REFINEMENT OF DYNAMIC EQUILIBRIUM USING SMALL RANDOM PERTURBATIONS

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Abstract. We propose a refinement process of dynamic equilibria based on small random perturbations (SRP) of the backward perfect foresight (bpf) equilibrium map in a class of one-step forward looking dynamic models. An equilibrium is selected if its stationary measure is the limit of the stationary measures associated with the processes generated by the SRP of the bpf maps, as the perturbation size approaches zero. We show that, for full measure sets of parameter values of a large class of one-parameter families of unimodal bpf maps, only determinate cycles or the chaotic sunspot equilibrium defined by Araujo and Maldonado (2000) are selected. Two examples are provided illustrating such refinement process.

Keywords: equilibrium selection; sunspot equilibrium; stochastic dynamical system; stochastic stability
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1. Introduction

The Rational Expectations Hypothesis requires not only individuals maximizers of their objective functions, but also the consistency between the perceived randomness of future variables and their actual distribution. A stronger concept is that of perfect foresight equilibrium, proposing that agents are able to have a perfect prevision of the exact value of the future state variable.

Even with the strong version of the Rational Expectations Hypothesis it is possible to find diversity of equilibria (more often in overlapping generations models than in infinitely lived consumers models, as shown by Kehoe and Levine (1985)). Some important equilibria are steady states, cycles and chaotic paths (Benhabib and Nishimura (1985); Brock and Hommes (1997, 1998); de Vilder (1996); Grandmont (1985); Hommes et al. (2005); Hommes (2018)). From them, it is possible to construct stochastic equilibria based on extrinsics, the so called "sunspot equilibria". The following works showed how sunspot equilibria arise from specific stationary equilibria in dynamic frameworks: from indeterminate stationary states, Peck (1988) and Chiappori et al. (1992); from regular cycles, Azariadis and Guesnerie (1986); from multiple steady states, Chatterjee et al. (1993); in models with memory, Woodford (1986); and from chaotic dynamics, Araujo and Maldonado (2000). If a multiplicity of equilibria is present in the model, the question of which should be selected as a robust equilibrium arises.

In this paper we provide a selection criterion for dynamic equilibria of a certain class of one-step forward looking economic models. Such selection criterion or *refinement of dynamic equilibria* is based on small random perturbations of the backward perfect foresight (bpf) map representing small stochastic errors on the response of the perfect prevision of the future state variable value. The stochastic processes generated in such way are stationary and, for a typical family of unimodal bpf maps, distinct outcomes may occur, depending on the parameter values of the model. If the unperturbed bpf dynamics exhibits a determinate (locally unique) cycle, the stationary measure associated with the small random perturbation is close to an atomic measure with support on that attracting cycle. In this case we will say that the refinement process selects that determinate cycle. More interestingly, if the bpf map exhibits ergodic chaos, namely, if it has an ergodic and absolutely continuous (with respect to Lebesgue) invariant measure, which is called *Bowen-Ruelle-Sinai (B-R-S) invariant measure of the map* , then the stationary measure associated with the small random perturbation is close to that measure. Such stationary measure is that corresponding to the global chaotic sunspot equilibrium presented in Araujo and Maldonado (2000). In this case, the refinement process selects that chaotic sunspot equilibrium. Neither indeterminate cycles (including indeterminate steady states) nor other kinds of sunspot equilibria other than the global chaotic sunspot equilibrium are selected under the refinement criterion proposed herein.

The term *sunspot* was coined by Cass and Shell (1983) defining it in the context of general equilibrium models, to study the influence of the agents expectations on market outcomes. More recently, Lucas and Stokey (2011) have argued that sunspots and contagion effects are sources of liquidity crises. They brought the argument of Cass and Shell (1983) on expectation coordination to explain bank runs and, consequently, the financial crises of 2008. Applications of the concept of sunspot equilibrium include the modelling of such bank-runs (Peck and Shell (2003)), restrictions on market participation (Balasko et al. (1995)), lotteries (Prescott and Shell (2002); Prescott and Townsend (1984)) and behavioral economics (Fehr et al. (2019)).

With respect to equilibria selection, in the literature we find other criteria, mostly applied to linear or monotonic dynamic models of rational expectations. Blanchard (1979) proposed requirements of consistency with economic behavior and stationarity as a way of selecting among a multiplicity of equilibria. The local stability criterion of Blanchard and Kahn (1980) and Benhabib and Farmer (1999) is well-known – it proposes that the chosen equilibrium must be stable under small perturbations of the initial condition. Another criterion is the minimal state variable criterion (McCallum (1983); Wallace (1978)) which states that the equilibrium selection must be done using forecasting functions with a minimal set of state variables and with parameters that are continuous functions around key values of the structural parameters. To this selection criterion belongs the least squares learning mechanism proposed by Marcet and Sargent (1989). Also, Sargent and Wallace (1985) and Woodford (1990) provided models with a continuum of sunspot equilibria, where any of them can be econometrically estimated. Finally, Driskill (2006) provided a new selection criterion of equilibria, the "finite-horizon" or "backward-induction" criterion, consisting in taking the finite horizon model associated to the original one and finding the limit of the finite-horizon equilibria as time goes to infinity. Our work contributes to the criteria already proposed in the literature, by focusing in the case of truly non-linear dynamics governing the state variable evolution.

The dynamics of unimodal maps plays a central role in our analysis (see e.g. de Melo and van Strien (1993) for an overview). Among these maps, the ones with the simplest dynamical behavior are called hyperbolic: those unimodal maps with a hyperbolic attractor, which is a cycle such that the modulus of the derivative of the map evaluated on it, is lower than one. A recent result of Kozlovski (2000, 2003) ensures that the set of hyperbolic unimodal maps is open and dense in the space of C^r unimodal maps for any $r \geq 2$. On the opposite side of the spectrum of dynamical complexity are those unimodal maps having an absolutely continuous invariant probability measure. In Blokh and Lyubich (1990); Keller (1990); Nowicki and van Strien (1991) the reader can find for more results regarding the existence of an absolutely continuous invariant measure for a unimodal map. A notion that will be used for the results herein is the stability of dynamical systems under independent and identically distributed random perturbations, known as *stochastic stability*, initially introduced by Kolmogoroff (1937) and Sinai (1972). For one-dimensional maps, Katok and Kifer (1986) proved stochastic stability for the quadratic family in the Misiurewicz case, i.e. for quadratic maps with no periodic attractors and such that the forward orbit of the critical point does not accumulate on the critical point itself. This result was later extended for sets of values of the parameter with positive Lebesgue measure by Benedicks and Young (1992), with respect to the convergence induced by the weak-topology, and by Baladi and Viana (1996), with respect to the norm topology and for a wider class of unimodal maps. More recently, Avila and Moreira (2003, 2005) proved that quadratic maps are stochastically stable for Lebesgue almost every parameter value, their results holding also for topologically generic parametric families of unimodal maps of the interval.

This paper is organized as follows. In Section 2, we specify the class of economic dynamic models under consideration herein, define the concept of refinement by small random perturbations of a perfect foresight equilibrium, and state our main results. In Section 3 we illustrate our findings through two examples. The first such example is the classical overlapping generations model with fiat money, while the second one is the Shapley-Shubik market game model. For a full measure set of values of the risk aversion parameter (for the first model) and of the market thickness parameter (for the second model), we find that the stationary measure associated with the small random perturbation of the backward perfect foresight map is close to: *i*) the stationary measure of the global chaotic sunspot equilibrium, whenever the bpf map is chaotic on a finite union of intervals, or *ii*) an atomic measure with support on a determinate cycle, whenever the bpf map possesses an attracting cycle.

2. Framework and main results

In this section we set the framework we will consider and define the refinement process of equilibria present in it. The definitions and concepts included in here are general and must be read as "provided that the functions and expressions are well-defined". In Subsection 2.1 we will be more specific regarding the class of functions we are going to deal with and the corresponding domains of each definition.

Let us consider one-period forward looking models of the form

$$F(x_t) = E_t[G(\tilde{x}_{t+1})] \ , \tag{1}$$

where F and G are differentiable functions defined on non-trivial intervals of $\mathbb{R}$, x_t is the value of the state variable of the model in period t, and $\tilde{x}_{t+1}$ is the random variable representing the possible values of the state variable of the model in period $t+1$. Denote by $E_t[\cdot]$ the mathematical expectation operator conditioned on the information available up to time t. The interpretation of the model (1) is the following: if the probability distribution of the state variable in period $t+1$ is given by that of the random variable $\tilde{x}_{t+1}$, then the current state variable value that equilibrates the individual decisions and the markets is x_t.

If F is an invertible function and the range of G is contained in the domain of F^{-1}, it is possible to define the backward perfect foresight map, as follows.

Definition The *backward perfect foresight map (bpf map)* associated with model (1) is the real valued map ϕ defined by $\phi(x) = F^{-1}(G(x))$ for all x in the domain of G such that $G(x)$ is in the domain of F^{-1}.

The map $\phi(x)$ represents the current value of the state variable that equilibrates the economy when the individuals' expectation regarding the future value of the state variable is the Dirac measure δ_x.

The bpf map allows us to define backward perfect foresight paths for the model.

Definition A *backward perfect foresight equilibrium path (bpf equilibrium path) through* $x_0 \in \mathbb{R}$ is a sequence of real numbers $\{x_t\}_{t\geq 0}$ such that $x_t = \phi(x_{t+1})$ for all $t \geq 0$.

Thus, a bpf equilibrium path is a feasible sequence $\{x_t\}_{t\geq 0}$ of state variable values such that, individuals having perfect foresight of the state variable equal to x_{t+1} for the period $t+1$, equilibrate their decisions in period t with the state variable value $x_t = \phi(x_{t+1})$.

When the bpf map is a unimodal function defined on an interval of $\mathbb{R}$, it results that for each current state value we will have two future states values that rationalizes that current state (i.e. there exist two temporary equilibria), therefore in general infinitely many forward perfect foresight paths can be obtained from a given initial state. If in addition, that bpf map exhibits chaotic behavior, Gardini et al. (2009) proved that the set of forward paths that are intertemporal equilibria has a fractal attractor. Characterizations and properties of those intertemporal perfect foresight equilibria were studied using the inverse limit approach by Medio and Raines (2007) and Mihailescu (2012).

The map ϕ brings into consideration a diversity of equilibria. A fixed point of ϕ, i.e. a value $\bar{x}$ in the domain of ϕ such that $\bar{x} = \phi(\bar{x})$, is a steady state of the economy. Analogously, a $k-$cycle of ϕ, meaning a sequence $x_1, x_2, \ldots, x_k$ of points in the domain of ϕ such that $x_1 = \phi(x_2)$, $x_2 = \phi(x_3)$, $\ldots, x_{k-1} = \phi(x_k)$, $x_k = \phi(x_1)$, is a $k-$cycle of the economy. If some of these equilibria are indeterminate, i.e. if there exists a continuum of bpf paths arbitrarily close to it, then stationary sunspots equilibria exist in neighborhoods of such equilibria (Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1991)). Moreover, whenever the map ϕ exhibits ergodic chaos, Araujo and Maldonado (2000) proved that there exist a chaotic sunspot equilibrium, i.e. a sunspot equilibrium whose stationary measure is given by the absolutely continuous invariant measure of ϕ.

The central question to our analysis is the following: if agents have perfect foresight regarding the future state of the economy, but there is a random perturbation (trembling selection) of the current state that equilibrates the economy, which of the existing equilibria, if any, proves to be robust when the random perturbations are small enough? This motivates the next definition.

Definition Let $\{\tilde{\epsilon}_t\}_{t\geq 0}$ be a sequence of independent and identically distributed random variables with density θ_ϵ and support $[-\epsilon, \epsilon]$. A *small random perturbation (SRP) of the bpf map* is a sequence $\{\tilde{x}_t\}_{t\geq 0}$ such that for all $t \geq 0$:

$$\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \tilde{\epsilon}_t \ . \tag{2}$$

Equation (2) must be read as follows: if the state variable in period $t+1$ assumes the value $\tilde{x}_{t+1} = z$, there is perfect foresight and in spite of this, there is a perturbation $\tilde{\epsilon}_t = \epsilon$ in the equilibrating state in t, then the observed state value in t is $\phi(z) + \epsilon$. Thus, a SRP of the bpf map provides a picture of what can happen if firstly, there is perfect foresight in the economy and nevertheless, second, the current state suffers a small random perturbation. It is worth noting that it is not a random perturbation of a perfect foresight path, rather it is the realization of two random processes ($\{\tilde{\epsilon}_t\}_{t\geq 0}$ and $\{\tilde{x}_t\}_{t\geq 0}$) that satisfy (2). The small errors introduced in each step can be interpreted as representing a round-off of the perfect foresight response $\phi(x_{t+1})$ or even recurrent approximations due to imprecision or uncertainty regarding some parameter value defining ϕ. Notice that unlike the definition 2, we do not fix the initial state value x_0, since it will depend on the values of $\tilde{x}_1$ and $\tilde{\epsilon}_0$, and as we will see later, it will be irrelevant for the refinement process.

The refinement process we will introduce here is based on the asymptotic statistical behavior of the perturbed dynamics realized by the path $\{\tilde{x}_t\}_{t\geq 0}$ as the maximum size ϵ of the stochastic errors goes to zero. An analogous process was initially introduced by Kolmogoroff (1937) and Sinai (1972) for the treatment of problems of statistical physics. It is not surprising that, whenever ϕ has an attracting cycle (or steady state), the stochastically perturbed dynamics visit only a neighborhood of such cycle; however, this is no longer the case if the cycles are sources. From the next subsection onward we will restrict ourselves to a family of unimodal bpf maps and we will analyze their asymptotic behavior under SRP.

The process (2) defines a *backward* Markovian stochastic process, namely, a process where the current state is a random variable, given the value of the next period state. To that Markovian process there is an associated transition function

$$P_\epsilon(x, A) \equiv \Pr(\tilde{x}_t \in A | \tilde{x}_{t+1} = x) = \int_A \theta_\epsilon \left(y - \phi(x)\right) \, \mathrm{d}y \ , \tag{3}$$

where $A \subset \mathbb{R}$ is a Borel set and x is an element in the domain of ϕ. Equation (3) gives the probability that the current value of the state variable belongs to a set A given the perfect foresight that the next time period state variable value is x. The transition function (3) has a stationary (or invariant) measure ν^ϵ if it satisfies:

$$\nu^\epsilon(A) = \int P_\epsilon(x, A) \, \mathrm{d}\nu^\epsilon(x) \ . \tag{4}$$

We will regard the limit of ν^ϵ as ϵ approaches 0 as an indication of which equilibrium of (1) will ultimately be selected by the SRP refinement process, namely, the only equilibrium of (1) consistent with such limiting stationary measures.

Remark 2.1. *In the sequel, we are particularly interested in stationary measures associated with:*

i) *a k-cycle $\{x_1, x_2, \ldots, x_k\}$: these are given by a convex linear combination of Dirac distributions $\mu = \frac{1}{k}\sum_{i=1}^{k} \delta_{x_i}$ supported on the cycle;*
ii) *a stationary sunspot equilibrium (X_0, Q): these are given by the stationary measure of the transition function Q, i.e. the probability measure μ satisfying $\mu(A) = \int Q(x, A) \, \mathrm{d}\mu(x)$.*

In item ii) of the remark above, we resort to the definition of sunspot equilibrium provided by Chiappori and Guesnerie (1991): A sunspot equilibrium for the model (1) is a subset $X_0 \subset \mathbb{R}$ with at least two elements and a transition function $Q : X_0 \times \mathcal{B}(X_0) \to [0, 1]$ such that:

i) $F(x) = E_{Q(x,\cdot)}[G(\tilde{x})]$ for all $x \in X_0$; and
ii) there exists $x_0 \in X_0$ such that $Q(x_0, \cdot)$ is truly stochastic.

We are now ready to introduce the central notion of this paper – an equilibrium selection criterion (or refinement of equilibria) based on small random perturbations.

Definition An equilibrium of the one-period forward looking model (1) is *selected by the criterion of the small random perturbations of the bpf map* if its associated stationary measure μ is the limit of the stationary measures ν^ϵ associated with the small random perturbation of bpf map given in (4), as the maximum size of the random perturbation tends to zero. Namely, if

$$\nu^\epsilon \to \mu$$

as $\epsilon \to 0$ in the weak topology[1].

[1]Let X be a metric space with Borel σ-algebra Σ. A sequence of probability measures $(\mu_n)_{n=1}^{\infty}$ on the measurable space (X, Σ) is said to converge in the weak topology to the probability measure μ, denoted here as $\mu_n \to \mu$, if $\displaystyle\int_X f \, \mathrm{d}\mu_n \to \int_X f \, \mathrm{d}\mu$ for all bounded, continuous functions $f : X \to \mathbb{R}$.

Definition 2 corresponds to that of stochastic stability originally introduced by Kolmogoroff (1937), and later extended by Sinai (1972) and applied to unimodal maps by Katok and Kifer (1986), Benedicks and Young (1992) and Baladi and Viana (1996). The intuition of that concept is that it seeks to relate the behavior of the dynamics of a function with that empirically observed, when the dynamics is stochastically perturbed in each iteration of the map. The novelty here is that we are using the same idea for the bpf map with the interpretation provided above and with its meaning being the refinement process of dynamic equilibria in models of the type (1).

2.1. **SRP Refinement for generic families of unimodal maps.**

In this subsection and the following we will be more specific regarding the class of functions and stochastic processes under consideration, as well as the domain containing the sequences generated by such processes.

Let us denote by $\mathcal{F}$ the class of C^3 unimodal maps $\phi : X \to X$ (the bpf map) defined on an interval of the form $X = [0, \alpha]$, with $0 < \alpha \leq +\infty$, and for which the following conditions hold:[2]

1. ϕ has a non-degenerate critical[3] point x^* with $\phi^2(x^*) > 0$;
2. ϕ has a repelling fixed point at zero, i.e. $\phi(0) = 0$ and $\phi'(0) > 1$;
3. ϕ has negative Schwarzian derivative[4].

Regarding the the random perturbations, we will assume the following.

Hypothesis 1. *The density* $\theta_\epsilon : \mathbb{R} \to \mathbb{R}_0^+$ *satisfies the conditions:*

(i) $\mathrm{supp}(\theta_\epsilon) \subset \Omega_\epsilon = [-\epsilon, \epsilon]$,
(ii) $M = \sup_{\epsilon > 0}(\epsilon \sup |\theta_\epsilon|) < \infty$,
(iii) $J_\epsilon := \{t |\ \theta_\epsilon(t) > 0\}$ *is an interval containing 0 and* $\eta_\epsilon := \log(\theta_\epsilon|_{J_\epsilon})$ *is a concave function.*

The truncated normal distribution and the uniform distribution on an interval provide examples of probability distributions whose densities satisfy Hypothesis 1.

Unimodal maps $\phi \in \mathcal{F}$ fit into three alternative topological types (see de Melo and van Strien (1993)):

i) The map ϕ has a periodic attractor $C \subset X$ whose basin of attraction[5] is big both from a topological point of view (open and dense set) and in a measure-theoretical sense (full measure). Both the periodic attractor and its basin are stable under deterministic C^1 perturbations of ϕ. Unimodal maps such as these are usually called *hyperbolic* or *regular*.

[2] From now on we will use the following notation: $\phi^0(x) = x$; $\phi^{n+1}(x) = \phi(\phi^n(x))$, for all $n \geq 0$.

[3] The critical point x^* is such that $\phi''(x^*) \neq 0$.

[4] The Schwarzian derivative of ϕ is defined as $S\phi = \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2}\left(\frac{\phi''(x)}{\phi'(x)}\right)^2$. See the textbook by de Melo and van Strien (1993) for a detailed treatment of one-dimensional dynamics.

[5] The *basin of attraction* of $C \subset X$ is the set of all points $x \in X$ for which the ω-limit set of x, $\omega(x) = \{y \in X :$ there exists a subsequence $n_i \to \infty$ with $\phi^{n_i}(x) \to y\}$, is such that $\omega(x) = C$.

ii) The map ϕ is transitive on some finite union of intervals, i.e. there exist orbits which are dense in these intervals, has a B-R-S measure which is absolutely continuous with respect to Lebesgue with support on that finite union of intervals, and has a positive Lyapunov exponent[6]. Even if ϕ may be unstable under deterministic perturbations (nearby maps may have a periodic attractor), the stochastic description given by the B-R-S measure is robust under stochastic perturbations in the sense that the perturbed system has a stationary measure whose density is close to B-R-S measure density with respect to the L^1 distance. This is the case into which the bpf maps of subsection 2.2 fall.

[6]If $\phi \in \mathcal{F}$ then there exists $\lambda_\phi \in \mathbb{R}$ such that for almost every $x \in X$ we have that $\lambda_\phi = \limsup \frac{1}{n} \log |D\phi^n(x)|$. Such number λ_ϕ is called the *Lyapunov exponent* of the map ϕ and characterizes the rate of separation of infinitesimally close trajectories.

iii) The map ϕ is infinitely renormalizable[7] and has a unique invariant probability measure μ, which is a B-R-S measure supported in the closure of the forward orbit of the critical point of ϕ. This set is an attracting Cantor set $\mathcal{C}$ and if x is in the basin of $\mathcal{C}$ then

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\phi^i(x)) = \int f(x)\, \mathrm{d}\mu$$

for every continuous function f.

In what concerns the "relative sizes" of the three alternative topological types to which $\phi \in \mathcal{F}$ may belong to, it should be remarked that the first two alternatives are observable for open sets of families of unimodal maps, with the first alternative also known to be dense within the class of unimodal maps under consideration here. However, that first alternative may not have full Lebesgue measure in the parameter space where the family is defined, as we will see in the examples of Section 3. Regarding the third alternative, this occurs only for a subset of Lebesgue measure zero of the parameter space.

The next theorem relies heavily on the result given in Avila and Moreira (2003) to provide a description of the eventual outcomes produced by random perturbation selection criterion of Definition 2. The proof is provided in the Appendix.

Theorem 2.2. *For topologically generic k-parameter families of bpf maps in the class $\mathcal{F}$, the random perturbation selection criterion of Definition 2 satisfying Hypothesis 1 will select a determinate cycle for an open and dense subset of the parameter space and, for Lebesgue almost every other parameter, it will select the chaotic sunspot equilibrium given in Araujo and Maldonado (2000).*

Thus, by Theorem 2.2, the random perturbation selection criterion of Definition 2 always picks a single equilibria for a large class of unimodal maps – the hyperbolic ones. For such maps, the selected equilibria is always a determinate cycle, since the attractor cycle for ϕ is a local source for its inverse, so it is locally unique. For almost every other value of the parameter (within generic parametric families), the chaotic sunspot equilibrium is selected. It is worth noting that, even though the latter is in the complement of an open and dense set, it may occur with positive Lebesgue measure, two contrasting ideas of *size* in mathematics and now in economics. In Subsection 2.2 we provide sufficient conditions for the occurrence of the latter alternative and in Section 3 we will illustrate in two economically relevant examples the non-triviality of the set of parameters values where it happens.

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[7]A closed proper subinterval J of X is called *restrictive* with period $n \geq 1$ for ϕ if
a) the interiors of $J, \dots, \phi^{n-1}(J)$ are disjoint;
b) $\phi^n(J) \subset J$, $\phi^n(\partial J) \subset \partial J$;
c) at least one of the intervals $J, \dots, \phi^{n-1}(J)$ contains the critical point;
d) J is maximal with respect to these properties.
The map $\phi^n : J \to J$ is called the *renormalization of ϕ to J*. A map is *infinitely renormalizable* if it has restrictive intervals of arbitrary high period.

We observe that whenever the bpf map ϕ exhibits chaotic behavior, it has infinitely many (repelling) cycles, which are indeterminate cycles indeed. Therefore, in this case, model (1) exhibits the following distinct equilibria:

i) infinitely many indeterminate cycles;
ii) infinitely many local sunspot equilibria in the neighborhood of the indeterminate cycles;
iii) infinitely many chaotic bpf equilibrium paths (realized by the bpf map ϕ);
iv) exactly one chaotic sunspot equilibrium (with stationary measure μ);

As noted earlier, out of all of the (infinitely many) equilibria listed above, only the last one emerges as a possible outcome of the refinement criterion of SRP whenever the original unperturbed bpf map exhibits chaotic dynamics.

2.2. **SRP refinement selects chaotic sunspot equilibria.** Our main result – Theorem 2.2 – establishes the possibility of selection of the chaotic sunspot equilibrium in the complement of an open and dense subset of the parameter space for topologically generic parametric families of bpf maps. Even though such set might be regarded as being small in a topological sense, it is important to stress that it may be significant from a metric point of view, i.e. it may occur as a positive Lebesgue measure subset of parameter space. In this part of the work we will provide conditions guaranteeing the convergence of the measure ν^ϵ to an absolutely continuous B-R-S measure μ.

Hypothesis 2. *The bpf map $\phi \in \mathcal{F}$ satisfies the following conditions:*

(**1**) Subexponential recurrence*: $|\phi^k(x^*) - x^*| \geq e^{-\alpha k}$ for all $k \geq H_0$,*
(**2**) Collet–Eckmann condition*: $|D\phi^k(\phi(x^*))| \geq e^{\gamma k}$ for all $k \geq H_0$,*
(**3**) Visiting property*: ϕ is topologically mixing in the dynamical interval $X = [\phi^2(x^*), \phi(x^*)]$, that is, given open sets $A, B \subset X$, there exists an integer N, such that, for all $n > N$, it holds that $\phi^n(A) \cap B \neq \emptyset$,*

where $H_0 \geq 1$, $\gamma > 0$ and $0 < \alpha < \gamma/4$ are fixed constants.

Under Hypothesis 2, the bpf map $\phi : X \to X$ has a Bowen-Ruelle-Sinai (B-R-S) invariant probability measure, namely, there exists an absolutely continuous measure μ with support in X such that $\mu(\phi^{-1}(A)) = \mu(A)$ for any Borel set $A \subset X$ (see Baladi and Viana (1996)). Those B-R-S measures are important because they allow a statistical description of the orbits of the map, namely, for any continuous function $f \in C(X)$, the average value of the function on the orbit generated by ϕ (namely, $n^{-1} \sum_{j=0}^{n-1} f(\phi^j(x))$) converges to $\int_X f(z)\,\mu(dz)$ when n goes to infinity, for Lebesgue almost all $x \in X$. The unimodal shape of ϕ and the existence of an invariant and absolutely continuous measure associated to it are conditions for the existence of a chaotic sunspot equilibrium, as shown by Araujo and Maldonado (2000).

It is important to specify the domain of ϕ allowing for well-defined processes originated from performing a small random perturbations of a bpf map. When a stochastic perturbation is introduced, the orbits of such stochastic dynamical system may leave the dynamical interval X associated with the bpf map ϕ with positive probability when the random perturbation size is large enough. However, since $\phi^2(x^*) > 0$, it is not difficult to prove that there exists $\epsilon_0 > 0$ such that for every probability density function θ_ϵ satisfying Hypothesis 1, where $\epsilon < \epsilon_0$, there exists a compact interval

$$X_\epsilon = [\phi(\phi(x^*) + \epsilon) - \epsilon, \phi(x^*) + \epsilon] \subset \mathbb{R}^+$$

such that X_ϵ is invariant under both the deterministic and stochastic dynamics associated to ϕ. Since we will study the limit where the maximum size of the random perturbation $\epsilon \to 0$, we can take the size of the random perturbation to be strictly smaller than $\epsilon_0 > 0$, thus yielding well defined random dynamics in the compact interval X_ϵ.

The next proposition provides conditions under which the chaotic SSE introduced by Araujo and Maldonado (2000) is the outcome of the SRP refinement process introduced here. Its proof uses some of the main results of Baladi and Viana (1996).

Proposition 2.3. *Suppose that:*

i) *model* (1) *has a bpf map* $\phi \in \mathcal{F}$ *satisfying Hypothesis 2; and*
ii) *the random perturbations* $\{\tilde{\epsilon}_j\}_{j\geq 0}$ *associated with the SRP of Definition 2 are sampled from a probability distribution with density satisfying Hypothesis 1.*

Then, the refinement process of Definition 2 will select the chaotic SSE of Araujo and Maldonado (2000). Namely, we have that $\nu^\epsilon \to \mu$ *as* $\epsilon \to 0$, *where* ν^ϵ *is the stationary measure* (4) *and* μ *is the absolutely continuous B-R-S invariant probability measure of* ϕ.

Despite the seemingly strong conditions – Hypothesis 2 – imposed on ϕ to guarantee the selection of the chaotic SSE, we will provide in the following section two economically relevant models exhibiting such behavior for a set of parameter values with positive Lebesgue measure.

3. Two examples

In this section we provide two examples to illustrate the refinement process given by the small random perturbation of the bpf map. Depending on the parameter values of these models, the refinement process will select either the chaotic sunspot equilibrium, provided the bpf map is chaotic on a finite union of intervals, or the determinate cycle, provided the bpf map has an attracting cycle. Neither indeterminate equilibria nor other types of classical local sunspot equilibria are selected by the SRP criterion proposed herein.

We will first introduce the theoretical model underlying each example, before summarizing the results of the numerical experiment performed to illustrate the abundance of parameter values leading to the selection of either a determinate cycle or the chaotic SSE, by means of the refinement process proposed herein. The first such example is the classical overlapping generation model with fiat money, while the second is the market game model of Shapley and Shubik.

3.1. **An OLG model with fiat money.** We will consider a two-period overlapping generations (OLG) model like the one introduced in Grandmont (1986). An analogous model was analyzed by Azariadis and Guesnerie (1986) to prove the existence of cycles and sunspots with finite support.

The economy is populated by a large number of young and old agents. The population sizes are the same and remain constant over time. We assume that there exists a representative agent with preferences given by a separable utility function

$$U(c_t, c_{t+1}) = \frac{c_t^{1-\alpha_1}}{1-\alpha_1} + \frac{c_{t+1}^{1-\alpha_2}}{1-\alpha_2} \ ,$$

where $\alpha_i > 0$ and $\alpha_i \neq 1$, $i = 1, 2$, are the coefficients of relative risk aversion of the agent, and (c_t, c_{t+1}) denotes the corresponding consumption plan. We suppose also that one unit of the good is produced with one unit of the unique productive factor (labor) and let l_1^* and l_2^* denote the agents labor endowments in the first and second periods of their lives, respectively. Finally, we assume that there is a risk-free asset (fiat money) that can be purchased by the agents providing a gross return $z_t = 1$, and the money supply is constant, i.e. $M_t = M_0$ for all $t \geq 0$.

The consumption-saving problem of the representative agent is as follows. Let p_t and p_{t+1} denote the prices of the unique good in the economy during the first and second periods of the individual's life. While p_t is known by the individual during her first stage life, she only knows a probability distribution μ_{t+1} of particular values of p_{t+1}. The agent must choose a consumption plan (c_t, c_{t+1}) and the first period saving m_t as the solution of the following optimization problem

$$\max_{\{c_t, c_{t+1}, m_t\}} \frac{c_t^{1-\alpha_1}}{1-\alpha_1} + E_t\left[\frac{c_{t+1}^{1-\alpha_2}}{1-\alpha_2}\right] \tag{5}$$

subject to the budget constraints

$$p_t c_t + m_t = p_t l_1^* \text{ and } p_{t+1} c_{t+1} = p_{t+1} l_2^* + m_t \ ,$$

where $E_t[\cdot]$ denotes the mathematical expectation with respect to the probability measure μ_{t+1}. The first order condition for an interior solution of (5) leads to

$$-\frac{1}{p_t}\left(l_1^* - \frac{m_t}{p_t}\right)^{-\alpha_1} + E_t\left[\frac{1}{p_{t+1}}\left(l_2^* + \frac{m_t}{p_{t+1}}\right)^{-\alpha_2}\right] = 0 \ . \tag{6}$$

The monetary equilibrium condition is $M_t = m = 1$. Defining the new variable $x_t = 1/p_t$, the first order condition (6) may be rewritten as

$$F(x_t) = E_t\left[G(x_{t+1})\right] \ , \tag{7}$$

where F and G are given by

$$F(x) = x(l_1^* - x)^{-\alpha_1} \qquad \text{and} \qquad G(x) = x(l_2^* + x)^{-\alpha_2} \ , \tag{8}$$

thus yielding an identity of form (1), defining the equilibrium dynamics for the state variable $x_t = 1/p_t$.

The map $F : [0, l_1^*) \to \mathbb{R}^+$ defined in (8) is strictly increasing, and thus invertible. In addition, for $\alpha_2 > 1$ the function G is unimodal. Hence, we obtain that the corresponding bpf map

$$\phi(x) = F^{-1}\left(G(x)\right) \tag{9}$$

is also unimodal.

3.2. **The Shapley-Shubik market game model.** We will now consider the one commodity overlapping generations version of the Shapley-Shubik market game (Shapley and Shubik (1977)), presented by Goenka et al. (1998).

This is a model of a pure exchange economy with overlapping generations of agents and a single good being traded. Time is discrete and labeled as $t = 1, 2, \ldots$. In each time period $t > 0$, a fixed number n of agents are born and live for two periods. Thus, young and old individuals coexist in each period t, with n elder individuals alive at time $t = 1$. Individuals are assumed to have identical utilities for consumption and identical initial endowments in each period of life. Denoting by (c_t^t, c_t^{t+1}) the consumption plan of an individual born in period t, her utility is given by

$$U(c_t^t, c_t^{t+1}) = u_1(c_t^t) + u_2(c_t^{t+1}) \ . \tag{10}$$

Thus, in the presence of uncertainty, an expected value operator $E_t[\cdot]$ is present in the second term of (10). The initial endowments in each period of life are denoted as ω_1 and ω_2.

The trade mechanism is as follows. There exists a fixed amount $\bar{m}$ of fiat money in the economy. Each individual i born at time t provides a (monetary) bid $b_{it} = (b_{it}^t, b_{it}^{t+1})$ to get monetary resources to buy goods, decides her savings m_t, and provides an offer of goods $q_{it} = (q_{it}^t, q_{it}^{t+1})$. Bids and offers operations are performed in trading posts and, in the presence of uncertainty, the negotiations are conditioned to each state of the nature. The aggregate bids and offers of individuals born at time $s = t, t+1$ are defined by

$$B_t^s = \sum_{i=1}^{n} b_{it}^s \qquad \text{and} \qquad Q_t^s = \sum_{i=1}^{n} q_{it}^s \ , \qquad s = t, t+1 \tag{11}$$

and the bids and offers in the market are

$$B_t = B_{t-1}^t + B_t^t \qquad \text{and} \qquad Q_t = Q_{t-1}^t + Q_t^t \ . \tag{12}$$

The terms of trade, that will be used as a “price” of the commodities in terms of fiat money, is defined by B_t/Q_t. Then, the lifetime budget constraints of each individual are given by

$$b_t^t + m_t = \frac{B_t}{Q_t} q_t^t \qquad \text{and} \qquad b_t^{t+1} = m_t + \frac{B_{t+1}}{Q_{t+1}} q_t^{t+1} \ . \tag{13}$$

If (13) is satisfied, then the lifetime consumption of each individual is

$$c_t^t = \omega_1 - q_t^t + \frac{Q_t}{B_t} b_t^t \qquad \text{and} \qquad c_t^{t+1} = \omega_2 - q_t^{t+1} + \frac{Q_{t+1}}{B_{t+1}} b_t^{t+1} \ . \tag{14}$$

To solve the individual problem in terms of the demand of money, we have to express the terms in (14) as functions of m_t. To do this, let us express the aggregate bids (offers) at time t and $t+1$ in terms of the individual bid (offer) and the other agents aggregate bids $(\hat{B}_t, \tilde{B}_{t+1})$ (offers $(\hat{Q}_t, \tilde{Q}_{t+1})$) at time t and $t+1$, respectively:

$$B_t = b_t^t + \hat{B}_t \qquad \left(Q_t = q_t^t + \hat{Q}_t\right) \ , \tag{15}$$

and

$$B_{t+1} = b_t^{t+1} + \tilde{B}_{t+1} \qquad \left(Q_{t+1} = q_t^{t+1} + \tilde{Q}_{t+1}\right) \ . \tag{16}$$

Substituting (15) into the first constraint in (13) and rearranging terms yields

$$b_t^t + m_t = \left[\frac{\hat{B}_t - m_t}{\hat{Q}_t}\right] q_t^t \ . \tag{17}$$

Similarly, substituting (16) into the second constraint in (13) results in

$$b_t^{t+1} = m_t + \left[\frac{\tilde{B}_{t+1} + m_t}{\tilde{Q}_{t+1}}\right] q_t^{t+1} \ . \tag{18}$$

Substituting (13) into (17) and (18), we obtain the terms of trade in each period, given by

$$\frac{B_t}{Q_t} = \frac{\hat{B}_t - m_t}{\hat{Q}_t} \qquad \text{and} \qquad \frac{B_{t+1}}{Q_{t+1}} = \frac{\tilde{B}_{t+1} + m_t}{\tilde{Q}_{t+1}} \ . \tag{19}$$

Finally, putting together (13), (14), (17) and (19) (as well as (13), (14), (18) and (19)), we obtain the individual's lifetime consumptions in terms of the money demand and the strategies of all other individuals:

$$c_t^t(m_t) = \omega_1 - \frac{\hat{Q}_t}{\hat{B}_t - m_t} m_t \qquad \text{and} \qquad c_t^{t+1}(m_t) = \omega_2 + \frac{\tilde{Q}_{t+1}}{\tilde{B}_{t+1} + m_t} m_t \ . \tag{20}$$

In period t the agents use beliefs μ_{t+1} regarding the aggregate bid value $\tilde{B}_{t+1}$ and the aggregate offer value $\tilde{Q}_{t+1}$ for the next period. Thus, a mathematical expectation operator with respect to the probability measure μ_{t+1} is included in the utility of the second life period. Therefore, the individual problem is to find the demand of money m_t solving the following:

$$\max_{m_t} \ u_1(c_t^t(m_t)) + E_t\left[u_2(c_t^{t+1}(m_t))\right] \ . \tag{21}$$

The first order condition for interior solutions of the optimization problem above is

$$\left(\frac{\hat{Q}_t}{\hat{B}_t - m_t} + \frac{\hat{Q}_t m_t}{(\hat{B}_t - m_t)^2}\right) u_1'\left(\omega_1 - \frac{\hat{Q}_t m_t}{\hat{B}_t - m_t}\right) =$$
$$E_t\left[\left(\frac{\tilde{Q}_{t+1}}{\tilde{B}_{t+1} + m_t} - \frac{\tilde{Q}_{t+1} m_t}{(\tilde{B}_{t+1} + m_t)^2}\right) u_2'\left(\omega_2 + \frac{\tilde{Q}_{t+1} m_t}{\tilde{B}_{t+1} + m_t}\right)\right] \ . \tag{22}$$

Peck et al. (1992) noticed that in this kind of models there is indeterminacy in either the offers or the bids. Therefore, we will suppose that the sequence of offers (q_t^t, q_t^{t+1}) is given and the amount of fiat money $\bar{m}$ is fixed. In this context we give the following definition of equilibrium.

Definition Given the exogenous offers (q_t^t, q_t^{t+1}) and the amount of fiat money $\bar{m}$, a *monetary Nash equilibrium* is a sequence of bids (b_t^t, b_t^{t+1}) that makes $m_t = \bar{m}$ the solution of (22).

To obtain the dynamics of the bids, notice that from (19) one can get

$$\frac{\tilde{Q}_t}{\tilde{B}_t + m_{t-1}} = \frac{\hat{Q}_t}{\hat{B}_t - m_t} \ ,$$

which, rearranging terms, yields

$$\tilde{B}_t = \frac{\tilde{Q}_t}{\hat{Q}_t}(\hat{B}_t - m_t) - m_{t-1} \ .$$

Substituting the two identities above into the right-hand side of (22), results in

$$\left(\frac{\hat{Q}_t}{\hat{B}_t - m_t} + \frac{\hat{Q}_t m_t}{(\hat{B}_t - m_t)^2}\right) u_1'\left(\omega_1 - \frac{\hat{Q}_t m_t}{\hat{B}_t - m_t}\right) = \tag{23}$$
$$E_t\left[\frac{\hat{Q}_{t+1}}{(\hat{B}_{t+1} - m_{t+1})^2}\left(\hat{B}_{t+1} - m_{t+1} - \frac{\hat{Q}_{t+1}}{\tilde{Q}_{t+1}} m_t\right) u_2'\left(\omega_2 + \frac{\hat{Q}_{t+1} m_t}{\hat{B}_{t+1} - m_{t+1}}\right)\right] \ .$$

Suppose now that $q_t^t = q_1$ and $q_t^{t+1} = q_2$ for all $t \geq 1$, and $\bar{m} = m$. Then, for all $t \geq 1$, one must have that

$$\hat{Q}_t = \hat{Q} = (n-1)q_1 + nq_2 \qquad \text{and} \qquad \tilde{Q}_t = \tilde{Q} = nq_1 + (n-1)q_2 \ .$$

Substituting the two identities above into (23) yields the dynamics in the aggregate bids:

$$\left(\frac{\hat{Q}}{\hat{B}_t - m} + \frac{\hat{Q} m}{(\hat{B}_t - m)^2}\right) u_1'\left(\omega_1 - \frac{\hat{Q} m}{\hat{B}_t - m}\right) =$$
$$E_t\left[\frac{\hat{Q}}{(\hat{B}_{t+1} - m)^2}\left(\hat{B}_{t+1} - \frac{m(\hat{Q} + \tilde{Q})}{\tilde{Q}}\right) u_2'\left(\omega_2 + \frac{\hat{Q} m}{\hat{B}_{t+1} - m}\right)\right] \ , \tag{24}$$

where the operator $E_t[\cdot]$ is now used to denote the expectation taken with respect to the probability distribution of the next period aggregate bids $\hat{B}_{t+1}$. Finally, set

$$x_t = \frac{\hat{Q} m}{\hat{B}_t - m}$$

and rewrite the equilibrium dynamics condition in (24) as

$$\frac{(x_t^2 + \hat{Q} x_t)}{\hat{Q}} u_1'(\omega_1 - x_t) = E_t\left[\frac{(\tilde{Q} x_{t+1} - x_{t+1}^2)}{\tilde{Q}} u_2'(\omega_2 + x_{t+1})\right] \ ,$$

where, by abuse of notation, the expectation with respect to the induced probability distribution on x_{t+1} is also denoted by $E_t[\cdot]$.

As in the example of Subsection 3.1, we will also use constant relative risk aversion utility functions

$$u_i(c) = \begin{cases} \frac{c^{1-\alpha_i}}{1-\alpha_i} & \text{if } \alpha_i > 0 \text{ and } \alpha_i \neq 1 \\ \ln(c) & \text{if } \alpha_i = 1 \end{cases}$$

Under these functional specifications, we are able to define

$$F(x) = \frac{(x^2 + \hat{Q}x)}{\hat{Q}}(\omega_1 - x)^{-\alpha_1}$$

and

$$G(x) = \frac{(\tilde{Q}x - x^2)}{\tilde{Q}}(\omega_2 + x)^{-\alpha_2} \ ,$$

yielding, in the general stochastic case, the dynamical system

$$F(x_t) = E_t\left[G(x_{t+1})\right] \ , \tag{25}$$

which is again of the form (1). Since the function F is strictly increasing in $[0, \omega_1]$ and the function G is a unimodal on the interval $[0, \tilde{Q}]$, the bpf map

$$\phi(x) = F^{-1}(G(x))$$

is also unimodal on $[0, \tilde{Q}]$.

Due to the striking similarities between the bpf maps (and corresponding dynamical systems) of the two models described in this section, we move on to perform a combined analysis for these two cases.

3.3. **Numerical simulations.** This is the main subsection of this part. We summarize the outcomes of some numerical experiments in order to illustrate the selection criterion given by the small random perturbations. Specifically, each bpf map of the models above will be parametrized using one of the parameters of the model. Then, we will track the values of that parameter for which the refinement process will select either a determinate cycle or a chaotic sunspot equilibrium according to Theorem 2.2.

Let $\phi_\lambda(x)$ denote the bpf in any of the models described in the previous subsections, where λ is one of the parameters defining the function, taking values in a given set Λ, as we describe below:

a) For the OLG model of Subsection 3.1, we fix all the parameter values except for $\lambda = \alpha_2$, which is allowed to vary in the set

$$\Lambda = \{\lambda \in [2, +\infty) : \, (l_1^*)^{\alpha_1} > (l_2^*)^{\lambda}\} \ .$$

Then, for every $\lambda \in \Lambda$, we can verify that ϕ_λ is a C^3 unimodal map with $\phi(0) = 0$, $\phi'(0) > 1$, a (positive) non-degenerate critical point $\bar{x}(\lambda)$ and negative Schwarzian derivative.

b) For the market game model of Subsection 3.2, we fix the relative risk aversion coefficients and initial endowments values in such a way that $\omega_1^{\alpha_1} > \omega_2^{\alpha_2}$. The parameter $\lambda = \tilde{Q}$ is allowed to vary in the set

$$\Lambda = (0, +\infty) \ .$$

Under these conditions, for every $\lambda \in \Lambda$, we can also verify that ϕ_λ is a C^3 unimodal map with $\phi(0) = 0$, $\phi'(0) > 1$, a (positive) non-degenerate critical point $\bar{x}(\lambda)$ and $S\phi(x) < 0$.

To check the validity of Hypothesis 2, let us introduce some additional concepts. Let Λ be an interval in $\mathbb{R}$ and denote by $\mathcal{F}_\Lambda$ a one-parameter family of bpf maps $\phi_\lambda \in \mathcal{F}$, depending on a real parameter $\lambda \in \Lambda$, and for which the map $(x, \lambda) \mapsto (\phi_\lambda(x), D_x\phi_\lambda(x), D_x^2\phi_\lambda(x))$ is C^1. Denote by $\bar{x}(\lambda_*)$ the critical point of the unimodal map ϕ_λ.

Definition We say that the one-parameter family ϕ_λ, $\lambda \in \Lambda$, has a *Misiurewicz parameter* $\lambda_* \in \Lambda$ *with generic unfolding* if the following conditions hold

a) $\lambda_* \in \Lambda$ is such that ϕ_{λ_*} is a Misiurewicz map, i.e. ϕ_{λ_*} has no periodic attractors and the forward critical orbit does not accumulate on its critical point;
b) the following transversality condition holds:

$$\lim_{n\to+\infty} \frac{D_\lambda \phi^n_{\lambda_*}(\bar{x}(\lambda_*))}{D_x \phi^{n-1}_{\lambda_*}(\phi_{\lambda_*}(\bar{x}(\lambda_*)))} \neq 0 \ .$$

Concerning the definition above, we remark that if ϕ_{λ_*} is a post-critically finite Misiurewicz map, i.e. ϕ_{λ_*} has no periodic attractors and some iterate N of the critical point $\bar{x}(\lambda_*)$ reaches a repelling periodic point $P(\lambda_*)$, then condition b) is equivalent to the transversality of the curves $\lambda \mapsto \phi^N_\lambda(\bar{x}(\lambda))$ and $\lambda \mapsto P(\lambda)$. We will use this equivalent geometrical condition in our numerical illustrations below.

The next result provides a set of sufficient conditions under which there exists a large set of bpf maps for which the strong stochastic stability of Proposition 2.3 holds; this avoids the verification of the conditions given in Hypothesis 2. The proof is in the Appendix.

Theorem 3.1. *Let ϕ_λ, $\lambda \in \Lambda \subset \mathbb{R}$, be a one-parameter family of bpf maps defined for one of the models above (the OLG model with fiat money of Section 3.1 or the market game model of Section 3.2).*

(i) *If ϕ_λ has a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, then there exists a positive measure set $A \subset \Lambda$ having λ_* as a density point such that for every $\lambda \in A$ there exists an invariant measure μ_λ which is an absolutely continuous B-R-S measure.*
(ii) *Moreover, if the dynamical system determined by ϕ_λ is perturbed by a random process $\{\tilde{\epsilon}_t\}_{t\geq 0}$ satisfying the hypothesis 1, then the results of Proposition 2.3 are also valid.*

We will now illustrate the large abundance of strong stochastic stable bpf maps for the OLG model of Subsection 3.1. We numerically determine values of parameters $(\alpha_1, \alpha_2, l_1^*, l_2^*)$ under which the map ϕ_λ is a post-critically finite Misiurewicz map, i.e. ϕ_λ has no periodic attractors and the critical orbit is pre-periodic to a repelling periodic orbit. To proceed with our numerical experiments, we fix the parameters $l_1^* = 3.51$ and $l_2^* = 0.55$ and work on the two parameter space of relative risk aversion coefficients $(\alpha_1, \alpha_2) \in (0,1) \times (2, +\infty)$. The results described below are robust with respect to changes in the values of l_1^* and l_2^*. We then numerically compute any intersections between the first N iterates of the critical point and the periodic points up to some finite period M, excluding all the non-transverse intersections since these do not satisfy item b) of Definition 3.3, as well as all the intersections with attracting periodic points. Checking the stability of the periodic points is relevant because in the case where the critical point is pre-periodic to a repelling periodic point there can be no stable or neutral cycles, since for unimodal maps with negative Schwarzian derivative, these would attract the critical orbit. In Figure 1 it is possible to observe the different dynamical limit behaviors of the bpf map as the risk aversion parameter α_2 increases from 2 to 7.5.

For small values of α_2, there exists a unique attracting fixed point of ϕ. Such attracting fixed point corresponds to a determinate steady state. As α_2 increases above 4.1 approximately, attracting periodic points of higher periods are generated by period-doubling bifurcations. Each attracting periodic point corresponds to a determinate cycle of the model. All those ϕ_λ lead to invariant measures supported on convex linear combination of Dirac measures supported on the determinate cycle, therefore, the refinement process will select the corresponding cycle. For large enough values of α_2 (greater than 6.4 approximately), Misiurewicz maps can be found. See Figure 2 shows the abundance of Misiurewicz maps within the parameter space $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.29)$. Such parameters correspond to bpf maps possessing an absolutely continuous invariant B-R-S measure. Therefore, the refinement process proposed here will select the chaotic sunspot equilibrium for such parameter values. Thus, we can conclude that the stability of the determinate cycle as selected equilibrium disappears as the relative risk aversion of the individual in her second period of life increases.

To stress the distinction between the behaviors associated with the two possibilities of convergence discussed above, for the parameter values used above and fixing additionally $\alpha_1 = 0.41$ and $\alpha_2 \in \{5.0; 6.5\}$, we plot in Figures 3 and 4 histograms associated with 10^6 iterations of paths generated by the bpf map ϕ for two different set of parameters values and varying sizes of the random perturbation.

Those are the densities of the stationary measure of the small random perturbations, which are close to the B-R-S measure μ, which is the stationary measure of the chaotic sunspot equilibrium.

The simulations discussed above can be criticized by the use of an excessively high relative risk aversion parameter. However, for the market game model of Section 3.2 with agents having constant relative risk aversion utility functions, we present the analogous analysis using more conservative values for that parameter. Specifically, fixing the parameter values $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and varying $\tilde{Q} \in (0.0, 0.135)$ we can observe in Figure 5 the existence of parameters values determining bpf maps with B-R-S measures for values of $\tilde{Q}$ below 0.098, namely, when there is scarcity of goods. For those parameter values we obtain the convergence of the stationary measures of small random perturbation to the absolutely continuous B-R-S measure of the bpf map, which is the stationary measure of the chaotic sunspot equilibrium. For $\tilde{Q}$ greater than 0.098, attracting cycles of the bpf map arise and therefore we obtain the convergence of the stationary measures of small random perturbations to convex linear combinations of Dirac measures supported on the corresponding deterministic cycle.

Figures 6 and 7 contain histograms associated with varying maximal sizes for the random perturbations for the parameter values listed above and, respectively, $\tilde{Q} = 0.09$ and $\tilde{Q} = 0.12$, enabling us to compare once again the chaotic dynamics case against the regular dynamics case associated with an attracting cycle.

We also run the same numerical analysis for the case of agents with logarithmic utilities and parameter values $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and varying $\tilde{Q} \in (0.0, 0.034)$. The results are quite similar to the previous analysis and Figure 8 shows that the refinement process will select the chaotic sunspot equilibrium for a wide range of values of $\tilde{Q}$ below 0.255. Due to the great similarity we do not report the histograms of this case. Therefore, we can conclude that the stability of the determinate cycle as selected equilibrium disappears as the total expected supply of goods for the second period of life becomes scarce.

4. Conclusions

In this paper we proposed a new selection criterion of dynamic equilibria in non-linear one-period forward looking economic models. The criterion is based on the limit behavior of the stationary measures of stochastic processes generated by a small random perturbation of the backward perfect foresight map of the model. This stability concept was originally introduced by Kolmogoroff (1937) and afterwards extended by Sinai (1972) to treat problems in statistical physics. Recently, its usage was broadly applied to the analysis of the unimodal maps dynamics by Katok and Kifer (1986), Baladi and Viana (1996) and Avila and Moreira (2003) and we use some of their findings to obtain our results.

It is important to highlight that the stochastic perturbations are not random deviations from a given perfect foresight path; rather, they can be interpreted as the rounding-off, in each date, of the equilibrating response to the perfect foresight, or even some approximations when there is uncertainty regarding some parameter defining the backward perfect foresight map. This "invisible trembling-hand" refinement adjusts the current value of the state variable that equilibrates the perfect foresight of the agents.

We apply that selection criterion to models exhibiting unimodal backward perfect foresight maps. For a large class of such models, we obtain that there exists a set of parameter values with full Lebesgue measure where the only equilibria that are selected under this refinement criterion are either determinate cycles (including deterministic steady states) or the Bowen-Ruelle-Sinai invariant measure of the backward perfect foresight map, which is the stationary measure of the chaotic sunspot equilibrium presented in Araujo and Maldonado (2000). Neither indeterminate cycles nor other types of sunspot equilibria are selected by the proposed criterion.

To illustrate the large set of parameter values where those results are valid, we performed numerical simulations of two economic dynamics models: the OLG model with fiat money and the Shapley-Shubick market game model. For both models we derived the equations defining the intertemporal equilibrium dynamics and studied the set of parameter values for which the stochastic processes under consideration possess an empirical measure converging to a Bowen-Ruelle-Sinai measure, as well as parameter values with the corresponding measure converging to an atomic measure with support on an attracting cycle. For the OLG model, the excess of risk aversion in the second period of life produces instability of the determinate cycle, triggering the arising of sunspots equilibria with absolutely continuous invariant measure supported in the whole relevant interval of the bpf map. In the Shapley-Shubick market game model, the scarcity of goods in the second period of individuals planning has the same destabilizing effect on the determinate cycles. Thus, the proposed analysis can be used as an indicator of stabilization policies by assessing the parameter values that allows for the robustness of those equilibria; namely the sunspot equilibrium of Araujo and Maldonado (2000) or the determinate cycles, with respect to the criterion of small random perturbation of perfect foresight equilibrium.

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Appendix

Proof of Theorem 2.2. The result relies heavily on the topological characterization of unimodal maps proposed in Avila-Moreira dichotomy for smooth unimodal maps (Avila and Moreira (2003)). Namely, for a topologically generic k-parameter family of bpf maps in $\mathcal{F}$, an open and dense set of parameters corresponds to hyperbolic maps. These are stochastically stable and, moreover, the stationary measure associated with a random perturbation of the corresponding bpf map will converge to an atomic measure with support on the (unique) attracting cycle of such hyperbolic map, which corresponds to a determinate cycle. Resorting to Avila-Moreira dichotomy, for almost every other parameter the bpf map satisfies the subexponential recurrence condition and the Collet-Eckmann condition (conditions 1 and 2 of hypothesis 2). These ensure the existence of an absolutely continuous stationary measure and strong stochastic stability. □

Proof of Proposition 2.3. Let $\epsilon > 0$ be small enough so that ν^ϵ is the unique invariant and ergodic probability measure associated with $P_\epsilon(\cdot, \cdot)$ (namely, satisfying (4)). Furthermore, such measure is absolutely continuous with respect to the Lebesgue measure. All those claims are proved in Benedicks and Young (1992), Part II.

Since the bpf map $\phi \in \mathcal{F}$ satisfies Hypothesis 2 and the probability density of the random perturbations satisfies Hypothesis 1, the main theorem in Baladi and Viana (1996) ensures that $\nu^\epsilon \to \mu$ in the weak-topology as $\epsilon \to 0$. □

Proof of Theorem 3.1. For the families of utility functions with constant relative risk aversion introduced in Subsections 3.1 and 3.2 and the choice of parameters discussed in items a) and b) in the second paragraph of Subsection 3.3, we have that for every $\lambda \in \Lambda$ the following holds:

- i) ϕ_λ is a C^3 unimodal map;
- ii) ϕ_λ has a (strictly positive) non-degenerate critical point $\bar{x}(\lambda)$;
- iii) ϕ_λ has a repelling fixed point at zero.

For one-parameter families of maps satisfying the conditions i), ii) and iii) above and having a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, there exists a positive Lebesgue measure set A in the space of parameters with λ^* as a density point and such that the subexponential recurrence and the Collet-Eckmann conditions of Hypothesis 2 hold for every $\lambda \in A$ (see Sections III.6, V.3 and V.6 of the textbook by de Melo and van Strien (1993) for further details). As a consequence, for every $\lambda \in A$, we have that

1) ϕ_λ admits an absolutely continuous invariant measure μ_λ, with a L^p density for any $p < 2$;
2) μ_λ is a B-R-S measure;
3) ϕ_λ has positive Lyapunov exponent almost everywhere.

Hence, the strong stochastic stability of ϕ_λ follows from Baladi and Viana (1996) by observing that condition (3) of Hypothesis 2 holds for unimodal maps with negative Schwarzian derivative and an absolutely continuous invariant measure. □

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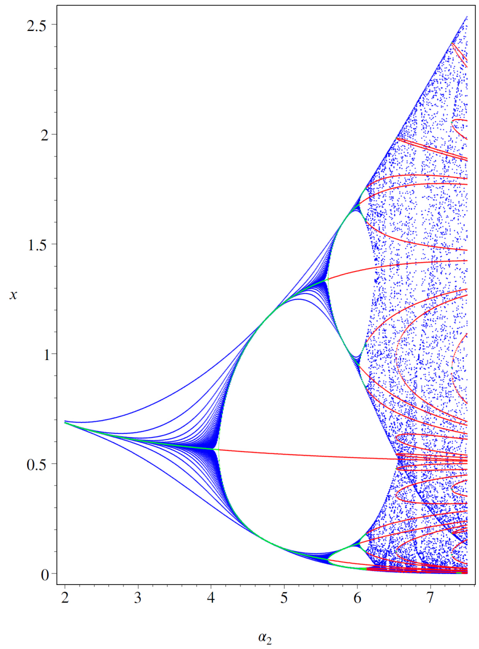

FIGURE 1. Plot in the (α_2, x) plane of the first 100 iterates of the critical point (in blue) and periodic points (of period 1, 2, 4 and 8) of the bpf map ϕ of Subsection 3.1. The stable periodic points are in green and the unstable ones in red. The other parameter values are $l_1^* = 3.51$, $l_2^* = 0.55$ and $\alpha_1 = 0.41$.

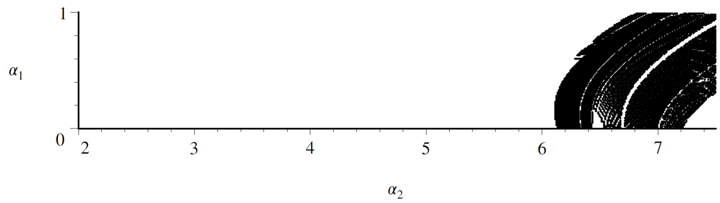

FIGURE 2. The distribution of Misiurewicz parameters for the family of bpf maps with $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.99)$ for fixed $l_1^* = 3.51$ and $l_2^* = 0.55$. These are obtained by considering intersections of the first 100 iterates of the critical point with unstable periodic points of periods 1, 2, 4 and 8.

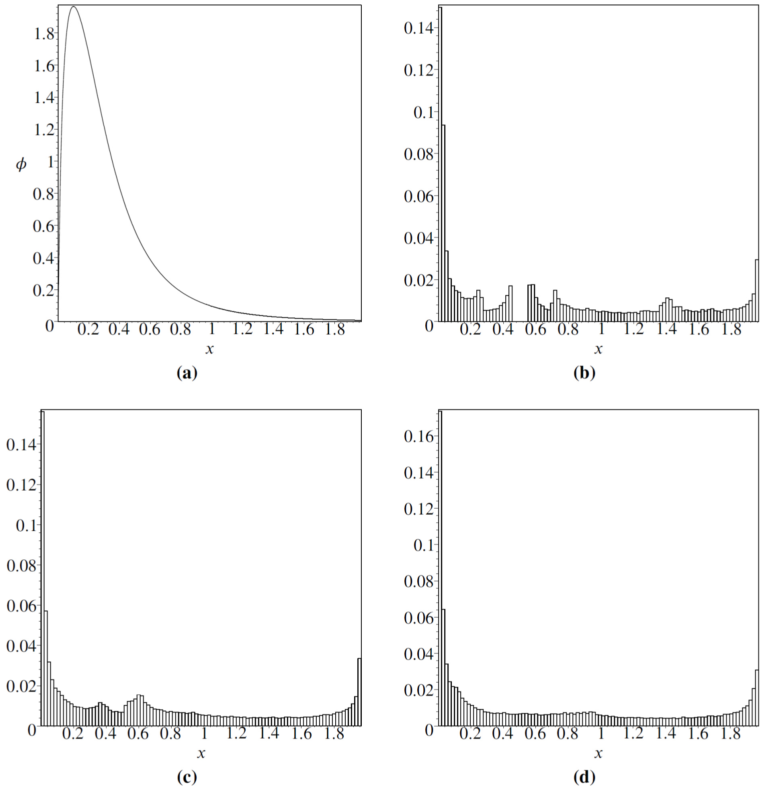

FIGURE 3. The bpf ϕ and the approximate densities associated with its stationary measure for the OLG model of Section 3.1 and parameter values $l_1^* = 3.51$, $l_2^* = 0.55$, $\alpha_1 = 0.41$ and $\alpha_2 = 6.5$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.001$ in Fig. (c) and $\epsilon = 0.00804199$ in Fig. (d).

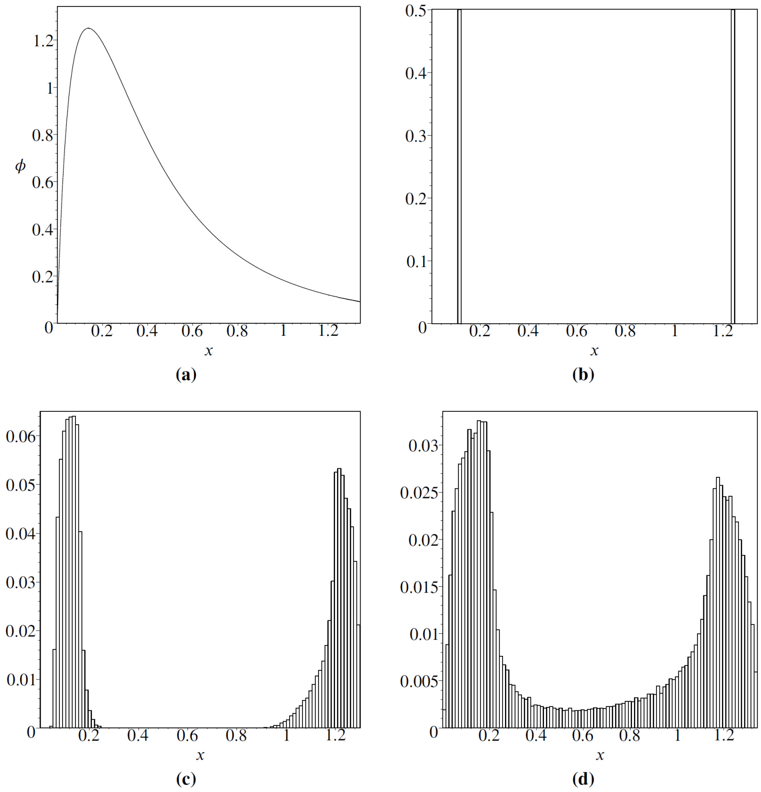

FIGURE 4. The bpf ϕ and the approximate densities associated with its stationary measure for the OLG model of Section 3.1 and parameter values $l_1^* = 3.51$, $l_2^* = 0.55$, $\alpha_1 = 0.41$ and $\alpha_2 = 5.0$ corresponding to deterministic bpf dynamics exhibiting an attracting cycle. Fig. (b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.05$ in Fig. (c) and $\epsilon = 0.09160628$ in Fig. (d).

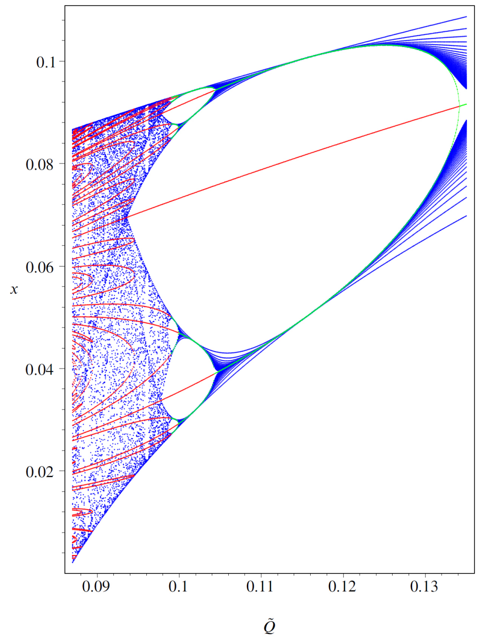

FIGURE 5. Plot in the $(\tilde{Q}, x)$ plane of the first 100 iterates of the critical point (in blue) and periodic points (of period 1, 2, 4 and 8) of the bpf map ϕ of Subsection 3.2 when agents have constant relative risk aversion utility functions. The stable periodic points are in green and the unstable ones in red. The other parameter values are $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$ and $\hat{Q} = 0.1$.

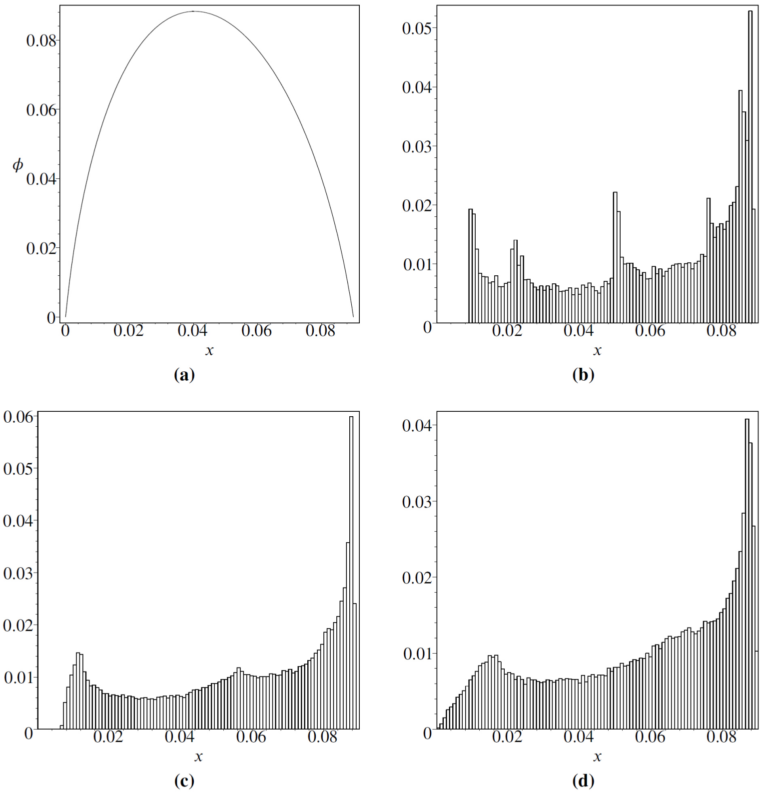

FIGURE 6. The bpf ϕ and the approximate densities associated with its stationary measure for the market game model of Subsection 3.2 when agents have constant relative risk aversion utility functions. The parameter values $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and $\tilde{Q} = 0.09$ corresponds to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.0005$ in Fig. (c) and $\epsilon = 0.00148461345$ in Fig. (d).

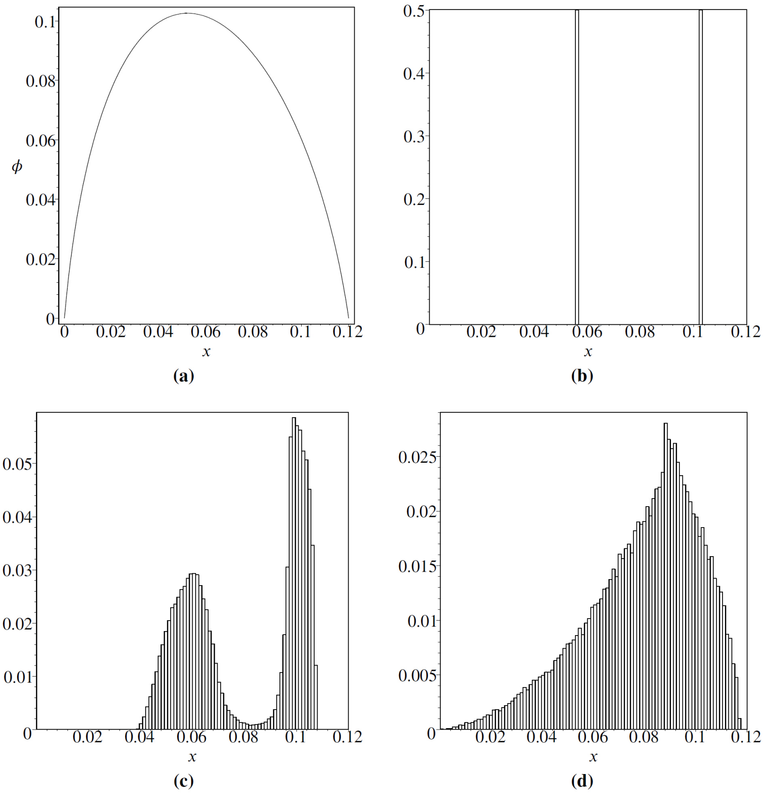

FIGURE 7. The bpf ϕ and the approximate densities associated with its stationary measure for the market game model of Subsection 3.2 when agents have constant relative risk aversion utility functions. The parameter values $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and $\tilde{Q} = 0.12$ corresponds to deterministic bpf dynamics exhibiting an attracting cycle. Fig. (b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.005$ in Fig. (c) and $\epsilon = 0.014360401$ in Fig. (d).

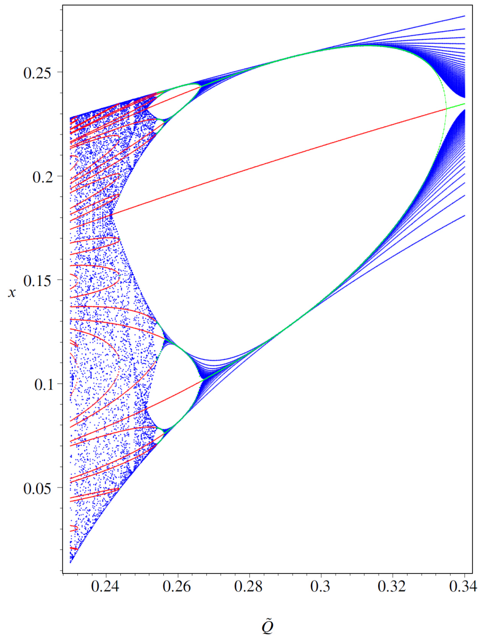

FIGURE 8. Plot in the $(\tilde{Q}, x)$ plane of the first 100 iterates of the critical point (in blue) and periodic points (of period 1, 2, 4 and 8) of the bpf map ϕ of Subsection 3.2 when agents have logarithmic utility functions. The stable periodic points are in green and the unstable ones in red. The other parameter values are $\omega_1 = 5.4$, $\omega_2 = 0.5$ and $\hat{Q} = 0.2$.