ON A VARIATIONAL SEQUENTIAL BARGAINING PRICING SCHEME

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Abstract. We propose a minimization problem as a model for the interaction between two agents trading a contingent claim in a incomplete discrete-time multiperiod financial market. The agents personal valuations for the contingent claim are assumed to depend on probability measures representing their beliefs concerning the future states of the world. The agents' goal is to achieve a common price for the contingent claim to be traded, while deviating as litle as possible from their initial beliefs. Under appropriate conditions, we prove that the minimization problem under consideration admits at least one solution. Furthermore, we provide a detailed description for the minimizers – orbits of a finite horizon discrete time dynamical system on the space of probability measures representing the agents beliefs.
MSC2000: 91B26, 91B24, 90C30, 90C25
Keywords: Variational scheme; Sequential bargaining; Real asset pricing; Incomplete markets.

1. Introduction

The pricing of contingent claims in complete markets is by now well understood in terms of the theory of general equilibrium. A contingent claim F which is traded in a complete market can be assigned a unique price in terms of the (unique) risk neutral measure (pricing kernel) which is defined by the market. This price is independent of the individual preferences towards risk of the agents involved in the transaction and reflects in some sense the information regarding the states of the world as filtered through the financial market. Unfortunately, this elegant theory does not cover many cases of practical financial interest, in which markets fail to be complete. The failure of the assumption of completeness of the market may occur in a variety of ways: transaction costs, market imperfections of various kinds, constraints in the allowed portfolios used to hedge the contingent claims, or absense of a well organized market for a contingent claim being just a few reasons.

When markets are incomplete, absense of arbitrage is no longer enough to provide a pricing rule for a contingent claim. The reason is that when markets are incomplete an infinity of pricing kernels (as opposed to a unique pricing kernel in the complete market case) is compatible with the assumption of absense of arbitrage, therefore, this assumption may at best provide an interval of prices at which the asset may be traded. The lower price in this interval can be thought of as a bid price whereas the higher price in this interval can be thought of as an ask price. However, any price within this interval is acceptable, and now the personal taste of agents towards risk plays an important role in the choice of price (see [2, 7, 9]). To this end utility pricing is employed, and the price of the asset is assessed as a reservation price, determined as the price that makes the agent indifferent between accepting a position in the asset and not accepting a position in it. In this valuation the underlying market plays an important role, since it is used in order to hedge the risk involved in taking up a position in the asset. Since the buyer and the seller of the asset are in principle having a different utility function, face different constraints with regards to the portfolios used in hedging their positions, and take opposite positions with respect to the contingent claim F, the reservation prices for the buyer and the seller of the asset are unlikely to coincide.

However, the transaction of the asset takes place at a commonly agreed price between the buyer and the seller. It is the aim of this paper to propose a bargaining approach to this problem that may offer a scenario under which a unique price for the asset can be reached. This scenario is based on two intertemporal optimization problems: (a) a portfolio optimization problem related to hedging of the risk involved in the different positions (open and long) of the contingent claim and (b) a belief updating problem for the two agents (concerning the probability of the future states of the world) which is expressed in terms of an optimization problem in a space of probability measures. During the course of the paper, we first state the relevant optimization problems and argue that it is a plausible scheme for determining a unique price of the asset, using arguments from economic theory and the theory of bargaining and then consider the well posedness and qualitative behaviour of the relevant optimization problems, using techniques from dynamic programming and the calculus of variations.

This paper is organized as follows. In Section 2 we motivate and describe the intuition behind our model, leaving the mathematical formulation of the model as an intertemporal minimization problem to Section 3. In Section 4 we study the existence of solutions to this minimization problem under general assumptions on the bidding functions used by the agents in the bargaining scheme, such as continuity or convexity, and provide qualitative results on the solution. The methods used are techniques from stochastic dynamic programming and convex optimization. In section 5 we provide examples of specific bidding rules compatible with economic theory that realize the assumptions used in Section 4 and comment upon their effects on the bargaining scheme. Finally, in Section 6 we provide our main conclusions.

2. Motivation

Consider two agents A and B, hereafter referred to as the seller and the buyer, respectively, that wish to trade an indivisible asset which is a contingent claim. The asset delivers an uncertain payoff F, at expiry T (a deterministic time). There is an underlying financial market, to which both of the agents have access, consisting of $N+1$ securities, one of which is assumed to be a riskless asset. The market is subject to uncertainty, modelled in terms of a filtered probability space $(\Omega, \mathcal{F}, P)$, where P is assumed to be the statistical measure for the market. The state of the market is modelled by a vector valued random variable $S(\cdot) = (S_0(\cdot), S_1(\cdot), \cdots, S_N(\cdot))$, where S_0 corresponds to the price of the riskless asset over the time period $[0, T]$. To simplify the model, we assume time to be discrete. The contingent claim, is assumed to depend on the state of the market, in the sense that its payoff F can be considered as a random variable on the probability space $(\Omega, \mathcal{F}, P)$, that may be partially correlated to the random variables $S(\cdot)$ or more simply $S(T)$. Note that this formulation allows us to consider contingent claims possibly different than financial assets, like for example real assets (e.g. real options) whose values are often only partially correlated with the underlying financial market. Furthermore, we consider the claim to be exchanged by the two agents only and not as being exchanged in a well organized market; it is an over the counter transaction. These two features naturally bring us into an incomplete market situation which is not covered by the standard theory of asset pricing.

The agents wish to reach a decision upon a common price for the exchange of the asset by time T_0 ($T_0 < T$). We assume here that T_0 is a deterministic time, which is pre-decided by both agents. In principle the valuation of the asset by the two agents will not coincide, so in order for the two agents to reach a commonly acceptable price, they have to get involved into a bargaining procedure. We assume that the two agents enter a sequential bargaining scheme that takes place at times $t = 1, \cdots, T_0 - 1$, during which they simultaneously state their bid and ask prices for the asset in question. These prices are updated on each period so that by time T_0 a common price is reached.

We assume that the price quotes (bid and ask prices) for the asset at each time t are determined as reservation prices by the two agents independently, decided upon by using their own expected utility functions, representing their preferences and attitudes towards risk. Importantly, we allow each agent to have her own subjective views towards the future states of the economy, and consequently towards the value of the payoff of the asset, F. This subjective view is modelled in terms of probability measures Q_A, Q_B respectively, which are used for the determination of the expected utility function. Then, at each bargaining time t, each agent can use the market to create a financial portfolio which is chosen optimally (i.e. as the one maximizing the expected utility of wealth at T) in two distinct situations, (a) when involved in a position in the asset and (b) when choosing not to get involved in the asset. The proposed price is chosen as the one leaving each agent indifferent between these two positions. Clearly, this price depends on the subjective probability of the agents as well as the state of the market. For the time being, we state this simply as $p_\beta(t) = P_\beta(t, Q_\beta(t), S(t))$ where $\beta = A, B$ and S denotes the state of the market (observable by both agents). We explicitly include the time dependence t to emphasize that the prices stated at time t depend on the beliefs that the two agents have at this time concerning the future states of the economy, $Q_\beta(t)$, as well as the state of the market, $S(t)$, which serves as a common signal concerning the state of the economy accessible to both agents. In principle $p_B(t) \leq p_A(t)$, with strict inequality meaning that the two agents fail to reach an agreement. Let us assume for the time being, that at time t, strict inequality holds.

The two agents are not very rigid about their beliefs concerning the future state of the economy and at the same time wish to reach an agreement. They are therefore, willing to change their subjective probabilities from $Q_\beta(t)$ to $Q_\beta(t+1)$, which in turn will lead to a new set of quoted prices $p_\beta(t+1)$. The new prices may be such that the two parties may reach an agreement or not, and the process is repeated until an agreement is reached at time T_0.

Some comments are in order concerning the mechanism of this belief updating process. Let us take agent A: At time t she has a subjective belief $Q_A(t)$ concerning the future states of the economy, and using that and her knowledge of the market by time $t-1$ she quotes a reservation price for the asset $p_A(t)$. Similarly for agent B. Now for the next offer, at time $t+1$, agent A has access to the information provided by the market by time t, but also knows the price $p_B(t)$ quoted by agent B at this time. By that quote she may get an idea of what agent's B subjective beliefs are, so she may get an estimate for $Q_B(t)$. Not being rigid about her beliefs, she is willing to change to a new measure $Q_A(t+1)$ so as to get closer to her idea of what $Q_B(t)$ was for two reasons: (a) because she is influenced by B's beliefs concerning the economy (as infered by the price quote she has made) and (b) because this is a way to get closer to an agreement. Of course this is made in such a way as to deviate as little as possible from her original belief $Q_A(t)$. Agent B is considered to act in a similar manner, and in this way price quotes are updated until agreement is reached. In this paper, the influence that the price quote of B has on the update of the beliefs for A is modelled in the following simple manner: Suppose that $p_A(t)$ and $p_B(t)$ are the price quotes of the two agents at time t. These obviously depend on $Q_A(t)$ and $Q_B(t)$. One way of reaching an agreement would be for agent A to settle for a price in the interval $[p_B(t), p_A(t)]$, say $\alpha p_B(t) + (1-\alpha)p_A(t)$ for some $\alpha \in [0, 1]$. This price would correspond to a new reservation price for agent A if the subjective probability Q_A^* was adopted instead of $Q_A(t)$. In some sense agent A is influenced by the price quote of B and decides to move her subjective beliefs towards Q_A^* but in such a way that she deviates as little as possible from her original belief $Q_A(t)$. Therefore, expressed in a loose fashion, $Q_A(t+1)$ is obtained by minimizing an appropriate distance between $Q_A(t)$ and Q_A^*. Agent B will act in a similar fashion. In other words, the deviation from the initial beliefs is done with some hesitation or reluctance that causes some disutility. A further source of disutility that has to be taken into account stems from the distance of the stated prices at each bargaining round; the further away the prices are, the more deviation from the initial beliefs has to take place (i.e., more compromise needs to be achieved and this causes additional disutility). Of course this disutility is perceived in a different way from each counterparty.

As a simple illustrating example, consider a transaction that involves the selling of a plant producing a particular commodity. The contract for the transaction between the buyer and the seller will have to take place at T_0. The two parties enter a sequential bargaining procedure at times $t = 1, \cdots, T_0 - 1$, by the end of which they reach an agreement for the price of the transaction. The price of the plant will depend upon the future value of its production, which in turn is subject to uncertainty and depends heavily on the price of the commodity that this plant produces in the relevant commodity market. The underlying financial market plays a dual role in determining the price of the plant. The first one is that the price of the commodity that the plant produces (hence its value) is correlated with some of the financial assets in the market (for example, if the product is related to copper, it is certainly correlated with futures on copper, as well as with the stocks of companies whose production relies on this commodity). The second role is that both parties will rely on the financial market, to either hedge away the risk involved in the investment or to gather the necessary funds to embark upon the transaction.

In the remaining of the paper, we formulate the above conceptual model of the bargaining procedure as a concrete optimization problem and consider its well posedness and its qualitative properties, using techniques from dynamic programming and convex optimization.

3. Mathematical formulation of the model

3.1. **The financial market.** Let $(\Omega, \mathbb{F}, P)$ be a probability space, with a finite sample space $\Omega = \{\omega_1, \ldots, \omega_K\}$ endowed with a σ-algebra $\mathbb{F}$ and a probability measure P on $\mathbb{F}$, assigning positive probability $P(\omega)$ to every $\omega \in \Omega$. We will assume that Ω contains all the possible states of the world of a multiperiod financial market model with $T+1$ trading dates, $\mathbb{T} = \{0, 1, \ldots, T\}$, one risk-free asset S_0 and an arbitrary but fixed number of risky-assets $S_1, \ldots, S_N$. The prices of all these assets are non-negative stochastic processes on the probability space $(\Omega, \mathbb{F}, P)$. The process $S_0 = \{S_0(t) : t \in \mathbb{T}\}$ is assumed to be risk-free, i.e. for every $\omega \in \Omega$ the sample path $S_0(t, \omega)$ is non-decreasing with respect to $t \in \mathbb{T}$. We denote by $\mathcal{F} = \{\mathcal{F}_t : t \in \mathbb{T}\}$ the filtration generated by the stochastic processes $S_0, S_1, \ldots, S_N$. The filtration $\mathcal{F}$ may be seen as encoding the flow of information made available to the observers of the financial market through observation of the security prices evolution with time.

We assume that the financial market under consideration is viable, i.e. there are no arbitrage opportunities. It is a well known fact [8] that this is equivalent to the existence of at least one risk neutral probability measure (or martingale measure), i.e. a probability measure Q in $(\Omega, \mathbb{F})$ such that Q is equivalent to P and for every $n \in \{1, \ldots, N\}$ and every $t, s \in \mathbb{T}$ such that $t + s \leq T$ we have that

$$E_Q[S_n^*(t+s)|\mathcal{F}_t] = S_n^*(t),$$

where $S_n^*(t)$ denotes the discounted price process

$$S_n^*(t) = \frac{S_n(t)}{S_0(t)}\ , \qquad n = 1, \ldots, N.$$

The discounted price processes are non-negative stochastic processes representing the value of the risky assets in units determined by the value of the risk-free asset at a given instant of time. Note that the stochastic processes $S_n^*(t)$ are martingales with respect to the measure Q.

A trading strategy is a set of rules that specifies an investor's position in each security at each point in time, and in each state of the world, i.e. a vector $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$ of stochastic processes $\pi_n = \{\pi_n(t) : t =$

$1, 2, \ldots, T\}$, $n = 0, 1, \ldots, N$, where each component $\pi_n(t)$ represents the number of units of each asset carried over from period $t-1$ to period t. We assume that each component of a trading strategy is predictable, i.e. $\pi_n(t)$ is $\mathcal{F}_{t-1}$-measurable for every $n = 0, 1, \ldots, N$. The value process $V = \{V_t : t = 0, 1, \ldots, T\}$ associated with the trading strategy $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$ is the stochastic process defined by $V_0(\pi) = \sum_{n=0}^{N} \pi_n(1) S_n(0)$ when $t = 0$, and $V_t(\pi) = \sum_{n=0}^{N} \pi_n(t) S_n(t)$ for $1 \leq t \leq T$.

A trading strategy $\pi = (\pi_0, \pi_1, \ldots, \pi_N)$ is said to be self-financing if the equality $V_t(\pi) = \sum_{n=0}^{N} \pi_n(t+1) S_n(t)$ holds for every $t = 1, \ldots, T-1$, i.e. no money is added or withdrawn from the portfolio between times $t = 0$ and $t = T$.

Throughout this paper we will denote the set of $\mathcal{F}_T$-measurable random variables on $(\Omega, \mathbb{F}, P)$ by $B_T(\Omega)$. A European-type contingent claim is a random variable $F \in B_T(\Omega)$. A multiperiod financial market is said to be complete if every contingent claim is marketable, i.e. there exists a self-financing trading strategy whose value process at time $t = T$ is equal to the payoff of the contingent claim. A classical result in mathematical finance ensures that a market is complete if and only if there exists a unique risk neutral probability measure Q (see [8]). Throughout this paper we will assume that the financial market under consideration is incomplete and in particular that F can not be reproduced. Thus, there exists an infinite family of risk neutral probability measures leading to an infinity of possible prices for the asset. Moreover, note that each risk neutral probability measure defines a (possibly) distinct price for a given contingent claim in $B_T(\Omega)$.

3.2. **A variational pricing problem.** The problem we adress in this paper is the one where a pair of agents, denoted by A and B from now on, need to agree on a price p for the trade at time $T_0 \in \{1, \ldots, T-1\}$ of a contingent claim with payoff $F \in B_T(\Omega)$ at time T. Let A be the seller and B be the buyer of such a contingent claim. We assume that the two agents have beliefs about the likelihood of the future states of the world, which are given by probability measures Q_A, $Q_B \in \Delta^K$, where Δ^K is the unit simplex in $\mathbb{R}^K$. We allow Q_A and Q_B to change with time, representing the fact that each agent is observing the evolution of the asset prices $S_0, S_1, ..., S_N$, and infering information from it, and furthermore being influenced by each other.

We assume that both agents have preferences described by utility functions $U_\beta : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, $\beta \in \{A, B\}$. For each fixed $t \in \mathbb{T}$, let $U_\beta(t, \cdot)$ satisfy the usual Inada conditions [3], i.e. the utility functions have value zero when $x = 0$, are strictly increasing, strictly concave and continuously differentiable, and their first derivatives satisfy the following asymptotic conditions

$$\lim_{x \to -\infty} U_\beta^{'}(t, x) = +\infty, \quad \lim_{x \to +\infty} U_\beta^{'}(t, x) = 0, \quad \beta = A, B.$$

Moreover, for each fixed $x \in \mathbb{R}$, let $U(\cdot, x)$ be a strictly decreasing function of $t \in \mathbb{T}$. The strict monotonicity with respect to time is modelling the effect of discounting [4]. A particular example is the standard exponential discounting factor whereupon $U(t, x) = \rho^t \mathcal{U}(x)$, $\rho < 1$.

Assume that agents A and B have initial wealths x_A and x_B. Let V_β, $\beta \in \{A, B\}$, be compact convex subsets of $\mathbb{R}^{N+1}$. We assume that the agents are allowed to allocate their wealth in the financial market by choosing trading strategies $\{\pi_\beta(t)\}_{t\in\{1,\dots,T\}} \in (V_\beta)^T$, $\beta \in \{A, B\}$. The sets V_β may be interpreted as constraints on the portfolio strategies of the agents. For each $\beta \in \{A, B\}$, we will denote by $X^{\beta,\pi_\beta}_{t_0,x_\beta}(t, \omega)$ the stochastic process representing agent's β wealth at time $t \in \{1, \dots, T\}$ and state of the world $\omega \in \Omega$, when choosing a trading strategy $\{\pi_\beta(t)\}_{t\in\{t_0+1,\dots,t\}}$ starting with wealth x_β at time t_0. It is easy to chek that

$$X^{\beta,\pi}_{0,x_\beta}(t) = x_\beta + l_\beta p I_{\{t\geq T_0\}} - l_\beta F I_{\{t=T\}} + \sum_{u=1}^{t}\sum_{j=0}^{N} \pi_j(u)\Delta S_j(u), \tag{3.1}$$

where $l_A = 1$, $l_B = -1$ and $I_A(x)$ denotes the indicator function of the set A.

Each agent's goal is to optimize her expected utility of wealth at the final period $t = T$ while trading at $t = T_0$ the contingent claim with payoff F at $t = T$. At each period of time $t \in \{0, \dots, T-1\}$, agents A and B allocate their wealth to the assets in the financial market by choosing trading strategies $\pi^*_{A,t} = \{\pi_A(u)\}_{u\in\{t+1,\dots,T\}} \in (V_\beta)^{T-t}$ and $\pi^*_{B,t} = \{\pi_B(u)\}_{u\in\{t+1,\dots,T\}} \in (V_\beta)^{T-t}$ in such a way that maximizes their expected utilities, i.e.

$$\pi^*_{\beta,t} = \operatorname*{argmax}_{\pi_{\beta,t}\in(V_\beta)^{T-t}} E_{Q_\beta}\Big[U_\beta\Big(T, X^{\beta,\pi_\beta}_{0,x_\beta}(T,\omega)\Big)\Big|\mathcal{F}_t\Big]. \tag{3.2}$$

Note that $\pi^*_{\beta,t}$ depends on the agent beliefs Q_β and initial wealth x_β, as well as the contingent claim price p and payoff F. This utility optimization problem will play an important role in determining bidding prices (see section 5).

Let Δ^K denote the K-dimensional simplex in $\mathbb{R}^K$ and for each $u \in \{0, \dots, T-1\}$ let $\mathbb{T}^+_u$ denote the set $\mathbb{T}^+_u = \{u, \dots, T\}$ and $\mathbb{T}^-_u$ denote the set $\mathbb{T}^-_u = \{0, \dots, u\}$.

For each $\beta \in \{A, B\}$, $x_\beta \in \mathbb{R}$ and $t \in \mathbb{T}^-_{T_0}$, let $p_{\beta,t,x_\beta} : \Delta^K \times B_T(\Omega) \to \mathbb{R}$ be agent β price function at time t for the contingent claim $F \in B_T(\Omega)$ under the beliefs $Q_\beta \in \Delta^k$. Whenever $F \in B_T(\Omega)$ is assumed to be fixed, we write $p_{\beta,t,x_\beta}(Q_\beta)$ for simplicity of notation.

For each $\beta \in \{A, B\}$, let $\psi_\beta : \mathbb{T} \times \Delta^K \times \Delta^K \to \mathbb{R}$ be such that for every $t \in \mathbb{T}^-_{T_0-1}$, $\psi_\beta(t, \cdot, \cdot)$ is a continuous function attaining its minimum value on the diagonal set

$$D = \{(x, y) \in \Delta^K \times \Delta^K : x = y\}. \tag{3.3}$$

Moreover, assume that the functions $\phi : \mathbb{R} \to \mathbb{R}$ and $\Gamma_\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in \{A, B\}$, are continuous functions with a unique mimimum at 0.

At each period of time $t \in \mathbb{T}^-_{T_0-1}$, the two agents observe the market and are allowed to update their beliefs about the future states of the world. The updated beliefs are choosen in a way that jointly minimizes the deviation form the agents' initial beliefs and, eventually, the difference between the agents' valuation for the contigent claim F. More precisely, let $x(u) = (Q_A(u), Q_B(u)) \in \Delta^K \times \Delta^K$, $u \in \mathbb{T}^-_{T_0}$, be the beliefs of the agents at time u

and let $L : \mathbb{T}^-_{T_0-1} \times (\Delta^K \times \Delta^K)^2 \to \mathbb{R}$ and $H : (\Delta^K \times \Delta^K)^2 \to \mathbb{R}$ be given by

$$
\begin{aligned}
L(u, x(u), x(u+1)) &= \alpha\psi_A\left(u, Q_A(u), Q_A(u+1)\right) \\
&\quad +(1-\alpha)\psi_B\left(u, Q_B(u), Q_B(u+1)\right) \\
H(x(u), x(u+1)) &= \phi(D_1(x(u+1))) \\
&\quad +\Gamma_A(D_2(x(u), x(u+1))) + \Gamma_B(D_3(x(u), x(u+1)))
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
D_1(x(u)) &= p_{B,u,x_B}(Q_B(u)) - p_{A,u,x_A}(Q_A(u)) \\
D_2(x(u), x(u+1)) &= \alpha p_{A,u,x_A}(Q_A(u)) \\
&\quad +(1-\alpha)p_{B,u,x_B}(Q_B(u)) - p_{A,u+1,x_A}(Q_A(u+1)) \\
D_3(x(u), x(u+1)) &= p_{B,u+1,x_B}(Q_B(u+1)) \\
&\quad -\alpha p_{A,u,x_A}(Q_A(u)) - (1-\alpha)p_{B,u,x_B}(Q_B(u)).
\end{aligned}
$$

The agents choose beliefs $\{Q_A(t)\}_{t\in\mathbb{T}^-_{T_0}}$, $\{Q_B(t)\}_{t\in\mathbb{T}^-_{T_0}}$ that minimize the functional

$$
J\left(\{x(u)\}_{u\in\mathbb{T}^-_{T_0}}\right) = \sum_{u=0}^{T_0-1} L(u, x(u), x(u+1)) + H(x(u), x(u+1)) \tag{3.5}
$$

subject to initial beliefs $Q_A(0)$, $Q_B(0) \in \Delta^K$ and under the constraint that the transaction takes place at T_0, i.e.

$$
p_{A,T_0,x_A}(Q_A(T_0)) \le p_{B,T_0,x_B}(Q_B(T_0)) \; .
$$

Function L models the reluctance (i.e., disutility) of the agents to deviate from their initial beliefs, whereas function H models the influence that the stated prices have on the beliefs update; agent B tries to infer the beliefs of agent A from the prices that have been stated in the previous period, and hence her beliefs are influenced by that. In our model this is captured by the function H by making, e.g. agent A trying to approach a new belief that would correspond to her stating a new price within the interval of bid and ask prices at the previous period. The effect of this influence is quantified by the parameter $\alpha \in (0,1)$, which can be "interpreted" as a measure of the relative bargaining power of the two agents. The functions ψ_β, $\beta \in \{A, B\}$, model the disutility the agents experience when updating their beliefs about the future states of the world, the function ϕ measures the disutility experienced by the agents when a common price for the contingent claim is not reached, and finally, the functions $\Gamma_\beta, \beta \in \{A, B\}$, provides a pricewise disutility for the beliefs update.

Note that the price functions $p_{\beta,t,x_\beta}(Q_\beta, F)$ play a key role in the minimization problem described above. We will discuss two relevant examples. In Section 5.2 and (3.5) we discuss two relevant examples of bidding rules, leading to price functions producing with distinct properties.

4. Solvability and qualitative properties for the minimization problem (3.5)

In this section we will study the existence of solution to the minimization problem (3.5) in the case where the agents' bidding price functions are continuous. Section 4.1 contains an existence result and an description of the minimizers main properties. In section 5.2 we consider the special case of indifference price functions.

4.1. **Existence of solution.** The main result in this section guarantees the existence of solutions to the minimization problem (3.5). We will use the following set of assumptions:

(A1) for each $\beta \in \{A, B\}$, the functions $\psi_\beta : \mathbb{T} \times \Delta^K \times \Delta^K \to \mathbb{R}$ are such that for every $t \in \mathbb{T}^-_{T_0-1}$, $\psi_\beta(t, \cdot, \cdot)$ is a continuous function attaining its minimum value on the diagonal set (3.3).

(A2) the functions $\phi : \mathbb{R} \to \mathbb{R}$ and $\Gamma_\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in \{A, B\}$ are continuous functions with a unique minimum at 0.

(A3) for every $u \in \mathbb{T}^-_{T_0}$ and every x_A, $x_B \in \mathbb{R}$ the price functions $p_{A,u,x_A}(Q_A)$ and $p_{B,u,x_B}(Q_B)$ are continuous with respect to $Q_A \in \Delta^K$ and $Q_B \in \Delta^K$, respectively.

(A4) the initial conditions $Q^0_A, Q^0_B \in \Delta^K$ are such that

$$p_{A,0,x_A}(Q^0_A, F) \geq p_{B,0,x_B}(Q^0_B, F)$$

for fixed levels of initial wealth $x_A, x_B \in \mathbb{R}$.

Theorem 1. *Let $\alpha \in [0, 1]$ be fixed and assume that (A1)-(A4) hold. Let L and H be as given in (3.4). Then the minimization problem*

$$\min_{\{x(u)\}_{u=0}^{T_0} \in (\Delta^K \times \Delta^K)^{T_0+1}} \sum_{u=0}^{T_0-1} L(u, x(u), x(u+1)) + H(x(u), x(u+1)) \quad (4.1)$$

subject to the constraints

$$p_{A,T_0,x_A}(Q_A(T_0)) \leq p_{B,T_0,x_B}(Q_B(T_0)), \quad Q_A(0) = Q^0_A \quad \text{and} \quad Q_B(0) = Q^0_B$$

has at least one solution.

Furthermore, for any such solution $Q^* = \{Q^*_A(t), Q^*_B(t)\}_{t=0}^{T_0}$ the following statements hold:

a) *for every $t \in \mathbb{T}^-_{T_0}$, the inequality holds*

$$p_{A,t,x_A}(Q^*_A(t), F) \geq p_{B,t,x_B}(Q^*_B(t), F);$$

b) *if there exists $t \in \mathbb{T}^-_{T_0}$ such that $p_{A,t,x_A}(Q^*_A(t), F) = p_{B,t,x_B}(Q^*_B(t), F)$, then for every u such that $u > t$ we have that*

$$p_{A,u,x_A}(Q^*_A(u), F) = p_{B,u,x_B}(Q^*_B(u), F);$$

c) *at $t = T_0$, we have that*

$$p_{A,T_0,x_A}(Q^*_A(T_0), F) = p_{B,T_0,x_B}(Q^*_B(T_0), F);$$

d) *$\{p_A(Q^*_A(u))\}_{u \in \mathbb{T}^-_{T_0}}$ is a non-increasing sequence and $\{p_B(Q^*_B(u))\}_{u \in \mathbb{T}^-_{T_0}}$ is a non-decreasing sequence.*

e) *the minimizers* Q^* *depend continuously on the relative bargaining power* $\alpha \in (0,1)$*, as well as on the initial beliefs* $(Q_A^0, Q_B^0) \in \Delta^K \times \Delta^K$ *and the initial wealth levels* $x_A, x_B \in \mathbb{R}$*.*

Proof. Throughout this proof, we will use the notation

$$x(t) = (Q_A(t), Q_B(t)) \in \Delta^K \times \Delta^K$$

and will denote p_{A,u,x_A} by p_A and p_{B,u,x_B} by p_B, whenever no confusion arises.

Let $\Pi_i : (\Delta^K \times \Delta^K)^{T_0+1} \to \Delta^K \times \Delta^K$ denote the projection onto the i-th component of $(\Delta^K \times \Delta^K)^{T_0+1}$, i.e. $\Pi_i(z_1, \ldots, z_{T_0+1}) = z_i$ for every $(z_1, \ldots, z_{T_0+1}) \in (\Delta^K \times \Delta^K)^{T_0+1}$. Let $\mathbb{S}$ be the subset of $(\Delta^K \times \Delta^K)^{T_0+1}$ given by

$$\mathbb{S} = \left\{ z \in (\Delta^K \times \Delta^K)^{T_0+1} : \Pi_1(z) = (Q_A^0, Q_B^0),\ (p_B - p_A) \circ \Pi_{T_0+1}(z) \geq 0 \right\} . \tag{4.2}$$

We will start by proving the existence of solutions to the optimization problem (4.1). Note that $\mathbb{S} \subseteq (\Delta^K \times \Delta^K)^{T_0+1}$ is a closed subset of a compact set. Thus, $\mathbb{S}$ is also a compact set. Moreover, note that the functional J introduced in (3.5) is continuous by assumptions $(A1)$-$(A2)$. The result follows by Weierstrass Theorem.

We will now prove item a). Let $Q^* = \{Q^*(u)\}_{u=0}^{T_0}$ be a solution of (4.1). By contradiction, assume that there exists $t \in \{1, \ldots, T_0 - 1\}$ such that

$$p_A(Q_A^*(t-1)) \geq p_B(Q_B^*(t-1)) \quad \text{and} \quad p_A(Q_A^*(t)) < p_B(Q_B^*(t)).$$

Let $x^\lambda(t) = \lambda Q^*(t-1) + (1-\lambda) Q^*(t)$, $\lambda \in [0,1]$, be a parametrization of the line segment connecting $Q^*(t-1)$ to $Q^*(t)$ on $(\Delta^K \times \Delta^K)$. Denote by X^λ the sequence given by

$$X^\lambda = \begin{cases} X^\lambda(u) = Q^*(u), & \text{if } u \neq t \\ X^\lambda(u) = x^\lambda(u), & \text{if } u = t \end{cases}$$

and consider the map $G : [0,1] \to \mathbb{R}$ given by

$$G(\lambda) = J(X^\lambda) \ ,$$

where J is the functional in (3.5). The proof follows by noting that there exists $0 < \epsilon < 1$ such that for $\lambda \in [0, \epsilon]$ the map G is decreasing, contradicting minimality of Q^*.

The proof of item b) is analogous. By contradiction, assume that Q^* is such that there exists $t \in \{1, \ldots, T_0 - 1\}$ such that

$$p_A(Q_A^*(t-1)) = p_B(Q_B^*(t-1)) \quad \text{and} \quad p_A(Q_A^*(t)) > p_B(Q_B^*(t)).$$

Let $x^\lambda(t) \in \Delta^K \times \Delta^K$ and $X^\lambda \in (\Delta^K \times \Delta^K)^{T_0+1}$ be as defined in the proof of item a). Again, it is possible to check that the map $G : [0,1] \to \mathbb{R}$ given by $G(\lambda) = J(X^\lambda)$ is decreasing in a neighbourhood of 0, contradicting minimality of Q^*.

Item c) follows from item a) and the constraint $p_A(Q_A^*(T_0)) \leq p_B(Q_B^*(T_0))$.

In what concerns the proof of item d), we consider only the case of the price function p_A, the proof for the price function p_B being similar. By contradiction, assume that Q^* is such that there exists $t \in \{1, \dots, T_0\}$ such that

$$p_A(Q_A^*(t-1)) < p_A(Q_A^*(t)).$$

Let $x^\lambda(t) \in \Delta^K \times \Delta^K$ and $X^\lambda \in (\Delta^K \times \Delta^K)^{T_0+1}$ be as defined in the proof of item a). Using item a), it is again possibly to check that the map $G : [0,1] \to \mathbb{R}$ given by $G(\lambda) = J(X^\lambda)$ is decreasing, contradicting minimality of Q^*.

Finally, we will prove item e). The continuity of the solution Q^* of the optimization problem (4.1) with respect to $(Q_A^0, Q_B^0, x_A, x_B, \alpha) \in \Delta^K \times \Delta^K \times \mathbb{R}^2 \times [0,1]$ is a consequence of Berge's maximum Theorem [1, Ch. VI, Sec. 3], which guarantees continuity of the minimal functional

$$J(Q_A^0, Q_B^0, x_A, x_B, \alpha, Q^*)$$

with respect $Q_A^0, Q_B^0 \in \Delta^K$, $\alpha \in [0,1]$, and $x_A, x_B \in \mathbb{R}$, and upper semicontinuity of the correspondence given by

$$(Q_A^0, Q_B^0, x_A, x_B, \alpha) \to Q^*(Q_A^0, Q_B^0, x_A, x_B, \alpha),$$

which completes the proof. □

4.2. **Existence and uniqueness of solution.** In this section we prove that, under appropriate convexity assumptions, the minimization problem associated with functional (3.5) admits a unique solution. In section 5.3, we show that the set of assumptions used here are reasonable by providing an example of an alternative bidding rule leading to price functions with the required convexity properties.

Consider the following set of assumptions:

- (B1) for each $\beta \in \{A, B\}$, the functions $\psi_\beta : \mathbb{T} \times \Delta^K \times \Delta^K \to \mathbb{R}$ are such that for every $t \in \mathbb{T}^-_{T_0-1}$, $\psi_\beta(t, \cdot, \cdot)$ are a strictly convex functions attaining its minimum value on the diagonal set (3.3).
- (B2) the functions $\phi : \mathbb{R} \to \mathbb{R}$ and $\Gamma_\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in \{A, B\}$ are convex functions with a unique minimum at 0.
- (B3) for each $\beta \in \{A, B\}$, the bidding price functions p_{β,t,x_β} are strictly convex.

Proposition 1. *Let $\alpha \in (0,1)$ be fixed and assume that $(B1)$-$(B3)$ and $(A4)$ hold. Then the minimization problem of Theorem 1 has a unique solution. Furthermore, items a), b), c), d) and e) of Theorem 1 apply.*

Proof. Recall the notation introduced in the proof of Theorem 1. Note that assumptions $(B3)$ and $(A4)$ guarantee that the set $\mathbb{S}$ defined in (4.2) is strictly convex and that assumptions $(B1)$ and $(B2)$ imply that the funtional J is strictly convex. The result then follows from standard results in convex optimization. □

The optimal solution of Proposition 1 will be the trajectory of a dynamical system defined by the Euler-Lagrange equation associated with the first order conditions given by Kuhn-Tucker Theorem applied to the minimization problem of Theorem 1.

4.3. **Qualitative properties.** The minimizers of (3.5) can be seen as the orbits of a finite-time dynamical system with a common initial condition $(Q_A(0), Q_B(0)) = (Q_A^0, Q_B^0)$ and a terminal condition on the set

$$\{(Q_A, Q_B) \in \Delta^K \times \Delta^K : p_{A,T_0,x_A}(Q_A) = p_{B,T_0,x_B}(Q_B)\}.$$

Note that due to the non-uniqueness of solutions, each solution starting from $(Q_A^0, Q_B^0) \in \Delta^K \times \Delta^K$ may be seen as a local minimizer of (3.5). A consequence of item $b)$ in the previous theorem is that the set

$$C = \{(Q_A, Q_B) \in \Delta^K \times \Delta^K : p_{A,u,x_A}(Q_A(u)) = p_{B,u,x_B}(Q_B(u))\}$$

is invariant under the finite-time dynamics.

The next result provides an additional characterization for the minimizers of (3.5).

Proposition 2. *Assume that the assumptions of Theorem 1 hold and let $Q^* = \{(Q_A^*(t), Q_B^*(t))\}_{t=0}^{T_0}$ be a minimizer of (3.5). If the functions ϕ, Γ_A and Γ_B are constant, then for every $t < T_0$ we have that*

$$Q_A^*(t) = Q_A^0 \qquad \textit{and} \qquad Q_B^*(t) = Q_B^0.$$

Proof. Let $t < T_0$ be such that

$$Q_A^*(t) \neq Q_A^0 \qquad \text{or} \qquad Q_B^*(t) \neq Q_B^0.$$

Consider the finite sequence

$$X^\lambda(u) = \begin{cases} X^\lambda(u) = Q^*(u), & \text{if } u \neq t \\ X^\lambda(u) = (1-\lambda)Q^*(u) + \lambda Q_A^0, & \text{if } u = t \end{cases}$$

and define the map $G : [0,1] \to \mathbb{R}$ given by $G(\lambda) = J(X^\lambda)$. Then G is decreasing with λ contradicting minimality of Q^*. □

We will assume that the conditions of Theorem 1 are satisfied for the remaining of this section. Fix $x_A, x_B \in \mathbb{R}$, let Δ^+ denote the set

$$\Delta^+ = \left\{(Q_A^0, Q_B^0) \in \Delta^K \times \Delta^K : p_{A,0,x_A}(Q_A^0) \geq p_{B,0,x_B}(Q_B^0)\right\},$$

and let $\mathcal{E}$ denote the set

$$\mathcal{E} = \Delta^+ \times [0,1].$$

We define the *common price* correspondence $p^* : \mathcal{E} \to \mathbb{R}$ by

$$\begin{aligned} p^*(Q_A^0, Q_B^0, \alpha) &= \left\{p_{A,T_0,x_A}(Q_A^*(T_0)) : \{Q^*(t)\}_{t\in\mathbb{T}_{T_0}^-} \text{ is a solution of (3.5)}\right\} \\ &= \left\{p_{B,T_0,x_B}(Q_B^*(T_0)) : \{Q^*(t)\}_{t\in\mathbb{T}_{T_0}^-} \text{ is a solution of (3.5)}\right\}. \end{aligned}$$

Note that p^* is a single-valued map whenever the price functions enjoy strict convexity properties (see Proposition 1). Since the common price correspondence is the evaluation of the bidding price functions at time T_0, when the bargaining procedure ends, it corresponds to the actual price that the asset is traded for.

Proposition 3. *The common price correspondence p^* is continuous on $\mathcal{E}$.*

Proof. Follows from assumption $(A3)$ and item $e)$ of Theorem 1. □

The next result describes the dependence of the common price correspondence p^* with respect to the relative bargaining power $\alpha \in [0, 1]$.

Proposition 4. *Assume that the assumptions of Theorem 1 hold and fix the agents initial beliefs $(Q_A^0, Q_B^0) \in \Delta^+$. Let $p^*(\alpha)$ denote the dependence of the common price correspondence on the agents relative bargaining power α. If the functions ϕ, Γ_A and Γ_B are non-constant, the following statements hold:*

(i) *if $\alpha = 0$ then $Q_B^*(t) = Q_B^0$ for every $t \in \mathbb{T}_{T_0}^-$ and $p_{B,T_0,x_B}(Q_B^0) \in p^*(0)$;*
(ii) *if $\alpha = 1$ then $Q_A^*(t) = Q_A^0$ for every $t \in \mathbb{T}_{T_0}^-$ and $p_{A,T_0,x_A}(Q_A^0) \in p^*(1)$;*
(iii) *the graph of $p^*(\alpha)$ is the union of graphs of increasing functions with α, one for each minimizer of* (3.5).

Proof. The proofs of items (ii) and (iii) are similar to Theorem 1 items $b)$ and $c)$ and we skip it. Item (iii) follows by Theorem 1, the form of functional J in (3.5) and the definition of the correspondence $p^*(\alpha)$. □

We remark that the final beliefs $Q_A^*(T_0), Q_B^*(T_0) \in \Delta^K$, and hence the price at which the asset F is finally traded, depend heavily on the relative bargaining power α. This confirms our intuition, as posed in the motivation (Section 2), that the relative rigidity of the beliefs of the two agents influences the price at which the trade is realized. In the limiting situations where α is either zero or one, which implies that either agent A or agent B is unwilling to change beliefs, whereas the counterparty is very flexible (or less powerful), the price is influenced by the party having stronger bargaining skills. This situation is reminiscent of the asymmetric Nash bargaining equilibrium [5, 6]. Finally, we note that, as seen in Proposition 2, the effect of the functions Γ_A and Γ_B quantifies the influence that the beliefs of agent B, as induced by the stated prices by agent A, have on the belief update of agent A, and vice-versa.

5. Alternative ways of determining the bidding price functions

In the previous section we have shown that our variational scheme for the belief update and bargaining procedure is well posed and has certain desirable qualitative properties as long as the price functions, which are used by the agents when bidding, are continuous with respect to the beliefs. Furthermore, we have seen that the prices are unique if this functions enjoy certain concavity properties. In this section we show that these assumptions are realistic and hold for a number of specific pricing schemes such as indifference pricing using the full financial market or indifference pricing using only the riskless asset. In particular, in Section 5.2 we offer an example of a pricing scheme based on indifference pricing using the full financial market that guarantees continuity of the price function with respect to the beliefs, whereas in Section 5.3 we offer an alternative pricing scheme based on indifference pricing for investors focusing only on the riskless asset which guarantees strict convexity properties for the price functions.

5.1. **Portfolio optimization under subjective beliefs.** In this subsection we provide a detailed analysis of the agents trading strategies and the corresponding maximum expected utility functions. We focus on results related to the dependence of the solution of this problem with respect to the beliefs of the agents. For the completeness of the paper, we also provide certain results related with the dependence of the solution to this problem with respect to the initial wealth and the chosen portfolio, which are well known, in appendix A.

For each $\beta \in \{A, B\}$, $x_\beta \in \mathbb{R}$ and $t \in \mathbb{T}^-_{T-1}$, we define the functional

$$L_{\beta,t,x_\beta} : (V_\beta)^{T-t} \to \mathbb{R}$$

as

$$L_{\beta,t,x_\beta}(Q, p, F; \pi) = E_Q\Big[U_\beta\Big(T, X^{\beta,\pi}_{0,x_\beta}(T,\omega)\Big)\Big|\mathcal{F}_t\Big] \ , \tag{5.1}$$

where the notation $L_{\beta,t,x_\beta}(Q, p, F; \pi)$ means that L is considered as a function of π, whereas Q, p, F are considered as fixed parameters.

Theorem 2. *Let $\beta \in \{A, B\}$. For each fixed $F \in B_T(\Omega)$, for every $t \in \mathbb{T}^-_{T-1}$ and each $x_\beta \in \mathbb{R}$, the optimal control problem*

$$\max_{\pi\in(V_\beta)^{T-t}} L_{\beta,t,x_\beta}(Q, p, F; \pi)$$

has a unique solution $\pi^*_{x_\beta,t} := \Pi(Q, p, x_\beta)$, where $\Pi : \Delta^K \times \mathbb{R}^2 \to (V_\beta)^{T-t}$ is a continuous function.

Proof. For simplicity of exposition, let us introduce the notation

$$z(t) = X^{\beta,\pi}_{0,x_\beta}(t).$$

We will prove that

$$\mathcal{V}_\beta(s, y) = \sup_{\pi\in V_{\beta,s,y}} E_Q\Big[\mathcal{V}_\beta(s', X^{\beta,\pi}_{s,y}(s'))|\mathcal{F}_s\Big] := \overline{\mathcal{V}_\beta}(s, y), \tag{5.2}$$

for every $s' \in \mathbb{T}$ such that $0 \le s \le s' \le T$. By definition of supremum, we have that

$$\mathcal{V}_\beta(s, y) \ge J_\beta(s, y, \pi) = E_Q[U_\beta(T, X^{\beta,\pi}_{s,y}(T))|\mathcal{F}_s]. \tag{5.3}$$

Recalling that $\mathcal{F}_s \subseteq \mathcal{F}_{s'}$, we have that

$$\begin{aligned} E_Q[U_\beta(T, X^{\beta,\pi}_{s,y}(T))|\mathcal{F}_s] &= E_Q\Big[E_Q[U_\beta(T, X^{\beta,\pi}_{s,y}(T))|\mathcal{F}_{s'}]\Big|\mathcal{F}_s\Big] \\ &= E_Q\Big[E_Q[U_\beta(T, X^{\beta,\pi}_{s',z(s')}(T))|\mathcal{F}_{s'}]\Big|\mathcal{F}_s\Big]. \end{aligned}$$

Recalling that the definition of the functional J, we obtain that

$$E_Q\Big[U_\beta\Big(T, X^{\beta,\pi}_{s,y}(T)\Big)\Big|\mathcal{F}_s\Big] = E_Q[J_\beta(s', z(s'), \pi)|\mathcal{F}_s] \tag{5.4}$$

and using (5.3) and (5.4), we obtain that

$$\mathcal{V}_\beta(s, y) \ge E_Q\Big[J_\beta(s', z(s'), \pi)\Big|\mathcal{F}_s\Big]$$

for every $\pi \in V_{\beta,s,y}$. Taking the supremum over $\pi \in V_{\beta,s,y}$, we get

$$\mathcal{V}_\beta(s, y) \ge \overline{\mathcal{V}_\beta}(s, y). \tag{5.5}$$

Conversely, for any $\epsilon > 0$ there exists $\pi_\epsilon \in V_{\beta,s,y}$ such that

$$\mathcal{V}_\beta(s,y) - \epsilon \le J_\beta(s,y,\pi_\epsilon).$$

Using (5.4) once more, we obtain that $J_\beta(s,y,\pi_\epsilon) = E_Q\Big[J_\beta(s', z_\epsilon(s'), \pi_\epsilon)\Big|\mathcal{F}_s\Big]$, where $z_\epsilon(s') = X^{\beta,\pi_\epsilon}_{0,x_\beta}(s')$. Since $J_\beta(s', z_\epsilon(s'), \pi_\epsilon) \le \mathcal{V}_\beta(s', z_\epsilon(s')) \le \overline{\mathcal{V}_\beta}(s,y)$, we obtain that

$$\mathcal{V}_\beta(s,y) - \epsilon \le \overline{\mathcal{V}_\beta}(s,y). \tag{5.6}$$

Combining (5.5) and (5.6), we conclude the proof of the dynamic programming equation (5.2).

The dynamic programming equation obtained above decomposes the optimal control problem (3.2) into a finite set of static optimization problems of the form

$$V(s,y) = \sup_{\pi \in V_{\beta,s,y}} E_Q\Big[V(s+1, X^{\beta;\pi}_{s,y}(s+1))\Big|\mathcal{F}_s\Big], \tag{5.7}$$

with $V(T,y) = U_\beta(T,y)$. Compactness of $V_{\beta,s,y}$ ensures that proceeding by backwards induction one obtains the optimal portfolio $\pi_{x_\beta,t}(u)$ for every $u \in \{t+1,\dots,T\}$. By Lemma A.2, we have that the value function V is strictly concave with respect to $y \in \mathbb{R}$. Hence, since $X^{\beta,\pi}_{s,y}(s+1)$ is an affine function of π, we obtain that $E_Q\Big[V(s+1, X^{\beta,\pi}_{s,y}(s+1)\Big|\mathcal{F}_s\Big]$ is also strictly concave with respect to $\pi \in V_{\beta,s,y}$, proving the uniqueness of $\pi^*_{\beta,t}$.

The continuity of the solution of the optimization problem (3.2) with respect to $Q \in \Delta^K$, $p \in \mathbb{R}$ and $x_\beta \in \mathbb{R}$ is a consequence of Berge's maximum Theorem [1, Ch. VI, Sec. 3], which guarantees continuity of the minimal functional

$$L_{\beta,t,x_\beta}(Q,p,F;\pi^*)$$

with respect to $(Q,p) \in \Delta^K \times \mathbb{R}$ and $x_\beta \in \mathbb{R}$, and upper semicontinuity of the correspondence given by

$$(Q,p,x_\beta) \to \pi^*(Q,p,x_\beta),$$

Since the solution $\pi^*_{x_\beta,t}$ is unique, we get that the previous correspondence is single-valued and therefore continuous. □

For each $\beta \in \{A,B\}$, $x_\beta \in \mathbb{R}$ and $t \in \mathbb{T}$, define agent β maximum expected utility function at time t, $\overline{U}_{\beta,t,x_\beta} : \Delta^K \times \mathbb{R} \times B_T(\Omega) \to \mathbb{R}$ to be

$$\overline{U}_{\beta,t,x_\beta}(Q,p,F) = \sup_{\pi \in (V_\beta)^{T-t}} E_Q\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}(T,\omega)\right)\Big|\mathcal{F}_t\right]. \tag{5.8}$$

Note that the properties obtained previously for the value function $\mathcal{V}_\beta$ hold for $\overline{U}_{\beta,t,x_\beta}$.

Corollary 1. *For each $\beta \in \{A,B\}$, every fixed $F \in B_T(\Omega)$ and $t \in \mathbb{T}^-_{T-1}$, the map $\overline{U}_{\beta,t,x_\beta} : \Delta^k \times \mathbb{R} \to \mathbb{R}$ is well defined as a function of $(Q,p) \in \Delta^K \times \mathbb{R}$. Moreover, $\overline{U}_{\beta,t,x_\beta}$ is a continuous function of its arguments, as well as the parameter $x_\beta \in \mathbb{R}$ denoting the initial wealth.*

Proof. This result is a straightforward consequence of the previous theorem. Note that $\overline{U}_{\beta,t,x_\beta}$ is a composition of the function L_{β,t,x_β} in (5.1) with the optimal control $\pi^*_{\beta,t}$ of (3.2). Since both are continuous with respect to $(Q,p) \in \Delta^K \times \mathbb{R}$, so is $\overline{U}_{\beta,t,x_\beta}$. Similar arguments apply for continuity with respect to x_β. □

Corollary 2. *For each $\beta \in \{A,B\}$, every fixed $F \in B_T(\Omega)$, $t \in \mathbb{T}^-_{T-1}$ and every $x_\beta \in \mathbb{R}$, the map $\overline{U}_{\beta,t,x_\beta}$ is strictly increasing and strictly concave with respect to $l_\beta p_\beta$, where $l_A = 1$ and $l_B = -1$.*

Proof. Consequence of Lemma A.2 and the definition of $\overline{U}_{\beta,t,x_\beta}$. □

Proposition 5. *For each $\beta \in \{A,B\}$, every fixed $F \in B_T(\Omega)$, $t \in \mathbb{T}^-_{T-1}$ and $x_\beta \in \mathbb{R}$, the map $\overline{U}_{\beta,t,x_\beta}$ is convex with respect to $Q \in \Delta^K$.*

Proof. Let $Q^1, Q^2 \in \Delta^K$ and $\lambda \in (0,1)$ be arbitrary, and let

$$Q = \lambda Q^1 + (1-\lambda)Q^2.$$

By definition of $\overline{U}_{\beta,t,x_\beta}$ we have that

$$\overline{U}_{\beta,t,x_\beta}(\lambda Q^1 + (1-\lambda)Q^2, p, F) = \sup_{\pi \in (V_\beta)^{T-t}} E_{\lambda Q^1 + (1-\lambda)Q^2}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right],$$

where the arguments of $X^{\beta,\pi}_{0,x_\beta}(T,\omega)$ have been dropped for simplicity of notation. Using the linearity of the expected value we obtain that

$$\begin{aligned}&\overline{U}_{\beta,t,x_\beta}(\lambda Q^1 + (1-\lambda)Q^2, p, F)\\ &\quad= \sup_{\pi \in (V_\beta)^{T-t}} \left(\lambda E_{Q^1}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right] + (1-\lambda)E_{Q^2}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right]\right).\end{aligned}$$

Applying the properties of the supremum

$$\begin{aligned}\overline{U}_{\beta,t,x_\beta}(\lambda Q^1 + (1-\lambda)Q^2, p, F) &\le \sup_{\pi \in (V_\beta)^{T-t}} \lambda E_{Q^1}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right]\\ &\quad + \sup_{\pi \in (V_\beta)^{T-t}} (1-\lambda)E_{Q^2}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right]\\ &= \lambda \sup_{\pi \in (V_\beta)^{T-t}} E_{Q^1}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right]\\ &\quad + (1-\lambda) \sup_{\pi \in (V_\beta)^{T-t}} E_{Q^2}\left[U_\beta\left(T, X^{\beta,\pi}_{0,x_\beta}\right)\Big|\mathcal{F}_t\right].\end{aligned}$$

Therefore, we conclude that

$$\overline{U}_{\beta,t,x_\beta}(\lambda Q^1 + (1-\lambda)Q^2, p, F) \le \lambda \overline{U}_{\beta,t,x_\beta}(Q^1, p, F) + (1-\lambda)\overline{U}_{\beta,t,x_\beta}(Q^2, p, F).$$

Thus, $\overline{U}_{\beta,t,x_\beta}$ is convex with respect to $Q \in \Delta^K$. □

5.2. **Indifference bidding price functions.** In this section we introduce indifference bidding price functions in the setup of our problem. Moreover, we provide a detailed analysis of its properties. Namely, we obtain that this class of price functions is continuous with respect to the agents' beliefs, thus providing an example of application of Theorem 1.

For each $\beta \in \{A, B\}$, $x_\beta \in \mathbb{R}$ and $t \in \mathbb{T}^-_{T_0}$, the agent β indifference price associated with the contingent claim $F \in B_T(\Omega)$ is the function $p_{\beta,t,x_\beta} : \Delta^K \times B_T(\Omega) \to \mathbb{R}$ implicitly defined through

$$\overline{U}_{\beta,t,x_\beta}(Q, 0, 0) = \overline{U}_{\beta,t,x_\beta}(Q, p_{\beta,t,x_\beta}, F)\ . \tag{5.9}$$

The concept of indifference price function can be interpreted in the following way: it is the exact value that leaves a given agent, trading in the underlying financial market, indifferent in what concerns buying or not buying the contingent claim with respect to maximum expected utility. It should be remarked that in a incomplete market the indifference prices for agents A and B need not coincide. Moreover, the indifference price functions may depend on the agents initial wealth.

Lemma 5.1. *For each $\beta \in \{A, B\}$, every $t \in \mathbb{T}^-_{T_0}$ and every $x \in \mathbb{R}$, the indifference price $p_{\beta,t,x} : \Delta^K \times B(\Omega) \to \mathbb{R}$ is well defined.*

Proof. Recall the definition of indifference price in (5.9). Note that $\overline{U}_{\beta,t,x_\beta}$ is a strictly increasing and strictly concave function of $l_\beta p_{\beta,t,x}$ by Corollary 2. Resorting to the Implicit Function Theorem, we obtain that there exists a unique function $p_{\beta,t,x} : \Delta^K \times B(\Omega) \to \mathbb{R}$ implicity defined by (5.9). □

Lemma 5.2. *For each $\beta \in \{A, B\}$, $t \in \mathbb{T}^-_{T_0}$ and $x_\beta \in \mathbb{R}$, the indifference price $p_{\beta,t,x_\beta} : \Delta^K \times B_T(\Omega) \to \mathbb{R}$ satisfies the inequalities*

$$\min_{\omega\in\Omega} F[\omega] \leq p_{\beta,t,x_\beta}(Q_\beta, F) \leq \max_{\omega\in\Omega} F[\omega].$$

for every $Q_A, Q_B \in \Delta^K$ and every $F \in B_T(\Omega)$.

Proof. We will only prove the case $\beta = A$, the proof of the case $\beta = B$ being similar. For simplicity of notation we will denote p_{A,T_0,x_A} by p_A and p_{B,T_0,x_B} by p_B. Recall that

$$X^{A,\pi}_{0,x_A}(t) = x_A + p_A I_{\{t\geq T_0\}} - F I_{\{t=T\}} + \sum_{u=1}^{t}\sum_{j=0}^{k} \pi_j(u)\Delta S_j(u).$$

Hence, we have that $X_F(T) = x_A + p_A - F + G_T$, where G_T is the gain process defined as

$$G_T = \sum_{u=1}^{t}\sum_{j=0}^{k} \pi_j(u)\Delta S_j(u).$$

Since Ω is a finite set and the contingent claim F is non-constant, it is easy to check that

$$\min_{\omega\in\Omega} F[\omega] \leq F[\omega] \leq \max_{\omega\in\Omega} F[\omega]\ , \qquad \omega \in \Omega.$$

For simplicity of exposition, let us introduce the notation

$$\underline{F} = \min_{\omega\in\Omega} F[\omega] \quad \text{and} \quad \overline{F} = \max_{\omega\in\Omega} F[\omega].$$

Clearly, we have that

$$x_A + p_A - \overline{F} + G_T \leq x_A + p_A - F + G_T \leq x_A + p_A - \underline{F} + G_T. \tag{5.10}$$

We choose $\pi^*_{0,0}, \pi^*_{p,\underline{F}} \in (V_\beta)^{T-t}$ in such a way that maximizes the functions $E_Q[U(x_A + G_T)|\mathcal{F}_t]$ and $E_Q[U(x_A + p_A - \underline{F} + G_T)|\mathcal{F}_t]$, respectively. These optimal trading strategies are guaranteed to exist by Theorem 2. We now observe that

$$\overline{U}_{\beta,t,x_A}(Q,p_A,\underline{F}) = \overline{U}_{\beta,t,x_A+p_A-\underline{F}}(Q,0,0) \ .$$

Moreover, from the definition of the value function $\mathcal{V}_\beta$ in (A.1) we have that

$$\overline{U}_{\beta,t,x_A+p_A-\underline{F}}(Q,0,0) = \mathcal{V}(x_A+p_A-\underline{F},t).$$

Similarly, we get that

$$\overline{U}_{\beta,t,x_B-p_B+\overline{F}}(Q,0,0) = \mathcal{V}(x_B-p_B+\overline{F},t).$$

Combining (5.10) with Lemma A.2, we obtain

$$\mathcal{V}(x_A+p_A-\overline{F},t) \le \mathcal{V}(x_A,t) \le \mathcal{V}(x_A+p_A-\underline{F},t).$$

The Inverse Function Theorem and Lemma A.2 guarantee that $\mathcal{V}(\cdot,t)$ has a unique inverse. Thus, solving the previous inequalities with respect to p_β, we conclude that

$$\underline{F} \le p_\beta \le \overline{F}.$$

□

Lemma 5.3. *For each $\beta \in \{A,B\}$, every fixed $F \in B_T(\Omega)$ and $t \in \mathbb{T}_{T_0}^-$, the indifference price functions p_{β,t,x_β} are continuous with respect to Q and differentiable with respect to x_β.*

Proof. Consequence of Lemma 5.1 and Corollary 2. □

We remark that the indifference bidding price functions p_A and p_B are not necessarily concave or convex with respect to Q_A and Q_B.

5.3. **Indifference bidding price functions focusing on the riskless asset.** We will now provide an example of a family of price functions satisfying the strict convexity assumptions of the price functions, which according to Proposition 1 guarantee a unique price for the asset, as obtained from the solution to the minimization problem (3.5).

For each $\beta \in \{A,B\}$, $x_\beta \in \mathbb{R}$ and $t \in \mathbb{T}_{T_0}^-$, define agent β price function $P_{\beta,t,x_\beta} : \Delta^K \times B_T(\Omega) \to \mathbb{R}$ through the implicit relation

$$U_\beta\left(T, \frac{S_0(T,\omega)}{S_0(t,\omega)} X_{0,x_\beta}^{\beta,\pi}(t)\right) = \\ E_{Q_\beta}\left[U_\beta\left(T, \frac{S_0(T,\omega)}{S_0(t,\omega)} X_{0,x_\beta}^{\beta,\pi}(t) + \frac{S_0(T,\omega)}{S_0(T_0,\omega)} l_\beta P_{\beta,t,x_\beta} - l_\beta F\right)\right] \tag{5.11}$$

This is an alternative price bidding rule where the investor used the full financial market up to time t where the bid is placed to reach a level of wealth $X_{0,x_\beta}^{\beta,\pi}(t)$ and then simply used the riskless asset to invest this wealth in order to estimate the reservation price for the asset being traded.

Lemma 5.4. *For each $\beta \in \{A,B\}$, $t \in \mathbb{T}_{T_0}^-$ and $x \in \mathbb{R}$, the price function $P_{\beta,t,x} : \Delta^K \times B_T(\Omega) \to \mathbb{R}$ is well defined.*

Proof. We prove the statement for the price function of agent A, the proof for agent B being similar. To simplify notation we will denote $X^{\beta,\pi}_{0,x_\beta}(t)$ by $X(t)$. Recall that the expected utility function of agent A for the contingent claim F, which we will denote by $\overline{V}$, is given by

$$\begin{aligned}\overline{V}(P) &= E_{Q_A}\left[U_A\left(T, \frac{S_0(T,\omega)}{S_0(t,\omega)}X(t) + \frac{S_0(T,\omega)}{S_0(T_0,\omega)}P - F\right)\right] \\ &= \sum_{k=1}^{K} Q_A^k U_A\left(T, \frac{S_0(T,\omega_k)}{S_0(t,\omega_k)}X(t) + \frac{S_0(T,\omega_k)}{S_0(T_0,\omega_k)}P - F[\omega_k]\right)\end{aligned}$$

Note that the factors $S_0(T,\omega)/S_0(t,\omega)$ and $S_0(T,\omega)/S_0(T_0,\omega)$ are both greater or equal than one. Since the functions

$$U_A\left(T, \frac{S_0(T,\omega_k)}{S_0(t,\omega_k)}X(t) + \frac{S_0(T,\omega_k)}{S_0(T_0,\omega_k)}P - F[\omega_k]\right)$$

are strictly increasing concave with respect to P for every $k \in \{1,\ldots,K\}$, we get that so is $\overline{V}(P)$, being a linear combination of such functions. Thus, the expected utility $\overline{V}$ is invertible. Therefore, the price P_A of agent A is well-defined as the solution of the equality

$$U_A(T, X(t)) = \overline{V}(P_A(Q_A)) \ , \tag{5.12}$$

for each $Q_A \in \Delta^K$. □

Lemma 5.5. *For each $\beta \in \{A,B\}$, $t \in \mathbb{T}^-_{T_0}$ and $x \in \mathbb{R}$, the price functions $l_\beta P_{\beta,t,x}$ are strictly concave with respect to $Q_\beta \in \Delta^K$.*

Proof. We prove the statement for the price function of agent A, the proof for agent B being similar. Let $Q_A^1, Q_A^2 \in \Delta^K$ and $0 \le \lambda \le 1$. We introduce the following simplifications to the notation

$$\begin{aligned}P_A^1 &= \frac{S_0(T,\omega)}{S_0(T_0,\omega)}P_A(Q_A^1) \\ P_A^2 &= \frac{S_0(T,\omega)}{S_0(T_0,\omega)}P_A(Q_A^2) \\ P_A^\lambda &= \frac{S_0(T,\omega)}{S_0(T_0,\omega)}P_A(\lambda Q_A^1 + (1-\lambda)Q_A^2) \ ,\end{aligned}$$

and

$$X = \frac{S_0(T,\omega)}{S_0(t,\omega)}X(t) \ .$$

Moreover, we will drop the explicit dependence of U_A on T.

From equality (5.11) we get that

$$\begin{aligned}U_A(X) &= E_{Q_A^j}\left[U_A(X + P_A^j - F)\right] \ , \qquad j = 1,2 \\ U_A(X) &= E_{\lambda Q_A^1 + (1-\lambda)Q_A^2}\left[U_A(X + P_A^\lambda - F)\right] \ .\end{aligned}$$

From the equalities above, we obtain that

$$\begin{aligned}&E_{\lambda Q_A^1 + (1-\lambda)Q_A^2}\left[U_A(X + P_A^\lambda - F)\right] \\ &= \lambda E_{Q_A^1}\left[U_A(X + A^1 - F)\right] + (1-\lambda)E_{Q_A^2}\left[U_A(X + P_A^2 - F)\right]\end{aligned}$$

which is equivalent to

$$\begin{aligned} & E_{\lambda Q_A^1+(1-\lambda)Q_A^2}\left[U_A(X+P_A^\lambda-F)\right] \qquad (5.13) \\ & \quad = E_{\lambda Q_A^1+(1-\lambda)Q_A^2}\left[U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F)\right]+I_{Q_A^1,Q_A^2}(\lambda)\ , \end{aligned}$$

where $I_{Q_A^1,Q_A^2}(\lambda)$ is given by

$$\begin{aligned} I_{Q_A^1,Q_A^2}(\lambda) &= \lambda E_{Q_A^1}\left[U_A(X+P_A^1-F)\right]+(1-\lambda)E_{Q_A^2}\left[U_A(X+P_A^2-F)\right] \\ & \quad -E_{\lambda Q_A^1+(1-\lambda)Q_A^2}\left[U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F)\right]\ . \end{aligned}$$

Assume that $I_{Q_A^1,Q_A^2}(\lambda)$ is non-negative for every $Q_A^1, Q_A^2 \in \Delta^K$ and every $\lambda \in [0,1]$. Then, we obtain from (5.13) that

$$\begin{aligned} & E_{\lambda Q_A^1+(1-\lambda)Q_A^2}\left[U_A(X+P_A^\lambda-F)\right] \\ & \quad \geq E_{\lambda Q_A^1+(1-\lambda)Q_A^2}\left[U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F)\right] \end{aligned}$$

Since the previous inequality holds for arbitrary measures Q_A^1 and Q_A^2, we get that

$$U_A(X+P_A^\lambda-F) \geq U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F)$$

and by monotonicity of the utility function U_A we obtain from the previous inequality that

$$P_A^\lambda \geq \lambda P_A^1+(1-\lambda)P_A^2\ ,$$

thus proving concavity of the seller price function P_A.

We will now see that $I_{Q_A^1,Q_A^2}(\lambda)$ is non-negative for every $Q_1, Q_2 \in \Delta^K$ and $\lambda \in [0,1]$. We first note that if $P_A^1 = P_A^2$, then $I_{Q_A^1,Q_A^2}(\lambda)$ is identically zero. It remains to see that $I_{Q_A^1,Q_A^2}(\lambda)$ is non-negative provided $P_A^1 \neq P_A^2$. Without loss of generality, we assume that $P_A^1 > P_A^2$. Rearranging terms, we write $I_{Q_A^1,Q_A^2}(\lambda)$ as

$$\begin{aligned} & I_{Q_A^1,Q_A^2}(\lambda) = \\ & \quad \lambda E_{Q_A^1}\left[U_A(X+P_A^1-F)-U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F)\right] \qquad (5.14) \\ & \quad +(1-\lambda)E_{Q_A^2}\left[U_A(X+P_A^2-F)-U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F)\right]\ . \end{aligned}$$

Since U_A is differentiable, we obtain that there exists $D_1 \in (\lambda P_A^1+(1-\lambda)P_A^2, P_A^1)$ such that

$$U_A(X+P_A^1-F)-U_A(X+\lambda P_A^1+(1-\lambda)P_A^2)-F) = (1-\lambda)P_A^1-P_A^2 U_A'(X+D_1-F). \qquad (5.15)$$

Similarly, there exists $C_2 \in (P_A^2, \lambda P_A^1+(1-\lambda)P_A^2)$ such that

$$U_A(X+P_A^2-F)-U_A(X+\lambda P_A^1+(1-\lambda)P_A^2-F) = -\lambda P_A^1-P_A^2 U_A'(X+C_2-F). \qquad (5.16)$$

Substituting (5.15) and (5.16) in (5.14), we get

$$I_{Q_A^1,Q_A^2}(\lambda) = \lambda(1-\lambda)P_A^1-P_A^2\left(E_{Q_A^1}\left[U_A'(X+C_1-F)\right]-E_{Q_A^2}\left[U_A'(X+C_2-F)\right]\right). \qquad (5.17)$$

Recalling that U_A is increasing, we obtain that for every $C_1 \in (\lambda P_A^1 + (1-\lambda)P_A^2, P_A^1)$ we have

$$E_{Q_A^1}\left[U_A(X + C_1 - F)\right] < E_{Q_A^1}\left[U_A(X + P_A^1 - F)\right] = U_A(X(t)), \qquad (5.18)$$

and for every $C_2 \in (P_A^2, \lambda P_A^1 + (1-\lambda)P_A^2)$ we have

$$E_{Q_A^2}\left[U_A(X + C_2 - F)\right] > E_{Q_A^2}\left[U_A(X + P_A^2 - F)\right] = U_A(X(t)). \qquad (5.19)$$

Combining concavity of U_A with (5.18) and (5.19), we obtain the inequality

$$E_{Q_A^1}\left[U_A'(X + C_1 - F)\right] > E_{Q_A^2}\left[U_A'(X + C_2 - F)\right],$$

which, combined with (5.17), guarantees that $I_{Q_A^1,Q_A^2}(\lambda)$ is non-negative. □

Corollary 3. *For each $\beta \in \{A, B\}$, $t \in \mathbb{T}_{T_0}^-$ and $x \in \mathbb{R}$, the price function $P_{\beta,t,x} : \Delta^K \times B(\Omega) \to \mathbb{R}$ satisfies the inequalities*

$$\min_{\omega\in\Omega} F[\omega] \le P_{\beta,t,x}(Q_\beta) \le \max_{\omega\in\Omega} F[\omega]$$

for every $Q_\beta \in \Delta^K$.

Proof. Recall the notation introduced in the proof of Lemma 5.5. Since Ω is a finite set and the contingent claim F is non-constant, it is easy to check that

$$\min_{\omega\in\Omega} F[\omega] \le F[\omega] \le \max_{\omega\in\Omega} F[\omega]\ , \qquad \omega \in \Omega.$$

For simplicity of exposition, as in Lemma 5.2 we use the notation

$$\underline{F} = \min_{\omega\in\Omega} F[\omega] \quad \text{and} \quad \overline{F} = \max_{\omega\in\Omega} F[\omega].$$

From the previous inequalities and monotonicity of the utility function U_A, we obtain that

$$U_A\left(X - P_A + \min_{\omega\in\Omega} F[\omega]\right) \le U_A\left(X - P_A + F[\omega]\right) \le U_A\left(X - P_A + \max_{\omega\in\Omega} F[\omega]\right)$$

where $\omega \in \Omega$. Thus, we get

$$U_A\left(X - P_A + \underline{F}\right) \le E_{Q_A}\left[U\left(X - P_A + F\right)\right] \le U_A\left(X - P_A + \overline{F}\right)\ . \qquad (5.20)$$

Recalling the definition of agent A reservation price, we get that

$$U_A(X) = E_{Q_A}\left[U\left(X - P_A + F\right)\right]\ . \qquad (5.21)$$

Combining (5.20) and (5.21), we obtain

$$U_A\left(X - P_A + \underline{F}\right) \le U_A(X) \le U_A\left(X - P_A + \overline{F}\right)\ .$$

Solving the previous inequalities with respect to P_A, we get the inequality in the statement. Clearly, one can obtain the inequality from agent B from analogous computation. □

6. Conclusion

We have used techniques from stochastic optimal control theory and optimization to study a minimization problem modelling the interaction between two agents trading a contingent claim in a incomplete discrete-time multi-period financial market. We have proved the existence of solutions to this minimization problem and provided a rather complete characterization for its minimizers.

Acknowledgments

D. Pinheiro's research was supported by FCT - Fundação para a Ciência e Tecnologia through the Program POCI 2010, CEMAPRE's strategic project PEst-OE/EGE/UI0491/2011 and the Project "Randomness in Deterministic Dynamical Systems and Applications" (PTDC-MAT-105448-2008). N. Azevedo's research was supported by FCT - Fundação para a Ciência e Tecnologia grant with reference SFRH-BD-67186-2009. S. Xanthopoulos and A.N. Yannacopoulos would like to thank CEMAPRE for its hospitality while completing this work.

Appendix A. Some auxiliary results on portfolio optimization

In this appendix, for completeness of the paper, we collect some results on portfolio optimization and in particular, in connection with the behaviour of the value functions with respect to the initial wealth

Lemma A.1. *For each $\beta \in \{A, B\}$, $x_\beta \in \mathbb{R}$ and $t \in \mathbb{T}_{T-1}^-$, the expected utility functions L_{β,t,x_β} are strictly concave with respect to $\pi \in (V_\beta)^{T-t}$.*

Proof. Let π^1, π^2 be arbitrary distinct elements of $(V_\beta)^{T-t}$. Recall the form of the wealth process $X_{0,x_\beta}^{\beta,\pi}(t)$ given in (3.1) and note that $X_{0,x_\beta}^{\beta,\pi}(t)$ depends linearly on the portfolio process $\pi = \{\pi(u)\}_{u\in\mathbb{T}_1^+}$. Since $U(t,\cdot)$ is strictly concave and using the linearity of the expected value, we obtain

$$L_{\beta,t,x_\beta}(Q,p,F;(\lambda\pi^1+(1-\lambda)\pi^2) > \lambda L_{\beta,t,x_\beta}(Q,p,F;\pi^1)+(1-\lambda)L_{\beta,t,x_\beta}(Q,p,F;\pi^2),$$

for every $\lambda \in (0,1)$. We conclude that $L_{\beta,t,x_\beta}(Q,p,F;\pi)$ is strictly concave with respect to π, as required. □

We resort to dynamic programming methods to deal with the discrete-time stochastic optimal control problem (3.2). Take $s \in \mathbb{T}_{T-1}^-$, $y \in \mathbb{R}$ and define the functional

$$J_\beta(s,y,\pi) = E_Q[U_\beta(T, X_{s,y}^{\beta,\pi}(T,\omega))|\mathcal{F}_s],$$

where $\pi \in V_{\beta,s,y}$ and $V_{\beta,s,y}$ is the set of admissable trading strategies

$$V_{\beta,s,y} = \{\pi \in (V_\beta)^{T-s} : X_{0,x_\beta}^{\beta,\pi}(s) = y\}\ .$$

Clearly, $V_{\beta,s,y}$ is a non-empty and convex set. To see this, let π^1, π^2 be arbitrary elements of $V_{\beta,s,y}$ and note that $\pi^1, \pi^2 \in (V_\beta)^{T-s}$ are such that $X_{0,x_\beta}^{\beta,\pi^1}(s) = y$ and $X_{0,x_\beta}^{\beta,\pi^2}(s) = y$. Then, $\pi^\alpha = \alpha\pi_1 + (1-\alpha)\pi_2 \in (V_\beta)^{T-s}$ is such that $X_{0,x_\beta}^{\beta,\pi^\alpha}(s) = y$. We now define the value function $\mathcal{V}_\beta : \mathbb{T}_{T-1}^- \times \mathbb{R} \to \mathbb{R}$ to be given by

$$\mathcal{V}_\beta(s,y) = \sup_{\pi\in V_{\beta,s,y}} J_\beta(s,y,\pi), \tag{A.1}$$

where J_β is the functional defined above.

Lemma A.2. *For each $\beta \in \{A, B\}$, if there exists an optimal control to the problem (3.2), the function $\mathcal{V}_\beta$ is strictly increasing and strictly concave.*

Proof. We start by proving that $\mathcal{V}_\beta$ is an increasing function of $y \in \mathbb{R}$. Fix $s \in \mathbb{T}^-_{T-1}$ and $y^1, y^2 \in \mathbb{R}$ such that $y^1 < y^2$. Then

$$X^{\beta,\pi}_{s,y^1}(T,\omega) < X^{\beta,\pi}_{s,y^2}(T,\omega)$$

for every $\pi \in (V_\beta)^{T-t}$. By assumption, the function $U_\beta(T,\cdot)$ is strictly increasing. Thus,

$$U_\beta\Big(T, X^{\beta,\pi}_{s,y^1}(T,\omega)\Big) < U_\beta\Big(T, X^{\beta,\pi}_{s,y^2}(T,\omega)\Big)$$

for every $\pi \in (V_\beta)^{T-t}$. As a consequence, we obtain that

$$E_Q\Big[U_\beta\Big(T, X^{\beta,\pi}_{s,y^1}(T,\omega)\Big)|\mathcal{F}_s\Big] < E_Q\Big[U_\beta\Big(T, X^{\beta,\pi}_{s,y^2}(T,\omega)\Big)\Big|\mathcal{F}_s\Big]$$

for every $\pi \in (V_\beta)^{T-t}$. Taking the supremum of the right-hand side over all $\pi \in V_{\beta,s,y^2}$, we obtain that

$$E_Q\Big[U_\beta\Big(T, X^{\beta,\pi}_{s,y^1}(T,\omega)\Big)\Big|\mathcal{F}_s\Big] < \mathcal{V}_\beta(s,y^2)$$

for every $\pi \in (V_\beta)^{T-t}$. Therefore, we conclude that

$$\mathcal{V}_\beta(s,y^1) \le \mathcal{V}_\beta(s,y^2). \tag{A.2}$$

Thus, $\mathcal{V}_\beta$ is increasing with respect to $y \in \mathbb{R}$.

We will now prove that $\mathcal{V}_\beta$ is concave with respect to $y \in \mathbb{R}$. Let $\pi^1, \pi^2 \in V_{\beta,s,y}$, and $\lambda \in (0,1)$. We introduce the notation

$$y^\lambda = \lambda y^1 + (1-\lambda)y^2 \qquad \text{and} \qquad \pi^\lambda = \lambda\pi^1 + (1-\lambda)\pi^2 \;.$$

From the linear dynamics of the wealth process we have that

$$X^{\beta,\pi^\lambda}_{s,y^\lambda}(T,\omega) = \lambda X^{\beta,\pi^1}_{s,y^1}(T,\omega) + (1-\lambda)X^{\beta,\pi^2}_{s,y^2}(T,\omega) \;. \tag{A.3}$$

To check this equality we will consider two cases: (i) $s < t_0$ and (ii) $s \ge t_0$. We will prove the case (i), case (ii) being similar. We have that

$$\begin{aligned}
&\lambda X^{\beta,\pi^1}_{s,y^1}(T,\omega) + (1-\lambda)X^{\beta,\pi^2}_{s,y^2}(T,\omega) = \\
&\quad = \lambda\Big(y^1 + l_\beta(p-F) + \sum_{u=t+1}^{T}\sum_{j=0}^{N}\pi^1_j(u)\Delta S_j(u)\Big) + \\
&\qquad + (1-\lambda)\Big(y^2 + l_\beta(p-F) + \sum_{u=t+1}^{T}\sum_{j=0}^{N}\pi^2_j(u)\Delta S_j(u)\Big) \\
&\quad = \lambda y^1 + (1-\lambda)y^2 + l_\beta(p-F) + \sum_{u=t+1}^{T}\sum_{j=0}^{N}\big(\lambda\pi^1_j(u) + (1-\lambda)\pi^2_j(u)\big)\,\Delta S_j(u)
\end{aligned}$$

By definition of y^λ and π^λ we check that

$$\begin{aligned}
&\lambda X^{\beta,\pi^1}_{s,y^1}(T,\omega) + (1-\lambda) X^{\beta,\pi^2}_{s,y^2}(T,\omega) = \\
&= \quad y^\lambda + l_\beta(p-F) + \sum_{u=t+1}^{T} \sum_{j=0}^{N} \pi_j^\lambda(u) \Delta S_j(u) \\
&= \quad X^{\beta,\pi^\lambda}_{y^\lambda,s}(T,\omega),
\end{aligned}$$

concluding the proof of the equality above. By equality (A.3) and strict concavity of the utility function U_β, we obtain

$$U_\beta(X^{\beta,\pi^\lambda}_{s,y^\lambda}) = U_\beta\Big(\lambda X^{\beta,\pi}_{s,y^1} + (1-\lambda) X^{\beta,\pi}_{s,y^2}\Big) > \lambda U_\beta(X^{\beta,\pi}_{s,y^1}) + (1-\lambda) U_\beta(X^{\beta,\pi}_{s,y^2}),$$

where the arguments of $X^{\beta,\pi}_{s,y}(T,\omega)$ have been dropped for simplicity of notation. Taking the supremum of the left-hand side over all $\pi \in V_{\beta,s,y^\lambda}$, we obtain that

$$\mathcal{V}_\beta(\lambda y^1 + (1-\lambda)y^2) > \lambda E_Q[U(X^{\beta,\pi}_{s,y^1})] + (1-\lambda) E_Q[U(X^{\beta,\pi}_{s,y^2})],$$

where the inequality holds for $\pi^1, \pi^2 \in V_{\beta,s,y}$. We conclude that

$$\mathcal{V}_\beta(\lambda y^1 + (1-\lambda)y^2) \geq \lambda \mathcal{V}_\beta(y^1) + (1-\lambda)\mathcal{V}_\beta(y^2), \tag{A.4}$$

ending the proof of concavity of $\mathcal{V}_\beta$ with respect to $y \in \mathbb{R}$.

In the case where an optimal control $\pi^*_{\beta,t}$ of (3.2) exists, the inequalities (A.2) and (A.4) are strict. □

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