FOCAL DECOMPOSITION, RENORMALIZATION AND SEMICLASSICAL PHYSICS

C. A. A. DE CARVALHO, M. M. PEIXOTO, D. PINHEIRO, AND A. A. PINTO

Abstract. We review some recent results concerning a connection between focal decomposition, renormalization and semiclassical physics. The dynamical behaviour of a family of mechanical systems that includes the pendulum at small neighbourhoods of the equilibrium but after long intervals of time can be characterized through a renormalization scheme acting on the dynamics of this family. We have proved that the asymptotic limit of this renormalization scheme is universal: it is the same for all the elements in the considered class of mechanical systems. As a consequence, we have obtained an asymptotic universal focal decomposition for this family of mechanical systems that can now be used to compute estimates for propagators in semiclassical physics.
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1. Introduction

In [5] we have introduced a renormalization scheme which acts on the dynamics of a family of mechanical systems to which the pendulum belongs, at small neighbourhoods of the equilibrium but after long intervals of time, so that the second order term of the period map can no longer be neglected. The main theorem in that paper states that this renormalization scheme converges under iteration to a family of *asymptotic trajectories*. As a corollary, we obtain a focal decomposition associated with this asymptotic trajectories to which we call *asymptotic universal focal decomposition*.

That was the first step towards a broader research program, proposed by Peixoto and Pinto, connecting renormalization techniques, focal decomposition of differential equations and semiclassical physics. The next steps of this research program include a proof of the convergence of renormalized focal decompositions to the asymptotic universal focal decomposition of [5] and an extension of this renormalization scheme to obtain a 4-dimensional asymptotic universal focal decomposition, i.e. with no restrictions on the base point of the boundary value problem. The ultimate goal of such program would be to deal with applications of this renormalization procedure to semiclassical physics.

We start this paper with some comments about focal decomposition and renormalization. Sections 2.1, 2.2 and 2.3 contain a short overview of the main concepts and results in [5]: namely, we define the renormalization scheme we work with, we state its main result, i.e. the convergence of such renormalization scheme to the asymptotic trajectories, and discuss the existence of the asymptotic universal focal decomposition induced by the asymptotic trajectories. In section 2.4 we define what we mean by a renormalized focal decomposition, state a new result concerning the convergence of the renormalized focal decompositions to the asymptotic universal focal decomposition, and give an idea of the main step towards the proof of this result. In sections 2.5 and 2.6 we state new results concerning the Hamiltonian character of the asymptotic trajectories and its action. Section 3 contains a description of the utility that the asymptotic universal focal decomposition may have in the computation of propagators in semiclassical physics. We conclude with some remarks regarding potential applications for the theoretical results we are developing.

1.1. **Focal decomposition.** The concept of focal decomposition was introduced by Peixoto in [8] (under the name of σ-decomposition), and further developed by Peixoto and Thom in [9]. The starting point was the 2-point boundary value problem for ordinary differential equations of the second order

$$\begin{aligned} \ddot{x} &= f(t, x, \dot{x}) \ , t, x, \dot{x} \in \mathbf{R} \\ x(t_1) &= x_1 \ , \ x(t_2) = x_2 \ , \end{aligned} \tag{1.1}$$

which was formulated for the equation of Euler at the beginnings of the calculus of variations in the first half of the 18th Century. This is the simplest and oldest of all boundary value problems. Accordingly, there is a vast literature, mostly in the context of applied mathematics where one frequently uses the methods of functional analysis. Here we adopt a different point of view: we look for the number $i \in \{0, 1, 2, \dots \infty\} = N$ of solutions of problem (1.1), and how this number varies with the endpoints (t_1, x_1) and (t_2, x_2).

If one of the endpoints in (1.1) is kept fixed, say $(t_1, x_1) = (0, 0)$, then we define σ_i, $i = 1, 2, ..., \infty$ as the set of points (t_2, x_2) such that problem (1.1) has exactly i solutions. We put $(0, 0) \in \sigma_\infty$ and $(0, y) \in \sigma_0$ for $y \neq 0$. Then we say that the restricted two-point problem with base point $(0, 0)$ consists of finding the corresponding focal decomposition of $\mathbf{R}^2$ by the sets σ_i:

$$\mathbf{R}^2 = \sigma_0 \cup \sigma_1 \cup ... \cup \sigma_\infty \ .$$

A notable example of a focal decomposition due to Peixoto and Thom [9], is provided by the focal decomposition of the pendulum equation $\ddot{x} + \sin x = 0$ with base point $(0, 0)$ (see Figure 1). This focal decomposition contains non-empty sets σ_i with all finite indices. Every set σ_{2k-1}, $k = 1, 2, ...$, consists of a 2-dimensional open set plus the cusp-point $(\pm k\pi, 0)$; they all have two connected components. All four connected components of the even-indexed sets σ_{2k} are open-arcs, asymptotic to one of the lines $x = \pm$ and incident to the cusp-points $(\pm k\pi, 0)$; the lines $x = \pm\pi$ are part of σ_1, except for the points $(0, \pm\pi)$ which belong to σ_0.

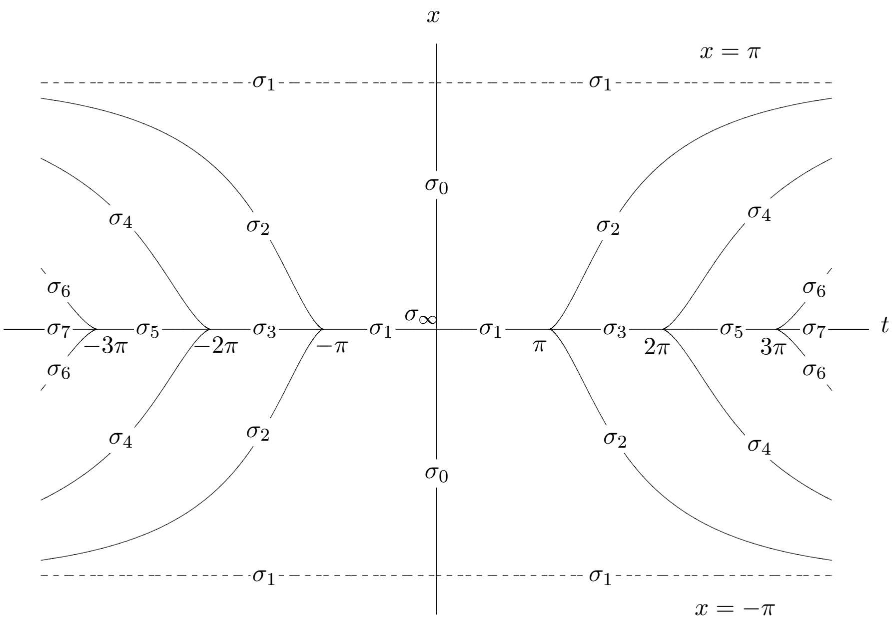

Figure 1: The pendulum's focal decomposition.

1.2. **Renormalization.** The main idea behind renormalization is the introduction of an operator — the renormalization operator — on a space of systems whose action on each system is to remove its small scale behaviour and to rescale the remaining variables to preserve some normalization. If a system converges to some limiting behaviour under iteration of the renormalization operator then we say that such behaviour is universal. Since the renormalization operator relates different scales, such universal behaviour is self-similar. See [5] and references therein for more details on renormalization.

The main subject of [5] is a renormalization scheme acting on the dynamics of a family of mechanical systems that include the pendulum. Our motivation for the introduction of such scheme comes from the restricted focal decomposition with base point $(0,0)$ of the pendulum equation $\ddot{x} + \sin x = 0$ in Figure 1. It turns out that the sequence formed by the even-indexed sets in the pendulum's focal decomposition is approximately self-similar. The renormalization scheme we introduce can then be described in the following way: for a large integer n, we consider the even-indexed set σ_{2n} and, contrary to previous renormalizations, we do not rescale time but just shift it so that its origin is at $t = n\pi$. We then restrict the initial velocities to a small interval so that that the index corresponding to the shifted even-indexed set is equal to one; we complete the procedure by normalizing space in such way a that the shifted even-indexed set is asymptotic to the lines $x = \pm 1$. Under iteration of this renormalization scheme, we obtain asymptotic trajectories that define an asymptotic focal decomposition. Both the asymptotic trajectories and focal decomposition are universal and self-similar.

2. The asymptotic universal focal decomposition and some consequences

In this section we characterize the action of the renormalization scheme introduced in [5] on the dynamics of a family of mechanical systems defined by a Lagrangian function $L(x,\dot{x}) = \dot{x}^2/2 - V(x)$, where $V(x)$ is a given non-isochronous potential, i.e. not all the periodic solutions of the corresponding Euler–Lagrange equation have the same period (see the paper [3] by Bolotin and MacKay and references therein). For more information on mechanical systems see the books [1, 2, 6, 7].

2.1. **Setting.** We consider mechanical systems defined by a Lagrangian function $\mathcal{L} : \mathbf{R}^2 \to \mathbf{R}$ of the form

$$\mathcal{L}\left(q, \frac{\mathrm{d}q}{\mathrm{d}\tau}\right) = \frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 - \mathcal{V}(q) \ , \tag{2.1}$$

where the potential function $\mathcal{V} : \mathbf{R} \to \mathbf{R}$ is a non–isochronous potential. Furthermore, we assume that the potential $\mathcal{V}$ is a C^κ map ($\kappa \geq 5$) with a Taylor expansion at a point $q^* \in \mathbf{R}$ given by

$$\mathcal{V}(q) = \mathcal{V}(q^*) + \frac{\mathcal{V}''(q^*)}{2}(q-q^*)^2 + \frac{\mathcal{V}^{(4)}(q^*)}{4!}(q-q^*)^4 \pm O\left(|q-q^*|^5\right) \ ,$$

where $\mathcal{V}''(q^*) > 0$ and $\mathcal{V}^{(4)}(q^*) \neq 0$. The Euler–Lagrange equation associated with (2.1) is

$$m \frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}q}(q) \ , \tag{2.2}$$

and the corresponding Hamilton equations are given by

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\tau} &= \frac{p}{m} \\ \frac{\mathrm{d}p}{\mathrm{d}\tau} &= -\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}q}(q) \ . \end{aligned} \tag{2.3}$$

Therefore, the point q^* is an elliptic equilibrium of (2.2) (or equivalently, $(q^*, 0)$ is an elliptic equilibrium of (2.3)) and thus, there is a 1–parameter family of periodic orbits covering a neighbourhood of the equilibrium point.

2.2. **Asymptotic universal behaviour for the trajectories.** Since q^* is an elliptic equilibrium of (2.2) there is $\epsilon > 0$ such that for all *initial velocity* $\nu \in [-\epsilon, \epsilon]$ the solutions $q(\nu; \tau)$ of the Euler–Lagrange equation (2.2) with initial conditions $q(\nu; 0) = q^*$ and $\mathrm{d}q/\mathrm{d}\tau(\nu; 0) = \nu$ are periodic. Thus, the *trajectories* $q : [-\epsilon, \epsilon] \times \mathbf{R} \to \mathbf{R}$ of (2.2) are well-defined by $q(\nu; \tau)$ for all $\tau \in \mathbf{R}$ and $\nu \in [-\epsilon, \epsilon]$. Furthermore, there exist $\alpha > 0$ small enough and $N \geq 1$ large enough such that, for every $n \geq N$, the *n-renormalized trajectories* $x_n : [-1, 1] \times [0, \alpha n] \to \mathbf{R}$ are well-defined by

$$x_n(v; t) = (-1)^n \; \Gamma_{n,t}^{-1} \; \mu^{-1} \left[q \left(\Gamma_{n,t} \; \mu \; \omega \; v; \frac{n\pi - \ell t}{\omega} \right) - q^* \right] \; ,$$

where $\Gamma_{n,t}$ is the (n, t)*-scaling parameter*

$$\Gamma_{n,t} = \left(\frac{8t}{3\pi n} \right)^{1/2} \; , \tag{2.4}$$

$\ell = \pm 1$ depending on the sign of $\mathcal{V}^{(4)}(q^*)$ and ω and μ are given by

$$\omega = \left(\frac{V''(q^*)}{m} \right)^{1/2} \; , \qquad \mu = \left(\frac{3! V''(q^*)}{|V^{(4)}(q^*)|} \right)^{1/2} \; . \tag{2.5}$$

Note that ω^{-1} and μ are the natural time and length scales for the dynamical system defined by (2.2). Furthermore, the variables v and t are dimensionless, as well as the (n, t)-scaling parameter $\Gamma_{n,t}$. Therefore, the n-renormalized trajectories $x_n(v; t)$ are dimensionless.

Definition The *asymptotic trajectories* $X_\ell : [-1, 1] \times \mathbf{R}_0^+ \to \mathbf{R}$ are defined by

$$X_\ell(v; t) = v \; \sin\left(\ell t \left(v^2 - 1\right)\right) \; ,$$

where $\ell = \pm 1$ depending on the sign of $\mathcal{V}^{(4)}(q^*)$.

The self-similarity of the asymptotic trajectories is exemplified in Figure 2. For fixed instants of time ($t = 3\pi/2$ in Fig. 2a, $t = 7\pi/2$ in Fig. 2b and $t = 11\pi/2$ in Fig. 2c), the plots correspond to the final position x of the asymptotic trajectories as function of the initial velocity v. Note that Figure 2a is a rescaling of the dashed box in Figure 2b and the smaller dashed box in Figure 2c, while Figure 2b is a rescaling of the larger dashed box in Figure 2c.

The main result in [5] states that the n-renormalized trajectories converge to the asymptotic trajectories as $n \to \infty$.

Theorem 2.1. *There exists* $\beta > 0$ *small enough, such that, for every* $0 < \epsilon < 1/3$*, we have that*

$$\|x_n(v; t) - X_\ell(v; t)\|_{C^0([-1,1] \times [0, \beta n^{1/3-\epsilon}], \mathbf{R})} \; < \; O\left(n^{-3\epsilon/2}\right) \; ,$$

where ℓ *is the sign of* $\mathcal{V}^{(4)}(q^*)$.

2.3. **Asymptotic universal focal decomposition.** The asymptotic trajectories $X_\ell(v; t)$ induce an asymptotic universal focal decomposition of the cylinder $\mathcal{C} = \mathbf{R}_0^+ \times [-1, 1]$ by the sets σ_i whose elements are pairs $(t, x) \in \mathcal{C}$ such that $X_\ell(v; t) = x$ has exactly i solutions $v(t, x) \in [-1, 1]$, each distinct solution corresponding to an asymptotic trajectory connecting the points $(0, 0) \in \mathcal{C}$ and $(t, x) \in \mathcal{C}$. Therefore, for each $i \in \{0, 1, ..., \infty\}$, the set $\sigma_i \subset \mathcal{C}$ contains all points in $(t, x) \in \mathcal{C}$ such that there exist exactly i asymptotic trajectories connecting $(0, 0) \in \mathcal{C}$ and $(t, x) \in \mathcal{C}$.

The following result is a consequence of theorem 2.1.

Theorem 2.2. *There exists an asymptotic universal focal decomposition for the Euler–Lagrange equation* (2.2) *induced by the asymptotic trajectories* $X_\ell(v; t)$.

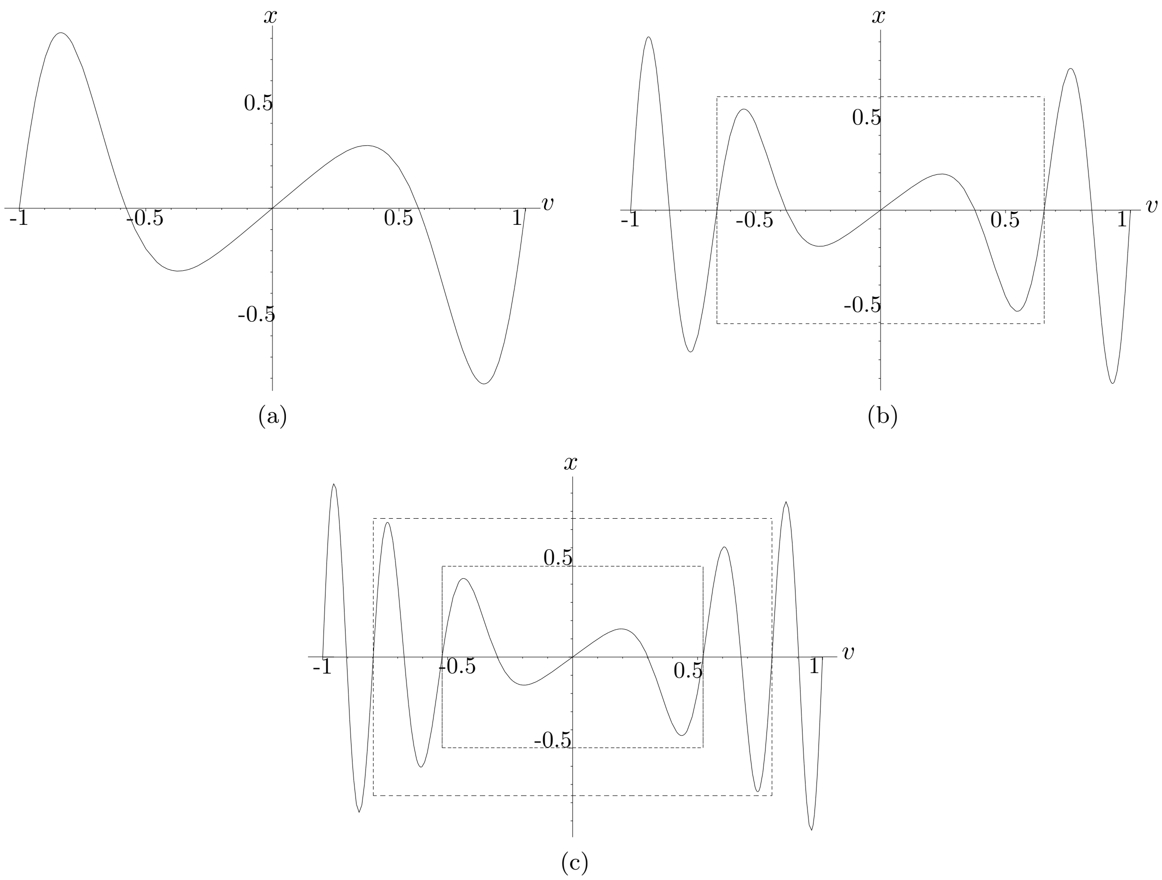

Figure 2: The asymptotic trajectories $X_\ell(v;t)$ plotted as functions of the initial velocity v for fixed instants of time: (a) $t = \frac{3\pi}{2}$, (b) $t = \frac{7\pi}{2}$ and (c) $t = \frac{11\pi}{2}$.

The asymptotic universal focal decomposition is shown in Figure 3. As in the case of the focal decomposition with base point $(0,0)$ of the pendulum equation $\ddot{x} + \sin x = 0$ (see [9, pp. 631, 197]), the asymptotic universal focal decomposition also exhibits non-empty sets σ_i with all finite indices. However, contrary to that focal decomposition, which gives a stratification for the whole $\mathbf{R}^2$, our asymptotic universal focal decomposition gives only a stratification of the half cylinder $\mathcal{C} = \mathbf{R}_0^+ \times [-1,1]$. There are two main reasons for this to happen which we pass to explain. Firstly, our renormalization scheme acts only on periodic orbits, neglecting the high-energy non-periodic orbits, which restrains $X_\ell(v;t)$ to the interval $[-1,1]$. Secondly, we have defined the renormalization operator only for positive times. Noticing that the mechanical systems we renormalize have time-reversal symmetry, one can extend the asymptotic universal focal decomposition to the cylinder $\mathbf{R} \times [-1,1]$ by a symmetry on the x axis.

Some comments about Figure 3 are appropriate here.

For every $k \in \mathbf{N}$, the set σ_{2k} is a 2-dimensional open set with two connected components. The odd-indexed sets σ_{2k-1} are the union of two open arcs, asymptotic to one of the lines $x = \pm 1$ and incident to the cusp-point $((k-1)\pi, 0)$, and a line segment joining the cusp points $((k-1)\pi, 0)$ and $(k\pi, 0)$; the lines $x = \pm 1$ are part of σ_0.

Thus, the even-indexed sets σ_{2k} are 2-dimensional manifolds, while the odd-indexed sets σ_{2k-1} are not manifolds because they contain the cusp-points $(k\pi, 0)$.

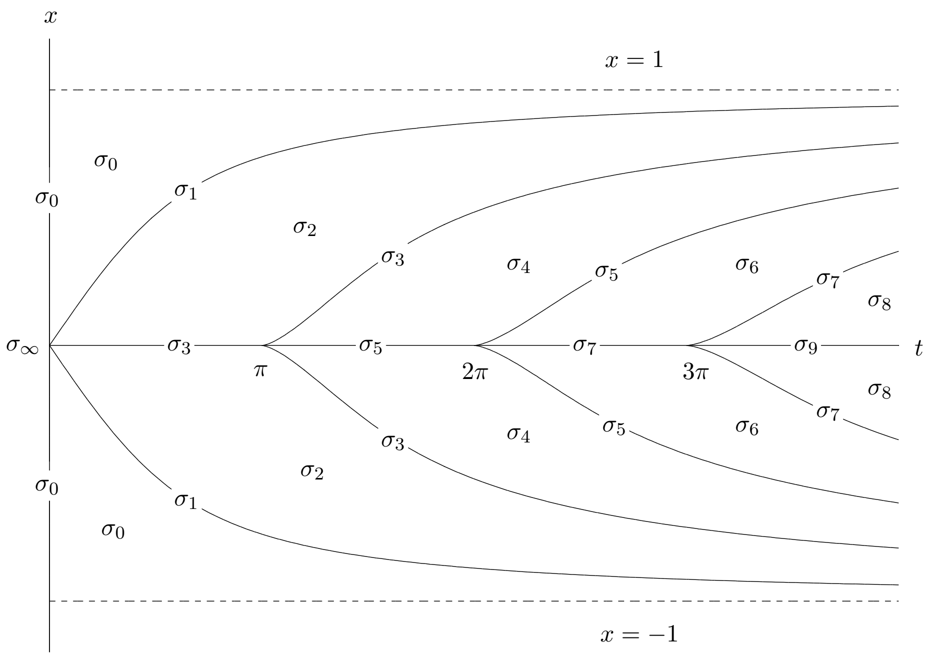

Figure 3: The asymptotic universal focal decomposition.

On the other hand, if we decompose the odd-indexed sets σ_{2k-1} into a cusp-point plus three 1-dimensional manifolds (two open arcs and one line segment), then we get a decomposition of the whole plane into a collection of disjoint connected manifolds. The above decomposition of $\mathcal{C}$ is an example of what is called a stratification of $\mathcal{C}$, the strata being the disjoint connected manifolds into which $\mathcal{C}$ was decomposed. Hence σ_1 consists of two 1-dimensional strata, σ_2 consists of two 2-dimensional strata, σ_3 consists of three 1-dimensional strata and one 0-dimensional strata and so on. To complete the picture, σ_0 consists of two 2-dimensional strata plus the x-axis minus the origin which belongs to σ_∞.

The differences in the indexes of the sets σ_i of the focal decompositions of Figure 1 and Figure 3 are due to the fact that the asymptotic trajectories $X_\ell(v,t)$ have $x = \pm 1$ as equilibria, and can be understood by an analysis of Figure 2.

2.4. **The renormalized focal decompositions.** Let us consider the asymptotic universal focal decomposition of the cylinder $\mathcal{C} = \mathbf{R}_0^+ \times [-1,1]$

$$\mathcal{C} = \cup_{k=0}^{\infty} \sigma_k \ .$$

For each $(t,x) \in \sigma_i$ we define the *index* $i(t,x)$ of (t,x) equal to i.

Note that each n-renormalized trajectory x_n induces a focal decomposition of the cylinder $\mathcal{C}$ by the sets σ_i^n whose elements are pairs $(t,x) \in \mathcal{C}$ such that $x_n(v;t) = x$ has exactly i solutions $v(t,x) \in [-1,1]$. Therefore, the n-renormalized focal decomposition is given by

$$\mathcal{C} = \cup_{k=0}^{\infty} \sigma_k^n \ .$$

Similarly, for each $(t,x) \in \sigma_i^n$ we define the *n-renormalized index* $i_n(t,x)$ of (t,x) as the integer i.

The following result states that the sequence of n-renormalized focal decompositions converges to the asymptotic universal focal decomposition.

Theorem 2.3. *For every $(t,x) \in \mathcal{C} = \mathbf{R}_0^+ \times [-1,1]$, the n-renormalized index $i_n(t,x)$ of (t,x) converges to the index $i(t,x)$.*

We should remark that unlike theorem 2.2, theorem 2.3 is not a consequence of theorem 2.1. The key point on its proof is an extension of the estimates in theorem 2.1 to the C^2 topology, which we leave for a future publication.

2.5. **The Hamiltonian character of the asymptotic trajectories.** The renormalization scheme of section 2.2 can be extended to act on the *velocities*

$$\dot{q}(\nu, \tau) = \frac{\mathrm{d}q}{\mathrm{d}\tau}(\nu; \tau)$$

associated with the periodic solutions $q(\nu; \tau)$ of the Euler–Lagrange equation (2.2) with initial conditions $q(\nu; 0) = q^*$ and $\mathrm{d}q/\mathrm{d}\tau(\nu; 0) = \nu$. By same reasoning of section 2.2, the *velocities* $\dot{q} : [-\epsilon, \epsilon] \times \mathbf{R} \to \mathbf{R}$ of (2.2) are well-defined by (2.5) for all $\tau \in \mathbf{R}$ and $\nu \in [-\epsilon, \epsilon]$. Furthermore, there exist $\alpha > 0$ small enough and $N \geq 1$ large enough such that, for every $n \geq N$, the *n-renormalized velocities* $y_n : [-1, 1] \times [0, \alpha n] \to \mathbf{R}$ are well-defined by

$$y_n(v; t) = (-1)^n \ \Gamma_{n,t}^{-1} \ \mu^{-1} \omega^{-1} \dot{q} \left(\Gamma_{n,t} \ \mu \ \omega \ v; \frac{n\pi - \ell t}{\omega} \right) \ ,$$

where $\Gamma_{n,t}$ is the (n, t)-*scaling parameter* defined in (2.4) and ω and μ are as given in (2.5). The n-renormalized velocities $y_n(v; t)$ are also dimensionless.

Definition The *asymptotic velocities* $Y_\ell : [-1, 1] \times \mathbf{R}_0^+ \to \mathbf{R}$ are defined by

$$Y_\ell(v; t) = v \ \cos\left(\ell t \left(v^2 - 1\right)\right) \ ,$$

where $\ell = \pm 1$ depending on the sign of $\mathcal{V}^{(4)}(q^*)$.

The following result is a natural extension of theorem 2.1. Its proof will be given in a future publication.

Theorem 2.4. *There exists $\beta > 0$ small enough, such that, for every $0 < \epsilon < 1/3$, we have that*

$$\| y_n(v; t) - Y_\ell(v; t) \|_{C^0([-1,1] \times [0, \beta n^{1/3 - \epsilon}], \mathbf{R})} \ < \ O\left(n^{-3\epsilon/2}\right) \ ,$$

where ℓ is the sign of $\mathcal{V}^{(4)}(q^*)$.

There is a strong geometrical and dynamical connection between the asymptotic trajectories $X_\ell(v; t)$ and the asymptotic velocities $Y_\ell(v; t)$. As stated in the following theorem, the pair formed by the asymptotic trajectories and the asymptotic velocities is the flow of a canonical Hamiltonian system.

Theorem 2.5. *Let $X_\ell(v; t)$ and $Y_\ell(v; t)$ denote, respectively, the asymptotic trajectories and the asymptotic velocities introduced above. The flow $\phi^t(0; v) = (X_\ell(v; t), Y_\ell(v; t))$ with initial condition $\phi^t(0; v) = (0, v)$ is the Hamiltonian flow of the canonical Hamiltonian system with Hamiltonian function $H : \mathbf{R}^2 \to \mathbf{R}$ given by*

$$H_\ell(x, y) = \ell \left(\left(\frac{x^2 + y^2}{2} \right)^2 - \frac{x^2 + y^2}{2} \right) \ .$$

2.6. **The action of the asymptotic trajectories.** Let N be a smooth manifold, $M = T^*N$ its cotangent bundle, Δ an interval in $\mathbf{R}$ and $H : M \times \Delta \to \mathbf{R}$ a smooth Hamiltonian function. It is well known that a path $\omega : [t_1, t_2] \to M$, from $p_1 \in M$ to $p_2 \in M$, starting at time $t_1 \in \Delta$ and ending at time $t_2 \in \Delta$, is a trajectory of the canonical Hamiltonian system (M, H) if it is a critical point of the action functional in phase space

$$F[\omega] = \int_\omega p \ \mathrm{d}q - H \ \mathrm{d}t \ .$$

An alternative approach is to write the action functional as the integral

$$F[\omega] = \int_\omega p\dot{q} - H \ \mathrm{d}t$$

and regard the integrand as a Lagrangian function $L : TM \times \Delta \to \mathbf{R}$, obtaining the action functional

$$F[\omega] = \int_\omega L \ \mathrm{d}t = \int_{t_1}^{t_2} L(\dot{\omega}(t), \omega(t), t) \ \mathrm{d}t \ .$$

Therefore, for any trajectory of the Euler-Lagrange equation determined by L, $\omega : [t_1, t_2] \to M$, one can compute its action $F[\omega]$. Moreover, noticing the dependence of a trajectory ω on its boundary conditions $\omega(t_1) = p_1$ and $\omega(t_2) = p_2$, one can define the *action correspondence* of the trajectories ω connecting (t_1, p_1) to (t_2, p_2) $S : (M \times \Delta)^2 \to \mathbf{R}$ by

$$S(t_1, p_1; t_2, p_2) = \{F[\omega] : \omega(t_1) = p_1, \ \omega(t_2) = p_2\} \ .$$

We remark that S is not a properly defined function, but a correspondence mapping each element of $(M \times \Delta)^2$ to a subset of $\mathbf{R}$. This is due to the fact that there might be more than one trajectory of the Euler-Lagrange equation determined by L connecting (t_1, x_1) to (t_2, x_2).

We denote by $S_\ell(t, x)$ the action correspondence of the asymptotic trajectories $X_\ell(v; t)$ connecting $(0, 0)$ to (t, x).

Theorem 2.6. *The action correspondence of the asymptotic trajectories $X_\ell(v; t)$ is given by*

$$S_\ell(t, x) = \left\{ \frac{\ell}{4} v^2 \left(\sin\left(2t(v^2 - 1)\right) + tv^2 \right) : v \in \hat{V}(t, x) \right\} \ ,$$

where, for each $(t, x) \in \mathcal{C} = \mathbf{R}_0^+ \times [-1, 1]$, the set $\hat{V}(t, x)$ is defined by

$$\hat{V}(t, x) = \{v(t, x) \in [-1, 1] : X_\ell(v(t, x); t) = x\} \ .$$

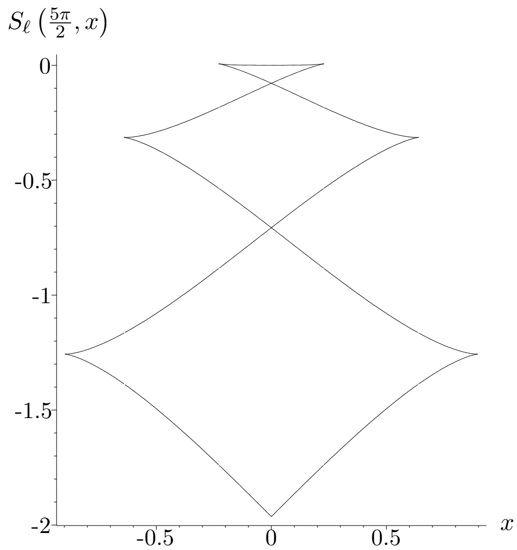

Figure 4: The action correspondence of the asymptotic trajectories for $t = \frac{5\pi}{2}$ and $\ell = -1$.

Note that the self-similarities of the asymptotic trajectories and the asymptotic universal focal decomposition are naturally carried over to the graph of the action correspondence in Figure 4.

3. Semiclassical physics

Focal decomposition is in fact a first step towards semiclassical quantization. This was already recognized in the semiclassical calculation of partition functions for quantum mechanical systems, where the need to consider a varying number of classical solutions in different temperature regimes became evident [4].

Either in quantum mechanics or in quantum statistical mechanics, the semiclassical approximation has to sum over all, or part of, the classical paths satisfying fixed point boundary conditions. The number and type of classical trajectories are the very ingredients which lead to a focal decomposition. It should, therefore, be no surprise that the focal decomposition can be viewed as the starting point for a semiclassical calculation.

As for the renormalization procedure, it was introduced to study the behavior of classical trajectories for very short space and very long time separations of the fixed endpoints. It maps those trajectories into n-renormalized ones, whose time separations are shifted by n half-periods, and whose space separations are scaled up to values of order one. As it has been shown in [5], this procedure converges to an asymptotic universal family of trajectories that have a well-defined and simple functional form, and which define an asymptotic universal focal decomposition self-similar to the original one.

The natural question to pose is whether the combination of focal decomposition and renormalization can be used to calculate semiclassical expansions for propagators in the short space, long time separation of the endpoints, or analogously, for thermal density matrices for short space separation and low temperatures (long euclidean time $\beta\hbar$ is equivalent to low temperatures $T = 1/(k_B\beta)$) by using the simple asymptotic forms alluded to in the previous paragraph (see [5] for a detailed discussion).

The conjecture to be investigated in a forthcoming article is that this can be done in a relatively simple way, thanks to the simple form of the asymptotes. This will bypass a much more difficult (if not impossible) calculation involving Jacobi's elliptic functions. Should our expectation be realized, we would obtain semiclassical estimates for both propagators and thermal density matrices in the short space/long time or short space/low temperature limits.

4. Conclusions

We study the dynamical behaviour of a family of mechanical systems that includes the pendulum. The analysis is based on the introduction of a renormalization scheme acting on the dynamics of this family of dynamical systems at small neighbourhoods of an elliptic equilibrium and after long intervals of time. We obtain interesting results concerning the existence of an asymptotic limit of this renormalization scheme, which is universal and self-similar. As a consequence we obtain an asymptotic universal focal decomposition for this family of mechanical systems. We review several new results concerning properties of the asymptotic limit of this renormalization scheme.

We believe that the existence of an asymptotic universal focal decomposition might be useful not only on the theory of boundary value problems of ordinary differential equations but also on several distinct fields of the physical sciences such as quantum statistical mechanics, optics, general relativity and even tsunami formation. Our belief on such applications is based on the relevance that the concept of focal decomposition may have on the study of caustic formation by focusing wavefronts, of such significance to those fields.

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References

[1] R. Abraham and J. Marsden. *Foundations of Mechanics*. Benjamin/Cummings, Reading, Massachusetts, 2nd edition, 1978.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer Verlag, 1989.

[3] S. V. Bolotin and R. S. MacKay. Isochronous potentials. In L. Vazquez, R. S. MacKay, and M. P. Zorzano, editors, *Localization and energy transfer in nonlinear systems*, pages 217–224. World Sci, 2003.

[4] C. A. A. de Carvalho and R. M. Cavalcanti. Tunneling catastrophes of the partition function. *Brazilian Journal of Physics*, 27:373–378, 1997.

[5] C. A. A. de Carvalho, M. M. Peixoto, D. Pinheiro, and A. A. Pinto. An asymptotic universal focal decomposition for non-isochronous potentials. Submitted. arXiv:math/0908.0778.

[6] R. S. MacKay and J. D. Meiss. *Hamiltonian Dynamical Systems: a reprint selection*. Adam Hilger (Bristol), 1987.

[7] J. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry*. Springer Verlag, 1999.

[8] M. M. Peixoto. On end point boundary value problems. *J. Differential Equations*, 44:273–280, 1982.

[9] M. M. Peixoto and R. Thom. Le point de vue énumératif dans les problèmes aux limites pour les équations différentielles ordinaires. *C. R. Acad. Sci., Paris, Sér. I*, 303:629–633; erratum, 307 (1988) 197–198; II, 303 (1986) 693–698, 1986.

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