\right)\ \tilde{N}(\mathrm{d}t,\mathrm{d}z)\ ,$$

with

$$\begin{aligned} b(t,x,a,\tau,u) &= (V_x(t,x,a,\tau))^T\sigma(t,x,a,\tau,u)\\ c(t,x,a,\tau,z) &= V(t,x,a+\Gamma_1(a,\tau,z),\tau-\Gamma_2(a,\tau,z))-V(t,x,a,\tau) \end{aligned}$$

and

$$\tilde{N}(\mathrm{d}t,\mathrm{d}z)=N(\mathrm{d}t,\mathrm{d}z)-\mathrm{d}t\mathrm{d}z$$

the compensated Poisson random measure associated with the semi-Markov process $\alpha(\cdot)$.

Taking the expected value in both sides of (14) and dividing by $\hat{s}-s$, we get

$$\frac{\mathbb{E}\left[V(\hat{s},\mathcal{X}(\hat{s}))-V(s,\bar{x})\right]}{\hat{s}-s}=\frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} a(t,\mathcal{X}(t),u(t))\,\mathrm{d}t\right]. \qquad (16)$$

Resorting to the Dynamic Programming Principle given in Theorem 3.3, we also obtain that

$$-\frac{\mathbb{E}\left[V(\hat{s},\mathcal{X}(\hat{s}))-V(s,\bar{x})\right]}{\hat{s}-s}\geq\frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} L\left(t,\mathcal{X}(t),u(t)\right)\ \mathrm{d}t\right].$$

From the previous inequality and identity (16), we get

$$\frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} L\left(t,\mathcal{X}(t),u(t)\right)+a(t,\mathcal{X}(t),u(t))\,\mathrm{d}t\right]\leq 0\ .$$

Letting $\hat{s}$ approach s from above, we obtain that

$$V_t+\mathcal{H}\left(s,x,a,\tau,u,V,V_x,V_{xx},V_\tau\right)\leq 0$$

for every $u\in U$, where the dependence of the derivatives of V on their arguments was dropped for notational convenience. Thus, taking the supremum over $u\in U$, we conclude that

$$V_t+\sup_{u\in U}\mathcal{H}\left(s,x,a,\tau,u,V,V_x,V_{xx},V_\tau\right)\leq 0\ . \qquad (17)$$

Conversely, for any $\epsilon>0$ and any $\hat{s}\in(s,T]$ with $\hat{s}-s$ small enough, there exists $\tilde{u}(\cdot):=u_{\epsilon,\hat{s}}(\cdot)\in\mathcal{U}^w[s,T]$ such that

$$V(s,\bar{x})-\epsilon(\hat{s}-s)\leq\mathbb{E}\left[\int_s^{\hat{s}} L\left(t,\mathcal{X}(t),\tilde{u}(t)\right)\ \mathrm{d}t+V\left(\hat{s},\mathcal{X}(\hat{s})\right)\right].$$

From the inequality above, after rearranging terms, we obtain

$$\epsilon \geq -\frac{\mathbb{E}\left[V(\hat{s},\mathcal{X}(\hat{s}))-V(s,\bar{x})\right]}{\hat{s}-s} - \frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} L\left(t,\mathcal{X}(t),\tilde{u}(t)\right)\,\mathrm{d}t\right] .$$

Combining identity (16) with the previous inequality we get that

$$\epsilon \geq -\frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} a(t,\mathcal{X}(t),\tilde{u}(t))\,\mathrm{d}t + \int_s^{\hat{s}} L\left(t,\mathcal{X}(t),\tilde{u}(t)\right)\,\mathrm{d}t\right] .$$

Therefore, we obtain

$$\epsilon \quad \geq \quad -\frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} V_t\left(t,\mathcal{X}(t)\right)+\widetilde{\mathcal{H}}\left(t,\mathcal{X}(t),\tilde{u}(t)\right)\,\mathrm{d}t\right] ,$$

where $\widetilde{\mathcal{H}}\left(t,\mathcal{X}(t),\tilde{u}(t)\right)$ is shorthand notation for

$$\mathcal{H}\Big(t,\mathcal{X}(t),\tilde{u}(t),V(t,\mathcal{X}(t)),V_x(t,\mathcal{X}(t)),V_{xx}(t,\mathcal{X}(t)),V_\tau(t,\mathcal{X}(t))\Big) .$$

Resorting to the uniform continuity of assumption ($A3$), we get

$$-\epsilon \leq V_t + \sup_{u\in U}\mathcal{H}\left(s,x,a,\tau,u,V,V_x,V_{xx},V_\tau\right) , \tag{18}$$

where the dependence of the derivatives of V on their arguments was again dropped for notational convenience.

The result follows from combining (17) with (18). □

Let $\bar{x}=(x,a,\tau)\in\mathcal{N}$ and let us use the notation $\widetilde{\mathcal{H}}(s,\bar{x},u)$ to represent the function

$$\widetilde{\mathcal{H}}(s,\bar{x},u) = \mathcal{H}(s,\bar{x},u,V(s,\bar{x}),V_x(s,\bar{x}),V_{xx}(s,\bar{x}),V_\tau(s,\bar{x})) .$$

The following verification theorem holds.

Proposition 4.2 (Verification Theorem)**.** *Assume that conditions (A1)–(A4) hold and that* $V(\cdot,\cdot,a,\cdot)\in C^{1,2,1}([0,T]\times\mathbb{R}^N\times\mathbb{R}_0^+;\mathbb{R})$ *for each* $a\in S$*. If* $V(s,\bar{x})$ *satisfies* (13)*, then the inequality*

$$V(s,\bar{x}) \geq J(s,\bar{x};u(\cdot))$$

holds for every $u(\cdot)\in\mathcal{U}^w[s,T]$ *and* $(s,\bar{x})\in[0,T)\times\mathcal{N}$*. Furthermore, an admissible pair* $(\bar{\mathcal{X}}(\cdot),\bar{u}(\cdot))$ *is optimal for* (6) *if and only if the equality*

$$V_t(t,\bar{\mathcal{X}}(t)) + \widetilde{\mathcal{H}}(t,\bar{\mathcal{X}}(t),\bar{u}(t)) = 0$$

holds for a.e. $t\in[s,T]$ *and* $\mathbb{P}-$a.s..

Proof. Using Itô's formula for semi-Markov modulated diffusive processes as given in Lemma A.1, for any control $u(\cdot)\in\mathcal{U}^w[s,T]$ and corresponding state trajectory $\mathcal{X}(\cdot)$, we get

$$V(s,\bar{x}) = \mathbb{E}\left[\Psi(\mathcal{X}(T)) - \int_s^T a(t,\mathcal{X}(t),u(t))\,\mathrm{d}t\right] ,$$

where $a(t,x,a,\tau,u)$ is as given in (15). Using (5) and the definition of the Hamiltonian function in the statement of Proposition 4.1, the last equality may be written as

$$V(s,\bar{x}) = J(s,\bar{x};u(\cdot)) - \mathbb{E}\left[\int_s^T V_t(t,\mathcal{X}(t)) + \widetilde{\mathcal{H}}(t,\mathcal{X}(t),u(t))\,\mathrm{d}t\right] . \tag{19}$$

Using the HJB equation (13), we conclude that

$$V(s,\bar{x}) \geq J(s,\bar{x};u(\cdot)) \ ,$$

completing the proof of the first part of the theorem.

To prove the second part of the theorem, let $(\bar{\mathcal{X}}(\cdot),\bar{u}(\cdot))$ be an optimal pair for (6). Applying equality (19) to $(\bar{\mathcal{X}}(\cdot),\bar{u}(\cdot))$, we get

$$V(s,\bar{x}) \geq J(s,\bar{x};\bar{u}(\cdot)) - \mathbb{E}\left[\int_s^T V_t(t,\bar{\mathcal{X}}(t)) + \widetilde{\mathcal{H}}(t,\bar{\mathcal{X}}(t),\bar{u}(t))\,\mathrm{d}t\right] \ .$$

The desired result follows immediately from the fact that

$$V_t(t,\bar{\mathcal{X}}(t)) + \widetilde{\mathcal{H}}(t,\bar{\mathcal{X}}(t),\bar{u}(t)) \leq 0 \ ,$$

a consequence of equation (13). □

Finally, we will show that the value function (7) is a viscosity solutions of the HJB equation (13). Before proceeding, let us recall the definition of viscosity solution used herein. A continuous function $v : [0,T]\times\mathcal{N} \to \mathbb{R}$ is called a *viscosity subsolution* of (13) (resp. *supersolution*) if for every $\bar{x} = (x,a,\tau) \in \mathcal{N}$ we have that

$$v(T,\bar{x}) \leq \Psi(\bar{x}) \quad (\text{resp. } v(T,\bar{x}) \geq \Psi(\bar{x})) \ ,$$

and, additionally,

$$\varphi_t(t_0,\bar{x}_0) + \sup_{u\in U}\mathcal{H}(t_0,\bar{x}_0,u,\varphi(t_0,\bar{x}_0),\varphi_x(t_0,\bar{x}_0),\varphi_{xx}(t_0,\bar{x}_0),\varphi_\tau(t_0,\bar{x}_0)) \geq 0$$

$$(\text{resp. } \varphi_t(t_0,\bar{x}_0) + \sup_{u\in U}\mathcal{H}(t_0,\bar{x}_0,u,\varphi(t_0,\bar{x}_0),\varphi_x(t_0,\bar{x}_0),\varphi_{xx}(t_0,\bar{x}_0),\varphi_\tau(t_0,\bar{x}_0)) \leq 0)$$

for every continuous function $\varphi : [0,T]\times\mathcal{N} \to \mathbb{R}$ such that $\varphi(\cdot,\cdot,a,\cdot) \in C^{1,2,1}([0,T]\times\mathbb{R}^N\times\mathbb{R}_0^+;\mathbb{R})$ for every $a \in S$ and any local maximum (resp. minimum) $(t_0,\bar{x}_0)$ of $v-\varphi$. If v is simultaneously a viscosity subsolution and viscosity supersolution of (13), then it is called a *viscosity solution* of (13).

Theorem 4.3. *Assume that conditions (A1)–(A4) hold. Then the value function V is a viscosity solution of* (13).

Proof. Let $\varphi : [0,T]\times\mathcal{N} \to \mathbb{R}$ be a continuous function such that $\varphi(\cdot,\cdot,a,\cdot) \in C^{1,2,1}([0,T]\times\mathbb{R}^N\times\mathbb{R}_0^+;\mathbb{R})$ for every $a \in S$. Start by supposing that $V-\varphi$ attains a local minimum at $(s,\bar{y}) \in [0,T)\times\mathcal{N}$, fix $u \in U$, and denote by $\mathcal{X}(\cdot)$ the state trajectory with initial condition $\bar{y} \in \mathcal{N}$ at time s under the control $u(t) \equiv u$. Using the Dynamic programming principle Theorem 3.3 we obtain that, for $\hat{s} > s$ with $\hat{s}-s > 0$ small enough, it holds that

$$\begin{aligned} 0 &\geq \frac{\mathbb{E}\left[V(s,\bar{y}) - \varphi(s,\bar{y}) - V(\hat{s},\mathcal{X}(\hat{s})) + \varphi(\hat{s},\mathcal{X}(\hat{s}))\right]}{\hat{s}-s} \\ &\geq \frac{1}{\hat{s}-s}\mathbb{E}\left[\int_s^{\hat{s}} L(t,\mathcal{X}(t),u)\,\mathrm{d}t - \varphi(s,\bar{y}) + \varphi(\hat{s},\mathcal{X}(\hat{s}))\right] \ . \end{aligned}$$

Applying Itô's formula for semi-Markov modulated diffusive processes (Lemma A.1) to the process $\varphi(t,\mathcal{X}(t))$ and combining the outcome with the inequality above, yields

$$\varphi_t(s,\bar{y}) + \mathcal{H}\left(s,\bar{y},u,\varphi(s,\bar{y}),\varphi_x(s,\bar{y}),\varphi_{xx}(s,\bar{y}),\varphi_\tau(s,\bar{y})\right) \leq 0$$

for all $u \in U$. Hence, we conclude that

$$\varphi_t(s,\bar{y}) + \sup_{u\in U} \mathcal{H}\left(s,\bar{y},u,\varphi(s,\bar{y}),\varphi_x(s,\bar{y}),\varphi_{xx}(s,\bar{y}),\varphi_\tau(s,\bar{y})\right) \le 0 \ . \qquad (20)$$

On the other hand, if $V-\varphi$ attains a local maximum at $(s,\bar{y}) \in [0,T)\times\mathcal{N}$ then, for any $\epsilon > 0$ and $\hat{s} > s$ with $\hat{s} - s > 0$ small enough, there exists $u_{\epsilon,\hat{s}}(\cdot) \in \mathcal{U}^\omega[s,T]$ such that

$$\begin{aligned} 0 &\le \mathbb{E}\left[V(s,\bar{y}) - \varphi(s,\bar{y}) - V(\hat{s},\mathcal{X}(\hat{s})) + \varphi(\hat{s},\mathcal{X}(\hat{s}))\right] \\ &\le \epsilon(\hat{s}-s) + \mathbb{E}\left[\int_s^{\hat{s}} L(t,\mathcal{X}(t),u_{\epsilon,\hat{s}}(t))\,\mathrm{d}t + \varphi(\hat{s},\mathcal{X}(\hat{s})) - \varphi(s,\bar{y})\right] \ , \end{aligned}$$

where $\mathcal{X}(\cdot)$ now denotes the state trajectory with initial condition $\bar{y} \in \mathcal{N}$ at time s under the control $u_{\epsilon,\hat{s}}(\cdot)$. Dividing the inequality above by $\hat{s}-s$ and applying Itô's formula for semi-Markov modulated diffusive processes (Lemma A.1) to the process $\varphi(t,\mathcal{X}(t))$, we get

$$\begin{aligned} -\epsilon &\le \frac{1}{\hat{s}-s}\mathbb{E}\Big[\int_s^{\hat{s}} \big\{\varphi_t(t,\mathcal{X}(t)) \\ &\quad + \mathcal{H}\left(t,\mathcal{X}(t),u_{\epsilon,\hat{s}}(t),\varphi(t,\mathcal{X}(t)),\varphi_x(t,\mathcal{X}(t)),\varphi_{xx}(t,\mathcal{X}(t)),\varphi_\tau(t,\mathcal{X}(t))\right)\big\}\,\mathrm{d}t\Big] \\ &\le \frac{1}{\hat{s}-s}\mathbb{E}\Big[\int_s^{\hat{s}} \big\{\varphi_t(t,\mathcal{X}(t)) \\ &\quad + \sup_{u\in U}\mathcal{H}\left(t,\mathcal{X}(t),u,\varphi(t,\mathcal{X}(t)),\varphi_x(t,\mathcal{X}(t)),\varphi_{xx}(t,\mathcal{X}(t)),\varphi_\tau(t,\mathcal{X}(t))\right)\big\}\,\mathrm{d}t\Big] \ . \end{aligned}$$

Hence, we obtain that

$$\varphi_t(s,\bar{y}) + \sup_{u\in U} \mathcal{H}\left(s,\bar{y},u,\varphi(s,\bar{y}),\varphi_x(s,\bar{y}),\varphi_{xx}(s,\bar{y}),\varphi_\tau(s,\bar{y})\right) \ge 0 \ . \qquad (21)$$

The result follows from combining (20) and (21). □

5. Application to consumption-investment problems

In what follows, we rely on the setup introduced in Section 2 with the additional simplifying assumption that the Brownian motion $(W(t) : t \in [0,T])$ is one-dimensional, and consider a continuous-time financial market consisting of one risk-free asset and one risky-asset. More precisely, we assume that the prices of the risk-free asset $(S_0(t) : t \in [0,T])$ and the risky asset $(S_1(t) : t \in [0,T])$ evolve according to the semi-Markov modulated SDEs

$$\begin{aligned} \mathrm{d}S_0(t) &= r(t,\alpha(t_-),\mathcal{T}(t_-))S_0(t)\mathrm{d}t, \\ \mathrm{d}S_1(t) &= \mu(t,\alpha(t_-),\mathcal{T}(t_-))S_1(t)\mathrm{d}t + \sigma(t,\alpha(t_-),\mathcal{T}(t_-))S_1(t)\mathrm{d}W(t) \end{aligned}$$

with positive initial conditions $S_0(0) = s_0$ and $S_1(0) = s_1$. Note that the financial market coefficients are joint functions of time, the current state of the semi-Markov process $\alpha(t)$ and the corresponding time component $\mathcal{T}(t)$ representing the time elapsed since the last switch of $\alpha(\cdot)$ (as observed at time t). More precisely, we assume that the risk-free interest rate $r(t,a,\tau)$, the risky-asset appreciation rate $\mu(t,a,\tau)$ and the risky-asset volatility $\sigma(t,a,\tau)$, are all deterministic continuous functions of $(t,\tau) \in [0,T] \times \mathbb{R}_0^+$ for every fixed $a \in S$. Additionally, we assume that the risk-free interest rate $r(t,a,\tau)$ is positive for every $(t,a,\tau) \in [0,T] \times S \times \mathbb{R}_0^+$.

We now introduce the control variables. The *consumption process* $(c(t) : t \in [0,T])$ is a $\mathbb{F}$-progressively measurable nonnegative process satisfying the following integrability condition for the investment horizon $T > 0$:

$$\int_0^T c(t)\,\mathrm{d}t < \infty \qquad \text{a.s.}\ .$$

Let $\theta(t)$ denote the fraction of the agent's wealth allocated to the risky asset S_1 at time $t \in [0,T]$. We assume that $(\theta(t) : t \in [0,T])$ is $\mathbb{F}$-progressively measurable and that, for the fixed maximum investment horizon $T > 0$, we have that

$$\int_0^T |\theta(t)|^2\,\mathrm{d}t < \infty \qquad \text{a.s.}\ .$$

Clearly, the agent invests $1 - \theta(t)$ of her wealth on the risk-free asset S_0.

The *wealth process* $X(t)$, $t \in [0,T]$, is defined by coupling (1) with the SDE

$$\begin{aligned} \mathrm{d}X(t) = X(0) + \Big[&(-c(t) + r(t,\alpha(t_-),\mathcal{T}(t_-))X(t))\,\mathrm{d}t \\ &+ \theta(t)\left(\mu(t,\alpha(t_-),\mathcal{T}(t_-)) - r(t,\alpha(t_-),\mathcal{T}(t_-))\right)X(t)\Big]\mathrm{d}t \\ &+ \theta(t)\sigma(t,\alpha(t_-),\mathcal{T}(t_-))X(t)\mathrm{d}W(t) \end{aligned} \tag{22}$$

and imposing initial conditions $X(0) = x$, $\alpha(0) = a$ and $\mathcal{T}(0) = \tau$ representing, respectively, the initial wealth, the initial state of the semi-Markov process $\alpha(\cdot)$, as well as the corresponding time component $\mathcal{T}(\cdot)$ measuring the time elapsed since the last switch of $\alpha(\cdot)$. Within the setup under consideration in this problem, the state space is now

$$\mathcal{N} = \mathbb{R} \times S \times \mathbb{R}_0^+\ .$$

Let $\bar{x} = (x, a, \tau) \in \mathcal{N}$. The consumption-investment problem is to find admissible consumption and investment strategies $(c,\theta) \in \mathcal{U}^s[0,T]$ which maximize the expected utility

$$J(\bar{x}; c(\cdot), \theta(\cdot)) = \mathbb{E}\left[\int_0^T U(t, c(t), \mathcal{X}^{c,\theta}_{0,\bar{x}}(t))\,\mathrm{d}t + \Psi(\mathcal{X}^{c,\theta}_{0,\bar{x}}(T))\right]\ , \tag{23}$$

where

$$\mathcal{X}^{c,\theta}_{0,\bar{x}}(t) = (X_{0,\bar{x}}(t; c(\cdot), \theta(\cdot)), \alpha_{0,\bar{x}}(t), \mathcal{T}_{0,\bar{x}}(t)) \in \mathcal{N}$$

denotes the solution of the system obtained by coupling (22) with the semi-Markov process dynamics (1), starting from $\bar{x} = (x, a, \tau) \in \mathcal{N}$ at time $t = 0$ under the strategies $(c(\cdot), \theta(\cdot)) \in \mathcal{U}^s[0,T]$. In the definition of the expected utility (23), $U(t, c, \bar{x})$ is the utility derived from a consumption rate $c \in [0,\infty)$ at time t while holding wealth x when the state of the semi-Markov process is a and its last switch occurred at time $t - \tau$. Similarly, $\Psi(\bar{x})$ is the utility obtained from holding wealth x at time T when the state of the semi-Markov process is a and its last switch occurred at time $T - \tau$.

Proceeding as described in Section 2, we rewrite the expected utility in dynamic programming form

$$J(s, \bar{x}; c(\cdot), \theta(\cdot)) = \mathbb{E}\left[\int_s^T U(t, c(t), \mathcal{X}^{c,\theta}_{s,\bar{x}}(t))\,\mathrm{d}t + \Psi(\mathcal{X}^{c,\theta}_{s,\bar{x}}(T))\right]\ , \tag{24}$$

where

$$\mathcal{X}^{c,\theta}_{s,\bar{x}}(t) = (X_{s,\bar{x}}(t; c(\cdot), \theta(\cdot)), \alpha_{s,\bar{x}}(t), \mathcal{T}_{s,\bar{x}}(t)) \in \mathcal{N}$$

denotes the solution of the system obtained by coupling (22) with the semi-Markov process dynamics (1), starting from $\bar{x} = (x, a, \tau) \in \mathcal{N}$ at time $t = s$ under the weak admissible controls $(c(\cdot), \theta(\cdot)) \in \mathcal{U}^w[s, T]$. We will employ dynamic programming techniques to obtain a rather complete description for the behavior of the maximum expected utility, or value function, given by

$$\begin{cases} V(t, \bar{x}) = \sup_{(c,\theta)\in\mathcal{U}^w[s,T]} J(t, \bar{x}; c(\cdot), \theta(\cdot)) \ , \quad (t, \bar{x}) \in [0, T) \times \mathcal{N} \\ V(T, \bar{x}) = \Psi(\bar{x}) \end{cases} .$$

In the next subsections we will illustrate the theoretical results obtained in the previous sections to study the optimal strategies for the expected utility given in (24) for the economically relevant cases of utility functions with constant coefficient of relative risk aversion - power type and logarithmic utility functions.

5.1. **The case of power utility functions.** In this subsection we assume that the utility functions are of the form

$$U(t, c, a, \tau) = \mathrm{e}^{-\rho t} u_a(\tau) \frac{c^\gamma}{\gamma} \qquad \text{and} \qquad \Psi(x, a, \tau) = \mathrm{e}^{-\rho t} v_a(\tau) \frac{x^\gamma}{\gamma} \ , \tag{25}$$

where $\gamma \in (0, 1)$ is the investor risk aversion coefficient, $\rho > 0$ is the discount rate, and the factors $u_a(\tau)$ and $v_a(\tau)$ are assumed to be strictly positive functions modeling the influence of the semi-Markov process state a and time component τ on the investor preferences induced by the utility functions U and Ψ.

In the next theorem, we compute the optimal strategies for the class of discounted utility functions (25). Before providing the precise statement, let us introduce the function $F : [0, 1] \times [0, T] \times S \times \mathbb{R}^+_0 \to \mathbb{R}$ given by

$$F(\theta; t, a, \tau) = \gamma \left[r(t, a, \tau) + \theta\big(\mu(t, a, \tau) - r(t, a, \tau)\big) - \frac{1}{2}(1 - \gamma)\theta^2\sigma^2(t, a, \tau) \right] \tag{26}$$

and note that

$$\frac{\partial F}{\partial \theta}(\theta; t, a, \tau) = \gamma \Big[\mu(t, a, \tau) - r(t, a, \tau) - (1 - \gamma)\theta\sigma^2(t, a, \tau) \Big] .$$

Theorem 5.1. *The maximum expected utility associated with* (24) *and the discounted utility functions* (25) *is given by*

$$V(t, x, a, \tau) = \xi_a(t, \tau) \frac{x^\gamma}{\gamma} \ , \tag{27}$$

the corresponding optimal strategies are of the form

$$c^*(t, x, a, \tau) = x \left(\frac{\mathrm{e}^{\rho t} \xi_a(t, \tau)}{u_a(\tau)} \right)^{-1/(1-\gamma)}$$

and

$$\theta^*(t,a,\tau) = \begin{cases} 1, & \text{if} \quad \mu(t,a,\tau) > r(t,a,\tau) \quad \text{and} \quad F'(1;t,a,\tau) \geq 0 \\ \hat{\theta}(t,a,\tau), & \text{if} \quad \mu(t,a,\tau) > r(t,a,\tau) \quad \text{and} \quad F'(1;t,a,\tau) < 0 \\ 0, & \text{if} \quad \mu(t,a,\tau) \leq r(t,a,\tau) \end{cases},$$

where $\hat{\theta}(t,a,\tau)$ *is given by*

$$\hat{\theta}(t,a,\tau) = \frac{\mu(t,a,\tau) - r(t,a,\tau)}{(1-\gamma)\sigma^2(t,a,\tau)}$$

and $\xi_a(t,\tau)$, $a \in S$, *are the solutions of the following coupled first order boundary value problem*

$$\begin{aligned} &\frac{\partial \xi_a}{\partial t}(t,\tau) + \frac{\partial \xi_a}{\partial \tau}(t,\tau) + (1-\gamma)\mathrm{e}^{-\rho t/(1-\gamma)}\xi_a(t,\tau)^{-\gamma/(1-\gamma)} \\ &+F(\theta^*(t,a);t,a,\tau)\xi_a(t,\tau) + \sum_{j\in S:j\neq a} \lambda_{aj}(\tau)(\xi_j(t,\tau) - \xi_a(t,\tau)) = 0 \\ &\xi_a(T,\tau) = e^{-\rho T} v_a(\tau) \ . \end{aligned}$$

Proof. Assume for the time being that the conditions of Proposition 4.1 hold. The Hamiltonian function $\mathcal{H}$ associated with the expected utility (24) and the discounted utility functions (25) is given by

$$\begin{aligned} &\mathcal{H}(t,x,a,\tau,c,\theta,V,V_x,V_{xx},V_\tau) = \\ &\quad \mathrm{e}^{-\rho t} u_a(\tau)\frac{c^\gamma}{\gamma} + \left(-c + \Big(r(t,a,\tau) + \theta\big(\mu(t,a,\tau) - r(t,a,\tau)\big)\Big)x\right) V_x(t,x,a,\tau) \\ &\quad + \frac{x^2}{2}\big(\theta\sigma(t,a,\tau)\big)^2 V_{xx}(t,x,a,\tau) + \sum_{j\in S:j\neq a}\lambda_{aj}(\tau)(V(t,x,j,0) - V(t,x,a,\tau)) \\ &\quad + V_\tau(t,x,a,\tau) \end{aligned}$$

and the corresponding Hamilton-Jacobi-Bellman equation is

$$V_t + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \mathcal{H}(t,x,a,\tau,c,\theta,V,V_x,V_{xx},V_\tau) = 0 \ .$$

Considering an ansatz of the form (27) and substituting in the HJB equation above, we get

$$\begin{aligned} &\left(\frac{\partial \xi_a}{\partial t}(t,\tau) + \frac{\partial \xi_a}{\partial \tau}(t,\tau)\right)\frac{x^\gamma}{\gamma} + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \Bigg\{ \mathrm{e}^{-\rho t}u_a(\tau)\frac{c^\gamma}{\gamma} \\ &\qquad + \left(-c + \Big(r(t,a,\tau) + \theta(\mu(t,a,\tau) - r(t,a,\tau))\Big)x\right)\xi_a(t,\tau)x^{\gamma-1} \\ &\qquad + \frac{\theta^2}{2}\sigma^2(t,a,\tau)(\gamma-1)\xi_a(t,\tau)x^\gamma \Bigg\} \\ &\qquad + \frac{x^\gamma}{\gamma}\sum_{j\in S:j\neq a}\lambda_{aj}(\tau)\big(\xi_j(t,0) - \xi_a(t,\tau)\big) = 0 \ . \end{aligned} \tag{28}$$

Note that the optimization problem (28) breaks down into two independent optimization problems and its solution can be obtained in a sequential way.

We start by optimizing (28) with respect to c, before proceeding to optimize with respect to the variable θ.

Since $\gamma \in (0,1)$, the quantity to be maximized in (28) is strictly concave with respect to the control variable c. Indeed, the first order condition associated with the optimization problem above provides a maximizer $c^*(t,x,a,\tau)$, which is given by

$$c^*(t,x,a,\tau) = x\left(\frac{\mathrm{e}^{\rho t}\xi_a(t,\tau)}{u_a(\tau)}\right)^{-1/(1-\gamma)} .$$

Replacing c by $c^*(t,x,a,\tau)$ in (28) and factoring out the term x^γ/γ, we obtain that

$$\frac{\partial \xi_a}{\partial t}(t,\tau) + \frac{\partial \xi_a}{\partial \tau}(t,\tau) + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)\big(\xi_j(t,0) - \xi_a(t,\tau)\big) \tag{29}$$

$$+ \sup_{\theta\in[0,1]}\left\{(1-\gamma)e^{-\rho t/(1-\gamma)}\left(\frac{u_a(\tau)}{(\xi_a(t,\tau))^\gamma}\right)^{1/(1-\gamma)} + F(\theta;t,a,\tau)\xi_a(t,\tau)\right\} = 0 ,$$

where $F(\theta;t,a,\tau)$ is as given in (26). Note that the first order condition with respect to θ is just $F'(\theta;t,a,\tau) = 0$ and that since $0 < \gamma < 1$, the second derivative of $F(\theta;t,a,\tau)$ with respect to θ, given by

$$F''(\theta;t,a,\tau) = -\gamma(1-\gamma)\sigma^2(t,a,\tau) ,$$

is negative for every $\theta \in [0,1]$. Taking into account the constraint $\theta \in [0,1]$ and the concavity of $F(\theta;t,a,\tau)$ with respect to θ, we conclude that the maximization problem in (29) has a unique solution $\theta^*(t,a,\tau)$. Moreover, from the definition of the function $F(\theta;t,a,\tau)$, it is possible to check that

i) if $\mu(t,a,\tau) - r(t,a,\tau) > 0$ and $F(1;t,a,\tau) < 0$, then
$$\theta^*(t,a,\tau) = \frac{\mu(t,a,\tau) - r(t,a,\tau)}{(1-\gamma)\sigma^2(t,a,\tau)} ;$$
ii) if $\mu(t,a,\tau) - r(t,a,\tau) > 0$ and $F(1;t,a,\tau) \geq 0$, then $\theta^*(t,a,\tau) = 1$;
iii) if $\mu(t,a,\tau) - r(t,a,\tau) \leq 0$, then $\theta^*(t,a,\tau) = 0$;

□

We remark that the optimal portfolio, determined by θ^*, depends on time $t \in [0,T]$, on the state of the semi-Markov process $a \in S$, and on the corresponding time component, measuring the time elapsed since the last switch of $\alpha(\cdot)$. However, θ^* does not depend on the wealth x, yielding a mutual fund theorem. In what concerns the optimal consumption c^*, it is clear that it is increasing with wealth and that, for choices of coefficients compatible with standard financial market behaviour, c^* is increasing with time t.

5.2. **The case of logarithmic utility functions.** Consider now utility functions of the form

$$U(t,c,a,\tau) = \mathrm{e}^{-\rho t}u_a(\tau)\ln c \qquad \text{and} \qquad \Psi(x,a,\tau) = \mathrm{e}^{-\rho t}v_a(\tau)\ln x , \tag{30}$$

where $\rho > 0$ is the discount rate and the functions $u_a(\tau)$ and $v_a(\tau)$ are as described in section 5.1.

The next theorem provides the optimal strategies for the class of discounted logarithmic utility functions of the form (30). Before providing the precise statement, let us introduce the function $F : [0,1]\times[0,T]\times S\times\mathbb{R}_0^+ \to \mathbb{R}$ given by

$$F(\theta; t, a, \tau) = r(t, a, \tau) + \theta\big(\mu(t, a, \tau) - r(t, a, \tau)\big) - \frac{1}{2}\theta^2\sigma^2(t, a, \tau) \qquad (31)$$

and note that

$$F'(\theta; t, a, \tau) = \mu(t, a, \tau) - r(t, a, \tau) - \theta\sigma^2(t, a, \tau)\ ,$$

where the derivative is taken with respect to θ.

Theorem 5.2. *The maximum expected utility associated with* (24) *and the discounted logarithmic utility functions* (30) *is defined by*

$$V(t, x, a, \tau) = \xi_a(t, \tau) \ln x + \zeta_a(t, \tau)\ ,$$

the corresponding optimal strategies are of the form

$$c^*(t, x, a, \tau) = \mathrm{e}^{-\rho t}\frac{u_a(\tau)}{\xi(t, \tau)}x$$

and

$$\theta^*(t, a, \tau) = \begin{cases} 1 & \text{if} \quad \mu(t,a,\tau) > r(t,a,\tau) \quad \text{and} \quad F'(1;t,a,\tau) \geq 0 \\ \hat{\theta}(t,a,\tau) & \text{if} \quad \mu(t,a,\tau) > r(t,a,\tau) \quad \text{and} \quad F'(1;t,a,\tau) < 0 \\ 0 & \text{if} \quad \mu(t,a,\tau) \leq r(t,a,\tau) \end{cases}\ ,$$

where $\hat{\theta}(t, a, \tau)$ *is given by*

$$\hat{\theta}(t, a, \tau) = \frac{\mu(t, a, \tau) - r(t, a, \tau)}{\sigma^2(t, a, \tau)}$$

and $\xi_a(t, \tau)$ *and* $\zeta_a(t, \tau)$*,* $a \in S$*, are the solutions of the following system of coupled first order boundary value problems*

$$\begin{aligned} &\frac{\partial \xi_a}{\partial t}(t,\tau) + \frac{\partial \xi_a}{\partial \tau}(t,\tau) + \mathrm{e}^{-\rho t}u_a(\tau) + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)(\xi_j(t,0) - \xi_a(t,\tau)) = 0 \\ &\frac{\partial \zeta_a}{\partial t}(t,\tau) + \frac{\partial \zeta_a}{\partial \tau}(t,\tau) - \mathrm{e}^{-\rho t}u_a(\tau)\left(\rho t + \ln\frac{\xi_a(t,\tau)}{u_a(\tau)} + 1\right) \\ &\qquad + F(\theta^*(t,a,\tau); t,a,\tau)\xi_a(t,\tau) + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)(\zeta_j(t,0) - \zeta_a(t,\tau)) = 0 \\ &\xi_a(T,\tau) = \mathrm{e}^{-\rho T}v_a(\tau)\ , \quad \zeta_a(T,\tau) = 0\ . \end{aligned}$$

Proof. Assume that the conditions of Proposition 4.1 hold. The Hamiltonian function $\mathcal{H}$ associated with the expected utility (24) and the discounted logarithmic utility functions (30) is defined by

$$
\begin{aligned}
&\mathcal{H}(t,x,a,\tau,c,\theta,V,V_x,V_{xx},V_\tau) = \\
&\quad \mathrm{e}^{-\rho t}u_a(\tau)\ln c + \left(-c + \Big(r(t,a,\tau) + \theta\big(\mu(t,a,\tau) - r(t,a,\tau)\big)\Big) x \right) V_x(t,x,a,\tau) \\
&\quad + \frac{x^2}{2}\big(\theta\sigma(t,a,\tau)\big)^2 V_{xx}(t,x,a,\tau) + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)(V(t,x,j,0) - V(t,x,a,\tau)) \\
&\quad + V_\tau(t,x,a,\tau)
\end{aligned}
$$

and the HJB equation is then

$$
V_t + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \mathcal{H}(t,x,a,\tau,c,\theta,V,V_x,V_{xx},V_\tau) = 0 \ .
$$

Considering an ansatz of the form

$$
V(t,x,a,\tau) = \xi_a(t,\tau)\ln x + \zeta_a(t,\tau)
$$

and substituting in the HJB equation above, we get

$$
\begin{aligned}
&\left(\frac{\partial \xi_a}{\partial t}(t,\tau) + \frac{\partial \xi_a}{\partial \tau}(t,\tau)\right)\ln x + \frac{\partial \zeta_a}{\partial t}(t,\tau) + \frac{\partial \zeta_a}{\partial \tau}(t,\tau) \\
&\quad + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \left\{ \mathrm{e}^{-\rho t}u_a(\tau)\ln c - \frac{c}{x}\xi_a(t,\tau) + F(\theta;t,a,\tau)\xi_a(t,\tau) \right\} \\
&\quad + \ln x \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)\big(\xi_j(t,0) - \xi_a(t,\tau)\big) \\
&\quad + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)\big(\zeta_j(t,0) - \zeta_a(t,\tau)\big) = 0 \ ,
\end{aligned}
\tag{32}
$$

where $F(\theta;t,a)$ is as given in (31).

We start by optimizing with respect to c, before proceeding to optimize with respect to the variable θ. The first-order condition associated with the optimization problem above provides a maximizer $c^*(t,x,a,\tau)$, given by

$$
c^*(t,x,a,\tau) = \frac{\mathrm{e}^{-\rho t}u_a(\tau)}{\xi_a(t,\tau)} x \ .
$$

Replacing c by $c^*(t,x,a,\tau)$ in (32), we obtain that

$$
\begin{aligned}
&\left(\frac{\partial \xi_a}{\partial t}(t,\tau) + \frac{\partial \xi_a}{\partial \tau}(t,\tau)\right)\ln x + \frac{\partial \zeta_a}{\partial t}(t,\tau) + \frac{\partial \zeta_a}{\partial \tau}(t,\tau) \\
&\quad + \mathrm{e}^{-\rho t}u_a(\tau)\left(-\rho t + \ln u_a(\tau) + \ln x - \ln \xi_a(t,\tau) - 1\right) \\
&\quad + \sup_{\theta\in[0,1]} \left\{ F(\theta;t,a,\tau)\xi_a(t,\tau) \right\} \\
&\quad + \ln x \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)\big(\xi_j(t,0) - \xi_a(t,\tau)\big) \\
&\quad + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)\big(\zeta_j(t,0) - \zeta_a(t,\tau)\big) = 0 \ .
\end{aligned}
\tag{33}
$$

As long as $\xi_a(t,\tau)$ is nonzero, the first-order condition for θ is $F'(\theta;t,a,\tau) = 0$, yielding the critical point

$$\hat{\theta}(t,a,\tau) = \frac{\mu(t,a,\tau) - r(t,a,\tau)}{\sigma^2(t,a,\tau)} \ .$$

Moreover, the second derivative of $F(\theta;t,a,\tau)$ with respect to θ is strictly negative for every $\theta \in [0,1]$. Taking into account the constraint $\theta \in [0,1]$ and the concavity of $F(\theta;t,a,\tau)$, we conclude that the maximization problem in (33) has a unique solution $\theta^*(t,a,\tau)$. Moreover, from the definition of the function $F(\theta;t,a,\tau)$, it is possible to check that

i) if $\mu(t,a,\tau) - r(t,a,\tau) > 0$ and $F(1;t,a,\tau) < 0$, then there exists $\hat{\theta}(t,a,\tau) \in (0,1)$ such that $F'(\hat{\theta}(t,a,\tau);t,a,\tau) = 0$ and, consequently, $\theta^*(t,a,\tau) = \hat{\theta}(t,a,\tau)$;
ii) if $\mu(t,a,\tau) - r(t,a,\tau) > 0$ and $F(1;t,a,\tau) \geq 0$, then $\theta^*(t,a,\tau) = 1$;
iii) if $\mu(t,a,\tau) - r(t,a,\tau) \leq 0$, then $\theta^*(t,a,\tau) = 0$.

The proof is completed by grouping terms in (33) to arrive at the sytem of partial differential equations describing $\xi_a(t,\tau)$ and $\zeta_a(t,\tau)$, $a \in S$. □

6. Conclusion

We employ dynamic programming techniques to characterize the solution of a stochastic optimal control problem with state variable dynamics determined by a diffusive SDE with semi-Markov modulated coefficients. After generalizing the classical Bellman's optimality principle to such framework, we derive the corresponding Hamilton-Jacobi-Bellman equation and show that the optimal control problem value function is a viscosity solution of such equation. We apply our results to an illustrative example: the famous Merton's consumption-investment problem for a financial market with assets whose prices evolve according to a semi-Markov modulated SDE.

We note that the class of stochastic optimal control problems studied here appears naturally in a number of real-life problems, all of which share the feature that both the state variable dynamics and the objective functional depend on a set of known "unknowns" occurring at random instants of time, encapsulated here by the components of the semi-Markov process. Potential examples of application include, for instance:

- the problem faced by an economic policy decision-maker such as, e.g. a central banker, in what concerns macroeconomic choices for an economy whose future growth depends on the outcomes of random future exogenous events, which may also influence the preference structures under which decisions are made;
- the problem faced by a manager of some manufacturing facility when deciding about the changes to the size of the facility work force, raw materials stocking and the level of production, given that the overall state of the economy may change at some random instants of time and that such changes may influence the prices at which the raw materials are available, as well as the price and demand for the manufacturing facility final product.

In what concerns future research, we plan to generalize our results to include larger families of state variable dynamics such as, for instance, semi-Markov modulated jump-diffusions. We will also consider extensions of Merton's optimal consumption-investment problem, as well as of optimal life-insurance purchase problems, to include financial market models driven by such classes of stochastic processes.

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Appendix A. Itô's Formula for a semi-Markov modulated diffusive SDE

In this section we state Itô's formula for a semi-Markov modulated diffusive SDE. The result follows from Itô's formula for semi-martingales. See [24, 28] for further details.

Lemma A.1 (Itô's rule for a semi-Markov modulated diffusion process)**.** *Let $\alpha(\cdot)$ be the semi-Markov process determined by* (1) *and $\tau(t)$ the corresponding time component. Suppose that $X(t)$ is given by the SDE*

$$\begin{aligned} \mathrm{d}X(t) &= f\left(t, X(t), \alpha(t_-), \mathcal{T}(t_-)\right) \mathrm{d}t \\ &\quad + \sigma\left(t, X(t), \alpha(t_-), \mathcal{T}(t_-)\right) \mathrm{d}W(t) \end{aligned} \tag{34}$$

Let $V(t,x,a,\tau)$ be such that $V(\cdot,\cdot,a,\cdot) \in C^{1,2,1}([0,T] \times \mathbb{R}^N \times \mathbb{R}_0^+;\mathbb{R})$ for every $a \in S$. Then, we have that

$$V(t, X(t), \alpha(t), \mathcal{T}(t)) - V(0, X(0), \alpha(0), \mathcal{T}(0)) = \int_0^t a\left(s, X(s), \alpha(s_-), \mathcal{T}(s_-)\right)\, \mathrm{d}s + M(t)$$

where

$$\begin{aligned} a(t,x,a,\tau) &= V_t(t,x,a,\tau) + V_\tau(t,x,a,\tau) \\ &\quad + \langle V_x(t,x,a,\tau), f(t,x,a,\tau)\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sigma^T(t,x,a,\tau) V_{xx}(t,x,a,\tau)\sigma(t,x,a,\tau)\right) \\ &\quad + \sum_{j\in S: j\neq a} \lambda_{aj}(\tau)\left(V(t,x,j,0) - V(t,x,a,\tau)\right) \end{aligned}$$

and

$$
\begin{aligned}
M(t) &= \int_0^t b(s, X(s), \alpha(s_-), \mathcal{T}(s_-)) \, \mathrm{d}W(s) \\
&\quad + \int_0^t \int_{\mathbb{R}} c\left(s, X(s), \alpha(s_-), \mathcal{T}(s_-), z\right) \, \mathrm{d}\tilde{J}(\mathrm{d}s, \mathrm{d}z) \\
b(t, x, a, \tau) &= (V_x(t, x, a, \tau))^T \sigma(t, x, a, \tau) \\
c(t, x, a, \tau, z) &= V(t, x, a + \Gamma_1(a, \tau, z), \tau - \Gamma_2(a, \tau, z)) - V(t, x, a, \tau)
\end{aligned}
$$

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