

# DYNAMIC PROGRAMMING FOR A MARKOV-SWITCHING JUMP-DIFFUSION

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**ABSTRACT.** We consider an optimal control problem with a deterministic finite horizon and state variable dynamics given by a Markov-switching jump-diffusion stochastic differential equation. Our main results extend the dynamic programming technique to this larger family of stochastic optimal control problems. More specifically, we provide a detailed proof of Bellman’s optimality principle (or dynamic programming principle) and obtain the corresponding Hamilton-Jacobi-Belman equation, which turns out to be a partial integro-differential equation due to the extra terms arising from the Lévy process and the Markov process. As an application of our results, we study a finite horizon consumption-investment problem for a jump-diffusion financial market consisting of one risk-free asset and one risky asset whose coefficients are assumed to depend on the state of a continuous time finite state Markov process. We provide a detailed study of the optimal strategies for this problem, for the economically relevant families of power utilities and logarithmic utilities.

**Keywords:** Stochastic Optimal Control, Jump-diffusion, Markov-switching, Optimal consumption-investment.

**AMS classification:** 93E20, 49L20, 91G10

## 1. INTRODUCTION

When speaking about the properties of real life as it expresses itself in nature, we learn that it is nonlinear rather than linear. One way to move between both is piecewise linearity, which we can easily generalize to piecewise-ness or “hybridicity” in general, e.g., in Engineering, specifically in Electrical Engineering and Electronics, and in Economics. But we also know that real life is, to some extent, discontinuous rather than continuous, e.g., in processes of Biology, Medicine, Engineering and Finance, so that phenomena of impulsiveness or “jumps” need to be taken into consideration, too. In a further step, we may allow both generalizations together in the sense that “regime” switches and jumps can occur randomly and, additionally, that the dynamics is stochastic in both the state space, that now permits jumps, and in some discrete space of states, which tell us in which discrete “mode” we are. An appropriate way to express this dynamics is stochastic hybrid systems with jumps, as represented by a stochastic differential equation (SDE) possibly equipped with conditional transition probabilities, or by a system of SDEs.

In this paper, we consider a decision making, or optimal control problem, subject to an underlying stochastic hybrid system with jumps. These problems are both relevant from the practical point of view and challenging mathematically (see, e.g., [31] and references therein for further details). To

have entered the areas of finance and insurance in the presence of stochastic hybrid systems with jumps herewith is a core achievement of those works and of ours. The time-continuous model in financial mathematics and actuarial sciences, expressed as portfolio optimization or, dually to that maximization, the minimization of expected costs, under finite maturity time (“finite horizon”), usually follows one of the following approaches: martingale duality methods, consisting of a static optimization problem and a representation problem, or stochastic control, consisting of a parametric optimization problem in the (deterministic) control space followed by a partial differential equation. In the latter approach, the necessary optimality conditions provided by the dynamic programming principle or the Hamilton-Jacobi-Bellman (HJB) equation need to be addressed. Herein, we are very close to Bellman’s dynamic programming technique, which we translate into our hybrid setting with jumps.

At this point we should mention that the dynamic programming technique was firstly introduced by Richard Bellman in the 1950s to deal with calculus of variations and optimal control problems [3, 4, 5, 6]. Further developments have been obtained since then by a number of scholars including Florentin [12, 13] and Kushner [22], among others. The approach introduced by Bellman relies on the description of the value function associated with a given optimal control problem through a backwards recursive relation, known currently as Bellman’s optimality principle. Under additional regularity conditions, it can be proved that such value function is also the solution of a partial differential equation, known as Hamilton-Jacobi-Bellman equation. A very complete treatment of the modern theory of optimal control problems can be found on the excellent monographs by Fleming and Soner [11], Yong and Zhou [33] and Oksendal and Sulem [26].

In the present paper, we demonstrate a dynamic programming principle for an optimal control problem with finite deterministic horizon and state variable dynamics given by a Markov-switching jump-diffusion stochastic differential equation. Moreover, we find the associated Hamilton-Jacobi-Bellman (HJB) equation, which in our case is a partial integro-differential equation due to the extra terms arising from the Lévy process terms and the Markov process driving the switching. The approach just described is distinct from the one of followed in [31]. The later paper introduces a numerical approach which is a multiple and prosperous extension of the one introduced by Koutsoukos in [21], concerning the aforementioned field, together with an application. Our contribution comes less from the numerical point of view, and is more centered on the theoretical framework, being of a widely analytical nature.

As an application of the abstract results presented here, we investigate a consumption-investment problem in a jump-diffusion financial market consisting of one risk-free and one risky asset whose coefficients are supposed to depend on the state of a continuous time finite state Markov process. Here, we present a detailed investigation of the optimal strategies, for power utilities and logarithmic utilities as well. The consumption-investment problem

was firstly studied by Merton in his seminal papers [24, 25]. This problem has been thoroughly studied ever since, including extensions to jump-diffusion financial markets (see, e.g., the series of papers by Framstad et al. [14, 15, 16] and references therein). In what concerns Markov-switching behaviour in economics and finance, this has been considered by Hamilton in [17] to explain shifts in growth rates of Gross National Product (GNP), Elliott et al. in [9, 10] to address problems related with option pricing and risk minimization, Zhang in [35] to determine optimal selling rules and by Zhang and Yin to deal with optimal asset allocation rules [36]. To the best of our knowledge, dynamic programming techniques have not yet been applied to the consumption-investment problem with an underlying Markov-switching jump-diffusion financial market.

We strongly believe that the methods and techniques developed here may be of interest to a wide range of topics in Applied Science, Computing and Engineering, eventually leading to future integration and comparison with other heuristic and model-free approaches and methods (see, e.g., [7, 8, 18, 30, 32, 34] and references therein).

This paper is organized as follows. In Section 2, we describe the setting we work with and formulate the problem we propose to address. Section 3 contains the dynamic programming principle and the HJB partial integro-differential equations, as well as the corresponding verification theorem and its proof. In Section 4, we address a consumption-investment problem and study the particular case of power utility functions and logarithmic utility functions. We conclude in Section 5.

## 2. SETUP AND PROBLEM FORMULATION

Let  $T > 0$  be a deterministic finite horizon and let  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  be a complete filtered probability space with filtration  $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$  satisfying the usual conditions, i.e.,  $\mathbb{F}$  is an increasing, right-continuous filtration and  $\mathcal{F}_0$  contains all  $P$ -null sets. For each  $d \in \mathbb{N}$ , let  $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$  and let  $\mathcal{B}_0^d$  be the Borel  $\sigma$ -field generated by the open subsets  $O$  of  $\mathbb{R}_0^d$  whose closure does not contain 0.

We will consider the following stochastic processes throughout this paper:

- (i) a standard  $M$ -dimensional Brownian motion  $W(t) = \{W(t) : t \in [0, T]\}$  defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ .
- (ii) a continuous time Markov process  $\{\alpha(t) : t \in [0, T]\}$  with a finite space state  $S = \{a_1, \dots, a_n\}$  and generator  $Q = (q_{ij})_{i,j \in S}$ . Let  $N_{ij}(t)$  denote the counting process given by

$$N_{ij}(t) = \sum_{0 < s \leq t} I_{\{\alpha(s_-)=i\}} I_{\{\alpha(s)=j\}},$$

where  $I_A$  denotes the indicator function of a set  $A$ . Note that  $N_{ij}(t)$  gives the number of jumps of the Markov process  $\alpha$  from state  $i$  to state  $j$  up to time  $t$ . Define the intensity process by

$$\lambda_{ij}(t) = q_{ij} I_{\{\alpha(t_-)=i\}},$$

and introduce the martingale process  $M_{ij}(t)$  given by

$$M_{ij}(t) = N_{ij}(t) - \int_0^t \lambda_{ij}(s) ds.$$

The process  $M_{ij}(t)$  is a purely discontinuous, square-integrable martingale which is null at the origin (see, e.g., [19, 29]).

- (iii) a  $K$ -dimensional Lévy process  $\{\eta(t) : t \in [0, T]\}$  with Poisson random measure  $J(t, A)$  with intensity (or Lévy measure)

$$\nu(A) = E[J(1, A)].$$

Recall that for each  $t > 0$ ,  $\omega \in \Omega$ ,  $J(t, \cdot)(\omega)$  is a counting measure on  $\mathcal{B}_0^K$ , and that for each  $A \in \mathcal{B}_0^K$ ,  $\{J(t, A) : t \in [0, T]\}$  is a Poisson process with intensity  $\nu(A)$ . For each  $t \in [0, T]$  and  $A \in \mathcal{B}_0^K$ , define the compensated Poisson random measure of  $\eta(\cdot)$  by

$$\tilde{J}(t, A) = J(t, A) - t\nu(A)$$

and notice that  $\{\tilde{J}(t, A) : t \in [0, T]\}$  is a martingale-valued measure [1]. Finally, notice that

$$\begin{aligned} \tilde{J}(dt, dz) &= (\tilde{J}_1(dt, dz_1), \dots, \tilde{J}_K(dt, dz_K)) \\ &= (J_1(dt, dz_1) - \nu_1(dz_1)dt, \dots, J_K(dt, dz_K) - \nu_K(dz_K)dt), \end{aligned}$$

where  $J_k$ ,  $k = 1, \dots, K$ , are independent Poisson random measures with Lévy measures  $\nu_k$  coming from  $K$  independent (1-dimensional) Lévy processes  $\eta_1, \dots, \eta_K$ .

We assume that the Brownian motion  $W(\cdot)$ , the Markov process  $\alpha(\cdot)$  and the Lévy process  $\eta(\cdot)$  are all independent and adapted to the filtration  $\mathbb{F}$ .

For each  $n \in \mathbb{N}$ , denote by  $L_{\mathcal{F}_t}^1([0, T]; \mathbb{R}^n)$  the set of all  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted  $\mathbb{R}^n$ -valued processes  $x(\cdot)$  such that

$$E \left[ \int_0^T \|x(t)\| dt \right] < \infty$$

and by  $L_{\mathcal{F}_T}^1(\Omega; \mathbb{R}^n)$  the set of  $\mathbb{R}^n$ -valued  $\mathcal{F}_T$ -measurable random variables  $Y$  such that  $E[\|Y\|]$  is finite, where  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^n$ . The spaces  $L_{\mathcal{F}_t}^1([0, T]; \mathbb{R}^n)$  and  $L_{\mathcal{F}_T}^1(\Omega; \mathbb{R}^n)$  are defined on the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ .

We introduce the following technical assumptions:

**(A1)**  $(U, d)$  is a Polish metric space, i.e., a complete separable metric space.

**(A2)** The maps  $f : [0, T] \times \mathbb{R}^N \times S \times U \rightarrow \mathbb{R}^N$ ,  $g : [0, T] \times \mathbb{R}^N \times S \times U \rightarrow \mathbb{R}^{N \times M}$ ,  $h : [0, T] \times \mathbb{R}^N \times S \times U \times \mathbb{R}_0^K \rightarrow \mathbb{R}^{N \times K}$ ,  $\Psi : [0, T] \times \mathbb{R}^N \times S \rightarrow \mathbb{R}$  and  $L : [0, T] \times \mathbb{R}^N \times S \times U \rightarrow \mathbb{R}$ , are such that:

- (i) each column  $h^{(k)}$  of the  $N \times K$  matrix  $h(t, x, a, u, z) = [h_{ik}]$  depends on  $z$  only through the  $k^{\text{th}}$  coordinate  $z_k$ , i.e.,

$$h^{(k)}(t, x, a, u, z) = h^{(k)}(t, x, a, u, z_k), \quad z = (z_1, \dots, z_K) \in \mathbb{R}_0^K;$$

- (ii) for each fixed  $a \in S$ ,  $f(\cdot, \cdot, a, \cdot)$ ,  $g(\cdot, \cdot, a, \cdot)$ ,  $h(\cdot, \cdot, a, \cdot, \cdot)$ ,  $\Psi(\cdot, \cdot, a)$ ,  $L(\cdot, \cdot, \cdot, a)$  are uniformly continuous and for each fixed  $a \in S$  and

$k = 1, \dots, K$  the function defined by

$$\int_{\mathbb{R}_0^1} h^{(k)}(\cdot, \cdot, a, \cdot, z_k) \nu_k(dz_k)$$

is also uniformly continuous;

(iii) for each fixed  $a \in S$ , there exists  $C > 0$  such that for  $\varphi(t, x, u) = f(t, x, a, u), g(t, x, a, u), \Psi(t, x, a), L(t, x, a, u)$ , we have

$$|\varphi(t, x, u) - \varphi(t, y, u)|^2 < C|x - y|^2,$$

$$|\varphi(t, 0, u)|^2 < C,$$

and for each  $k = 1, \dots, K$ , we have

$$\int_{\mathbb{R}_0^1} \left| h^{(k)}(t, x, a, u, z_k) - h^{(k)}(t, y, a, u, z_k) \right|^2 \nu_k(dz_k) < C|x - y|^2,$$

$$\int_{\mathbb{R}_0^1} \left| h^{(k)}(t, 0, a, u, z_k) \right|^2 \nu_k(dz_k) < C.$$

We consider a stochastic controlled system of the form

$$\begin{aligned} dX(t) &= f(t, X(t_-), \alpha(t_-), u(t_-)) dt + g(t, X(t_-), \alpha(t_-), u(t_-)) dW(t) \\ &\quad + \int_{\mathbb{R}_0^K} h(t, X(t_-), \alpha(t_-), u(t_-), z) \tilde{J}(dt, dz) \quad (t \in [0, T]), \quad (1) \\ X(0) &= x, \quad \alpha(0) = a, \end{aligned}$$

together with an objective functional of the form

$$\begin{aligned} J(x, a; u(\cdot)) &= E \left[ \int_0^T L(t, X_{0,x,a}(t; u(\cdot)), \alpha_{0,a}(t), u(t)) dt \right. \\ &\quad \left. + \Psi(T, X_{0,x,a}(T; u(\cdot)), \alpha_{0,a}(T)) \right], \quad (2) \end{aligned}$$

where  $(X_{0,x,a}(t; u(\cdot)), \alpha_{0,a}(t)) \in \mathbb{R}^N \times S$  denotes the state trajectory associated with a control trajectory  $u(\cdot)$  and starting from  $(x, a)$  when  $t = 0$ .

Note that the components of Eqn. (1) take the form

$$\begin{aligned} dX_i(t) &= f_i(t, X(t_-), \alpha(t_-), u(t_-)) dt \\ &\quad + \sum_{j=1}^M g_{ij}(t, X(t_-), \alpha(t_-), u(t_-)) dW_j(t) \\ &\quad + \sum_{k=1}^K \int_{\mathbb{R}_0^1} h_{ik}(t, X(t_-), \alpha(t_-), u(t_-), z_k) \tilde{J}_k(dt, dz_k), \end{aligned}$$

where  $f_i$  denotes the  $i^{\text{th}}$  component of  $f$ ,  $g_{ij}$  denotes the  $(i, j)$  entry of matrix  $g$ , and  $h_{ik}$  denotes the  $(i, k)$  entry of matrix  $h$ .

We say that the control process  $u : [0, T] \times \Omega \rightarrow U$  is a *strong admissible control* if  $u$  is measurable and  $\{\mathcal{F}_t\}$ -adapted, the stochastic differential

equation (1) has a unique strong solution and

$$\begin{aligned} E \left[ \int_0^T |L(t, X_{0,x,a}(t; u(\cdot)), \alpha_{0,a}(t), u(t))| dt \right] &< \infty, \\ E \left[ |\Psi(T, X_{0,x,a}(T; u(\cdot)), \alpha_{0,a}(T))| \right] &< \infty. \end{aligned}$$

We denote the set of all strong admissible controls by  $\mathcal{U}^s[0, T]$ .

The stochastic optimal control problem is to find a control  $u \in \mathcal{U}^s[0, T]$  which maximizes the objective functional  $J(x, a; u(\cdot))$  subject to the state equation (1) over the set of admissible controls  $\mathcal{U}^s[0, T]$ . Assumptions (A1) and (A2) provide an appropriate generalization to the setup under consideration here of standard assumptions in the Stochastic Differential Equations and Optimal Control Theory literature (see [1, 26, 28]). These conditions are used to ensure well-posedness of the Optimal Control problem associated with (1) and (2), the existence and uniqueness of solutions of (1) and the Markovian property of the solutions of (1), key for the strategy to be developed below.

We will use dynamic programming techniques to address the maximization problem described above. We remark that this is not the only available method to address this problem, e.g., one could use Pontryagin's stochastic maximum principle [33] or martingale duality methods [20]. However, the Markovian property of the solutions of (1), ensured by assumptions (A1) and (A2) combined with standard results in the theory of stochastic differential equations [1, 26, 28], make the dynamic programming method particularly suitable to address this problem. This is due to the fact that the Markovian property enables one to reduce the initial optimal control problem to a two-parameter family of related problems, from which one is able to extract a recursive relation leading to Bellman's optimality principle and the HJB equation. In order to proceed, we need to consider the weak formulation of the stochastic control problem under consideration as an auxiliary tool.

For any  $(s, y, i) \in [0, T] \times \mathbb{R}^N \times S$ , consider the state equation:

$$\begin{aligned} dX(t) &= f(t, X(t_-), \alpha(t_-), u(t_-)) dt + g(t, X(t_-), \alpha(t_-), u(t_-)) dW(t) \\ &\quad + \int_{\mathbb{R}_0^K} h(t, X(t_-), \alpha(t_-), u(t_-), z) \tilde{J}(dt, dz) \quad (t \in [s, T]), \quad (3) \\ X(s) &= y, \quad \alpha(s) = i, \end{aligned}$$

along with the objective functional

$$\begin{aligned} J(s, y, i; u(\cdot)) &= E \left[ \int_s^T L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad \left. + \Psi(T, X_{s,y,i}(T; u(\cdot)), \alpha_{s,i}(T)) \right], \quad (4) \end{aligned}$$

where  $(X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t)) \in \mathbb{R}^N \times S$  is the solution of Eqn. (3) associated with the control  $u(\cdot)$  and starting from  $(y, i)$  when  $t = s$ .

For each  $s \in [0, T)$ , we denote by  $\mathcal{U}^w[s, T]$  the set of 8-tuples

$$(\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot))$$

for which the following conditions hold:

- (i)  $(\Omega, \mathcal{F}, \mathbb{P})$  is a complete probability space;
- (ii)  $\mathbb{F} = \{\mathcal{F}_t^s\}_{t \geq s}$  is a right-continuous filtration;
- (iii)  $\{W(t) : t \in [s, T]\}$  is a  $M$ -dimensional standard Brownian motion defined on  $(\Omega, \mathcal{F}, P)$  over  $[s, T]$  and adapted to the filtration  $\mathbb{F}$ ;
- (iv)  $\{\alpha(t) : t \in [s, T]\}$  is a continuous time Markov process on  $(\Omega, \mathcal{F}, P)$  with finite state space  $S$  and adapted to the filtration  $\mathbb{F}$ ;
- (v)  $\{\eta(t) : t \in [s, T]\}$  is a  $K$ -dimensional Lévy process defined on  $(\Omega, \mathcal{F}, P)$  over  $[s, T]$  and adapted to the filtration  $\mathbb{F}$ ;
- (vi)  $u : [s, T] \times \Omega \rightarrow U$  is an  $\{\mathcal{F}_t^s\}_{t \geq s}$ -adapted process on  $(\Omega, \mathcal{F}, P)$ ;
- (vii) under  $u(\cdot)$ , for any  $y \in \mathbb{R}^N$  and  $i \in S$ , the stochastic differential equation (3) admits a unique solution  $X(\cdot)$  on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, P)$ .

We call  $\mathcal{U}^w[s, T]$  the set of *weak admissible controls*. Whenever the meaning is clear from the context, we will use the shorter notation  $u(\cdot) \in \mathcal{U}^w[s, T]$  for the 8-tuple  $(\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot)) \in \mathcal{U}^w[s, T]$ .

The optimal control problem under consideration can be restated in dynamic programming form as follows. For any  $(s, y, i) \in [0, T) \times \mathbb{R}^N \times S$ , find  $\bar{u}(\cdot) \in \mathcal{U}^w[s, T]$  such that

$$J(s, y, i; \bar{u}(\cdot)) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y, i; u(\cdot)). \quad (5)$$

Note that under assumptions (A1)-(A2), for any  $(s, y, i) \in [0, T) \times \mathbb{R}^N \times S$  and  $u(\cdot) \in \mathcal{U}^w[s, T]$ , SDE (3) admits a unique solution  $X(\cdot) = X_{s, y, i}(\cdot; u(\cdot))$  (see [1, 26, 28]), and the objective functional in Eqn. (4) is well-defined. Thus, the *value function* is well-defined by

$$\begin{cases} V(s, y, i) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y, i; u(\cdot)), \\ V(T, y, i) = \psi(T, y, i) \end{cases} \quad ((s, y, i) \in [0, T) \times \mathbb{R}^N \times S). \quad (6)$$

In Section 3, we will state and prove a dynamic programming principle for the value function  $V$ . Furthermore, we will obtain the Hamilton-Jacobi-Bellman equation associated with the optimal control problem under consideration and the corresponding verification theorem. Section 4 contains an application of the results obtained in Section 3 to an optimal consumption-investment problem for a financial market with asset prices described by Markov-switching linear jump-diffusions.

### 3. DYNAMIC PROGRAMMING PRINCIPLE AND HJB EQUATION

The goal of this section is to obtain a dynamic programming principle for (6) and to derive the associated HJB equation. We start by providing a well-known property of the value function that will be useful in the proof of the dynamic programming principle.

Recall that the paths of the state variable component  $X(\cdot)$  consist of that of an Lévy process interlaced by switches of the Markov process  $\{\alpha(t) : t \in [0, T]\}$ . Standard results on stochastic differential equations (see, e.g., [1])

guarantee that for any  $s \in [0, T]$ ,  $y, \hat{y} \in \mathbb{R}^N$ ,  $i \in S$ , and any  $w$ -admissible control  $u(\cdot) \in \mathcal{U}^w[s, T]$ , there exists  $C > 0$  such that

$$E \left( \sup_{t \in [s, T]} |X_{s, y, i}(t; u(\cdot)) - X_{s, \hat{y}, i}(t; u(\cdot))|^2 \right) \leq C|y - \hat{y}|^2. \quad (7)$$

Combining assumptions (A1) and (A2) with the estimate in Eqn. (7), it is possible to obtain the existence of a positive constant  $C'$  such that

$$|J(s, y, i; u(\cdot)) - J(s, \hat{y}, i; u(\cdot))|^2 \leq C'|y - \hat{y}|^2$$

for every  $u(\cdot) \in \mathcal{U}^w[s, T]$ . Taking the supremum in  $u(\cdot) \in \mathcal{U}^w[s, T]$ , we obtain the following property for the value function  $V$  defined in Eqn. (6).

**Lemma 3.1.** *Let conditions (A1)-(A2) hold. Then, there exists a constant  $C' > 0$  such that for every  $s \in [0, T]$ ,  $y, \hat{y} \in \mathbb{R}^N$  and  $i \in S$ , we have*

$$|V(s, y, i) - V(s, \hat{y}, i)|^2 \leq C'|y - \hat{y}|^2.$$

Let  $\mathbb{F}^s = \{\mathcal{F}_t^s\}_{t \geq s}$  be the filtration jointly generated by the Brownian motion  $W(\cdot)$ , the Markov process  $\alpha(\cdot)$  and the Lévy process  $\eta(\cdot)$  over a time interval  $[s, t]$  augmented by all the  $P$ -null sets in  $\mathcal{F}$ . For any  $s \in [0, T]$  and any  $(\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot)) \in \mathcal{U}^w[s, T]$ , take  $\hat{s} \in [s, T]$ . Since for every  $\hat{s} \in [s, T]$  the pair  $(X_{s, y, i}(\hat{s}; u(\cdot)), \alpha_{s, i}(\hat{s}))$  is  $\mathcal{F}_{\hat{s}}^s$ -measurable, the solutions  $X_{s, y, i}(t; u(\cdot))$  and  $X_{\hat{s}, X_{s, y, i}(\hat{s}; u(\cdot)), \alpha_{s, i}(\hat{s})}(t; u(\cdot))$  agree a.s. for every  $t \in [\hat{s}, T]$ .

**Lemma 3.2.** *Let  $(s, y, i) \in [0, T] \times \mathbb{R}^N \times S$  and  $u(\cdot) \in \mathcal{U}^w[s, T]$ . For any  $\hat{s} \in [s, T]$ , the following equality holds  $P$ -a.s.  $\omega$ :*

$$\begin{aligned} & J(\hat{s}, X_{s, y, i}(\hat{s}; u(\cdot)), \alpha_{s, i}(\hat{s}); u(\cdot)) \\ &= E \left[ \int_{\hat{s}}^T L \left( t, X_{\hat{s}, X_{s, y, i}(\hat{s}; u(\cdot)), \alpha_{s, i}(\hat{s})}(t; u(\cdot)), \alpha_{\hat{s}, \alpha_{s, i}(\hat{s})}(t), u(t) \right) dt \right. \\ & \quad \left. + \Psi \left( T, X_{\hat{s}, X_{s, y, i}(\hat{s}; u(\cdot)), \alpha_{s, i}(\hat{s})}(T; u(\cdot)), \alpha_{\hat{s}, \alpha_{s, i}(\hat{s})}(T) \right) \mid \mathcal{F}_{\hat{s}}^s \right] (\omega). \end{aligned}$$

We will now use the two lemmas above to prove a dynamic programming principle for the value function in Eqn. (6).

**Theorem 3.3** (Dynamic programming principle). *Assume that conditions (A1)-(A2) hold. Then, for any  $(s, y, i) \in [0, T] \times \mathbb{R}^N \times S$  we have that*

$$V(s, y, i) = \sup_{u \in \mathcal{U}^w[s, T]} E \left[ \int_s^{\hat{s}} L(t, X_{s, y, i}(t; u(\cdot)), \alpha_{s, i}(t), u(t)) dt \right. \\ \left. + V(\hat{s}, X_{s, y, i}(\hat{s}; u(\cdot)), \alpha_{s, i}(\hat{s})) \right] \quad (8)$$

for all  $\hat{s} \in [s, T]$ .

*Proof.* Denote the right-hand side of (8) by  $\bar{V}(s, y, i)$ . Start by noting that for any  $\epsilon > 0$  there exists  $u(\cdot) \in \mathcal{U}^w[s, T]$  such that

$$V(s, y, i) - \epsilon < J(s, y, i; u(\cdot)).$$



Recalling that the definition of the objective functional in Eqn. (4) and letting  $\hat{s} \in [s, T]$ , we obtain

$$\begin{aligned} V(s, y, i) - \epsilon &< E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad + E \left[ \int_{\hat{s}}^T L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad \left. \left. + \Psi(T, X_{s,y,i}(T; u(\cdot)), \alpha_{s,i}(T)) \mid \mathcal{F}_{\hat{s}}^s \right] \right]. \end{aligned}$$

Since the solutions of Eqn. (3) have the Markov property, we get

$$\begin{aligned} V(s, y, i) - \epsilon &< E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad + E \left[ \int_{\hat{s}}^T L(t, X_{\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})}(t; u(\cdot)), \alpha_{\hat{s}, \alpha_{s,i}(\hat{s})}(t), u(t)) dt \right. \\ &\quad \left. \left. + \Psi(T, X_{\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})}(T; u(\cdot)), \alpha_{\hat{s}, \alpha_{s,i}(\hat{s})}(T)) \mid \mathcal{F}_{\hat{s}}^s \right] \right]. \end{aligned}$$

Using the representation provided by Lemma 3.2, we obtain from the previous inequality that

$$\begin{aligned} V(s, y, i) - \epsilon &< E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad \left. + J(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s}); u(\cdot)) \right]. \end{aligned}$$

Hence, combining the definition of the value function of Eqn. (6) with the previous inequality, we get

$$\begin{aligned} V(s, y, i) - \epsilon &< E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad \left. + V(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})) \right] \leq \bar{V}(s, y, i). \end{aligned}$$

We will now prove the converse. Let  $(s, y, i) \in [0, T) \times \mathbb{R}^n \times S$  and fix an arbitrary control  $u(\cdot) \in \mathcal{U}^w[s, T]$ . Using Lemma 3.1 and the comments preceding its statement, for any  $\hat{s} \in [s, T]$  and  $\epsilon > 0$ , there exists a weak admissible control  $u_\epsilon(\cdot) \in \mathcal{U}^w[\hat{s}, T]$  such that

$$V(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})) - \epsilon \leq J(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s}); u_\epsilon(\cdot)). \quad (9)$$

Define the process

$$\tilde{u}_\epsilon(t, \omega) = \begin{cases} u(t, \omega) & \text{if } t \in [s, \hat{s}), \\ u_\epsilon(t, \omega) & \text{if } t \in [\hat{s}, T]. \end{cases}$$

Using the measurable selection theorem it is possible to guarantee that  $\tilde{u}_\epsilon(\cdot)$  is progressively measurable, and thus  $\tilde{u}_\epsilon(\cdot) \in \mathcal{U}^w[s, T]$ .

From the definition of the value function and the weak admissible control  $\tilde{u}_\epsilon(\cdot)$ , we get

$$\begin{aligned} V(s, y, i) &\geq J(s, y, i; \tilde{u}_\epsilon(\cdot)) \\ &= E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad + E \left[ \int_{\hat{s}}^T L\left(t, X_{\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})}(t; u_\epsilon(\cdot)), \alpha_{\hat{s}, \alpha_{s,i}(\hat{s})}(t), u_\epsilon(t)\right) dt \right. \\ &\quad \left. \left. + \Psi\left(T, X_{\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})}(T; u_\epsilon(\cdot)), \alpha_{\hat{s}, \alpha_{s,i}(\hat{s})}(T)\right) \mid \mathcal{F}_{\hat{s}}^s \right] \right]. \end{aligned}$$

Combining Lemma 3.2 with the previous inequality, we obtain

$$\begin{aligned} V(s, y, i) &\geq E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad \left. + J(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s}); u_\epsilon(\cdot)) \right]. \end{aligned}$$

Hence, inequality (9) implies that

$$\begin{aligned} V(s, y, i) &\geq E \left[ \int_s^{\hat{s}} L(t, X_{s,y,i}(t; u(\cdot)), \alpha_{s,i}(t), u(t)) dt \right. \\ &\quad \left. + V(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})) - \epsilon \right]. \end{aligned}$$

The proof is completed by taking the supremum over  $u(\cdot) \in \mathcal{U}^w[s, T]$ .  $\square$

**Proposition 3.4.** *Assume that conditions (A1)-(A2) hold. If the triple  $(\bar{X}(\cdot), \bar{\alpha}(\cdot), \bar{u}(\cdot))$  is optimal for problem of Eqn. (5), then*

$$\begin{aligned} V(t, \bar{X}(t), \bar{\alpha}(t)) &= E \left[ \int_t^T L(r, \bar{X}(r), \bar{u}(r), \bar{\alpha}(r)) dr \right. \\ &\quad \left. + \Psi(T, \bar{X}(T), \bar{\alpha}(T)) \mid \mathcal{F}_t^s \right] \quad P - \text{a.s.} \end{aligned}$$

for every  $t \in [s, T]$ .

*Proof.* Let  $(\bar{X}(\cdot), \bar{u}(\cdot), \bar{\alpha}(\cdot))$  be an optimal solution for problem of Eqn. (5). Then, we have that

$$V(s, y, i) = J(s, y, i; \bar{u}(\cdot)).$$

Letting  $t \in [s, T]$  and using Lemma 3.2, we obtain

$$J(s, y, i; \bar{u}(\cdot)) = E \left[ \int_s^t L(r, \bar{X}(r), \bar{\alpha}(r), \bar{u}(r)) dr + J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t)) \right].$$

Using the dynamic programming principle, Theorem 3.3, we obtain the following sequence of inequalities:

$$\begin{aligned} V(s, y, i) &\leq E \left[ \int_s^t L(r, \bar{X}(r), \bar{\alpha}(r), \bar{u}(r)) dr \right] + E [J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t))] \\ &\leq E \left[ \int_s^t L(r, \bar{X}(r), \bar{\alpha}(r), \bar{u}(r)) dr \right] + E [V(t, \bar{X}(t), \bar{\alpha}(t))] \\ &\leq V(s, y, i). \end{aligned}$$

Thus, all the inequalities above are indeed equalities. Therefore, we obtain that

$$E[J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t))] = E[V(t, \bar{X}(t), \bar{\alpha}(t))].$$

The result then follows by observing that

$$V(t, \bar{X}(t), \bar{\alpha}(t)) \geq J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t)) \quad P - \text{a.s.},$$

completing the proof.  $\square$

We will now use the dynamic programming principle to obtain the corresponding HJB equation, a sequence of partial integro-differential equation, indexed by the state of the Markov process  $\alpha(\cdot)$ , whose “solution” is the value function of the optimal control problem under consideration here. Let  $I \subseteq \mathbb{R}$  be an interval and denote by  $C^{1,2}(I \times \mathbb{R}^N; \mathbb{R})$  the set of all continuous functions  $V : I \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $V_t$ ,  $V_X$ , and  $V_{XX}$  are all continuous functions of  $(t, x) \in I \times \mathbb{R}^N$ . Moreover, let  $\text{tr}(A)$  denote the trace of a  $N \times N$  symmetric real matrix  $A$  and let  $\langle \cdot, \cdot \rangle$  denote the inner product in  $\mathbb{R}^n$ .

**Theorem 3.5.** *Suppose that conditions (A1)-(A2) hold and that the value function  $V$  is such that  $V(\cdot, \cdot, \alpha) \in C^{1,2}([0, T] \times \mathbb{R}^N; \mathbb{R})$  for every  $\alpha \in S$ . Then, for each  $\alpha \in S$ , the value function  $V(\cdot, \cdot, \alpha)$  defined on  $[0, T] \times \mathbb{R}^N$  is the solution of the Hamilton-Jacobi-Bellman equation:*

$$\begin{cases} V_t + \sup_{u \in U} \mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) = 0, \\ V(T, X, \alpha) = \Psi(T, X, \alpha) \quad ((t, X, \alpha) \in [0, T] \times \mathbb{R}^N \times S), \end{cases} \quad (10)$$

where the Hamiltonian function  $\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX})$  is defined by

$$\begin{aligned} \mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) &= L(t, X, \alpha, u) + \langle V_X(t, X, \alpha), f(t, X, \alpha, u) \rangle \\ &\quad + \frac{1}{2} \text{tr} (g^T(t, X, \alpha, u) V_{XX}(t, X, \alpha) g(t, X, \alpha, u)) \\ &\quad + \sum_{j \in S: j \neq \alpha} q_{\alpha j} (V(t, X, j) - V(t, X, \alpha)) \\ &\quad + \sum_{k=1}^K \int_{\mathbb{R}_0^1} W_k(t, X, \alpha, u, V, V_X, z_k) \nu_k(dz_k) \end{aligned}$$

and the auxiliary functions  $W_k(t, X, \alpha, u, V, V_X, z_k)$ ,  $k = 1, \dots, K$ , are defined by

$$\begin{aligned} W_k(t, X, \alpha, u, V, V_X, z_k) &= V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha) \\ &\quad - \langle V_X(t, X, \alpha), h^{(k)}(t, X, \alpha, u, z_k) \rangle. \end{aligned}$$

*Proof.* Fix  $(s, y, i) \in [0, T] \times \mathbb{R}^N \times S$  and  $u \in U$ . Let  $(X(\cdot), \alpha(\cdot))$  be the state trajectory associated with the control  $u(\cdot) \in \mathcal{U}^w[s, T]$ , where  $u(t)$  is constant such that  $u(t) \equiv u$ . Take  $\hat{s} \in [s, T]$ . Using Itô's formula (see Lemma A.1 in Appendix A), we obtain

$$\begin{aligned} V(\hat{s}, x(\hat{s}), \alpha(\hat{s})) - V(s, y, i) &= \int_s^{\hat{s}} a(t, X(t), \alpha(t), u(t)) dt \\ &+ \int_s^{\hat{s}} b(t, X(t), \alpha(t), u(t)) dW(t) + \int_s^{\hat{s}} c(t, X(t), \alpha(t), u(t)) dM(t) \\ &+ \int_s^{\hat{s}} \sum_{k=1}^K \int_{\mathbb{R}_0^1} d_k(t, X(t), \alpha(t), u(t), z_k) \tilde{J}_k(dt, dz_k), \end{aligned} \quad (11)$$

where

$$\begin{aligned} a(t, X, \alpha, u) &= V_t(t, X, \alpha) + \langle V_X(t, X, \alpha), f(t, X, \alpha, u) \rangle \\ &+ \frac{1}{2} \text{tr} (g^T(t, X, \alpha, u) V_{XX}(t, X, \alpha) g(t, X, \alpha, u)) \\ &+ \sum_{j \in S: j \neq \alpha} q_{\alpha j} (V(t, X, j) - V(t, X, \alpha)) \\ &+ \sum_{k=1}^K \int_{\mathbb{R}_0^1} W_k(t, X, \alpha, u, V, V_X, z_k) \nu_k(dz_k) \end{aligned} \quad (12)$$

and

$$\begin{aligned} b(t, X, \alpha, u) &= (V_X(t, X, \alpha))^T g(t, X, \alpha, u) \\ c(t, X, \alpha, u) &= \sum_{j \in S: j \neq \alpha} (V(t, X, j) - V(t, X, \alpha)) \\ d_k(t, X, \alpha, u, z_k) &= V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha). \end{aligned}$$

Since  $W(\cdot)$ ,  $M(\cdot)$  and  $\tilde{J}(\cdot, dz)$  are all martingales, the expected value of the stochastic integrals against  $dW$ ,  $dM$ , and  $\tilde{J}$  all vanish. Using the observation above, dividing the identity of Eqn. (11) by  $\hat{s} - s$ , where  $s \neq \hat{s}$ , and taking the expectation in both sides, we obtain

$$\begin{aligned} \frac{E[V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) - V(s, y, i)]}{\hat{s} - s} &= \\ &= \frac{1}{\hat{s} - s} E \left[ \int_s^{\hat{s}} a(t, X(t), \alpha(t), u(t)) dt \right]. \end{aligned} \quad (13)$$

Using Theorem 3.3, we get that

$$\begin{aligned} -\frac{E[V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) - V(s, y, i)]}{\hat{s} - s} &\geq \\ &= \frac{1}{\hat{s} - s} E \left[ \int_s^{\hat{s}} L(t, X(t), \alpha(t), u(t)) dt \right]. \end{aligned}$$

Combining identity of Eqn. (13) with the last inequality, we obtain

$$\frac{1}{\hat{s} - s} E \left[ \int_s^{\hat{s}} L(t, X(t), \alpha(t), u(t)) + a(t, X(t), \alpha(t), u(t)) dt \right] \leq 0.$$

Letting  $\hat{s} \downarrow s$ , we conclude that

$$V_t(s, y, i) + \mathcal{H}(s, y, i, u, V(s, y, i), V_X(s, y, i), V_{XX}(s, y, i)) \leq 0$$

for every  $u \in U$ . This results in

$$V_t(s, y, i) + \sup_{u \in U} \mathcal{H}(s, y, i, u, V(s, y, i), V_X(s, y, i), V_{XX}(s, y, i)) \leq 0.$$

On the other hand, for any  $\epsilon > 0$  and any  $\hat{s} \in (s, T]$  with  $\hat{s} - s$  small enough, there exists a  $\tilde{u}(\cdot) := u_{\epsilon, \hat{s}}(\cdot) \in \mathcal{U}^w[s, T]$  such that

$$V(s, y, i) - \epsilon(\hat{s} - s) \leq E \left[ \int_s^{\hat{s}} L(t, X(t), \alpha(t), \tilde{u}(t)) dt + V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) \right].$$

Rearranging terms in the previous inequality, we get

$$\begin{aligned} \epsilon \geq & - \frac{E[V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) - V(s, y, i)]}{\hat{s} - s} \\ & - \frac{1}{\hat{s} - s} E \left[ \int_s^{\hat{s}} L(t, X(t), \alpha(t), \tilde{u}(t)) dt \right]. \end{aligned}$$

Combining identity of Eqn. (13) with the previous inequality, we obtain

$$\epsilon \geq - \frac{1}{\hat{s} - s} E \left[ \int_s^{\hat{s}} a(t, X(t), \alpha(t), \tilde{u}(t)) dt + \int_s^{\hat{s}} L(t, X(t), \alpha(t), \tilde{u}(t)) dt \right].$$

Therefore, we conclude that

$$\epsilon \geq - \frac{1}{\hat{s} - s} E \left[ \int_s^{\hat{s}} V_t(t, X(t), \alpha(t)) + \tilde{\mathcal{H}}(t, X(t), \alpha(t), \tilde{u}(t)) dt \right],$$

where  $\tilde{\mathcal{H}}(t, X(t), \alpha(t), \tilde{u}(t))$  is notation for

$$\mathcal{H}(t, X(t), \alpha(t), \tilde{u}(t), V(t, X(t), \alpha(t)), V_X(t, X(t), \alpha(t)), V_{XX}(t, X(t), \alpha(t))).$$

Using the uniform continuity in assumption (A2), we obtain

$$-\epsilon \leq V_t(s, y, i) + \sup_{u \in U} \mathcal{H}(s, y, i, u, V(s, y, i), V_X(s, y, i), V_{XX}(s, y, i)),$$

completing the proof.  $\square$

Before proceeding to the statement and proof of the next result, let us introduce the notation  $\tilde{H}(s, y, i, u)$  for the function

$$\tilde{H}(s, y, i, u) = \mathcal{H}(s, y, i, u, V(s, y, i), V_X(s, y, i), V_{XX}(s, y, i)).$$

We will now state the verification theorem associated with the dynamic programming principle, Theorem 3.3, and the corresponding HJB equation (10).

**Proposition 3.6** (Verification Theorem). *Assume that conditions (A1)-(A2) hold and that  $V(\cdot, \cdot, a) \in C^{1,2}([0, T] \times \mathbb{R}^N; \mathbb{R})$  for each  $a \in S$ . Let  $V(s, y, i)$  be a solution of the HJB equation (10). Then, the inequality*

$$V(s, y, i) \geq J(s, y, i; u(\cdot))$$

*holds for every  $u(\cdot) \in \mathcal{U}^w[s, T]$  and  $(s, y, i) \in [0, T] \times \mathbb{R}^N \times S$ . Furthermore, an admissible pair  $(\bar{X}(\cdot), \bar{\alpha}(\cdot), \bar{u}(\cdot))$  is optimal for (5) if and only if the equality*

$$V_t(t, \bar{X}(t), \bar{\alpha}(t)) + \tilde{\mathcal{H}}(t, \bar{X}(t), \bar{\alpha}(t), \bar{u}(t)) = 0 \quad (14)$$

holds for a.e.  $t \in [s, T]$  and  $\mathbb{P} - \text{a.s.}$

*Proof.* For any  $u(\cdot) \in \mathcal{U}^w[s, T]$  and corresponding state trajectory  $(X(\cdot), \alpha(\cdot))$ , using Itô's formula we get

$$V(s, y, i) = E \left[ \Psi(T, X(T), \alpha(T)) - \int_s^T a(t, X(t), \alpha(t), u(t)) dt \right],$$

where  $a(t, X, \alpha, u)$  is as given in (12). Using (4) and the definition of the Hamiltonian function in the statement of Theorem 3.5, the last equality may be written as

$$V(s, y, i) = J(s, y, i; u(\cdot)) - E \left[ \int_s^T V_t(t, X(t), \alpha(t)) + \tilde{\mathcal{H}}(t, X(t), \alpha(t), u(t)) dt \right]. \quad (15)$$

Using the HJB equation (10), we conclude that

$$V(s, y, i) \geq J(s, y, i; u(\cdot)),$$

completing the proof of the first part of the theorem.

To prove the second part of the theorem, let  $(\bar{X}(\cdot), \bar{u}(\cdot), \bar{\alpha}(\cdot))$  be an optimal solution for (5). Applying equality (15) to  $(\bar{X}(\cdot), \bar{u}(\cdot), \bar{\alpha}(\cdot))$ , we get

$$V(s, y, i) \geq J(s, y, i; \bar{u}(\cdot)) - E \left[ \int_s^T V_t(t, \bar{X}(t), \bar{\alpha}(t)) + \tilde{\mathcal{H}}(t, \bar{X}(t), \bar{\alpha}(t), \bar{u}(t)) dt \right].$$

The desired result follows immediately from the fact that

$$V_t(t, \bar{X}(t), \bar{\alpha}(t)) + \tilde{\mathcal{H}}(t, \bar{X}(t), \bar{\alpha}(t), \bar{u}(t)) \leq 0,$$

due to the HJB equation (10).  $\square$

#### 4. APPLICATION TO CONSUMPTION-INVESTMENT PROBLEMS

We will now discuss the application of the results in the previous section to the topic of optimal consumption-investment problems in Markov-switching jump-diffusion financial markets.

The Brownian motion can be interpreted as small random shocks that influence the market dynamics. The jump process models large changes in the asset price, which can be interpreted as a consequence of abrupt market events or news with large impact. Finally, a transition of the Markov process to a different state models a shift in the financial market behaviour, e.g., a shift from a “bull” day to a “bear” day, or a shift from a “volatile” day to a “choppy” trading day. Similarly, for a longer term point of view, the shifts in the Markov process may be seen as modelling global changes in market behaviours such as bull or bear market periods, as well as range-bound markets.

**4.1. Problem formulation.** Consider the setup introduced in Section 2. Namely, let  $T > 0$  be a deterministic finite horizon and let  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  be a complete filtered probability space. Define on  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  a standard one-dimensional Brownian motion  $W(t) = \{W(t) : t \in [0, T]\}$ , a continuous time Markov process  $\{\alpha(t) : t \in [0, T]\}$  with a finite space state  $S = \{a_1, \dots, a_n\}$  and a generator  $Q = (q_{ij})_{i,j \in S}$ , and a one-dimensional Lévy process  $\{\eta(t) :$

$t \in [0, T]$  with Poisson random measure  $J(t, A)$  with intensity  $\nu(A) = E[J(1, A)]$ . Throughout this section, we assume that the three stochastic processes,  $\alpha(\cdot)$ ,  $W(\cdot)$  and  $\eta(\cdot)$ , are independent.

Using the stochastic processes described above, we define a continuous-time financial market consisting of one risk-free asset and one risky-asset. More precisely, we assume that the prices of the risk-free asset  $\{S_0(t) : t \in [0, T]\}$  and the risky asset  $\{S_1(t) : t \in [0, T]\}$  evolve according to the stochastic differential equations

$$\begin{aligned} dS_0(t) &= r(t, \alpha(t_-))S_0(t_-)dt, \\ dS_1(t) &= \mu(t, \alpha(t_-))S_1(t_-)dt + \sigma(t, \alpha(t_-))S_1(t_-)dW(t) \\ &\quad + S_1(t_-) \int_{\mathbb{R}_0} h(t, \alpha(t_-), z) \tilde{J}(dt, dz) \end{aligned} \quad (16)$$

with positive initial conditions  $S_0(0) = s_0$  and  $S_1(0) = s_1$ . Note that the financial market coefficients depend both on time and the state of the Markov process  $\{\alpha(t) : t \in [0, T]\}$ . More precisely, we assume that the riskless interest rate  $r(t, a)$ , the risky-asset appreciation rate  $\mu(t, a)$  and the risky-asset volatility  $\sigma(t, a)$ , are deterministic continuous functions on the interval  $[0, T]$  for every fixed  $a \in S$ , and that  $h(t, a, z)$  is a deterministic continuous function of  $t \in [0, T]$  for every  $z \in \mathbb{R}_0^1$  and  $a \in S$ . Additionally, we assume that the risk-free interest rate  $r(t, a)$  is positive for every  $(t, a) \in [0, T] \times S$ .

In order to ensure that the risky asset price  $S_1(t)$  remains positive for every  $t \in [0, T]$ , we impose that

$$h(t, a, z) > -1,$$

for every  $t \in [0, T]$  and  $a \in S$ . Finally, we also assume that

$$E \left[ \int_0^T \left( |\sigma(t, a)|^2 + \int_{\mathbb{R}_0^1} |h(t, a, z)|^2 \nu(dz) \right) dt \right] < \infty$$

for every  $a \in S$ .

We now introduce the control variables. The *consumption process*  $\{c(t) : t \in [0, T]\}$  is a  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -progressively measurable non-negative process satisfying the following integrability condition for the investment horizon  $T > 0$ :

$$\int_0^T c(t) dt < \infty \quad \text{a.s..}$$

Let  $\theta(t)$  denote the fraction of the agent's wealth allocated to the risky asset  $S_1$  at time  $t \in [0, T]$ . We assume that  $\{\theta(t) : t \in [0, T]\}$  is  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -progressively measurable and that, for the fixed maximum investment horizon  $T > 0$ , we have that

$$\int_0^T |\theta(t)|^2 dt < \infty \quad \text{a.s..}$$

Clearly, the agent invests  $1 - \theta(t)$  of her wealth on the risk-free asset  $S_0$ . We now define the *wealth process*  $X(t)$ , for  $t \in [0, T]$ , through the stochastic

differential equation

$$\begin{aligned} dX(t) = & \left( -c(t) + \left( (1 - \theta(t))r(t, \alpha(t_-)) + \theta(t)\mu(t, \alpha(t_-)) \right) X(t_-) \right) dt \\ & + \theta(t)X(t_-) \left( \sigma(t, \alpha(t_-))dW(t) + \int_{\mathbb{R}_0^d} h(t, \alpha(t_-), z)\tilde{J}(dt, dz) \right) \end{aligned} \quad (17)$$

with initial conditions  $X(0) = x$  and  $\alpha(0) = a$  representing, respectively, the initial wealth and the initial state of the Markov process  $\alpha(\cdot)$ .

Let us denote by  $\mathcal{A}(x, a)$  the set of all admissible decision strategies, i.e., all admissible choices for the control variables  $(c, \theta) \in [0, +\infty) \times [0, 1]$  such that the wealth process defined by (17) is square integrable with respect to  $dt \times dP$  over  $[0, T] \times \Omega$ . The dependence of  $\mathcal{A}(x, a)$  on  $(x, a) \in \mathbb{R} \times S$  denotes the restriction imposed on the wealth process by the boundary conditions  $X(0) = x$  and  $\alpha(0) = a$ . Similarly, let us denote by  $\mathcal{A}(t, x, a)$  the set of all admissible decision strategies  $(c, \theta) \in [0, +\infty) \times [0, 1]$  for the dynamics of the wealth process of Eqn. (17) with boundary condition  $X(t) = x$  and  $\alpha(t) = a$ .

The consumption-investment problem is to find consumption and investment strategies,  $(c, \theta) \in \mathcal{A}(x, a)$  which maximize the expected utility

$$\begin{aligned} J(x, a; c(\cdot), \theta(\cdot)) = & E \left[ \int_0^T U(t, c(t), \alpha_{0,a}(t)) dt \right. \\ & \left. + \Psi(T, X_{0,x,a}(T; c(\cdot), \theta(\cdot)), \alpha_{0,a}(T)) \right]. \end{aligned}$$

Here,  $U(t, c, a)$  is the utility obtained from a consumption level  $c \in [0, \infty)$  at time  $t$  when the state of the Markov process  $\alpha(\cdot)$  is  $a$ ,  $\Psi(T, x, a)$  is the utility obtained from holding wealth  $x$  at time  $T$  when the state of the Markov process  $\alpha(\cdot)$  is  $a$ , and  $(X_{s,y,i}(t; c(\cdot), \theta(\cdot)), \alpha_{s,i}(t)) \in \mathbb{R} \times S$  is the solution of Eqn. (17) associated with the strategies  $c(\cdot), \theta(\cdot)$  and initial condition from  $(x, a)$  when  $t = 0$ .

Proceeding as described in Section 2, we introduce the expected utility

$$\begin{aligned} J(s, y, i; c(\cdot), \theta(\cdot)) = & E \left[ \int_s^T U(t, c(t), \alpha_{s,i}(t)) dt \right. \\ & \left. + \Psi(T, X_{s,y,i}(T; c(\cdot), \theta(\cdot)), \alpha_{s,i}(T)) \right], \end{aligned} \quad (18)$$

where  $(X_{s,y,i}(t; c(\cdot), \theta(\cdot)), \alpha_{s,i}(t)) \in \mathbb{R} \times S$  is the solution of Eqn. (17) associated with the strategies  $c(\cdot), \theta(\cdot)$  and starting from  $(y, i)$  when  $t = s$ . Using dynamic programming techniques it is possible to obtain a rather complete description for the behaviour of the maximum expected utility, or value function, given by

$$\begin{cases} V(t, x, a) = \sup_{(c,\theta) \in \mathcal{A}(t,x,a)} J(t, x, a; c(\cdot), \theta(\cdot)), \\ V(T, x, a) = \psi(T, x, a) \end{cases} \quad ((t, x, a) \in [0, T] \times \mathbb{R} \times S).$$



Indeed, in the next sections we will use the theory built for the proof of Theorems 3.3 and 3.5 to study the optimal strategies for the expected utility in Eqn. (18) both in the case where the utility functions are of power type and in the case of logarithmic utilities. Our choice for these families of utility functions is related with the fact that both have a constant coefficient of relative risk aversion or Arrow-Pratt-De Finetti measure of relative risk-aversion (firstly introduced in [2, 27]). This property makes such utility functions key examples in Economic Theory [23] and makes the computation of closed form solutions for the HJB equation easier in our case. Alternative choices could include the exponential family of utility functions, known to have constant coefficient of absolute risk aversion, or even families of non-concave utility functions, thoroughly used in Behavioural Economics. While we expect that all the qualitative properties of the results contained in the next sections still hold for the family of exponential utility functions, this is not necessarily the case for non-concave utility functions, which require further study.

**4.2. The case of power utility functions.** In this subsection we assume that the utility functions belong to the following class of power utilities. Let

$$U(t, c, a) = e^{-\rho t} \frac{c^{\gamma_a}}{\gamma_a}, \quad \Psi(t, x, a) = e^{-\rho t} \frac{x^{\gamma_a}}{\gamma_a}, \quad (19)$$

where  $\gamma_a \in (0, 1)$  is the risk aversion coefficient associated with the state of the Markov process  $\alpha(t) = a \in S$ , and  $\rho > 0$  is the discount rate.

In the next theorem, we will compute the optimal strategies for the class of discounted utility functions of Eqn. (19). Before providing the precise statement, let us introduce the function  $F : [0, 1] \times [0, T] \times S \rightarrow \mathbb{R}$  given by

$$\begin{aligned} F(\theta; t, a) &= \gamma_a \left[ r(t, a) + \theta(\mu(t, a) - r(t, a)) - \frac{1}{2}(1 - \gamma_a)\theta^2\sigma^2(t, a) \right] \\ &\quad + \int_{\mathbb{R}_0^1} \left( (1 + \theta h(t, a, z))^{\gamma_a} - 1 - \gamma_a \theta h(t, a, z) \right) \nu(dz) \end{aligned} \quad (20)$$

and note that

$$\begin{aligned} F'(\theta; t, a) &= \gamma_a \left[ \mu(t, a) - r(t, a) - (1 - \gamma_a)\theta\sigma^2(t, a) \right. \\ &\quad \left. + \int_{\mathbb{R}_0^1} \left( (1 + \theta h(t, a, z))^{\gamma_a - 1} - 1 \right) h(t, a, z) \nu(dz) \right], \end{aligned}$$

where the derivative is taken with respect to  $\theta$ .

**Theorem 4.1.** *The maximum expected utility associated with Eqn. (18) and the discounted utility functions of Eqn.(19) is given by*

$$V(t, x, a) = \xi_a(t) \frac{x^{\gamma_a}}{\gamma_a}, \quad (21)$$

the corresponding optimal strategies are of the form

$$c^*(t, x, a) = x(e^{\rho t} \xi_a(t))^{-1/(1-\gamma_a)}$$

and

$$\theta^*(t, a) = \begin{cases} 1, & \text{if } \mu(t, a) > r(t, a) \text{ and } F'(1; t, a) \geq 0, \\ \hat{\theta}(t, a), & \text{if } \mu(t, a) > r(t, a) \text{ and } F'(1; t, a) < 0, \\ 0, & \text{if } \mu(t, a) \leq r(t, a), \end{cases}$$

where  $\hat{\theta}(t, a)$  is the unique solution of  $F'(\theta; t, a) = 0$  in  $(0, 1)$  and  $\xi_a(t)$ ,  $a \in S$ , are the solutions of the following coupled ordinary differential equations terminal value problem

$$\begin{aligned} \xi'_a(t) + (1 - \gamma_a)e^{-\rho t/(1-\gamma_a)}\xi_a(t)^{-\gamma_a/(1-\gamma_a)} + F(\theta^*(t, a); t, a)\xi_a(t) \\ + \sum_{j \in S: j \neq a} q_{aj}(\xi_j(t) - \xi_a(t)) = 0, \end{aligned} \quad (22)$$

$$\xi_a(T) = e^{-\rho T}.$$

*Proof.* Assume for the time being that the conditions of Theorem 3.5 hold. The Hamiltonian function  $\mathcal{H}$  associated with the expected utility (18) and the discounted utility functions (19) is defined by

$$\begin{aligned} \mathcal{H}(t, x, a, c, \theta, V, V_x, V_{xx}) = \\ e^{-\rho t} \frac{c^{\gamma_a}}{\gamma_a} + \left( -c + \left( r(t, a) + \theta(\mu(t, a) - r(t, a)) \right) x \right) V_x(t, x, a) \\ + \frac{x^2}{2} (\theta \sigma(t, a))^2 V_{xx}(t, x, a) + \sum_{j \in S: j \neq a} q_{aj} (V(t, x, j) - V(t, x, a)) \\ + \int_{\mathbb{R}_0^1} \left( V(t, x + \theta x h(t, a, z)) - V(t, x, a) - \theta x V_x(t, x, a) h(t, a, z) \right) \nu(dz) \end{aligned}$$

and the Hamilton-Jacobi-Bellman equation is defined by

$$V_t + \sup_{(c, \theta) \in [0, \infty) \times [0, 1]} \mathcal{H}(t, x, a, c, \theta, V, V_x, V_{xx}) = 0.$$

Considering an ansatz of the form of Eqn. (21) and substituting in the HJB equation above, we get

$$\begin{aligned} \xi'_a(t) \frac{x^{\gamma_a}}{\gamma_a} + \sup_{(c, \theta) \in [0, \infty) \times [0, 1]} \left\{ e^{-\rho t} \frac{c^{\gamma_a}}{\gamma_a} \right. \\ \left. + \left( -c + \left( r(t, a) + \theta(\mu(t, a) - r(t, a)) \right) x \right) \xi_a(t) x^{\gamma_a - 1} \right. \\ \left. + \frac{\theta^2}{2} \sigma^2(t, a) (\gamma_a - 1) \xi_a(t) x^{\gamma_a} + \frac{x^{\gamma_a}}{\gamma_a} \sum_{j \in S: j \neq a} q_{aj} (\xi_j(t) - \xi_a(t)) \right. \\ \left. + \xi_a(t) \frac{x^{\gamma_a}}{\gamma_a} \int_{\mathbb{R}_0^1} \left( (1 + \theta h(t, a, z))^{\gamma_a} - 1 - \gamma_a \theta h(t, a, z) \right) \nu(dz) \right\} = 0. \end{aligned} \quad (23)$$

Note that the optimization problem in Eqn. (23) breaks down into two independent optimization problems and its solution can be obtained in a sequential way. We start by optimizing the Eqn. (23) with respect to  $c$ , before proceeding to optimize with respect to the variable  $\theta$ .

Since  $\gamma_a \in (0, 1)$  for every  $a \in S$ , the quantity to be maximized in (23) is strictly concave with respect to the control variable  $c$ . Indeed, the first order condition associated with the optimization problem above provides a maximizer  $c^*(t, x, a)$ , which is given by

$$c^*(t, x, a) = x(e^{\rho t} \xi_a(t))^{-1/(1-\gamma_a)}.$$

Replacing  $c$  by  $c^*(t, x, a)$  in Eqn. (23) and factoring out the term  $x^{\gamma_a}/\gamma_a$ , we obtain that

$$\xi'_a(t) + \sup_{\theta \in [0,1]} \left\{ (1 - \gamma_a) e^{-\rho t/(1-\gamma_a)} \xi_a(t)^{-\gamma_a/(1-\gamma_a)} + F(\theta; t, a) \xi_a(t) + \sum_{j \in S: j \neq a} q_{aj} (\xi_j(t) - \xi_a(t)) \right\} = 0, \quad (24)$$

where  $F(\theta; t, a)$  is as introduced in Eqn. (20). Note that the first order condition with respect to  $\theta$  is just  $F'(\theta; t, a) = 0$  and that since  $0 < \gamma_a < 1$ , the second derivative of  $F(\theta; t, a)$  with respect to  $\theta$ , represented by

$$F''(\theta; t, a) = -\gamma_a(1 - \gamma_a) \left[ \sigma^2 + \int_{\mathbb{R}_0^1} (1 + \theta h(t, a, z))^{\gamma_a-2} h^2(t, a, z) \nu(dz) \right],$$

is negative for every  $\theta \in [0, 1]$ . Taking into account the constraint  $\theta \in [0, 1]$  and the concavity of  $F(\theta; t, a)$ , we conclude that the maximization problem in (24) has a unique solution  $\theta^*(t, a)$ . Moreover, from the definition of the function  $F(\theta; t, a)$ , it is possible to check that

- i) if  $\mu(t, a) - r(t, a) > 0$  and  $F(1; t, a) < 0$ , then there exists  $\hat{\theta}(t, a) \in (0, 1)$  such that  $F'(\hat{\theta}(t, a); t, a) = 0$  and, consequently,  $\theta^*(t, a) = \hat{\theta}(t, a)$ ;
- ii) if  $\mu(t, a) - r(t, a) > 0$  and  $F(1; t, a) \geq 0$ , then  $\theta^*(t, a) = 1$ ;
- iii) if  $\mu(t, a) - r(t, a) \leq 0$ , then  $\theta^*(t, a) = 0$ .

We will now check that the system of ordinary differential equations in Eqn. (22) admits a unique positive solution. For that purpose, we consider the new variable

$$z_a(t) = (e^{\rho t} \xi_a(t))^{1/(1-\gamma_a)} : \quad (a \in S).$$

A straightforward computation shows that  $z_a(t)$  must satisfy the system of ordinary differential equations

$$\begin{aligned} z'_a(t) &= \frac{\rho}{1 - \gamma_a} z_a(t) - 1 - \frac{1}{1 - \gamma_a} + F(\theta^*(t, a); t, a) z_a(t) \\ &\quad - z_a^{\gamma_a} \sum_{j \in S: j \neq a} q_{aj} (z_j^{1-\gamma_a}(t) - z_a^{1-\gamma_a}(t)) = 0, \quad (25) \end{aligned}$$

$$z_a(T) = 1.$$

A simple analysis shows that the solutions of Eqn. (25) are positive and bounded away from zero for every  $t \in [0, T]$ . Clearly, the same statement also holds for the solutions  $\xi_a(t)$  of Eqn. (22).  $\square$

A few remarks concerning the optimal strategies determined in Theorem 4.1 are in order. We start by noting that  $\theta^*$  does not depend on the wealth

$x \in \mathbb{R}$ , and depends on time  $t \in [0, T]$  only through its coefficients dependence (i.e. for a financial market model with constant coefficients,  $\theta^*$  is independent of  $t$ ). However, as can be seen from its form,  $\theta^*$  depends strongly on the state  $a \in S$  of the Markov process state  $\alpha(\cdot)$ .

As noted in [14], the occurrence of jumps in the financial market leads to more conservative optimal strategies when compared with purely diffusive markets. More precisely, agents allocate a smaller proportion of their wealth to the risky asset and consume more compared to their current wealth.

In what concerns the optimal consumption  $c^*$ , it is clearly that it is increasing with wealth and that, for choices of coefficients compatible with standard financial market behaviour,  $c^*$  is increasing with time  $t$ .

**4.3. The case of logarithmic utility functions.** Consider the utility functions

$$U(t, c) = e^{-\rho t} \ln c, \quad \Psi(t, x) = e^{-\rho t} \ln x, \quad (26)$$

where  $\rho > 0$  is the discount rate.

In the next theorem, we will compute the optimal strategies for the class of discounted logarithmic utility functions of Eqn. (26). Before providing the precise statement, let us introduce the function  $F : [0, 1] \times [0, T] \times S \rightarrow \mathbb{R}$  given by

$$\begin{aligned} F(\theta; t, a) &= r(t, a) + \theta(\mu(t, a) - r(t, a)) - \frac{1}{2}\theta^2\sigma^2(t, a) \\ &+ \int_{\mathbb{R}_0^1} \left( \ln(1 + \theta h(t, a, z)) - \theta h(t, a, z) \right) \nu(dz) \end{aligned} \quad (27)$$

and note that

$$F'(\theta; t, a) = \mu(t, a) - r(t, a) - \theta\sigma^2(t, a) - \int_{\mathbb{R}_0^1} \frac{\theta h^2(t, a, z)}{(1 + \theta h(t, a, z))} \nu(dz),$$

where the derivative is taken with respect to  $\theta$ .

**Theorem 4.2.** *The maximum expected utility associated with Eqn. (18) and the discounted logarithmic utility functions (26) is defined by*

$$V(t, x, a) = \xi(t) \ln x + \zeta_a(t), \quad (28)$$

the corresponding optimal strategies are of the form

$$c^*(t, x, a) = \frac{e^{-\rho t}}{\xi(t)} x$$

and

$$\theta^*(t, a) = \begin{cases} 1, & \text{if } \mu(t, a) > r(t, a) \text{ and } F'(1; t, a) \geq 0, \\ \hat{\theta}(t, a), & \text{if } \mu(t, a) > r(t, a) \text{ and } F'(1; t, a) < 0, \\ 0, & \text{if } \mu(t, a) \leq r(t, a), \end{cases}$$

where  $\hat{\theta}(t, a)$  is the unique solution of  $F'(\theta; t, a) = 0$  in  $(0, 1)$ ,  $\xi(t)$  is determined by

$$\xi(t) = \frac{(\rho - 1)}{\rho} e^{-\rho T} + \frac{e^{-\rho t}}{\rho}$$

and  $\zeta_a(t)$ ,  $a \in S$ , are the solutions of the following coupled ordinary differential equations terminal value problem

$$\begin{aligned} \zeta'_a(t) &- e^{-\rho t} (\rho t + \ln \xi(t) + 1) + F(\theta^*(t, a); t, a) \xi(t) \\ &+ \sum_{j \in S: j \neq a} q_{aj} (\zeta_j(t) - \zeta_a(t)) = 0, \\ \zeta_a(T) &= 0. \end{aligned}$$

*Proof.* Assume that the conditions of Theorem 3.5 hold. The Hamiltonian function  $\mathcal{H}$  associated with the expected utility of Eqn. (18) and the discounted logarithmic utility functions (26) is defined by

$$\begin{aligned} \mathcal{H}(t, x, a, c, \theta, V, V_x, V_{xx}) = & \\ & e^{-\rho t} \ln c + \left( -c + \left( r(t, a) + \theta(\mu(t, a) - r(t, a)) \right) x \right) V_x(t, x, a) \\ & + \frac{x^2}{2} (\theta \sigma(t, a))^2 V_{xx}(t, x, a) + \sum_{j \in S: j \neq a} q_{aj} (V(t, x, j) - V(t, x, a)) \\ & + \int_{\mathbb{R}_0^1} \left( V(t, x + \theta x h(t, a, z)) - V(t, x, a) - \theta x V_x(t, x, a) h(t, a, z) \right) \nu(dz) \end{aligned}$$

and the Hamilton-Jacobi-Bellman equation is determined by

$$V_t + \sup_{(c, \theta) \in [0, \infty) \times [0, 1]} \mathcal{H}(t, x, a, c, \theta, V, V_x, V_{xx}) = 0.$$

Considering an ansatz of the form

$$V(t, x, a) = \xi_a(t) \ln x + \zeta_a(t)$$

and substituting in the HJB equation above, we get

$$\begin{aligned} \xi'_a(t) \ln x + \zeta'_a(t) + \sup_{(c, \theta) \in [0, \infty) \times [0, 1]} & \left\{ e^{-\rho t} \ln c \right. \\ & + \left( -\frac{c}{x} + \left( r(t, a) + \theta(\mu(t, a) - r(t, a)) \right) \right) \xi_a(t) - \frac{1}{2} (\theta \sigma(t, a))^2 \xi_a(t) \\ & + \ln x \sum_{j \in S: j \neq a} q_{aj} (\xi_j(t) - \xi_a(t)) + \sum_{j \in S: j \neq a} q_{aj} (\zeta_j(t) - \zeta_a(t)) \\ & \left. + \int_{\mathbb{R}_0^1} \left( \xi_a(t) \ln(1 + \theta h(t, a, z)) - \xi_a(t) \theta h(t, a, z) \right) \nu(dz) \right\} = 0. \end{aligned} \quad (29)$$

We start by optimizing with respect to  $c$ , before proceeding to optimize with respect to the variable  $\theta$ . The first-order condition associated with the optimization problem above provides a maximizer  $c^*(t, x, a)$ , given by

$$c^*(t, x, a) = \frac{e^{-\rho t}}{\xi_a(t)} x.$$

Replacing  $c$  by  $c^*(t, x, a)$  in Eqn. (29) we obtain that

$$\begin{aligned} \xi'_a(t) \ln x + \zeta'_a(t) + \sup_{\theta \in [0,1]} \left\{ e^{-\rho t} (-\rho t + \ln x - \ln \xi_a(t) - 1) + F(\theta; t, a) \xi_a(t) \right. \\ \left. + \ln x \sum_{j \in S: j \neq a} q_{aj} (\xi_j(t) - \xi_a(t)) + \sum_{j \in S: j \neq a} q_{aj} (\zeta_j(t) - \zeta_a(t)) \right\} = 0, \end{aligned} \quad (30)$$

where  $F(\theta; t, a)$  is as given in (27). Note that as long as  $\xi_a(t)$  is nonzero, the first-order condition to  $\theta$  is just  $F'(\theta; t, a) = 0$ . Moreover, the second derivative of  $F(\theta; t, a)$  with respect to  $\theta$ , given by

$$F''(\theta; t, a) = -\sigma^2 - \int_{\mathbb{R}_0^1} \frac{h^2(t, a, z)}{(1 + \theta h(t, a, z))^2} \nu(dz),$$

is negative for every  $\theta \in [0, 1]$ . Taking into account the constraint  $\theta \in [0, 1]$  and the concavity of  $F(\theta; t, a)$ , we conclude that the maximization problem in Eqn. (30) has a unique solution  $\theta^*(t, a)$ . Moreover, from the definition of the function  $F(\theta; t, a)$ , it is possible to check that

- i) if  $\mu(t, a) - r(t, a) > 0$  and  $F(1; t, a) < 0$ , then there exists  $\hat{\theta}(t, a) \in (0, 1)$  such that  $F'(\hat{\theta}(t, a); t, a) = 0$  and, consequently,  $\theta^*(t, a) = \hat{\theta}(t, a)$ ;
- ii) if  $\mu(t, a) - r(t, a) > 0$  and  $F(1; t, a) \geq 0$ , then  $\theta^*(t, a) = 1$ ;
- iii) if  $\mu(t, a) - r(t, a) \leq 0$ , then  $\theta^*(t, a) = 0$ .

From Eqn. (30), we obtain

$$\begin{aligned} \xi'_a(t) + e^{-\rho t} + \sum_{j \in S: j \neq a} q_{aj} (\xi_j(t) - \xi_a(t)) &= 0, \\ \zeta'_a(t) - e^{-\rho t} (\rho t + \ln \xi_a(t) + 1) + F(\theta^*(t, a); t, a) \xi_a(t) \\ + \sum_{j \in S: j \neq a} q_{aj} (\zeta_j(t) - \zeta_a(t)) &= 0, \\ \xi_a(T) = e^{-\rho T}, \quad \zeta_a(T) &= 0. \end{aligned} \quad (31)$$

From Eqn. (31) we get that  $\xi_i(t) = \xi_j(t)$  for every  $i, j \in S$  and every  $t \in [0, T]$ . Let  $\xi(t) = \xi_a(t)$ ,  $a \in S$ . Then,  $\xi(t)$  must be a solution of the boundary value problem

$$\begin{aligned} \xi'(t) + e^{-\rho t} &= 0, \\ \xi(T) &= e^{-\rho T}, \end{aligned}$$

which has solution

$$\xi(t) = \frac{(\rho - 1)}{\rho} e^{-\rho T} + \frac{e^{-\rho t}}{\rho}.$$

We conclude the proof by substituting  $\xi(t)$  in the equation for  $\zeta_a(t)$  in Eqn. (31). Finally, we note that the ordinary differential equations describing  $\zeta_a(t)$  are linear and thus have a unique solution.  $\square$

**4.4. A simple example.** We consider now, as an example, a Markov-switching jump-diffusion financial market with one risk-free security and one risky asset. The Markov process  $\{\alpha(t) : t \in [0, T]\}$  driving the switching is assumed to have a finite space state  $\mathcal{S} = \{+, 0, -\}$ . We think of the state “+” as representing an upward trending market, in which case the financial markets and the underlying economy are healthy, whereas the state “0” represents a period where the financial markets are range bound, i.e., with no precise direction. On the contrary, we associate the state “-” to periods of great stress in the markets, such as a large economic or financial crisis. We restrict ourselves to the case where the financial market has constant coefficients throughout this example. Let us use the indices “+”, “0” and “-” to distinguish between the market’s coefficients on each state of the Markov process  $\{\alpha(t) : t \in [0, T]\}$ . A reasonable choice of parameters should satisfy:

- 1)  $r_+ > r_0 > r_- \geq 0$ , i.e., interest rates tend to be higher during expansion periods and lower during recessions.
- 2)  $\mu_+ > \mu_0 > \mu_-$ , i.e., risky assets appreciation rates tend to be higher during expansion periods and lower during recession periods.
- 3)  $\mu_+ > r_+$ ,  $\mu_0 \approx r_0$ ,  $\mu_- < 0$ , i.e., risky asset appreciation rates tend to be larger than interest rates during expansions, negative during recessions, and close to zero for range bound markets.
- 4)  $\sigma_+ < \sigma_0 < \sigma_-$ , meaning that the degree of uncertainty associated with the financial assets return is smaller in a upward trading market when compared with a distressed market.

Moreover, it would be reasonable to also assume that  $h_+$ ,  $h_0$  and  $h_-$  are such that the tail of  $h_-$  is heavier than the tail of  $h_0$ , and the tail of  $h_0$  is heavier than the tail of  $h_+$ , i.e., it is more likely to find large jumps in distressed markets when compared to upward trending markets or range bound markets.

For concreteness of exposition, we take the deterministic fixed horizon to be  $T = 20$ . In accordance with the discussion above, we will use the following choice of parameters:

- (iii) State +: the risk-free interest rate is  $r_+ = 0.05$ , the risky asset mean appreciation rate is  $\mu_+ = 0.20$  and its volatility is  $\sigma_+ = 0.11$ ;
- (ii) State 0:  $r_0 = 0.03$ ,  $\mu_0 = 0.04$  and  $\sigma_0 = 0.21$ ;
- (i) State -:  $r_- = 0.01$ ,  $\mu_- = -0.1$  and  $\sigma_- = 0.40$ .

In what concerns the jump process, for the sake of simplicity, we pick a setup where the size and waiting time for the jumps do not depend on the state of the Markov process  $\alpha(\cdot)$ . Specifically, we take  $h(t, a, z) = z$ , let  $\nu(dz)$  be the measure associated with a uniform distribution with support  $[-0.25, 0.25]$ , and let the waiting time between jumps be exponentially distributed with mean  $\lambda = 1$ . The generator of the Markov process  $\alpha(\cdot)$ ,  $Q = (q_{ij})_{i,j \in \mathcal{S}}$ , is determined by  $q_{+0} = 3$ ,  $q_{+-} = 6$ ,  $q_{0+} = 1$ ,  $q_{0-} = 1$ ,  $q_{-+} = 2$  and  $q_{-0} = 1$ . The quantities  $q_{ij}$ ,  $i, j \in \mathcal{S}$ , are the instantaneous transition rates for the switching from state  $i \in \mathcal{S}$  to state  $j \in \mathcal{S}$ , and the waiting times for such switches to occur are exponentially distributed with mean  $q_{ij}$ . Thus, in light

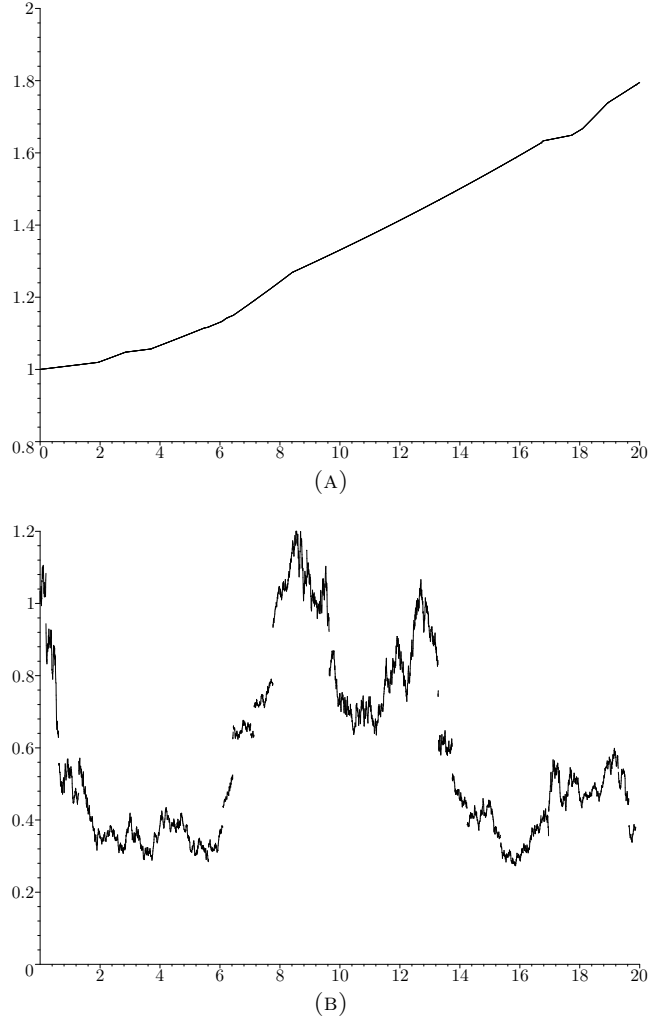


FIGURE 1. Sample path of the risk free asset price evolution with time (top) and the corresponding path for the risky asset (bottom) for the example of Section 4.4. The initial value of the two assets is equal to 1.

of this interpretation for the entries of the generator matrix  $Q$ , the values of the entries of  $Q$  are picked in such a way that on average the market remains in the state  $+$  longer than in the remaining states, being more likely to see a switch from  $+$  to  $0$ , than it is to see one from  $+$  to  $-$ . The waiting time to leave the state  $0$  is shorter (in average), and it is as likely to move from  $0$  to  $+$  as it is to move from  $0$  to  $-$ . Finally, the waiting times to leave the state  $-$  were picked in such a way that it is more likely to observe a transition from  $-$  to  $0$  than it is to observe a transition from  $-$  to  $+$ . In what concerns the agent preferences, we assume that these are described by discounted power utilities of the form (19) with risk aversion parameters  $\gamma_+ = 0.6$ ,  $\gamma_0 = 0.5$  and  $\gamma_- = 0.4$  and fixed discount rate equal to  $\rho = 0.03$ . The choice of parameters described above is illustrative,



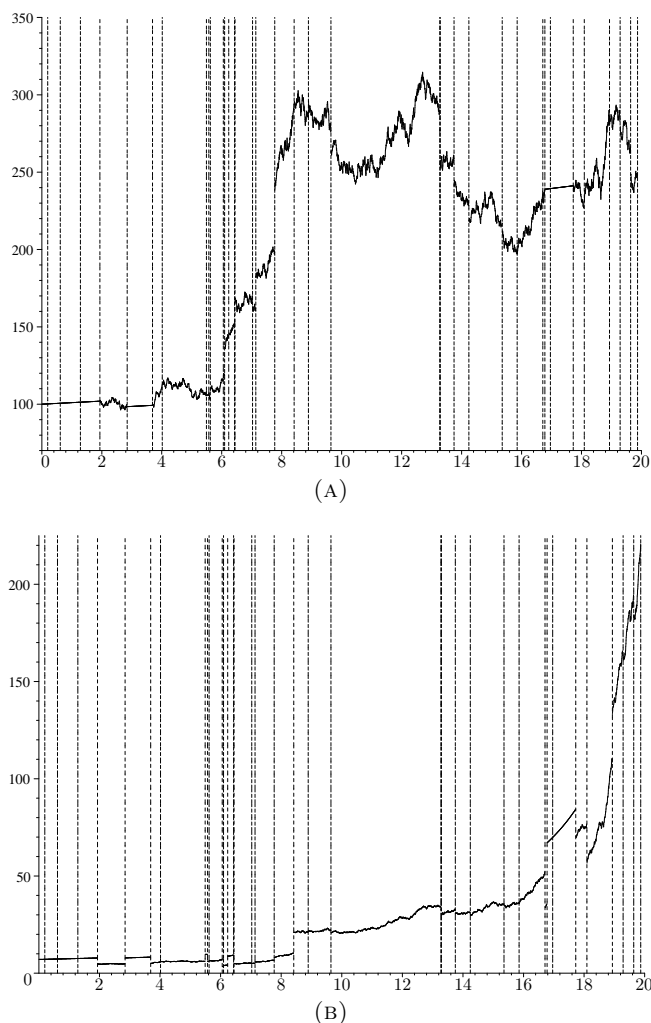


FIGURE 2. Sample path of the wealth process associated with the optimal portfolio (top) and the corresponding optimal consumption path (bottom) for the consumption investment problem with parameters as described in Section 4.4. The vertical dashed lines indicate instants of time of jumps or switches of the Markov process  $\alpha(\cdot)$ . The initial wealth is taken to be equal to 100.

and we aimed at making it as realistic as possible while trying to keep it as simple as possible for the sake of presentation. A complete statistical fitting of the model introduced in this paper to real financial data still needs to be produced and is outside of the scope of the current paper.

In Figure 1 one can see an example of a sample path obtained from the jump-diffusion Markov-switching financial market described above. Note that as the Markov process  $\alpha(\cdot)$  switches between states, the rate of growth of the risk free asset in Figure 1a changes accordingly, as well as the mean direction of growth of the risky asset in Figure 1b. In Figure 2 we present

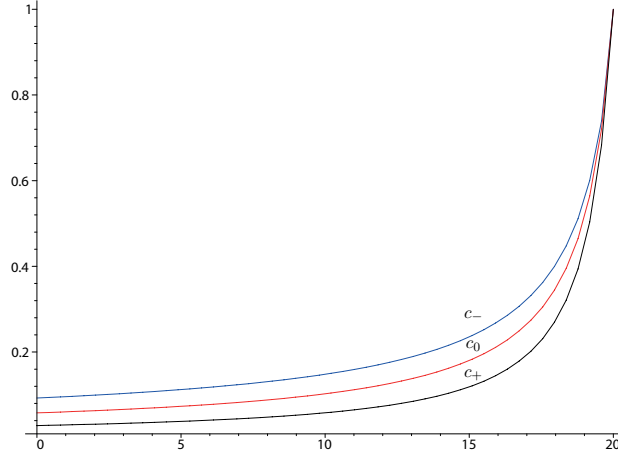


FIGURE 3. Plot of the proportion of wealth consumed as a function of time for a fixed level of wealth  $x = 1$ . The horizontal axis represents time, and the vertical axis represents the consumption rate  $c$ .

the corresponding paths for the wealth process associated with the optimal strategies (Figure 2a) and the optimal consumption level (Figure 2b). Note that even though the risky asset loses roughly 60% of its initial value as time increases from 0 to 20, and the risk free asset gains about 80% in value over the same time span, the optimal wealth process value increases by roughly 150%, with corresponding adjustments in the optimal consumption level. It seems to us that this behaviour illustrates the advantages that may arise from an appropriate understanding of financial markets properties in what concerns portfolio management, in the obvious sense that a deeper knowledge of the current “state” of the financial markets should lead to an increased performance of financial portfolios. Hence, we believe that there may be merit in approaches connecting the optimal control of jump-diffusion Markov-switching financial markets (as described here) with statistics and econometrics techniques that may provide appropriate detection techniques for the financial markets switching times.

Finally, the plot in Figure 3 illustrates the proportion of wealth consumed in each one of the 3 states  $\mathcal{S} = \{+, 0, -\}$  as a function of time and for a fixed level of wealth, as opposed to Figure 2b for which the level of wealth changes with time. Note that the consumption rate is an increasing function of time and wealth. In a growing economy, the agent has a lower risk aversion and allocates most of his wealth to the financial market. Moreover, when financial markets are not healthy, agent risk aversion increases and a larger proportion of their wealth is devoted to consumption. Although the proportion of wealth consumed is lower in upward trending markets, this does not mean that the total amount of wealth consumed is lower, since the total amount of wealth tends to increase more in upward trending markets. This strategy is optimal for a rational investor that maximizes expected utility (18), i.e., consumption in  $[0, T]$  and final wealth at time  $T$ .

Naturally, consumption decreases with an increase of the risk aversion parameter  $\gamma$ , whereas  $\theta^*$  increases with an increase of  $\gamma$ . Finally,  $\theta^*$  also grows with the risky asset appreciation rate  $\mu$  and decreases with the volatility coefficient  $\sigma$ . Such qualitative properties for the optimal consumption and investment strategies are robust in what concerns (reasonable) changes in the model under consideration here.

## 5. CONCLUSION AND OUTLOOK

In this paper, we introduced *stochastic hybrid systems with jumps* and contributed to their optimization in the presence of control variables, as decision variables in addition to the state variable. In fact, we analysed and solved a *stochastic optimal control* problem for a *Markov-switching jump-diffusion stochastic differential equation*. We employed techniques from dynamic programming to extend Bellman's optimality principle to the setup under consideration here and derived the corresponding family of *Hamilton-Jacobi-Bellman equations* which implicitly describe the value function that is associated with the stochastic optimal control problem under consideration. We concluded with an application to a problem of *consumption-investment type*. The present model with Markov-switching may be of great use in portfolio management and economics, since it seems to be very appropriate and useful for a representation of markets' movements between various states (e.g., growth, crisis, range bound or bubbles) and of regime switches, respectively. For the practical implementation of such models it would be interesting to develop statistical inference methods that are able to determine a reasonable number of states and to detect the state of the market at a given moment in time. The development of new numerical techniques [31], i.e., approximation schemes rather than closed-form solutions for the treatment of our optimal control problems, seems to us of great interest, too. In this respect, our paper may stimulate rich future research and collaboration, in theory and methodology, which eventually aim at an overcoming of emerging real-world challenges in decision making under multiple uncertainties and different tendencies of change.

## APPENDIX A. ITÔ'S FORMULA

In this section we state Itô's formula for a jump-diffusion with Markov-switching. The result follows from Itô's formula for semi-martingales. See [28] for further details.

**Lemma A.1** (Itô's rule for a Markov-switching jump-diffusion). *Suppose that  $X(t)$  is a Markov-switching jump-diffusion process given by*

$$\begin{aligned} dX(t) &= f(t, X(t_-), \alpha(t_-), u(t))dt + \sum_{m=1}^M g(t, X(t_-), \alpha(t_-), u(t))dW_m(t) \\ &\quad + \sum_{k=1}^K \int_{\mathbb{R}_0^1} h^{(k)}(t, X(t_-), \alpha(t_-), z_k) \tilde{J}_k(dt, dz). \end{aligned}$$

Let  $V(t, x, \alpha)$  be such that  $V(\cdot, \cdot, \alpha) \in C^{1,2}([0, T] \times \mathbb{R}^N)$  for every  $\alpha \in S$ . Then, we have that

$$\begin{aligned} V(T, X(T), \alpha(T)) - V(0, X(0), \alpha(0)) &= \int_0^T a(t, X(t), \alpha(t), u(t)) dt \\ &+ \int_0^T b(t, X(t), \alpha(t), u(t)) dW(t) + \int_0^T c(t, X(t), \alpha(t), u(t)) dM(t) \\ &+ \int_0^T \sum_{k=1}^K \int_{\mathbb{R}_0^1} d_k(t, X(t), \alpha(t), u(t), z_k) \tilde{J}_k(dt, dz_k), \end{aligned}$$

where

$$\begin{aligned} a(t, X, \alpha, u) &= V_t(t, X, \alpha) + \langle V_X(t, X, \alpha), f(t, X, \alpha, u) \rangle \\ &+ \frac{1}{2} \text{tr}(g^T(t, X, \alpha, u) V_{XX}(t, X, \alpha) g(t, X, \alpha, u)) \\ &+ \sum_{j \in S: j \neq \alpha} q_{\alpha j} (V(t, X, j) - V(t, X, \alpha)) \\ &+ \sum_{k=1}^K \int_{\mathbb{R}_0^1} W_k(t, X, \alpha, u, V, V_X, z_k) \nu_k(dz_k) \end{aligned}$$

with

$$\begin{aligned} W_k(t, X, \alpha, u, V, V_X, z_k) &= V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha) \\ &- \langle V_X(t, X, \alpha), h^{(k)}(t, X, \alpha, u, z_k) \rangle \end{aligned}$$

and

$$\begin{aligned} b(t, X, \alpha, u) &= (V_X(t, X, \alpha))^T g(t, X, \alpha, u) \\ c(t, X, \alpha, u) &= \sum_{j \in S: j \neq \alpha} (V(t, X, j) - V(t, X, \alpha)) \\ d_k(t, X, \alpha, u, z_k) &= V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha). \end{aligned}$$

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