WHO WOULD INVEST ONLY IN THE RISK-FREE ASSET?

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Abstract. Within the setup of continuous-time semimartingale financial markets, we show that a multiprior Gilboa-Schmeidler minimax expected utility maximizer forms a portfolio consisting only of the riskless asset if and only if among the investor's priors there exists a probability measure under which all admissible wealth processes are supermartingales. Furthermore, we show that under a certain attainability condition (which is always valid in finite or complete markets) this is also equivalent to the existence of an equivalent (local) martingale measure among the investor's priors. As an example, we generalize a no betting result due to Dow and Werlang.

1. Introduction

Expected utility maximization plays a prominent role in mathematical finance as a decision-making tool. According to this paradigm, out of a family of wealth processes $(X_t(\pi))_{t\in[0,T]}$, with regard to a market S, and indexed in terms of portfolio processes π in a certain set $\mathcal{X}(x)$ of admissible portfolios with initial wealth x, a decision maker (or investor) chooses the one corresponding to portfolio processes $\pi^* \in \mathcal{X}(x)$ for which the following variational principle, known as *portfolio optimization problem*, holds:

$$\mathbb{E}_P[U(X_T(\pi^*))] = \sup_{\pi\in\mathcal{X}(x)} \mathbb{E}_P[U(X_T(\pi))] \ .$$

Interestingly, the solvability of the portfolio optimization problem is closely related to qualitative and quantitative properties of the financial market described by S. In particular, it is well known that the well-posedness of the utility maximization problem is related to issues regarding the existence of an equivalent (local) martingale measure (also known as risk neutral measure) and the corresponding availability of a linear pricing rule. This discussion has been initiated, for a variety of market models, in the seminal works of [19] and [20, 21], where the concept of market viability is defined as the precise setting under which the above portfolio optimization problem has a solution in terms of a net trade compatible with certain constraints. It should also be remarked that the notion of market viability is closely related with absence of arbitrage, as well as with the existence of equivalent martingale measures, which can then be reinterpreted as appropriate pricing kernels as mentioned

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above. Indeed, the two notions are equivalent for finite markets. However, when considering continuous-time markets or models in infinite probability spaces, the situation becomes delicate on account of various mathematical intricacies. While market viability remains robust as a concept, the notion of arbitrage, as well as this of equivalent martingale measure or pricing kernel, have to be refined. This contributed to the appearance of various alternative definitions, the connection among which is clear but complicated, a fact that has led to the development of interesting literature by leading experts in the field. In particular, various definitions have been introduced to express what should be meant by "absence of arbitrage", among which, just to mention a few, one should list those of *no arbitrage* (NA), *no unbounded profit with bounded risk* (NUPBR), *no free lunch* (NFL), *no free lunch with bounded risk* (NFLBR), and *no free lunch with vanishing r isk* (NFLVR). Links between these definitions have been thoroughly investigated. For instance, it is known that $NFL \implies NFLBR \implies NFLVR \implies NUPBR$ and $NFLVR \iff LUPBR + NA$. For a unified perspective of the most significant no arbitrage conditions, in the context of continuous semimartingale models, see [15] and references therein. It shouldn't be surprising then that similar observations hold when discussing the existence of equivalent martingale measures. Indeed, each notion of absence of arbitrage is related to a generalized (often weaker) concept of equivalent martingale measure, each one of which having consequences on the behaviour of the portfolio optimization problem. For instance, as shown in [22] within a general semimartingale setting, the condition NUPBR is necessary for the solvability of the portfolio optimization problem, confirming and generalizing a similar result of [25]. Quite recently, [8] have shown that the condition NUPBR is equivalent to the solvability of the portfolio optimization problem (but possibly up to an equivalent change of measure; see also [9] for a counter example). It is worth noting that these considerations have led to elegant formulations of the portfolio optimization problem in terms of duality methods, allowing for semi-explicit representations of the solutions (see e.g. [24], [10]).

However, the traditional von Neumann-Morgenstern expected utility framework fails to address the problem of model uncertainty and ambiguity aversion that has been dictated by the famous Ellsberg paradox ([14]), related with the distinction introduced in 1921 by Frank Knight between risk and uncertainty ([23]). According to this distinction, "risk" refers to the situation where the unique probability distribution of a random experiment is assumed to be known, while the term (Knightian) "uncertainty" is reserved for situations where such a unique probability assignment does not exist. It is precisely the fact that decision makers are not always capable of attaching a unique probability measure to the relevant state space that was manifested by the Ellsberg paradox, reflecting the fact that when the information available is not "sufficient" to form a single probability distribution assumption, a decision maker may consider a whole set of alternative distributions as plausible models and then act on this consideration. The model uncertainty and ambiguity aversion paradigm has been employed to provide some explanation of various empirical observations that did not comply to the more traditional expected utility model, such as for example the failure of the two-fund separation theorem, the equity premium and the risk-free rate puzzles, the trading freezes, etc..

There are two basic strands in the literature that extend the expected utility paradigm, to cope with model uncertainty. The first, introduced by [27], is based on the use of non-additive probabilities (capacities) to represent the decision maker's beliefs, while the second, introduced by [16], allows for beliefs to be represented by a set of probabilities, while preferences are expressed by the "maxmin" on the set of expected utilities. Under the Gilboa-Schmeidler approach, the traditional utility is replaced by the robust utility

$$\mathcal{U}(X) := \inf_{P \in \mathcal{P}} \mathbb{E}_P[U(X)] \ ,$$

where $\mathcal{P}$ is a relevant set of priors concerning the distribution of the random variable X. Now, the aim of the investor is to maximize this robust utility over all admissible trading strategies, leading to an optimization problem of the form

$$\sup_{X \in \mathcal{X}(x)} \inf_{P \in \mathcal{P}} \mathbb{E}_P[U(X_T)] \ ,$$

where X takes values in the set $\mathcal{X}(x)$ of all wealth processes associated with admissible portfolios. Knightian decision theory plays nowadays a prominent role in economic theory (see e.g. [3] for a rigorous formulation of Knight's ideas), and finance in particular, as ambiguity introduces interesting effects in the portfolio optimization problem. For example, in the context of a simple one period model with a single asset, [12] have shown that, although within the expected utility paradigm trades occur generically, ambiguity may generate no betting intervals. This no betting effect is further addressed and reconfirmed in later studies, see e.g. [5], [13] and [17]. For a detailed recent review of the literature see [18] and references therein.

Furthermore, there has been some recent interesting academic activity with regard to the characterization of the non arbitrage condition under model uncertainty (e.g. [2], [7], [4]). For example, in [2] it is shown in a discrete time non-dominated model uncertainty setting that NA holds if and only if there exists a family of probability measures such that any admissible value process is a local supermartingale under these measures.

The goal of this work is to extend the discussion concerning the connection between the existence of martingale and supermartingale measures and the portfolio optimization problem in the framework of minimax utility. In particular, we show that for a general class of minimax utilities of the Gilboa-Schmeidler type, the optimal portfolio consists purely of investment in the riskless asset if and only if among the investor's priors there exists a probability measure under which all admissible wealth processes are supermartingales. In addition, we show that under a certain attainability condition (which is always valid in finite or complete markets) this is also equivalent to the existence of an equivalent (local) martingale measure among the investor's priors. Furthermore, we show in a simple example a potentially interesting link of our results to the no betting or "market freezes" phenomenon as described in [12].

This paper is organized as follows. In Section 2 we describe the setting we work with. Section 3 is devoted to the analysis of the particular, yet very relevant, case of von Neuman-Morgernstern utilities and in Section 4 we state and prove our main results.

2. Setup and problem formulation

We consider a securities market consisting of $d+1$ assets, one riskless and d risky assets. The riskless asset is assumed to bear a deterministic instantaneous risk free interest rate. Then, without loss of generality and to ease notation, we may assume that the rate of return of the riskless asset is $r = 0$. Otherwise, we may simply use the (non-zero) price of the riskless asset as numéraire. Let $T > 0$ be a fixed finite horizon. The risky assets price process will be denoted by $S = (S_t)_{t\in[0,T]}$ with $S_t = \left(S_t^1, \dots, S_t^d\right)$, $t \in [0,T]$, where the coordinate processes S^i, $i = 1, 2, \dots, d$, are assumed to be locally bounded semimartingales on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, for any probability measure P under consideration.

A portfolio $\pi \in \mathcal{X}(x)$ is a pair (x, H) where the constant x is the initial wealth and $H = (H_t)_{t\in[0,T]}$ with $H_t = \left(H_t^1, \dots, H_t^d\right)$, $t \in [0,T]$, is a predictable process specifying the amount of risky assets held in the portfolio. The wealth of an investor with a self financing portfolio $\pi = (x, H)$ is given by the stochastic process $X = (X_t)_{t\in[0,T]}$, defined as

$$X_t = x + \int_0^t H_u \, \mathrm{d}S_u$$

for every $t \in [0,T]$.

On the filtered measurable space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$ we assign a set of priors $\mathcal{P}$ consisting of the subjective probability laws that may govern the market according to the beliefs of the investor. Decisions concerning optimal portfolio choice are made using a minimax utility framework, to accommodate the effects of robustness and model uncertainty. Hence, the investor solves a minimax problem of the form

$$\sup_{X\in\mathcal{X}(x)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X_T)] \ . \tag{1}$$

where U is the utility function of the investor and $\mathcal{X}(x)$ is the set of all admissible wealth processes X with initial wealth x, defined as

$$\mathcal{X}(x) = \{X \geq 0 \ : \ X_t = x + \int_0^t H_u \, \mathrm{d}S_u \ \text{ for } t \in [0,T] \ \} \ .$$

Before advancing, we need to define the set of equivalent supermartingale measures and the set of equivalent local martingale measures

Definition Let $P \in \mathcal{P}$. We define

$$\mathcal{S}_P = \{Q \sim P \ : \ X \text{ is } Q\text{-supermartingale for all } X \in \mathcal{X}(x)\}$$

to be the set of supermartingale measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$ that are equivalent to P and set

$$\mathcal{S}_P = \cup_{P\in\mathcal{P}} \mathcal{S}_P \ .$$

Similarly, we define

$$\mathcal{M}_P = \{Q \sim P \ : \ Q \text{ is a local martingale measure}\}$$

to be the set of local martingale measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$ that are equivalent to P and set

$$\mathcal{M}_{\mathcal{P}} = \cup_{P\in\mathcal{P}}\mathcal{M}_P \ .$$

It is clear that $\mathcal{M}_P \subset \mathcal{S}_P$ since all admissible processes are uniformly bounded from below. In what concerns the utility function in the minimax optimization problem (1), we impose that:

Assumption $U : \mathbb{R}_+ \to \mathbb{R}$ is a strictly concave, continuously differentiable, strictly increasing function satisfying the Inada conditions $\lim_{x\to 0^+} U'(x) = \infty$, $\lim_{x\to\infty} U'(x) = 0$ and the asymptotic elasticity inequality

$$AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1 \ .$$

Furthermore, one needs to impose sufficient hypotheses on the set of priors $\mathcal{P}$ such that a saddle point for problem (1) exists. The fundamental condition is:

Assumption The set of priors $\mathcal{P}$ is convex and weakly compact.

This fundamental assumption may have to be complemented with further technical conditions which are dependent on the specific choice of model. For example, in a non-dominated continuous semimartingale setting one can adopt the additional [11] condition concerning the existence of a non empty set $\mathcal{P}_0$ of orthogonal martingale laws satisfying Hölder continuity conditions, such that (i) for any $P \in \mathcal{P}$, there exists $P_0 \in \mathcal{P}_0$ with $\mathbb{E}_{P_0}\left[(\mathrm{d}P/\mathrm{d}P_0)^2\right] \le C$ for some constant C, and (ii) for any $P_0 \in \mathcal{P}_0$, there exists a $P \in \mathcal{P}$ such that $P \sim P_0$ (see Hypothesis (H) in [11]). In the context of Lévy models and the particular case of log or power utilities one can adopt the additional [26] condition that the uncertainty about drift, volatility and jumps is parametrized by a non empty set $\Theta \subset \mathbb{R}^d \times \mathbb{S}^d_+ \times \mathcal{L}$ where $\mathcal{L}$ is the set of Lévy measures on $\mathbb{R}^d$, which is convex and satisfies specific boundedness conditions (see Assumption 2.1 in [26]).

Finally, we impose a non arbitrage condition:

Assumption For each $P \in \mathcal{P}$ we assume that $\mathcal{M}_P \neq \emptyset$.

It should also be made clear that we do not require the members of the set $\mathcal{P}$ to be mutually equivalent or even dominated.

We will show that, under the assumptions listed above, an investor with preferences described by a Gilboa-Schmeidler minimax utility will place all of his wealth on the riskless asset if and only if the set of priors $\mathcal{P}$ contains a supermartingale measure (i.e. $\mathcal{P} \cap \mathcal{S}_{\mathcal{P}} \neq \emptyset$). Furthermore, we will show that, under a certain attainability condition (which holds when Ω is finite or the market is complete), the investor will invest only in the risk free asset if and only if the set of priors $\mathcal{P}$ contains an equivalent martingale measure (i.e. $\mathcal{P} \cap \mathcal{M}_{\mathcal{P}} \neq \emptyset$). As a first step towards this goal, we will show that the same conclusion holds whenever $\mathcal{P} = \{P\}$ is a singleton, corresponding to the case of von Neumann-Morgernstern utilities.

3. An auxiliary result: the case of von Neumann-Morgernstern utilities

In the case of von Neumann-Morgernstern utilities, the minimax problem (1) reduces to the optimization problem

$$u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}_P[U(X_T)] \ . \tag{2}$$

This problem has been studied in the seminal work of [24] using duality techniques. As in [24], we also assume that $u(x) < \infty$ for some $x > 0$. They have solved (2) in terms of the dual problem

$$v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}_P[V(Y_T)] \ , \tag{3}$$

where $\mathcal{Y}(y)$ is defined as

$$\begin{aligned}\mathcal{Y}(y) = \{Y = (Y_t)_{t\in[0,T]} \geq 0 \ : \ Y_0 = y, \ XY = (X_tY_t)_{t\in[0,T]} \\ \text{is a supermartingale for all } X \in \mathcal{X}(1)\} \ ,\end{aligned}$$

and V is the Fenchel-Legendre conjugate of the utility function U (see [6]), defined by

$$V(y) = \sup_{x>0}\{U(x) - yx\} \ .$$

Under the assumptions stated in Section 2, and in particular under the assumption that $AE(U) < 1$, both the primal and the dual problem admit unique solutions, $\hat{X} := (\hat{X}_t)_{t\in[0,T]} \in \mathcal{X}(x)$ and $\hat{Y} := (\hat{Y}_t)_{t\in[0,T]} \in \mathcal{Y}(y)$ respectively, related through the identity

$$\hat{X}_T = I(\hat{Y}_T) \ \text{ for } \ y = u'(x), \ \ I = (U')^{-1}, \tag{4}$$

whereas [24, Thm 2.2]

$$\hat{X}\hat{Y} = \{\hat{X}_t\hat{Y}_t, \ \ t \in [0,T]\}, \ \text{ uniformly integrable martingale.} \tag{5}$$

Furthermore, it holds that

$$v(y) = \inf_{Q \in \mathcal{M}_P} \mathbb{E}_P\left[V\left(y\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right] , \tag{6}$$

where $\mathrm{d}Q/\,\mathrm{d}P$ denotes the Radon-Nikodym derivative of Q with respect to P on $(\Omega, \mathcal{F}_T)$.

We should stress here that the infimum in (6) may or may not be attained. For example, the infimum is always attained if Ω is a finite probability space (see e.g. [10, Ch. 3]), or if the market is complete. On the other hand, if Ω is not finite and the market is incomplete, one can find counter examples as detailed in [24].

Proposition 3.1. *Suppose that Assumptions 2 and 2 hold and that $\mathcal{P} = \{P\}$ is a singleton. Then:*

- **(i):** *The optimal portfolio is non-random if and only if $P \in \mathcal{S}_P$.*
- **(ii):** *If moreover the infimum in (6) is attained, then the optimal portfolio is non-random if and only if $P \in \mathcal{M}_P$.*

Proof. (i) If $P \in \mathcal{S}_P$, then clearly $\mathbb{E}_P[X_T] \leq x$ for any admissible portfolio. Jensen's inequality and the fact that U is increasing readily imply that $\mathbb{E}_P[U(X_T)] \leq U(\mathbb{E}_P[X_T]) \leq U(x)$ and thus the non random portfolio of constant value x is the maximizer.

For the converse assume that $\hat{X} = x$ is the maximizer. From (5) we have that $\hat{X}_t \hat{Y}_t = \mathbb{E}_P[\hat{X}_T \hat{Y}_T \mid \mathcal{F}_t]$ for any $t \in [0, T]$ which implies that $\hat{Y}_t = \mathbb{E}_P[\hat{Y}_T \mid \mathcal{F}_t]$ since $\hat{X} = x$ by assumption. This in turn implies that $\hat{Y}_t = U'(x)$ for every $t \in [0, T]$, since by (4) $\hat{Y}_T = U'(x)$ which is also non random. Being optimal, $\hat{Y} = U'(x) \in \mathcal{Y}(y)$ and thus the definition of $\mathcal{Y}(y)$ implies that $XU'(x)$ is supermartingale for all $X \in \mathcal{X}(1)$. Therefore, since $U'(x) > 0$, we conclude that any arbitrary admissible process $X \in \mathcal{X}(x)$ is a P-supermartingale, which means that $P \in \mathcal{S}_P$.

(ii) If $P \in \mathcal{M}_P$ then $P \in \mathcal{S}_P$, since the admissible processes X are bounded from below and the proof follows as in case (i) above.

For the converse, suppose that the non random portfolio is the maximizer, i.e. $\hat{X} = x$. Then, the solution of the dual problem is given by $\hat{Y} = y$ (indeed, we can argue again as in part (i) of the proof to show that if $\hat{X} = x$ then $\hat{Y}$ is constant and $\hat{Y} \in \mathcal{Y}(y)$). Then, (3) implies that

$$v(y) = \mathbb{E}_P[V(\hat{Y}_T)] = \mathbb{E}_P[V(y)] = V(y)\ . \tag{7}$$

On the other hand, under the assumption that the infimum in (6) is attained, there exists a measure $Q^* \in \mathcal{M}_P$ such that

$$v(y) = \mathbb{E}_P\left[V\left(y\frac{\mathrm{d}Q^*}{\mathrm{d}P}\right)\right]. \tag{8}$$

However, since $\mathbb{E}_P[\,\mathrm{d}Q^*/\,\mathrm{d}P] = 1$, we also have that

$$V(y) = V\left(\mathbb{E}_P\left[y\frac{\mathrm{d}Q^*}{\mathrm{d}P}\right]\right)\ . \tag{9}$$

Combining (7), (8) and (9) we deduce that we have a Jensen's equality with regard to the convex function V:

$$\mathbb{E}_P\left[V\left(y\frac{\mathrm{d}Q^*}{\mathrm{d}P}\right)\right] = V\left(\mathbb{E}_P\left[y\frac{\mathrm{d}Q^*}{\mathrm{d}P}\right]\right)\ .$$

Therefore, the random variable $\frac{\mathrm{d}Q^*}{\mathrm{d}P}$ is a.e. constant with regard to P and thus $P = Q^* \in \mathcal{M}_P$. □

Remark We note that:

- **1):** The result of Proposition 3.1 can be rephrased as: The non random portfolio is optimal if and only if $1 \in \mathcal{Y}(1)$.
- **2):** One can easily check that the statement and proof of Proposition 3.1 hold if the set $\mathcal{S}_P$ is replaced by the set $\mathcal{S}_P^T = \{Q \sim P\ :\ \mathbb{E}_Q[X_T] \leq x \text{ for all } X \in \mathcal{X}(x)\}$.
- **3):** Proposition 3.1 holds for the special case of models with finite probability spaces, for both single period and multiple periods models. In such cases the infimum in (6) is always attained (see [10, Theorem 3.2.1]). It should be clear that in such a setting a simpler proof of our result may be obtained by treating directly the portfolio optimization problem and the resulting variational inequality.

4): In market models where the infimum in (6) is attained for any probability measure P such that $\mathcal{M}_P \neq \emptyset$, one is also tempted to think along the following lines: If $Q \in \mathcal{S}_P$ is a supermartingale measure, then any investor who would believe that the market is governed by Q, would choose the non random portfolio, according to Proposition 3.1 (i). But then Proposition 3.1 (ii) implies that $Q \in \mathcal{M}_P$. Therefore, one has that $\mathcal{S}_P \subset \mathcal{M}_P$ and since the inverse inclusion holds as well, we obtain that $\mathcal{S}_P = \mathcal{M}_P$. Taking this thought one step further, one wonders with regard to the infimum in (6), whether is it (in general) true that $\inf_{Q\in\mathcal{M}_P} \mathbb{E}_P\left[V\left(y\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right] = \inf_{Q\in\mathcal{S}_P} \mathbb{E}_P\left[V\left(y\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right]$.

Example For CARA utilities of the form

$$U(x) = \frac{1}{\alpha}x^{\alpha} \ , \qquad \alpha < 1 \ ,$$

one can easily compute the Fenchel-Legendre conjugate to obtain

$$V(y) = -\frac{1}{\nu}y^{\nu} \ , \text{with } \nu = \frac{\alpha}{\alpha-1} \ .$$

In this case, assuming that the infimum is attained for a measure $Q^* \in \mathcal{M}_P$ such that

$$\mathrm{d}Q^*/\,\mathrm{d}P = \phi > 0$$

and, additionally, that P is absolutely continuous with respect to Lebesgue measure with density $f \geq 0$, then identity (7) assumes the form

$$1 = \int_0^{\infty} \phi(s)^{\nu} f(s)\,\mathrm{d}s \ ,$$

where it also holds that

$$\int_0^{\infty} f(s)\,\mathrm{d}s = \int_0^{\infty} \phi(s) f(s)\,\mathrm{d}s = 1 \ .$$

As a consequence of Hölder's inequality, the only function ϕ with this property is the constant function. Indeed, we can write

$$\begin{aligned} 1 = \int_0^{\infty} f(s)\,\mathrm{d}s &= \int_0^{\infty} f^{\frac{\nu}{\nu-1}} \phi^{\frac{\nu}{\nu-1}} f^{-\frac{1}{\nu-1}} \phi^{-\frac{\nu}{\nu-1}}\,\mathrm{d}s \\ &\leq \left\{\int_0^{\infty} (f^{\frac{\nu}{\nu-1}}\phi^{\frac{\nu}{\nu-1}})^{\frac{\nu-1}{\nu}}\,\mathrm{d}s\right\}^{\frac{\nu}{\nu-1}} \left\{\int_0^{\infty} (f^{\frac{1}{1-\nu}}\phi^{\frac{\nu}{1-\nu}})^{1-\nu}\,\mathrm{d}s\right\}^{\frac{1}{1-\nu}} \\ &= \left\{\int_0^{\infty} f(s)\phi(s)\,\mathrm{d}s\right\}^{\frac{\nu}{\nu-1}} \left\{\int_0^{\infty} f(s)\phi(s)^{\nu}\,\mathrm{d}s\right\}^{\frac{1}{1-\nu}} = 1. \end{aligned}$$

Since the equality is achieved in Hölder's inequality, there exists a constant C such that

$$(f^{\frac{\nu}{\nu-1}}\phi^{\frac{\nu}{\nu-1}})^{\frac{\nu-1}{\nu}} = C(f^{\frac{1}{1-\nu}}\phi^{\frac{\nu}{1-\nu}})^{1-\nu}$$

from which it follows that ϕ is a constant function and, thus, equal to 1. Therefore, we obtain that $P = Q^*$, as required.

4. Gilboa-Schmeidler minimax utilities

We now proceed to treat the general case where $\mathcal{P}$ is not a singleton.

Theorem 4.1. *Suppose that assumptions 2, 2 and 2 hold for the utility function U and the set of priors $\mathcal{P}$. Consider an investor reporting minimax utility of the form*

$$\mathcal{U}(X) = \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X)]$$

for random wealth X. The solution to the robust final wealth optimization problem

$$\sup_{X\in\mathcal{X}(x)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X_T)]$$

is a non-random portfolio if and only if $\mathcal{P}\cap\mathcal{S}_P \neq \emptyset$. Furthermore, if the infimum in (6) *is attainable for every $P\in\mathcal{P}$ then the solution to the optimization problem is the non random portfolio if and only if $\mathcal{P}\cap\mathcal{M}_P \neq \emptyset$.*

Proof. Throughout the proof we will use the notation $X^0 = \{X^0_t\}_{t\in[0,T]}$ for the wealth process of the non random portfolio that has constant value $X^0_t = x$ for every $t\in[0,T]$.

Assume first that $\mathcal{P}\cap\mathcal{S}_P \neq \emptyset$ and that $Q\in\mathcal{P}\cap\mathcal{S}_P$. Applying Jensen's inequality to the concave function U and using the fact that $Q\in\mathcal{S}_P$ we obtain that for any $X\in\mathcal{X}(x)$, we must have that

$$U(x) \geq U(\mathbb{E}_Q[X_T]) \geq \mathbb{E}_Q[U(X_T)] \geq \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X_T)] \ ,$$

where the last inequality follows from the fact that $Q\in\mathcal{P}$. Taking the supremum over all $X\in\mathcal{X}(x)$ in the above we conclude that

$$U(x) \geq \sup_{X\in\mathcal{X}(x)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X)] \ .$$

For the non-random portfolio $X^0(x)$ the equality is attained, hence X^0 is a maximizer. Since $\mathcal{M}_P\subset\mathcal{S}_P$, we would have clearly obtained the same result if we had started from the assumption $\mathcal{P}\cap\mathcal{M}_P\neq\emptyset$.

For the converse, assume that X^0 is a maximizer. Then,

$$u(x) = \sup_{X\in\mathcal{X}(x)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X_T)] = \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X^0_T)] = U(x) \ , \tag{10}$$

since

$$\mathbb{E}_P[U(X^0_T)] = U(x)$$

for every $P\in\mathcal{P}$.

By the saddle point property [11, Thm. 1], it holds that

$$\sup_{X\in\mathcal{X}(x)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(X_T)] = \inf_{P\in\mathcal{P}} \sup_{X\in\mathcal{X}(x)} \mathbb{E}_P[U(X_T)] \ ,$$

and hence, by (10), we have that

$$U(x) = \inf_{P\in\mathcal{P}} \sup_{X\in\mathcal{X}(x)} \mathbb{E}_P[U(X_T)] \ . \tag{11}$$

Since $\mathcal{P}$ is weakly compact, by the lopsided minimax theorem of Aubin and Ekeland [1, Ch. 6, Sec. 2, Thm. 7] (see also [11, Lem. 9]), there exists $\hat{P}\in\mathcal{P}$ for which

$$\inf_{P\in\mathcal{P}} \sup_{X\in\mathcal{X}(x)} \mathbb{E}_P[U(X_T)] = \sup_{X\in\mathcal{X}(x)} \mathbb{E}_{\hat{P}}[U(X_T)] \ . \tag{12}$$

Combining (11) with (12) we conclude that

$$U(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}_{\hat{P}}[U(X_T)] \ .$$

We have thus reduced the robust problem to the single prior problem of the previous section with $P = \hat{P}$. Hence, by applying Proposition 3.1 we obtain the desired result. □

In the result above, we have considered the market and the prices as given, and the problem of optimal portfolio selection of an uncertainty averse investor with a set of priors concerning the market was studied. It turns out that Theorem 4.1 allows one to relate to the well known results of [12], concerning the effects of uncertainty on the net demand of risky assets, and thus contribute to a better understanding of the phenomenon of the existence of market freezes, which refers to situations where the market endogenously stops as the following example shows.

Remark Consider an one period market starting at $t = 0$ and ending at $t = T$ and an investor contemplating positioning on a set of risky assets with payoffs $A = (A_1, \ldots, A_N)$ at time T. The investor reports a minimax utility with a set of priors $\mathcal{P}$ concerning the random variable A. Then, the no betting set $\mathcal{N}$, consisting of those asset prices for which the net demand of the assets is zero, is the convex set $\mathcal{N} = \{\mathbb{E}_P[A] \ : \ P \in \mathcal{P}\}$.

Indeed, let $\pi \in \mathcal{N}$. This means that the investor will not take up a position in this market, hence by Theorem 4.1, restricted in the one period case, this is equivalent to the existence of some $P \in \mathcal{P}$ such that $\pi = \mathbb{E}_P[A]$.

In the special case where the number of risky assets is $N = 1$, the no betting set $\mathcal{N}$ is an interval. For instance, Dow and Werlang provide a simple illustrative example in section 2 of [12], where two possible outcomes H and L have respective non additive probabilities π and π'. This is equivalent to considering the set of additive probabilities $(q,\, 1-q)$ corresponding to $(H,\, L)$ where q ranges from π to $1 - \pi'$. Then they prove that the interval of no betting prices ranges between $\pi H + (1-\pi)L$ and $(1-\pi')H + \pi' L$ which obviously coincides with the interval of prices that our generalized methodology suggests.

Acknowledgments

We would like to acknowledge useful discussions with Professors Zhenyu Cui, F. Santambrogio and A. Tsekrekos.

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