ON SOME PROBLEMS CONCERNING PLANAR RANDOM WALKS

by

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Dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics
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ABSTRACT

(Mathematics-Probability)

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Abstract

We consider two questions related to planar simple random walk. Our first result confirms an observation made by Mandelbrot about the number of large holes made by planar simple random walk S. We show that if N_δ is the number of components of $\mathbb{C} \setminus S[0, 2n]$ of area greater than $n^{1-\delta}$, then for all δ less than or equal to some $\delta_0 > 0$,

$$\frac{\log^2(n^\delta)}{n^\delta} N_\delta \xrightarrow{P} 2\pi, \text{ as } n \to \infty.$$

In the second part of the thesis, we establish some of the basic estimates needed to extend the main result of [23], where it is shown that the scaling limit of loop-erased random walk from an interior point of a domain to the boundary is the radial Schramm-Löwner Evolution with parameter 2 (SLE_2), to the chordal case. The expected result is that the scaling limit of loop-erased random walk (LERW) excursion in the upper half-plane $\mathbb{H}$ is chordal SLE_2. The natural time parameter for chordal SLE is the half-plane capacity, hcap, as introduced in [17]. We define the discrete half-plane capacity, denoted by dhcap, an analogous quantity for discrete subsets of the discrete upper half-plane $\mathcal{H}$, as well as a natural correspondence between discrete sets $A \subset \mathcal{H}$ and continuous sets $\tilde{A} \subset \mathbb{H}$. We show that for a large class of such sets dhcap(A) is close to hcap($\tilde{A}$). We estimate very precisely a discrete Green's function in $\mathcal{H} \setminus A$ and express it in terms of various parameters of the set A. Applying this to LERW excursion and using the relationship between hcap and dhcap should provide information on the driving process of the LERW path, whose scaling limit is expected to be Brownian motion with variance 2. In the event that the scaling limit of LERW excursion is indeed chordal SLE_2, this work should give us the necessary background to study another question: how does SLE relate to general Laplacian random walks,

a family of random walks of which LERW is a special case?

In both problems we work with coupling methods for random walk and Brownian motion, namely Skorokhod embedding and the so-called KMT approximation. Other tools used are bounds on derivatives of conformal transformations, the Beurling projection theorem, and ideas involving Brownian disconnection exponents.

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A mes parents

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Introduction

Simple random walk, which we will denote by S throughout this thesis, and standard Brownian motion, denoted by B, are two of the most common stochastic processes and they appear naturally in a variety of areas, such as physics, biology, or finance. The first is a discrete process, whereas the second is continuous. It has been known for a long time that Brownian motion is the scaling limit not only of simple random walk, the most elementary random walk, but also of a much larger class of random walks. Brownian motion is the limit of random walk very much in the same sense as the normal distribution is the limit of normalized sums of random variables. Many properties are shared by Brownian motion and random walk and knowledge about one has often helped understand the other better. There is still a multitude of open questions related to these processes. Moreover, it has recently been observed that Brownian motion is related in a particularly strong way to a whole range of other discrete stochastic models arising in the field of statistical mechanics, when placed into Löwner's ordinary differential equation

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{e^{iU_t} + g_t(z)}{e^{iU_t} - g_t(z)}, \quad g_0(z) = z,$$

defined for z in $\mathbb{U}$, the open unit disk centered at the origin, and some real function U_t.

The one-parameter family of processes obtained when letting $U_t = B_{\kappa t}$, where $\kappa \geq 0$ was introduced by Oded Schramm in [29] and is now commonly called the radial Schramm-Löwner Evolution with parameter κ (SLE_κ).

One obtains a similar process in the upper half-plane, called chordal SLE, by solving the equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - B_{\kappa t}}, \quad g_0(z) = z,$$

defined on $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Depending on the value of κ, this process has been found to be related to a variety of discrete models in probability, such as the loop-erased random walk, the uniform spanning tree, and the critical percolation cluster interface. Furthermore, connections with the self-avoiding walk, the Ising model, domino tilings, the Potts model, are still at the conjecture level.

In this thesis, we address two questions. The first is on planar simple random walk. We prove that an observation by Mandelbrot on the behavior of the number of large holes made by planar simple random walk is correct. This problem is discussed in Chapter 2. The second problem is related to SLE. In Chapter 4, we define a capacity for discrete subsets of the upper half-plane $\{(x, y) \in \mathbb{Z}^2 : y > 0\}$ and prove some important properties of this object. We also provide some estimates for a particular Green's function. Applying this work to loop-erased random walk (LERW) excursion in the upper half-plane should allow us to prove that the scaling limit of LERW excursion in the upper-half plane is chordal SLE_2 (it has already been shown in [23] that radial SLE_2 is the scaling limit of LERW from the interior of a domain to its boundary), and provide the tools needed to investigate the relationship between the one-parameter family of Laplacian random walks (as introduced in [25]) and the one-parameter family of SLE curves.

A few of the techniques we use are common to the two problems. We therefore treat them in separate chapters.

In Chapter 1, we present two well-known coupling methods between random walk and Brownian motion, namely Skorokhod embedding and the KMT approximation, which we use in the two problems mentioned above.

Appendix A contains a series of standard estimates for random walk and Brownian motion. These are all well-known.

In Appendix B, we provide a few interesting estimates for random walk, which we could not find anywhere in the literature.

Finally, Chapter 3 gives the background on SLE needed to understand the relevance of our work in Chapter 4.

Chapter 0

Definitions and Notation

This thesis will be concerned with two problems in which standard Brownian motion and simple random walk play a central role. Many objects will appear in all the parts of the thesis. To keep the text as compact as possible and to make the reader's work a little bit easier, we give the essential definitions of the thesis in this chapter.

First, we give definitions which do not involve probability:

$\mathbb{Z} = \{.., -2, -1, 0, 1, 2, ..\}$ will denote the set of integers and we choose the definition $\mathbb{N} = \{1, 2, ..\}$ for the natural numbers. $\mathbb{Q}$ and $\mathbb{R}$ will stand for the set of rational numbers and real numbers, respectively. In general, if A is a subset of $\mathbb{R}$, then we let $A_+ = A \cap [0, \infty), A_- = A \cap (-\infty, 0],$ and $A^* = A \setminus \{0\}$. For instance, $\mathbb{R}_+^* = (0, \infty)$.

If $x, y \in \mathbb{R}^d, d(x, y)$ denotes the Euclidean distance between x and y. For any space Ω and any measurable set $A \subset \Omega$, the complement of A is $A^c = \Omega \setminus A$. If $\Omega = \mathbb{C}, |A|$ will denote the area of A and $\mathrm{diam}(A) = \sup_{x,y \in A} d(x, y)$ the diameter of A. $\bar{A}$ will stand for the closure of A. The distance from a point x to a set A is $d(x, A) = \inf_{y \in A} d(x, y)$. For a countable set A, $\#(A)$ is the cardinality of A. If A is a subset of $\mathbb{Z}^2$, we let $\partial A = \{z \in \mathbb{Z}^2 : d(z, A) = 1\}$ be the boundary of A, $\partial_{int}(A) = \{z \in \mathbb{Z}^2 : d(z, A^c) = 1\}$ the inner boundary of A, and write $\bar{A}$ for

$A \cup \partial A = \{z \in \mathbb{Z}^2 : d(z, A) \leq 1\}$. Note that we use the notation ∂A and $\bar{A}$ for continuous and discrete sets alike. It will be clear from context which definition is being used.

We denote the real and the imaginary part of a point $z \in \mathbb{C}$ by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

Disks will be particularly important in many of the estimates we need. For $x \in \mathbb{C}$ and $d \in \mathbb{R}_+$, we let

$$D(x, d) := \{y \in \mathbb{C} : |y - x| \leq d\}, \qquad \tilde{D}(x, d) = \{z \in \mathbb{Z}^2 : |z - x| \leq d\}.$$

If $x = 0$, we will just write $D(d)$ and $\tilde{D}(d)$. If moreover $d = 1$, we write $\mathbb{D}$ for $D(1)$, the closed unit disk centered at the origin. We will write $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for the open unit disk centered at the origin.

Similarly, we define the continuous and discrete circles

$$C(x, d) = \partial D(x, d), \qquad \tilde{C}(x, d) = \partial_{int} \tilde{D}(x, d),$$

and write $C(d)$ and $\tilde{C}(d)$ if $x = 0$.

For any $z \in \mathbb{C}$, we define $\mathrm{Sq}(z)$ to be the closed region bounded by the square centered at z, whose sides are parallel to the axes and of length 1.

K, C_1, C_2 will denote arbitrary positive constants and will be independent of all other quantities involved in a given equation, unless stated otherwise. Their value will be allowed to vary from a line to the next.

Many of the results in this thesis are about the asymptotic behavior of various quantities. We describe the symbols we will need to denote different forms of asymptotic behavior. We write $h(x) \sim g(x)$ if

$$\lim_{x \to \infty} \frac{h(x)}{g(x)} = 1$$

and $h(x) \asymp g(x)$ if there exist constants C_1, C_2, x_0 such that for all $x \geq x_0$,

$$C_1 g(x) \leq h(x) \leq C_2 g(x).$$

$h(x) = \mathcal{O}\left(g(x)\right)$ means that there exist C_1, x_0 such that for all $x \geq x_0$,

$$h(x) \leq C_1 g(x).$$

At times, the constant C_1 will depend on parameters that are present in the functions $h(x)$ and $g(x)$. When we state this explicitly, we will refer to C_1 as "the constant of the $\mathcal{O}$". Finally, $h(x) = o(g(x))$ if

$$\lim_{x\to\infty} \frac{h(x)}{g(x)} = 0.$$

All these definitions are concerned with the behavior of functions as $x \to \infty$. They can be made for other limits, such as $\lim_{x\to 0}$, as well, and it will be clear from context which one is being considered.

We will now define objects involving probability: The letter B is reserved for standard Brownian motion and so is S for simple random walk. The dimension will always be one or two and it will be clear from context which it is. It is assumed that the reader is familiar with these objects. Except in Chapter 1, where we work in dimensions 1 and 2, whenever we write "Brownian motion", we will mean planar standard Brownian motion and "random walk" will implicitly refer to two-dimensional simple random walk.

If Ω is a probability space and $\{X_n\}_{n\geq 1}$ and X are random variables, we say that X_n converges to X **in probability** if for every $\epsilon > 0, \mathbb{P}\left\{|X_n - X| > \epsilon\right\} \to 0$ as $n \to \infty$. In that case, we write

$$X_n \xrightarrow{P} X.$$

If $\mathbb{P}\left\{\omega \in \Omega : X_n(\omega) \overset{n\to\infty}{\to} X(\omega)\right\} = 1$, we say that X_n converges to X **almost surely** and write

$$X_n \overset{\text{a.s.}}{\to} X.$$

We will be working with various stochastic processes in this thesis, but will use only the letter $\mathbb{P}$ to refer to their probability measure. Again, it will always be clear from context to which process(es) $\mathbb{P}$ refers. For a point z in $\mathbb{R}^d$ ($d \in \{1,2\}$), $\mathbb{P}^z$ will denote the probability measure associated to a process started at z and $\mathbb{E}^z$ will denote the expectation associated to a process started at z.

In general, stopping times for random walk will be denoted by lower-case Greek letters and stopping times for Brownian motion by upper-case Greek letters. We will be particularly interested in hitting times of a set. For a set $A \subset \mathbb{C}$ (which may be discrete), we let

$$\tau_A = \inf\{n \geq 1 : S_n \in A\} \quad \text{and} \quad T_A = \inf\{t \geq 0 : B_t \in A\}.$$

The exit times from the disk of radius $r \in \mathbb{R}_+^*$, centered at the origin, are the following:

$$\xi_r = \inf\{k \geq 0 : |S_k| \geq r\}, \qquad \Xi_r = \inf\{t \geq 0 : |B_t| \geq r\}.$$

We will also need to consider the first time a certain distance is reached by the Brownian motion or the random walk:

$$\hat{\xi}_r = \inf\{k \geq 0 : |S_k - S_0| \geq r\}, \qquad \hat{\Xi}_r = \inf\{t \geq 0 : |B_t - B_0| \geq r\}.$$

Note that $\hat{\xi}_r = \xi_r$ if $S_0 = 0$ and $\hat{\Xi}_r = \Xi_r$ if $B_0 = 0$.

Important ideas from complex analysis will be used in the forthcoming chapters and we will give the necessary background later. However, we just give two definitions for reference here. A function $h : \mathbb{R}^k \to \mathbb{R}$ is called C^k if all its kth partial derivatives exist. If D, D' are domains, we will call a bijective conformal map $f : D \to D'$ a **conformal transformation**. If such a map exists, we say that D and D' are conformally equivalent.

Chapter 1

Coupling Methods

Since the 1960's, much has been written about ways to couple random walk with Brownian motion and the subject is now quite well understood. Usually the goal is to put the two processes on a same probability space in such a way that with large probability they remain close to each other at all times of a given time interval. For an extensive discussion of this problem, see [5]. Two such couplings of somewhat different nature will be needed in this thesis and we discuss them in this chapter. The first is based on the Skorokhod embedding scheme. The second is much sharper and is the so-called KMT approximation.

1.1 Skorokhod embedding

We state here Skorokhod's theorem on how to embed general random walk in Brownian motion but will not need it directly since we will make an explicit construction of the coupling. Note that in this section, random walk and Brownian motion are one-dimensional.

Theorem 1.1.1 (Skorokhod Embedding). Suppose that $(X_i)_{i\geq 1}$ are independent, identically distributed real-valued random variable with mean 0 and variance 1. Then there exist a probability space containing a Brownian motion $\{B_t : t \geq 0\}$, the random variables $(X_i)_{i\geq 1}$, a sequence of stopping times $0 = T_0 \leq T_1 \leq T_2 \leq ...$, such that the increments $T_n - T_{n-1}$ are independent, identically distributed, $\mathbb{E}[T_n] = n$, and the sequence $\{B(T_n)\}_{n\geq 1}$ has the same distribution as the random walk $\{S_n\}_{n\geq 1}$ associated with $\{X_i\}_{i\geq 1}$.

For more details, see for instance [7]. Note that although this theorem is a beautiful result, it does not say anything about how close the random walk and the Brownian motion actually are if we look at them path by path. For theorems addressing this question, we again refer the reader to [5].

We want to make the random walk S_n a continuous process and define for all $t \geq 0$,

$$S_t = S_{[t]} + (t - [t])(S_{[t]+1} - S_{[t]}),$$

where $[t]$ denotes the integer part of t. In what follows, the subscripts k, n will refer to positive integers, whereas s, t will be positive real numbers.

For simple random walk, it is easy to see that by defining $T_0 = 0$ and for $i \geq 1, T_i = \inf\{t \geq T_{i-1} : |B_t - B(T_{i-1})| = 1\}$, the conditions of Theorem 1.1.1 are satisfied. From now on we denote the simple random walk $\{B(T_n)\}_{n\geq 1}$ by S_n and keep in mind that $\{S_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are not independent.

This section's main result is the following theorem:

Theorem 1.1.2. There exists a coupling of standard Brownian motion B and simple random walk S such that $\forall\, g(n) \geq 1$ satisfying $g(n) = \mathcal{O}\left(n^{1/4}\right)$, there exist constants $b, K > 0$ such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq n} |B_t - S_t| \geq n^{1/4} g(n)\}\right\} \leq Kne^{-bg(n)}.$$

Proof. For notational purposes, we let $h(n) = g(n)\sqrt{n}$ and for $k \geq 1$,

$$I_k = [(k-1)[h(n)], k[h(n)]] \,.$$

For $n \geq 4$, we have the covering $[0, n] \subset \bigcup_{k=1}^{[n/g(n)]+3} I_k$. Also, if $0 \leq t - s \leq h(n)$, the interval $[s, t]$ covers at most 3 of the I_k's. We use this and Lemma A.1.5 to see that there exist constants $C_1, a > 0$ such that

$$\mathbb{P}\{\sup_{0\leq k\leq n} |B_k - S_k| \geq \frac{1}{2} n^{1/4} g(n)\}$$

$$\leq \mathbb{P}\left\{\max_{1\leq k\leq n} |T_k - k| \geq h(n)\right\} + \mathbb{P}\left\{\sup_{\substack{s\leq n \\ |t-s|\leq h(n)}} |B_t - B_s| \geq \frac{1}{2} n^{1/4} g(n)\right\}$$

$$\leq C_1 n e^{-ag(n)} + \mathbb{P}\left\{\sup_{\substack{1\leq k\leq [\sqrt{n}/g(n)]+3 \\ t\in I_k}} |B_t - B_{(k-1)[h(n)]}| \geq \frac{1}{8} n^{1/4} g(n)\right\}$$

$$\leq C_1 n e^{-ag(n)} + 2\frac{\sqrt{n}}{g(n)} \mathbb{P}\left\{\sup_{0\leq t\leq [h(n)]} |B_t| \geq \frac{1}{8} n^{1/4} g(n)\right\}$$

$$\leq C_1 n e^{-ag(n)} + C_1' \frac{\sqrt{n}}{g(n)^{3/2}} e^{-a' g(n)} \leq K_1 n e^{-b_1 g(n)},$$

where we use Lemma A.1.1 in the last step and $K_1 = \max\{C_1, C_1'\}, b_1 = \min\{a, a'\}$. Now it suffices to observe that

$$\mathbb{P}\{\sup_{0\leq t\leq n} |B_t - S_t| \geq n^{1/4} g(n)\}$$

$$\leq \mathbb{P}\left\{\sup_{0\leq k\leq n} |B_k - S_k| \geq \frac{1}{2} n^{1/4} g(n)\right\}$$

$$+ n\mathbb{P}\left\{\sup_{0\leq t\leq 1} |B_t - S_t| \geq n^{1/4} g(n); |B_0 - S_0| \leq \frac{1}{2} n^{1/4} g(n)\right\}$$

$$\leq K_1 n e^{-b_1 g(n)} + K_2 n e^{-b_2 n^{1/2}} \leq K n e^{-bg(n)},$$

where $K = \max\{K_1, K_2\}, b = \min\{b_1, b_2\}$.

□

Remark 1. The theorem implies that for any ϕ such that $\phi(n) \to \infty$ as $n \to \infty$,

$$\sup_{0 \le t \le n} \frac{|B_t - S_t|}{n^{1/4} \log n \phi(n)} \xrightarrow{P} 0.$$

This is close to being the best we can hope for when using Skorokhod's method. In [11], Kiefer shows that the best possible result for simple random walk with Skorokhod's method is

$$\frac{|S_n - B_n|}{(n \log\log n)^{1/4} (\log n)^{1/2}} \stackrel{\text{a.s.}}{=} \mathcal{O}(1).$$

1.2 Extending the result to the plane

The construction we made in the previous section does not work in dimensions other than 1. However, there is a nice way of getting around this problem, which is as follows:

Let B_t^1, B_t^2 be independent one-dimensional Brownian motions. Then

$$B_t = (B_t^1, B_t^2)$$

is a planar Brownian motion. For $i \ge 0, j = 1, 2$, let T_i^j be the stopping time for B_t^j as defined in the previous section and for $i \ge 0$, define $S_N^j = B^j(T_n^j)$. Then S^1, S^2 are independent one-dimensional random walks. We let $L_i = (L_i^1, L_i^2)$ be independent random vectors, independent of B^1, B^2, with distribution

$$\mathbb{P}\{L_i = (1,0)\} = \mathbb{P}\{L_i = (0,1)\} = \frac{1}{2}.$$

If we define $T_n^j = \sum_{i=1}^n L_i^j$, it is easy to check that $S_n := (S^1_{T_n^1}, S^2_{T_n^2})$ is a planar simple random walk. The statement and the proof of the main result are essentially the same as in one dimension. The only difference is that now, the time-parameter is different for the two processes. A one-line heuristic argument is that Brownian motion moves a little bit faster since it is not restricted to moving along the lines of the lattice, but can take "diagonal shortcuts".

Theorem 1.2.1. There exists a coupling of standard Brownian motion B and simple random walk S in the plane such that $\forall\, g(n) \geq 1$ satisfying $g(n) = \mathcal{O}\left(n^{1/4}\right)$, there exist constants $b, K > 0$ such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq n} |B_t - S_{2t}| \geq n^{1/4} g(n)\}\right\} \leq Kne^{-bg(n)}.$$

Proof.

$$\mathbb{P}\left\{\max_{1\leq k\leq n} |B_k - S_{2k}| \geq n^{1/4} g(n)\}\right\} \leq 2\mathbb{P}\left\{\max_{1\leq k\leq n} |B_k^1 - S^1_{T^1_{2k}}| \geq \frac{1}{\sqrt{2}} n^{1/4} g(n)\}\right\}.$$

Since T_{2k} is a sum of $2k$ random variables of mean $1/2$ and finite variance, the exact same argument as in Theorem 1.1.2 can be used to show this result. □

Remark 2. Obviously, the extension we just did works in higher dimensions as well. Since we are working in the plane, we could also have used the method which we discuss in the next section.

1.3 The KMT approximation

In [12], Komlós, Major, and Tusnády provide a coupling between random walk and Brownian motion, in which the typical distance between the two paths is significantly smaller from that obtained by the Skorokhod embedding. The coupling is constructed in a "non-Markovian" way and more caution is needed when computing hitting times for the coupling. The statement we present here is in fact a particular case of their result.

Theorem 1.3.1 (KMT Approximation). There exist a probability space containing a Brownian motion B and a simple random walk S, constants $K, \lambda, C > 0$ such that for all $x > 0$ and every n,

$$\mathbb{P}\left\{\max_{1\le k\le n}|S_k - B_k| > C\log n + x\right\} \le K\exp\{-\lambda x\}.$$

An immediate consequence is that there exists $K > 0$ such that for every $\lambda > 0$ we can find a $C = C(\lambda) > 0$ such that $\forall n \ge 1$,

$$\mathbb{P}\left\{\max_{1\le k\le n}|S_k - B_k| > C\log n\right\} \le Kn^{-\lambda}. \tag{1.1}$$

We want the paths to be close not only at integer times, but at all real times up to n.

Corollary 1.3.2. There exist a probability space containing a Brownian motion and a simple random walk and a constant $K' > 0$ such that for every $\lambda > 0$ we can find a $C' = C'(\lambda) > 0$ such that for every $n \ge 1$,

$$\mathbb{P}\left\{\sup_{0\le t\le n}|S_t - B_t| > C'\log n\right\} \le K'n^{-\lambda}.$$

Proof. We let $C' = 2C$, where C is as in (1.1).

$$\begin{aligned}
&\mathbb{P}\{\sup_{0\le t\le n}|S_t - B_t| > 2C\log n\}\\
&\le \mathbb{P}\left\{\max_{1\le k\le n}|S_k - B_k| > C\log n\right\}\\
&\quad + n\mathbb{P}\left\{\sup_{0\le t\le 1}|S_t - B_t| \ge 2C\log n; |S_0 - B_0| \le C\log n\right\}\\
&\le Kn^{-\lambda} + n\mathbb{P}\left\{\sup_{0\le t\le 1}|B_t| \ge \frac{C}{2}\log n\right\} \le K'n^{-\lambda}.
\end{aligned}$$

∎

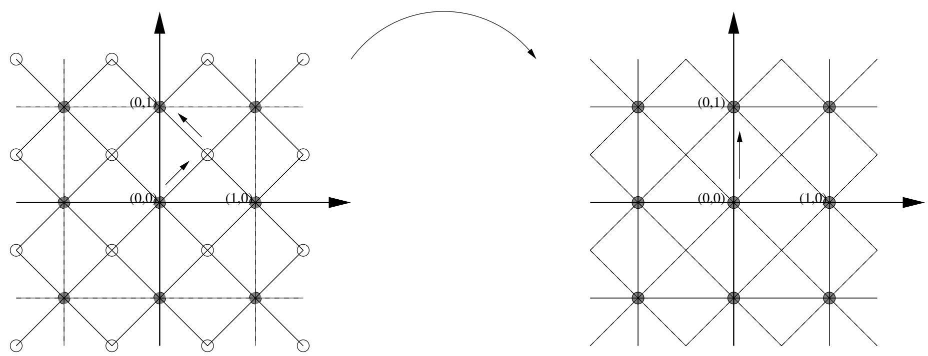

Figure 1.1: Two independent one-dimensional simple random walks on the diagonal lattice (dotted lines) give a simple random walk in $\mathbb{Z}^2$.

Again, we need to extend the result to two dimensions. The coupling is so fine that extending the result as we did with the Skorokhod embedding does not work. We will use a very simple trick which works only in two dimensions. Fortunately, this is all we need. The idea is to note that if we take two independent random walks, one on the horizontal axis and one on the vertical axis and add up the components at each step, we get a simple random walk in the plane on a "diagonal lattice".

More precisely, for $j \geq 1, k = 1, 2$, let X_j^k be independent random vectors with distribution

$$\mathbb{P}\left\{X_j^1 = \pm\frac{1}{\sqrt{2}}e^{i\pi/4}\right\} = \mathbb{P}\left\{X_j^2 = \pm\frac{1}{\sqrt{2}}e^{i3\pi/4}\right\} = \frac{1}{2}.$$

Then if we let $S_n^k = \sum_{j=1}^{n} X_j^k$, S_n^1 and S_n^2 are independent simple random walks on $\frac{1}{\sqrt{2}}e^{i\pi/4} \cdot \mathbb{Z} = \{k \cdot \frac{1}{\sqrt{2}}e^{i\pi/4} : k \in \mathbb{Z}\}$ and $\frac{1}{\sqrt{2}}e^{i3\pi/4} \cdot \mathbb{Z}$, respectively. In particular, we can couple each one with a Brownian motion with speed $1/2$ in such a way that

$$\mathbb{P}\{\sup_{0\leq t\leq n} |S_t^j - B_{t/2}^j| > C' \log n\} \leq K' n^{-\lambda}.$$

Then $S_n = (S_n^1, S_n^2)$ is a planar simple random walk and, using the rotational invariance of planar Brownian motion, $B_t = (B_t^1, B_t^2)$ is a planar standard Brownian motion. Moreover,

$$\mathbb{P}\{\sup_{0\le t\le n}|S_{2t}-B_t|>\sqrt{2}C'\log n\}$$
$$\le \mathbb{P}\left\{\sup_{0\le t\le n}|S^1_{2t}-B^1_t|^2+|S^2_{2t}-B^2_t|^2\ge 2(C')^2\log^2 n\right\}$$
$$\le \mathbb{P}\left\{\sup_{0\le t\le n}|S^1_{2t}-B^1_t|^2\ge (C')^2\log^2 n\right\}+\mathbb{P}\left\{\sup_{0\le t\le n}|S^2_{2t}-B^2_t|^2\ge (C')^2\log^2 n\right\}$$
$$\le 2K'n^{-\lambda}.$$

This gives:

Corollary 1.3.3. There exist a probability space containing a planar standard Brownian motion and a two-dimensional simple random walk and a constant $K''>0$ such that for every $\lambda>0$ we can find a $C''=C''(\lambda)>0$ such that for every $n\ge 1$,

$$\mathbb{P}\left\{\sup_{0\le t\le n}|S_{2t}-B_t|>C''\log n\right\}\le K''n^{-\lambda}.$$

1.4 Reparametrizing the random walk

To conclude this chapter, we reparametrize the random walks defined by the Skorokhod embedding and the KMT approximation in such a way that $|B_t-S_t|$ is small (instead of $|B_t-S_{2t}|$ in the original coupling). We do this mainly for aesthetic reasons. This time-change will not affect any of the forthcoming results, as it does not affect hitting distributions which will be the object of our interest. **From now on, whenever we write S_t, it will be understood that we are looking at planar simple random walk with twice the usual speed; B_t will be planar standard Brownian motion.** We rewrite the two main results of this chapter in the form in which we will use them in the next chapters.

Theorem 1.4.1. [Skorokhod Approximation] There exists a probability space containing a Brownian motion B and a random walk S such that $\forall\, g(n) \geq 1$ satisfying $g(n) = \mathcal{O}\left(n^{1/4}\right)$, there exist constants $b, K > 0$ such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq n} |B_t - S_t| \geq n^{1/4} g(n)\}\right\} \leq Kne^{-bg(n)}.$$

Theorem 1.4.2. [KMT Approximation] There exist a probability space containing a Brownian motion B and a random walk S, a constant K, such that for every $\lambda > 0$ there is a $C = C(\lambda) > 0$ such that for every $n \geq 1$,

$$\mathbb{P}\left\{\sup_{0\leq t\leq n} |S_t - B_t| > C\log n\right\} \leq Kn^{-\lambda}.$$

Chapter 2

The Distribution of Large Random Walk Holes

2.1 Introduction

In [26], Benoît B. Mandelbrot, suggests the interesting behavior of an exponent related to planar simple random walk. Let the "holes" or "gaps" made by a continuous planar stochastic process X over a time interval $[a, b]$ be the connected components of $\mathbb{C} \setminus X[a, b]$. Mandelbrot observed the following behavior for planar simple random walk loops (random walks of even length with same starting and ending point): Run planar simple random walk for a fixed large time n. For $A \leq 1$, let

$$N(A) = \text{ number of holes of area } \geq An.$$

Then there exists an A_0, possibly depending on n, such that

- For $A \geq A_0$,

$$N \asymp A^{-1}.$$

- For $A \leq A_0$, there is a broad range of hole sizes for which

$$N \asymp A^{-\frac{5}{6}},$$

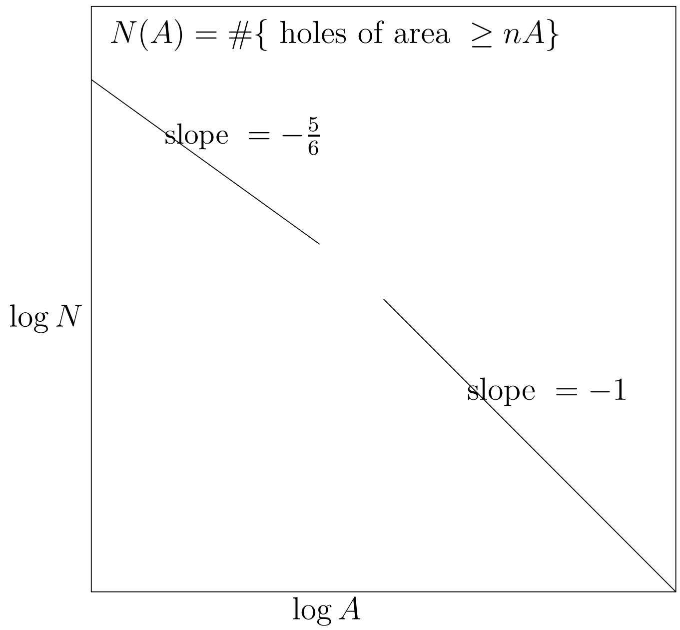

Figure 2.1: Two regimes

where $N(A) \asymp f(A)$ means that there exist $C_1, C_2 > 0$ such that $C_1 f(A) \leq N(A) \leq C_2 f(A)$.

There is no reason to believe that the exponent should be different for simple random walk, where we do not require that the starting and ending point be the same, so we will consider that process instead. Two papers treating a similar problem were written at the end of the 80's. They are central to our approach. In [27], Mountford proved the following result:

If $N_{[a,b]}$ is the number of components of $\mathbb{C} \setminus B[0,1]$ whose areas lie in the interval $[a, b]$, then for all $c > 1$,

$$u \log^2(cu/\pi) N_{[u,cu)} \xrightarrow{P} 2\pi \left(1 - \frac{1}{c}\right) \text{ as } u \downarrow 0.$$

Mountford's paper is important for us as it contains some of the ideas which we use in our approach to the problem. However, we will only use directly the results of another paper ([24]), written by Le Gall, which is an extension and improvement

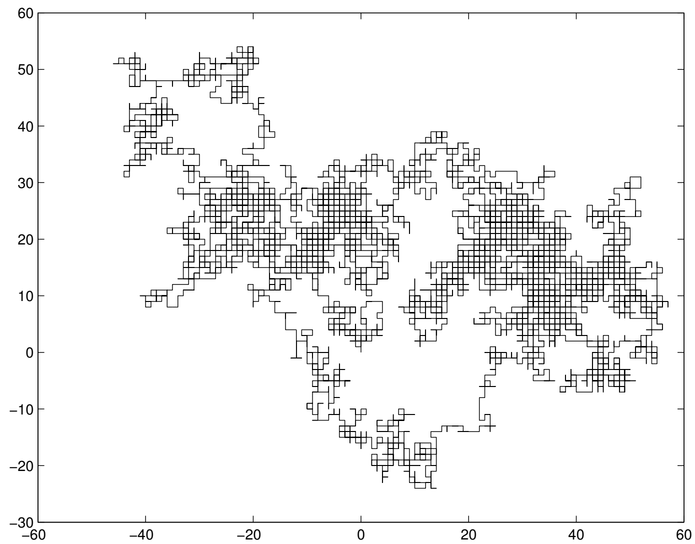

Figure 2.2: Planar simple random walk of 10000 steps

of Mountford's result. We will write more on Le Gall's estimates in a forthcoming section.

This chapter's main result is the following:

Theorem 2.1.1. If S is simple random walk and

$$\tilde{N}(\delta) = \#\{\text{connected components of } \mathbb{C} \setminus S[0,n] \text{ of area greater than } n^{1-\delta}\},$$

then for all $0 < \delta \leq \delta_0 = \frac{1}{200}$,

$$\frac{\log^2(n^\delta)}{n^\delta} \tilde{N}(\delta) \xrightarrow{P} 2\pi, \text{ as } n \to \infty.$$

Remark 3. Recall that we reparametrized S in Chapter 1, so that it takes two steps every unit of time.

As a general rule, in this chapter, a tilde will refer to a quantity related to random walk.

Here are the key ideas of the argument (formal definitions of all the quantities involved are made in the next section):

1. Couple random walk and Brownian motion via Skorokhod embedding and use Le Gall's result to find information about the large random walk holes.

2. Decompose $[n^{1-\delta}, \infty)$ into a union of smaller intervals $I_j = [n^{1-\delta}c^j, n^{1-\delta}c^{j+1})$, where $j \geq 1$ and $c = 1 + \epsilon > 1$, and show that on each one, for every $\epsilon > 0$ small enough,

$$\mathbb{P}\left\{|\tilde{N}_j - N_j| > K\epsilon N_j\right\} \to 0 \text{ fast enough, as } n \to \infty.$$

Here,

$$N_j = \#\{\text{ connected components of } \mathbb{C} \setminus B[0,n] \text{ with area in } I_j\},$$

$$\tilde{N}_j = \#\{\text{ connected components of } \mathbb{C} \setminus S[0,n] \text{ with area in } I_j\},$$

and K is some positive constant.

3. Reduce the problem to a question about $\Delta(z) := \big||C(z)| - |\tilde{C}(z)|\big|$ for each $z \in \mathbb{Z}^2$, where $C(z)$ is the connected component of $\mathbb{C} \setminus B[0,n]$ containing z, $\tilde{C}(z)$ is the connected component of $\mathbb{C} \setminus S[0,n]$ containing z. The idea is to show that on "good configurations" and for large components, $\Delta(z)$ is small compared to the size of the components.

4. Eliminate the "bad configurations" by showing that their probabilities go to 0 as $n \to \infty$. This involves

   - Handling the case where z is close to $\partial C(z)$ or $\partial\tilde{C}(z)$. We do this with the help of ideas relating the two-sided disconnection exponent for Brownian motion and random walk to the fractal dimension of the Brownian frontier.
   - Look at other "bad cases" which can occur even if z is far from $\partial C(z)$ and $\partial\tilde{C}(z)$: $C(z)$ closing very late before time n or $C(z)$ being a very thin

component, and the same for $\tilde{C}(z)$. The ideas involved for these cases involve the one-sided disconnection exponent for Brownian motion and random walk, and Beurling estimates.

In Section 2, we give all the definitions needed for this chapter. In Section 3, we discuss Le Gall's results and give a list of the consequences which will be essential for us. The fourth section looks at how "thick" the boundary of a Brownian motion or random walk component is. The idea is based on the method of [15], which exhibits a relationship between the two-sided disconnection exponent of Brownian motion to the Hausdorff dimension of its frontier. Section 5 contains a sequence of preparatory lemmas which compare the areas of the Brownian motion and random walk component containing a given lattice point. In the last section, we use the results of all the previous sections to prove Proposition 2.6.1 from which the main result of the chapter, Theorem 2.1.1, follows directly.

2.2 Definitions

In what follows, all multiplicative constants will be denoted by K. It will be understood that they may be different from one line to the next. Also, all inequalities will be limiting inequalities in the sense that they are valid for n large enough.

Most of the definitions we make in this section will only come into play in Section 2.5, but to keep the sequence of lemmas in that section as compact as possible we write the required definitions here together with those needed in Section 2.3.

We know from Chapter 1 that there exists a coupling of standard Brownian motion B and (re-parametrized) simple random walk S such that for every $g(n) \geq 1$ satisfying $g(n) = \mathcal{O}\left(n^{1/4}\right)$, there exist constants $b, K > 0$ such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq n} |B_t - S_t| \geq n^{1/4} g(n)\right\} \leq K n e^{-b g(n)}. \tag{2.1}$$

From here on, we will be working with this coupling. We define

$$\mathcal{P} = \left\{ \sup_{0 \le t \le n} |B_t - S_t| \le n^{1/4} \log^2 n \right\}. \tag{2.2}$$

(2.1) implies that we can find an appropriate coupling of Brownian motion and random walk and constants $b, K > 0$, so that

$$\mathbb{P}(\mathcal{P}^c) \le K n^{1 - b \log n}. \tag{2.3}$$

Note that this means that $\mathbb{P}(\mathcal{P}^c)$ decays faster than any power function. For any $z \in \mathbb{C}$ and any $t \ge 0$, we let

$$C_t(z) = \text{ the connected component of } \mathbb{C} \setminus B[0,t] \text{ containing } z \tag{2.4}$$

and

$$\tilde{C}_t(z) = \text{ the connected component of } \mathbb{C} \setminus S[0,t] \text{ containing } z. \tag{2.5}$$

If $z \in B[0,t]$ we let $C_t(z) = \emptyset$ and if $z \in S[0,t], \tilde{C}_t(z) = \emptyset$. If $t = n$, we just write $C(z)$ and $\tilde{C}(z)$.

For a curve $\gamma : [a,b] \to \mathbb{C}$ and a point $z \notin \gamma([a,b])$, we define $\arg_z(\gamma(t))$ to be the continuous argument of γ around z, with the convention that $\arg_z(\gamma(0)) = 0$. Note that $\arg_z(\cdot)$ is defined on the parametric interval $[a,b]$, not on the image of γ. For a proof of the fact that a continuous choice of the argument exists, see for instance [31].

It is a well-known property of Brownian motion that for a given $z \in \mathbb{C}$ other than the starting point, and any $t > 0$, $\mathbb{P}\{z \in B[0,t]\} = 0$, so that for $z \in \mathbb{C} \setminus (0,0)$,

$$\mathbb{P}\{C_t(z) = \emptyset\} = 0 \quad \text{and} \quad \mathbb{P}\{\arg_z(B_t) \text{ is well-defined}\} = 1.$$

In particular, $\mathbb{P}\{B[0,n] \cap (\mathbb{Z}^2 \setminus \{(0,0)\}) = \emptyset\} = 1$. In order to avoid overloading the equations in the next sections, we assume from now on that every $z \in \mathbb{Z}^2 \setminus \{(0,0)\}$ satisfies $z \notin B[0,n]$.

The quantities we define now arise naturally in the treatment of the problem. We will need to consider the first time at which points lie in a finite component and also what happens between a time at which they "almost" lie in a finite component and the time at which they actually do.

We define the **closing times** for z by Brownian motion and random walk:

$$T_z = \inf\{t \geq 0 : C_t(z) < \infty\} \qquad \text{and} \qquad \tilde{T}_z = \inf\{t \geq 0 : \tilde{C}_t(z) < \infty\},$$

as well as the **closing points** for z:

$$x_z = B(T_z) \qquad \text{and} \qquad \tilde{x}_z = S(\tilde{T}_z).$$

Another characterization of T_z is

$$T_z = \inf\{t \geq 0 : \exists\, 0 \leq s < t \text{ with } B_s = B_t, |\arg_z(B_s) - \arg_z(B_t)| \neq 0\},$$

and similarly for $\tilde{T}_z$. Note that T_z and $\tilde{T}_z$ are stopping times.

The **last call** for z by Brownian motion is

$$T_z^l = T_{z,n}^l = \inf\{t \geq 0 : \exists\, 0 \leq s \leq t : |\arg_z(B_s) - \arg_z(B_t)| \geq 3\pi/2; d(B_s, B_t) \leq 3n^{1/4} \log^2(n)\},$$

and

$$\tilde{T}_z^l = \tilde{T}_{z,n}^l = \inf\{t \geq 0 : \exists\, 0 \leq r \leq t : |\arg_z(S_r) - \arg_z(S_t)| \geq 3\pi/2; d(S_r, S_t) \leq 3n^{1/4} \log^2(n)\}$$

is the last call for z by random walk. The **last call points** are just

$$x_z^l = B(T_z^l) \text{ and } \tilde{x}_z^l = S(\tilde{T}_z^l).$$

Note that

$$T_z^l < T_z \qquad \text{and} \qquad \tilde{T}_z^l < \tilde{T}_z.$$

We define

$$N_{[a,b)} = N_{[a,b)}(n) = \#\{\text{ connected components of } \mathbb{C} \setminus B[0,n] \text{ with area in } [a,b)\},$$

$$\tilde{N}_{[a,b)} = \#\{\text{ connected components of } \mathbb{C} \setminus S[0,n] \text{ with area in } [a,b)\}.$$

Throughout this chapter, δ will always refer to a real number $\in (0, \frac{1}{200}]$. We let

$$c = 1 + \epsilon, \tag{2.6}$$

where $\epsilon > 0$ is small. Eventually, we will let $\epsilon \to 0$. Most of the quantities with which we will work below depend on n, δ, and ϵ. To simplify the notation, we will not keep the indexes which refer to them, but the reader should be aware of their implicit presence. Let

$$I = [n^{1-\delta}, \infty), \qquad N = N_I, \qquad \tilde{N} = \tilde{N}_I,$$

and define for all $j \geq -1$,

$$I_j^R = [n^{1-\delta} c^{j+1} (1+\epsilon^2)^{-1}, n^{1-\delta} c^{j+1}),$$

for all $j \geq 0$,

$$I_j = [n^{1-\delta} c^j, n^{1-\delta} c^{j+1}), \qquad I_j^L = [n^{1-\delta} c^j, n^{1-\delta} c^j (1+\epsilon^2)),$$
$$I_j^- = I_j \setminus (I_j^L \cup I_j^R), \qquad I_j^+ = I_j \cup (I_{j-1}^R \cup I_{j+1}^L).$$

The number of components in the corresponding intervals will be

$$N_j = N_{I_j}, \qquad \tilde{N}_j = \tilde{N}_{I_j}, \qquad N_j^L = N_{I_j^L}, \qquad N_j^R = N_{I_j^R}, \qquad N_j^{\pm} = N_{I_j^{\pm}}.$$

We let

$$Z_j^{\pm} = \text{ the set of all components of } \mathbb{C} \setminus B[0,n] \text{ with area in } I_j^{\pm},$$

$$\tilde{Z}_j = \text{ the set of all components of } \mathbb{C} \setminus S[0,n] \text{ with area in } I_j,$$

$$Z_j = \text{ the set of all components of } \mathbb{C} \setminus B[0,n] \text{ with area in } I_j,$$

and define the random variables

$$\Delta_j^+ = |\tilde{Z}_j| - |Z_j^+|, \qquad \Delta_j^- = |Z_j^-| - |\tilde{Z}_j|.$$

In the analysis of the problem, we will need to assume that the Brownian paths do not go too far and it is therefore natural to introduce

$$\mathcal{N} = \{ \sup_{0 \le t \le n} |B_t| \le \sqrt{n} \log n \}.$$

In Appendix A, we show that $\mathbb{P}(\mathcal{N}^c)$ decays faster than any power function.

We let $\beta = \frac{1}{25}$ and define for $y, z \in \mathbb{C}$,

$$\mathcal{B}_z(y) = \{d(y, \partial C(z)) \ge n^{1/3-\beta}\}, \quad \tilde{\mathcal{B}}_z(y) = \{d(y, \partial \tilde{C}(z)) \ge n^{1/3-\beta}\},$$
$$\mathcal{B}_z^-(y) = \{d(y, \partial C(z)) \ge \tfrac{1}{2} n^{1/3-\beta}\}, \quad \tilde{\mathcal{B}}_z^-(y) = \{d(y, \partial \tilde{C}(z)) \ge \frac{1}{2} n^{1/3-\beta}\}. \quad (2.7)$$

Remark 4. The $1/3$ exponent is $1-2/3$, where $2/3$ comes from the quantities related to the boundary of the components. See Section 2.4. β is chosen in such a way that, among other things, we have $n^{1/3-\beta} \cdot n^{2/3+\delta} = o(n)$.

$\mathcal{B}(z)$ will be short for $\mathcal{B}_z(z)$ and similarly for the other definitions of (2.7). The following events will also appear naturally in the analysis of the problem and we define them here:

$$\mathcal{E}_1(z) = \{|C(z)| < \infty\}, \quad \mathcal{E}_2(z) = \{|\tilde{C}(z)| < \infty\}, \quad \mathcal{E}(z) = \mathcal{E}_1(z) \cap \mathcal{E}_2(z),$$

$$\mathcal{L}_1(z) = \{n^{1-\delta} \le |C(z)|\}, \quad \mathcal{L}_2(z) = \{n^{1-\delta} \le |\tilde{C}(z)|\}, \quad \mathcal{L}(z) = \mathcal{L}_1(z) \cap \mathcal{L}_2(z).$$

To keep the notation as simple as possible in the next sections, we will use the abbreviation

$$\mathcal{G} = \mathcal{G}(z) = \mathcal{N} \cap \mathcal{P} \cap \mathcal{B}(z) \cap \mathcal{E}(z) \cap \mathcal{L}(z).$$

$\mathcal{G}$ stands for "good" and is an intersection of events on which the result we wish to show will be more likely to hold.

Finally, we define the difference in area of the Brownian motion and random walk component containing a given point $z \in \mathbb{Z}^2$:

$$\Delta(z) = \left| |C(z)| - |\tilde{C}(z)| \right|. \tag{2.8}$$

2.3 Consequences of Le Gall's result

In [24], Le Gall computes the expectation and an upper bound for the variance of $|A_u|$, where

$$A_u = \{y \in \mathbb{C} : \pi(\lambda u)^2 \leq |C_1(y)| \leq \pi u^2\}$$

$\lambda < 1$, and $C_1(z)$ is defined as in (2.4). In particular, he shows that the variance is of smaller order of magnitude than the second moment. His two estimates which are relevant to us are the following:

1. $$\mathbb{E}\left[|A_u|\right] = \frac{\pi|\log\lambda|}{|\log u|^2}\left(1 + \mathcal{O}\left(\frac{\log|\log u|}{|\log\lambda||\log u|^{1/2}}\right)\right), \tag{2.9}$$

   where $\mathcal{O}\left(\cdot\right)$ is for $u \to 0$, but the corresponding constant may depend on λ.

2. There exists a constant $K > 0$ such that $\forall\, u \in (0, 1/4)$,

   $$Var[|A_u|] \leq K|\log u|^{-11/2}. \tag{2.10}$$

As an intermediate step towards the estimates we need, we rewrite these equations for a slightly different quantity. Let $\hat{A}_v = \{y \in \mathbb{C} : vn \leq |C(y)| \leq cvn\}$, where $c = 1 + \epsilon > 1$ is the same constant as defined in the previous section. (Recall that $C(y)$ is short for $C_n(y)$.) Then a change of variables and scaling properties of Brownian motion allow us to deduce the following from (2.9) and (2.10):

$$\mathbb{E}\left[|\hat{A}_v|\right] = \frac{2\pi\log c}{\log^2(cv/\pi)} n\left(1+\mathcal{O}\left(\frac{\log\log(cv)}{\log c(\log(cv))^{1/2}}\right)\right), \tag{2.11}$$

where, again, $\mathcal{O}(\cdot)$ is for $v \to 0$, but the corresponding constant may depend on c.

$$Var\left[|\hat{A}_v|\right] \leq Kn^2|\log(cv)|^{-11/2}, \tag{2.12}$$

for all v with $cv \in (0, 1/4)$.

We can now easily translate these facts into the results we need for our problem, namely results about the number of components of area lying in a certain interval, rather than the total area covered by these components. This just requires dividing the total area by the area of one component, as well as choosing the appropriate values of v. Since the areas are not a determined number, but lie in an interval, this generates an additional error term.

We let $m = \left[\frac{\delta\log n}{2\log c}\right]$. The motivation for this definition will become clear in Section 2.6. What matters for now is that for every $j \leq m-1$, components of area in I_j have area less than $n^{1-\delta/2}$, which is smaller than $\frac{1}{4}n$ for n large enough, so that we can use (2.12). Then, for $0 \leq j \leq m-1$,

$$\mathbb{E}\left[N_j\right] = \gamma_j\left(1-\epsilon(1+\mathcal{O}(\epsilon)) + \mathcal{O}\left(\frac{\log(\frac{1}{2}|\log(\frac{c^{j+1}n^{-\delta}}{\pi})|)}{|\log(\frac{c^{j+1}n^{-\delta}}{\pi})|^{1/2}}\right)\right),$$

$$Var[N_j] \leq K\gamma_j^2\frac{1}{|\log(\frac{c^{j+1}n^{-\delta}}{\pi})|^{3/2}},$$

$$\mathbb{E}\left[N_j^-\right] = \gamma_j^-\left(1+\mathcal{O}\left(\epsilon^2\right) + \mathcal{O}\left(\frac{\log\log(c^{j+1}n^{-\delta}(1+\epsilon^2)^{-1})}{|\log(c^{j+1}n^{-\delta}(1+\epsilon^2)^{-1})|^{1/2}}\right)\right),$$

$$Var[N_j^-] \leq K(\gamma_j^-)^2\frac{1}{|\log(\frac{c^{j+1}}{\pi n^{\delta}(1+\epsilon^2)})|^{3/2}},$$

$$\mathbb{E}\left[N_j^L\right] = \gamma_j^{LR}\left(1+\mathcal{O}\left(\frac{\log(\frac{1}{2}|\log(\frac{c^j n^{-\delta}}{\pi})|)}{|\log(\frac{c^j n^{-\delta}}{\pi})|^{1/2}}\right)+r\epsilon^2\right),$$

$$Var[N_j^L] \leq K(\gamma_j^{LR})^2 \frac{1}{|\log(\frac{c^j n^{-\delta}}{\pi})|^{3/2}},$$

and for $-1 \leq j \leq m-1$,

$$\mathbb{E}\left[N_j^R\right] = \gamma_j^{LR}\left(1+\mathcal{O}\left(\frac{\log(\frac{1}{2}|\log(\frac{c^j n^{-\delta}}{\pi})|)}{|\log(\frac{c^j n^{-\delta}}{\pi})|^{1/2}}\right)-r\epsilon^2\right),$$

$$Var[N_j^R] \leq K(\gamma_j^{LR})^2 \frac{1}{|\log(\frac{c^j n^{-\delta}}{\pi})|^{3/2}},$$

where

$$\gamma_j = \frac{2\pi\log(c)}{c^j\log^2\left(\frac{c^{j+1}}{\pi n^\delta}\right)}n^\delta,\ \gamma_j^{LR} = \frac{2\pi\log(1+\epsilon^2)}{c^j\log^2\left(\frac{c^j}{\pi n^\delta}\right)}n^\delta,\ \gamma_j^- = \frac{2\pi\log(c)}{c^j\log^2\left(\frac{c^{j+1}}{\pi n^\delta(1+\epsilon^2)}\right)}n^\delta, \quad (2.13)$$

$r \in (0,1)$, and K and $\mathcal{O}$ may depend on ϵ. $\mathcal{O}(\cdot)$ is for $n \to \infty$. These estimates are useless if $c^j n^{-\delta}$ does not decay uniformly in j as $n \to \infty$. Fortunately, as mentioned before, we will only need them for $j \leq m-1$, for which $c^j n^{-\delta} \leq K n^{-\delta/2}$, with some constant K uniform in j, δ, ϵ.

The most important consequence of these estimates for us is that for every $C_1 > 0, \delta > 0$, and $\epsilon > 0$, there is a constant $K = K(C_1, \delta, \epsilon) > 0$ such that for all $-1 \leq j \leq \left[\frac{\delta \log n}{2\log c}\right]$,

$$\mathbb{P}\left\{|N_j - \gamma_j| \geq C_1\gamma_j\right\} \leq \frac{K}{(\log n)^{3/2}}. \quad (2.14)$$

We will also need the main result of Le Gall's paper, which is the following: If $N_B = N_B(u)$ is the number of connected components of $\mathbb{C} \setminus B[0,1]$ of area greater than u, then

$$\lim_{u\to 0} u(\log u)^2 N_B = 2\pi, \text{ a.s.} \quad (2.15)$$

Recall that $N = N_{[n^{1-\delta},\infty)}$ and note that different scales are involved in Le Gall's results and the problem which we wish to study here: while N_B is concerned with holes made by $B[0,1]$, N and $\tilde{N}$ are concerned with holes made by $B[0,n]$. Using the scaling properties of Brownian motion, the following is a trivial consequence of (2.15): For every $C_1 > 0$,

$$\mathbb{P}\left\{ |\frac{N}{\gamma} - 1| \geq C_1 \right\} \to 0 \text{ as } n \to \infty, \tag{2.16}$$

where

$$\gamma = \frac{2\pi n^\delta}{\log^2(n^\delta)}. \tag{2.17}$$

In particular, for every $\delta > 0, K_1 < 2\pi, K_2 > 2\pi$,

$$\mathbb{P}\{N < K_1\gamma\} \to 0 \text{ as } n \to \infty \text{ and } \mathbb{P}\{N > K_2\gamma\} \to 0 \text{ as } n \to \infty. \tag{2.18}$$

Of course, (2.16) and (2.18) still hold if we replace the pair (N, γ) by (N_j, γ_j), (N_j^L, γ_j^{LR}), (N_j^R, γ_j^{LR}), or (N_j^-, γ_j^-).

We conclude this section by stating the following fact which will be useful later and which can be easily deduced from (2.13) if one remembers that $c = 1 + \epsilon$:

$$\gamma_j^{LR} = \epsilon\gamma_j^-(1 + \mathcal{O}(\epsilon)). \tag{2.19}$$

2.4 Disconnection exponents and the components' boundaries

In [20], [21], and [22], Lawler, Schramm, and Werner computed a whole class of planar Brownian intersection exponents and disconnection exponents. In this section, we introduce two of these exponents, the one-sided and two-sided disconnection exponents and use the exact value of the latter to estimate the expected number of lattice squares intersected by the boundary of the "large components" of a planar Brownian motion. More precisely, we give an upper bound for the expected box counting dimension (see [9] for a definition) of the boundary of

$$\mathcal{C}_n = \mathcal{C}_n(\epsilon) = \{z \in \mathbb{C} \setminus B[0,n] : |C(z)| \geq n^{1-\epsilon}\}$$

and

$$\tilde{\mathcal{C}}_n = \tilde{\mathcal{C}}_n(\epsilon) = \{z \in \mathbb{C} \setminus S[0,n] : |\tilde{C}(z)| \geq n^{1-\epsilon}\}.$$

Burdzy and Lawler showed in [4] that the intersection exponents for random walk and Brownian motion are the same. Lawler and Puckette extended this work and showed in [18] and [19] that the two exponents which we discuss in this section are the same for Brownian motion and random walk. This allows us to estimate the expected number of lattice squares intersected by the boundary of the "large components" of a planar random walk as well.

We use the notation of [15] and let $\mathbb{P}^{x^1,x^2}$ be the probability measure associated to two independent planar Brownian paths B^1 and B^2 with $B^1(0) = x^1$ and $B^2(0) = x^2$, where $x^1, x^2 \in \mathbb{C}$. Similarly, $\mathbb{P}^{x^1}$ will denote the probability measure associated to the lone Brownian path B^1 started at x^1. For $i = 1, 2$ and $n \in \mathbb{N}$, we define $T^i_n(x) = \inf\{t \geq 0 : |B^i(t) - x| = n\}$; if $x = 0$, we just write T^i_n.

We give the analogous definitions for random walk: $\mathbb{P}^{x^1,x^2}$ is the probability measure associated to two independent planar random walks S^1 and S^2 with $S^1(0) = x^1$ and $S^2(0) = x^2$, where $x^1, x^2 \in \mathbb{Z}^2$. Also, $\mathbb{P}^{x^1}$ is the probability measure associated to S^1 started at x^1. It will be clear from context whether $\mathbb{P}$ refers to Brownian motion or random walk. We also let $\tau^i_n(x) = \inf\{k > 0 : |S^i(k) - x| = n\}$, and write $\tau^i_n(0) = \tau^i_n$, where, again, $i = 1, 2$ and $n \in \mathbb{N}$.

For any compact $A \subset \mathbb{C}$, we let $\bar{Q}(A)$ be the unbounded component of $\mathbb{C} \setminus A$.

2.4.1 One-sided disconnection exponent

Let

$$A_n = \{D(0,1) \cap \bar{Q}(B^1[0, T_n^1]) \neq \emptyset\} \qquad \text{and} \qquad \mathbb{P}(A_n) = \sup \mathbb{P}^x(A_n),$$

where the sup is over all x with $|x| \leq 1$, and

$$\tilde{A}_n = \{(0,0) \cap \bar{Q}(S^1[1, \tau_n^1]) \neq \emptyset\}.$$

The following lemma is a consequence of the three papers mentioned at the beginning of this section and of [18], which gives part (b).

Lemma 2.4.1. There exists a constant $K > 0$ such that for all $n \geq 1$,

$$(a)\, \mathbb{P}(A_n) \leq K n^{-1/4}, \qquad (b)\, \mathbb{P}(\tilde{A}_n) \leq K n^{-1/4}.$$

2.4.2 Two-sided disconnection exponent and the boundary of $\mathcal{C}_n$

Let

$$F_n = \{D(0,1) \cap \bar{Q}(B^1[0,n] \cup B^2[0,n]) \neq \emptyset\},$$

$$D_n = \{D(0,1) \cap \bar{Q}(B^1[0, T_n^1] \cup B^2[0, T_n^2]) \neq \emptyset\},$$

$$\tilde{F}_n = \{(0,0) \cap \bar{Q}(S^1[1,n] \cup S^2[1,n]) \neq \emptyset\},$$

$$\tilde{D}_n = \{(0,0) \cap \bar{Q}(S^1[1, \tau_n^1] \cup S^2[1, \tau_n^2]) \neq \emptyset\},$$

where $D(0,1)$ is the closed disk of radius 1, centered at 0. We will write

$$\mathbb{P}(F_n) = \sup \mathbb{P}^{x^1, x^2}(F_n) \qquad \text{and} \qquad \mathbb{P}(D_n) = \sup \mathbb{P}^{x^1, x^2}(D_n),$$

where the sup is over all $|x^1| \leq 1, |x^2| \leq 1$. In the same way, we let

$$\mathbb{P}(\tilde{F}_n) = \sup \mathbb{P}(\tilde{F}_n) \qquad \text{and} \qquad \mathbb{P}(\tilde{D}_n) = \sup \mathbb{P}(\tilde{D}_n),$$

where the sup is over all possible pairs of vectors $(S^1(1), S^2(1))$.

The following Lemma is a consequence of [20], [21], and [22], where the value of the Brownian disconnection exponent is computed and [19], where the equality of the Brownian exponent and the random walk exponent is shown.

Lemma 2.4.2. There exists a constant K such that for all $n \geq 1$,

$$(a)\ \mathbb{P}(F_n) \leq Kn^{-1/3}, \qquad (b)\ \mathbb{P}(D_n) \leq Kn^{-2/3}$$

$$(c)\ \mathbb{P}(\tilde{F}_n) \leq Kn^{-1/3}, \qquad (d)\ \mathbb{P}(\tilde{D}_n) \leq Kn^{-2/3}.$$

We now turn to the question in which we are interested for our specific problem, which concerns the "size" (fractal dimension) of the boundary of large components. Lawler first showed in [15] that there is a strong link between the two-sided disconnection exponent and the Hausdorff dimension of the Brownian frontier. The proof of Lemma 2.4.3 below is essentially based on ideas of his paper.

Recall the definition of $\mathcal{C}_n$ and $\tilde{\mathcal{C}}_n$ at the beginning of this chapter. The main lemma of this section gives an upper bound for the expected number of squares of the dual lattice of $\mathbb{Z}^2$ (i.e. squares of side-length 1, centered at the points of $\mathbb{Z}^2$, and whose sides are parallel to the coordinate axes) which are intersected by the boundary of $\mathcal{C}_n$. Whenever we write "squares" below, we mean the squares of the dual lattice. We will say that a square $\mathcal{S}$ is *hit* by a set A if $A \cap \mathcal{S} \neq \emptyset$.

Lemma 2.4.3. There exists a constant $K > 0$ such that for every $\epsilon > 0$, every $n \geq 1$,

$$\mathbb{E}\left[\#\{y \in \mathbb{Z}^2 : \mathrm{Sq}(y) \cap \partial\mathcal{C}_n(\epsilon) \neq \emptyset\}\right] \leq K\, n^{\frac{2}{3}+\frac{\epsilon}{2}}$$

and

$$\mathbb{E}\left[\#\{y \in \mathbb{Z}^2 : \mathrm{Sq}(y) \cap \partial\tilde{\mathcal{C}}_n(\epsilon) \neq \emptyset\}\right] \leq K\, n^{\frac{2}{3}+\frac{\epsilon}{2}}.$$

Proof. The general strategy of the proof is essentially to bound the expected number of time segments $[j-1, j]$ over which the Brownian path intersects $\partial\mathcal{C}_n(\epsilon)$. This will suffice thanks to the fact that the expected number of squares hit by $B([j-1,j])$ is finite. For $1 \le j \le n$, let d_j be the diameter of $B[j-1,j]$ and $\mathcal{B}_j$ the closed ball of radius d_j, centered at B_j, so that $B[j-1,j] \subset \mathcal{B}_j$. We let $\tilde{n} = [n^{1-\epsilon}(\log n)^{-3}]$ and call

$$\mathcal{A}_j^L = B[0 \vee (j-1-\tilde{n}), j-1] \text{ and } \mathcal{A}_j^R = B[j, (2j-1) \wedge (j+\tilde{n})]$$

the *left arm* and *right arm*, respectively, of $\mathcal{B}_j$. The left and right arm span time intervals of same length and that length is $j-1$ if $j \le \tilde{n}+1$ and $\tilde{n}$ if $j \ge \tilde{n}+1$.

With high probability, for every $1 \le j \le n$, both arms of $\mathcal{B}_j$ are completely contained in a disk of radius $n^{(1-\epsilon)/2}(\log n)^{-1}$. The idea which we will use below is that if $B[j-1,j]$ is to intersect $\partial\mathcal{C}_n(\epsilon)$, then on one of these events of high probability, the arms of $\mathcal{B}_j$ cannot disconnect $\mathcal{B}_j$ from infinity. Indeed, since the arms span a time interval of at most $\tilde{n}$, we expect them to cover roughly a distance of at most $\sqrt{\tilde{n}} = \frac{n^{(1-\epsilon)/2}}{(\log n)^{3/2}}$. If they disconnect $\mathcal{B}_j$ from infinity, then assuming that d_j is bounded, $B[j-1,j]$ can intersect a component of area at most $K\frac{n^{1-\epsilon}}{(\log n)^3}$. We now write out this argument formally. The main difficulty is to handle the cases where either d_j is very large or the arms of $\mathcal{B}_j$ are unusually long.

We define $m = [n/2]+1$. By symmetry, we have

$$\mathbb{E}[\#(\text{squares hit by } \partial\mathcal{C}_n)] \le 2\sum_{j=1}^{m} \mathbb{E}\left[\#(\text{squares hit by } \partial\mathcal{C}_n \cap B[j-1,j])\right]. \qquad (2.20)$$

We first look at the terms for $j \le \tilde{n}$. It is easy to check that if $B[j-1,j]$ intersects l squares, then $d_j \ge \left[\frac{\sqrt{l}}{10}\right]$. This is not optimal but enough for our needs. Therefore, for $j \le \tilde{n}$, we have

$$\mathbb{E}[\#(\text{squares hit by } \partial\mathcal{C}_n \cap B[j-1,j])]$$

$$= \sum_{l \ge 1} \mathbb{P}\left\{\#(\text{squares hit by } \partial\mathcal{C}_n \cap B[j-1,j]) = l\right\} l$$

$$\leq \sum_{l\geq 1} \mathbb{P}\left\{B[j-1,j]\cap\partial\mathcal{C}_n \neq \emptyset; d_j \geq \left[\frac{\sqrt{l}}{10}\right]\right\} l$$

$$= \sum_{l\geq 1} l \sum_{k\geq 0} \mathbb{P}\left\{B[j-1,j]\cap\partial\mathcal{C}_n \neq \emptyset; d_j \in \left[[\frac{\sqrt{l}}{10}]+k, [\frac{\sqrt{l}}{10}]+k+1\right]\right\}$$

$$\leq \sum_{l\geq 1} l \sum_{k\geq 0} \mathbb{P}\left\{\mathcal{B}_j\cap\partial\mathcal{C}_n \neq \emptyset \,\middle|\, d_j \in \left[[\frac{\sqrt{l}}{10}]+k, [\frac{\sqrt{l}}{10}]+k+1\right]\right\}$$

$$\cdot\, \mathbb{P}\left\{d_j \geq [\frac{\sqrt{l}}{10}]+k\right\}.$$

We now separate the last term into the case where the arms are not unusually long and the case where they are. This gives

$$\leq \sum_{l\geq 1}\left[l \sum_{k\geq 0} \mathbb{P}\left\{\mathcal{B}_j\cap\bar{Q}(\mathcal{A}_j^L\cup\mathcal{A}_j^R) \neq \emptyset \,\middle|\, d_j \in \left[[\frac{\sqrt{l}}{10}]+k, [\frac{\sqrt{l}}{10}]+k+1\right]\right\}\right. \tag{2.21}$$

$$\left.\cdot\, \mathbb{P}\left\{d_j \geq [\frac{\sqrt{l}}{10}]+k\right\}\right] \tag{2.22}$$

$$+\,\mathbb{P}\left\{\operatorname{diam}(\mathcal{A}_j^L) \geq n^{\frac{1-\epsilon}{2}}\log^{-1} n\right\} + \mathbb{P}\left\{\operatorname{diam}(\mathcal{A}_j^R) \geq n^{\frac{1-\epsilon}{2}}\log^{-1} n\right\} \tag{2.23}$$

By looking at the time-reversal of $\mathcal{A}_j^L$ and Lemma 2.4.2 (a), scaling tells us that the probability in (2.21) is bounded above by $K(\frac{j}{(\sqrt{l}/10+k)^2})^{-1/3}$. Since for any $a \geq 0$ and $b \geq 1, a+b \leq 4(a+1)b$, this is smaller than $Kj^{-1/3}l^{1/3}k^{2/3}$. For the probability in (2.22), we use Lemma A.1.1 to get an upper bound of $K\exp\{-\frac{1}{2}(\frac{1}{100}l+k^2)\}$. The same lemma also implies that the probabilities in (2.23) are bounded above by $Kn^{-1/2}$. This gives the following:

$$\mathbb{E}[\#(\text{squares hit by } \partial\mathcal{C}_n \cap B[j-1,j])]$$

$$\leq Kn^{-1/2} + Kj^{-1/3}\sum_{l\geq 1} l\cdot l^{1/3}\exp\{-l/200\}\sum_{k\geq 0}k^{2/3}\exp\{-k^2/2\}$$

$$\leq Kj^{-1/3}, \tag{2.24}$$

since both sums are finite.

In the same way, if $\tilde{n} \le j \le m$, then the arms of $\mathcal{B}_j$ are very likely to reach distance at least $n^{(1-\epsilon)/2}(\log n)^{-1}$. Using Lemma 2.4.2 ($b$), we get

$$\mathbb{E}[\#(\text{squares hit by } \partial\mathcal{C}_n \cap B[j-1,j])]$$
$$\le K(n^{(1-\epsilon)/2}(\log n)^{-1})^{-2/3}. \tag{2.25}$$

Now all that is left to do is use (2.24), (2.25), and (2.20) to conclude that

$$\begin{aligned}
&\mathbb{E}[\#(\text{squares hit by } \partial\mathcal{C}_n] \\
&\le K\left[\sum_{j=1}^{\tilde{n}} j^{-1/3} + \sum_{j=\tilde{n}}^{m} (n^{(1-\epsilon)/2}(\log n)^{-1})^{-2/3}\right] \\
&\le K\left[\tilde{n}^{2/3} + n \cdot (n^{(1-\epsilon)/2}(\log n)^{-1})^{-2/3}\right] \\
&\le K\left[n^{\frac{2}{3}(1-\epsilon)} + n^{\frac{2}{3}+\frac{\epsilon}{3}}(\log n)^{2/3}\right] \le K n^{\frac{2}{3}+\frac{\epsilon}{2}}.
\end{aligned}$$

This gives the lemma. □

The exact same ideas combined with Lemma 2.4.2 (c) and (d) give

Lemma 2.4.4. There exists a constant $K > 0$ such that for every $\epsilon > 0$, every $n \ge 1$,

$$\mathbb{E}\left[\#\{y \in \mathbb{Z}^2 : \mathrm{Sq}(y) \cap \partial\mathcal{C}_n(\epsilon) \neq \emptyset\}\right] \le K\, n^{\frac{2}{3}+\frac{\epsilon}{2}}$$

and

$$\mathbb{E}\left[\#\{y \in \mathbb{Z}^2 : \mathrm{Sq}(y) \cap \partial\tilde{\mathcal{C}}_n(\epsilon) \neq \emptyset\}\right] \le K\, n^{\frac{2}{3}+\frac{\epsilon}{2}}.$$

2.5 Preliminary lemmas

The work of this section leads to Proposition 2.6.1, of which Theorem 2.1.1 is an almost direct consequence. The general strategy of the proof of the proposition is to eliminate all the "bad cases" (events on which $\{|\tilde{N} - N| > \epsilon N\}$ is likely to hold) by

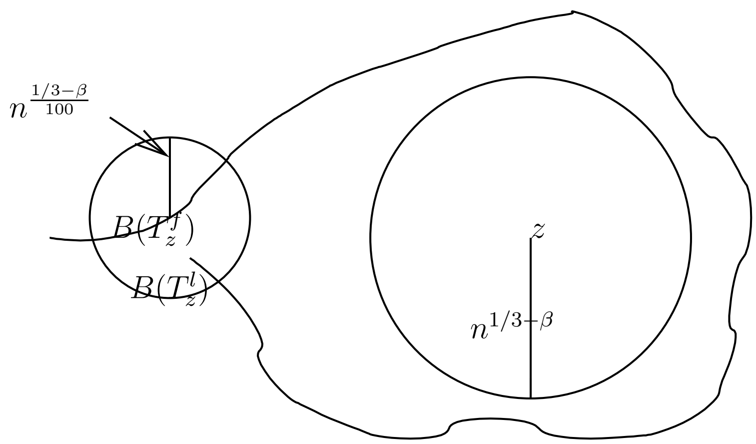

Figure 2.3: Proof of Lemma 2.5.1: T_z^l is the first time at which $D(z, n^{1/3-\beta})$ is "almost" disconnected from ∞.

showing that they have a probability which goes to 0 as n gets large. This is handled in the sequence of lemmas of this section, as well as in Section 2.4.

The first two lemmas are results which concern either Brownian motion or random walk, but do not consider the coupling. Lemmas 2.5.3 and 2.5.4 address questions on the joint behavior of the coupled random walk and Brownian motion.

The first lemma of this section shows that outside of an unlikely set, the closing time by Brownian motion for a point which is not too close to the Brownian path comes soon after the last call. Recall that $\beta = \frac{1}{25}$.

Lemma 2.5.1. There exists a constant $K > 0$ such that for any $z \in \mathbb{C}$,

$$(a)\ \mathbb{P}\left\{T_z - T_z^l > n^{2/3}; \mathcal{B}(z)\right\} \le \mathbb{P}\left\{T_z - T_z^l > n^{2/3}; \mathcal{B}^-(z)\right\} \le K n^{\frac{\beta}{2}-\frac{1}{24}} \log n.$$

$$(b)\ \mathbb{P}\left\{\tilde{T}_z - \tilde{T}_z^l > n^{2/3}; \tilde{\mathcal{B}}(z)\right\} \le \mathbb{P}\left\{\tilde{T}_z - \tilde{T}_z^l > n^{2/3}; \tilde{\mathcal{B}}^-(z)\right\} \le K n^{\frac{\beta}{2}-\frac{1}{24}} \log n.$$

Proof. [See Figure 2.3]

(a) We start by pointing out that if $x \in (D(z, n^{1/3-\beta}))^c$ and $0 \le s \le t$ are such that $B[s,t] \subset D(x, \frac{n^{1/3-\beta}}{100})$, then we have the obvious rough bound

$$|\arg_z(B_s) - \arg_z(B_t)| \le \frac{\pi}{8}. \tag{2.26}$$

We define

$$T_z^f = \inf\{t \geq 0 : |\arg_z(B_t) - \arg_z(B_{T_z^l})| \geq 3\pi/2, d(B_t, B_{T_z^l}) \leq 3n^{1/4}\log^2 n\},$$

$x_z^f = B(T_z^f)$, and $\Phi_z^f = \inf\{t \geq T_z^f : B_t \in \partial D_z^f\}$, where $D_z^f = D(x_z^f, \frac{n^{1/3-\beta}}{100})$.$x_z^f$ can be thought of as being the point "across from" x_z^l on the Brownian path up to time T_z^l (see Figure 2.3).

We first note that on the event $\mathcal{B}(z)$, the definition of x_z^l implies that the connected random set $A_z = B[T_z^f, \Phi_z^f]$ contains x_z^f, intersects ∂D_z^f, and satisfies

1. $d(x_z^l, A_z) \leq 3n^{1/4}\log^2(n)$.
2. For any $t \in [T_z^f, \Phi_z^f], |\arg_z(B_t) - \arg_z(B(T_z^l))| > \pi$.

The second point is true because $|\arg_z(B(T_z^f)) - \arg_z(B(T_z^l))| \geq 3\pi/2$ and inside $D_z^f, \arg_z(B_t)$ does not vary by more than $\frac{\pi}{8}$, by (2.26), since on $\mathcal{B}(z), x_z^f \in (D(z, n^{1/3-\beta}))^c$. In fact, as n increases, $|\arg_z(B(T_z^f)) - \arg_z(B(T_z^l))|$ becomes arbitrarily close to $2k\pi$ for some strictly positive integer k.

If we let $\Phi_z^l = \inf\{t \geq T_z^l : B_t \in \partial D_z^f\}$, the fact that T_z^l is a stopping time and that A_z is measurable with respect to T_z^l allows us to use the strong Markov property and write

$$\begin{aligned}
&\mathbb{P}\{T_z - T_z^l > n^{2/3}\} \\
&\qquad \leq \mathbb{P}\left\{B[T_z^l, T_z^l + n^{2/3}] \cap A_z = \emptyset\right\} \\
&\qquad \leq \mathbb{P}\left\{B[T_z^l, T_z^l + n^{2/3}] \cap \partial D_z^f = \emptyset\right\} + \mathbb{P}\left\{B[T_z^l, \Phi_z^l] \cap A_z = \emptyset\right\} \\
&\qquad \leq \mathbb{P}\left\{\sup_{0 \leq t \leq n^{2/3}} |B_t| \leq 2n^{1/3-\beta}\right\} + \mathbb{P}^{3n^{1/4}\log^2(n)}\left\{B[0, \Xi_{n^{1/3-\beta}}] \cap \Sigma = \emptyset\right\},
\end{aligned}$$

by Beurling's projection theorem (see Appendix A) and translation invariance of Brownian motion. Here, Σ is the straight line from $(0,0)$ to $(-\frac{n^{1/3-\beta}}{100}, 0)$ and for $a \geq 0$, $\Xi_a = \inf\{t \geq 0 : |B_t| \geq a\}$.

Thus, $\mathbb{P}\left\{T_z - T_z^l > n^{2/3}\right\} \leq \exp\{-Cn^{2\beta}\} + Kn^{\frac{\beta}{2}-\frac{1}{24}} \leq Kn^{\frac{\beta}{2}-\frac{1}{24}}$, by Lemmas A.1.2 and A.3.2.

The proof of (b) is virtually the same, but in this case we use the discrete Beurling estimate.

$\square$

It should be intuitively clear that many points are disconnected from infinity in the time interval $[n - n^a, n]$, where $a \leq 1$, but most of these points will lie in rather small components. Lemma 2.5.2 shows that in fact it is unlikely that the closing time for a point which lies in a large component occur very late.

Lemma 2.5.2. There exists a constant $K > 0$ such that for any $z \in \mathbb{Z}^2, 0 < \delta \leq \frac{1}{200}$, and $0 \leq a \leq 1 - \delta$,

$$(a)\ \mathbb{P}\left\{T_z \in [n - n^a, n]; \mathcal{L}_1(z)\right\} \leq Kn^{\frac{a+\delta-1}{8}}(\log n)^{1/4}.$$

$$(b)\ \mathbb{P}\left\{\tilde{T}_z \in [n - n^a, n]; \mathcal{L}_2(z)\right\} \leq Kn^{\frac{a+\delta-1}{8}}(\log n)^{1/4}.$$

Remark 5. The result holds, in fact, for any $a \geq 0$, but ceases to be interesting if $a \geq 1 - \delta$, since for those values of a, $Kn^{\frac{a+\delta-1}{8}}(\log n)^{1/4} \to \infty$, as $n \to \infty$.

Proof. [See Figure 2.4]

(a) Recall that $\mathcal{L}_1(z)$ means that z lies in a Brownian motion component of area $\geq n^{1-\delta}$. In Section 2.4 we defined, for any compact $A \subset \mathbb{C}$, $\bar{Q}(A)$ to be the unbounded component of $\mathbb{C} \setminus A$. We call Γ the first time at which $D(n^{a/2}\log n)$ is disconnected from ∞. If we let $\Lambda = \sup\{t \geq 0 : D(B_n, n^{a/2}\log n) \cap \bar{Q}(B[t, n]) \neq \emptyset\}$ be the greatest time at which $B[t, n]$ disconnects $D(B_n, n^{a/2}\log n)$ from infinity (with the convention $\sup\emptyset = 0$), then if $\operatorname{diam}(B[\Lambda, n]) \leq n^{(1-\delta)/2}$ and $B[n - n^a, n] \subset D(B_n, n^{a/2}\log n)$, $B[n - n^a, n]$ cannot intersect the boundary of a component of area $\geq n^{1-\delta}$. Thus,

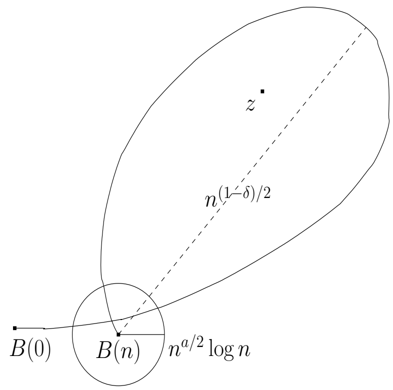

Figure 2.4: Proof of Lemma 2.5.2: Time-reversed Brownian motion must get from $B(n)$ to distance $n^{(1-\delta)/2}$ without disconnecting $D(B(n), n^{a/2}\log n)$ from infinity.

$$\begin{aligned}
&\mathbb{P}\{T_z \in [n-n^a, n]; \mathcal{L}_1(z)\} \\
&\quad \leq \mathbb{P}\left\{B[n-n^a, n] \cap \partial C(z) \neq \emptyset; \mathcal{L}_1(z)\}\right\} \\
&\quad \leq \mathbb{P}\left\{\sup_{n-n^a \leq t \leq n} |B_t - B_n| \geq n^{a/2}\log n\right\} + \mathbb{P}\left\{\text{diam}(B[\Lambda, n]) \geq n^{(1-\delta)/2}\right\} \\
&\quad = \mathbb{P}\left\{\sup_{0 \leq t \leq n^a} |B_t| \geq n^{a/2}\log n\right\} + \mathbb{P}\left\{\Xi_{n^{1-\delta}} < \Gamma\right\} \\
&\quad \leq Kn^{-\log n/2} + Kn^{(a+\delta-1)/8}(\log(n))^{1/4},
\end{aligned}$$

by Lemmas A.1.1 and 2.4.1 (a). The equality can be seen by time-reversing Brownian motion.

Again, the proof of (b) is the same but uses the corresponding results for random walk, namely Corollary A.1.4 and Lemma 2.4.1 (b).

□

Lemma 2.5.3 shows that the chance of $C(z)$ being finite and $\tilde{C}(z)$ being infinite is small when z is not too close to the boundary of $C(z)$. The condition that z be away from the boundary of $C(z)$ is essential, since otherwise, it could easily happen that the Brownian path passes on one "side" of z and the random walk on the other (see Figure 2.5). To avoid this, we need z to be at a distance from $\partial C(z)$ which is greater

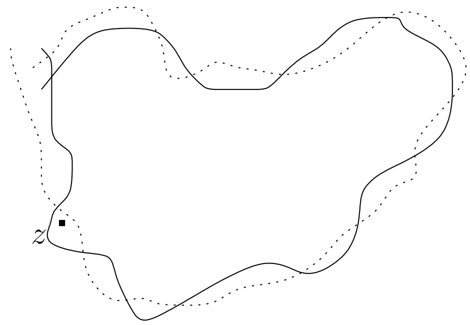

Figure 2.5: z close to the boundary of $C(z)$

than the maximal distance between the coupled random walk and Brownian motion, i.e. $n^{1/4}\log^2 n$. In fact, the condition $\mathcal{B}(z) = \{d(z, \partial C(z)) \geq n^{1/3-\beta}\}$ which we use to restrict ourselves to points away from the boundary gives us more room than we really need. Recall that $(\mathcal{E}_1(z))^c = \{|C(z)| = \infty\}$ and $(\mathcal{E}_2(z))^c = \{|\tilde{C}(z)| = \infty\}$. $\mathcal{P}$ is still the condition associated with the coupling, i.e. that B and S be close to each other in the sense of (2.2).

Lemma 2.5.3. There exists a constant $K > 0$ such that for every $z \in \mathbb{C}, j \geq 0$, and $n \geq 1$,

$$(a)\ \mathbb{P}\left\{|C(z)| \in I_j; \mathcal{B}(z); (\mathcal{E}_2(z))^c; \mathcal{P}\right\} \leq K n^{\frac{\beta}{2}-\frac{1}{24}} \log n.$$

$$(b)\ \mathbb{P}\left\{|\tilde{C}(z)| \in I_j; \tilde{\mathcal{B}}(z); (\mathcal{E}_1(z))^c; \mathcal{P}\right\} \leq K n^{\frac{\beta}{2}-\frac{1}{24}} \log n.$$

Proof. [See Figure 2.6]

(a) The key idea of the proof is that when the Brownian motion component closes, the last call for random walk has already occurred. At that instant, either it is very late, i.e. a time very close to n, which is unlikely by Lemma 2.5.2 (a), or the random walk has plenty of time left to disconnect z from infinity and will do it with high probability by Lemma 2.5.1 (b). We can easily check that $\mathcal{B}(z) \cap \mathcal{P} \subset \tilde{\mathcal{B}}^{-}(z)$ and $\mathcal{B}(z) \cap \mathcal{P} \subset \{\tilde{T}^l_z \leq T_z\}$. The first inclusion is obvious. The second follows from an argument similar to the one we used at the beginning of the proof of Lemma 2.5.1:

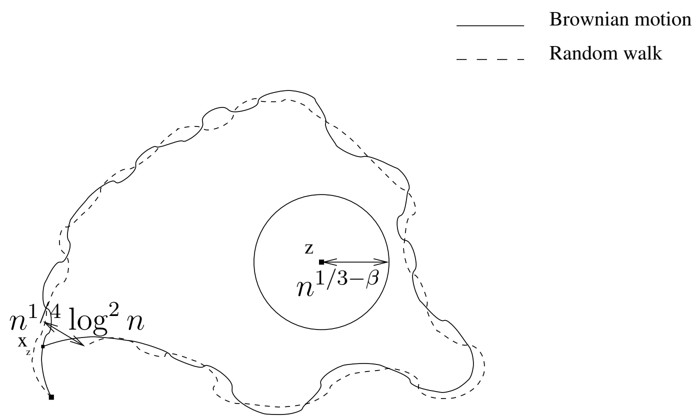

Figure 2.6: Proof of Lemma 2.5.3: When Brownian motion disconnects z from ∞, random walk is close to doing so as well.

By definition of T_z, there is a time $t \le T_z$ such that $B_t = B_{T_z}$ and $|\arg_z(B_t) - \arg_z(B_{T_z})| \ge 2\pi$. The conditions $\mathcal{B}(z)$ and $\mathcal{P}$ ensure that $|\arg_z(B_t) - \arg_z(S_t)| \le \frac{\pi}{8}$ and $|\arg_z(B_{T_z}) - \arg_z(S_{T_z})| \le \frac{\pi}{8}$, so that $|\arg_z(S_t) - \arg_z(S_{T_z})| \ge \frac{3\pi}{4}$. Also, the condition $\mathcal{P}$ guarantees that $d(S_t, B_t) \le n^{1/4}\log^2 n$ and $d(S_{T_z}, B_{T_z}) \le n^{1/4}\log^2 n$, which implies $d(S_t, S_{T_z}) \le 2n^{1/4}\log^2 n$. By the definition of $\tilde{T}_z^l$ it is now clear that $\tilde{T}_z^l \le T_z$. Therefore,

$$\begin{aligned}
&\mathbb{P}\{|C(z)| \in I_j; |\tilde{C}(z)| = \infty; \mathcal{B}(z); \mathcal{P}\} \\
&\qquad \le \mathbb{P}\left\{|C(z)| < \infty; |\tilde{C}(z)| = \infty; \mathcal{B}(z); \mathcal{P}; \tilde{T}_z^l < n - n^{2/3}\right\} \\
&\qquad\qquad + \mathbb{P}\left\{T_z \in [n - n^{2/3}, n]; \mathcal{L}_1(z)\right\} \\
&\qquad \le \mathbb{P}\left\{\tilde{T}_z - \tilde{T}_z^l > n^{2/3}; \mathcal{B}(z); \mathcal{P}\right\} + \mathbb{P}\left\{T_z \in [n - n^{2/3}, n]; \mathcal{L}_1(z)\right\} \\
&\qquad \le n^{\frac{\beta}{2} - \frac{1}{24}} \log n + K n^{-\frac{1}{6} + \frac{\delta}{4}} \\
&\qquad \le K n^{\frac{\beta}{2} - \frac{1}{24}} \log n,
\end{aligned}$$

by Lemma 2.5.1 (b) and Lemma 2.5.2 (a).

Part (b) is done in the same way, but we use the other parts of Lemmas 2.5.1 and Lemma 2.5.2.

□

The last preparatory lemma shows that if $y \in C(z)$, but y and z are not too close to the boundary of $C(z)$, it is unlikely that $y \notin \tilde{C}(z)$ if $\tilde{C}(z) < \infty$.

Lemma 2.5.4. For any given $y, z \in \mathbb{C}$,

$$\mathbb{P}\left\{y \in C(z) \setminus \tilde{C}(z); \mathcal{B}(z); \mathcal{B}_z(y); \mathcal{E}(z); \mathcal{P}\right\} \leq K n^{\frac{\beta}{2}-\frac{1}{24}}(\log n).$$

Proof. For $y, z \in \mathbb{C}$, we introduce stopping times similar to those defined in Section 2.2:

$$\begin{aligned} T_{y,z} &= \inf\{t \geq 0 : \exists 0 \leq s \leq t \text{ with } B_s = B_t\}, \\ &\qquad |\arg_y(B_s) - \arg_y(B_t)| \neq |\arg_z(B_s) - \arg_z(B_t)|\}, \\ \tilde{T}_{y,z} &= \inf\{t \geq 0 : \exists 0 \leq r \leq t \text{ with } S_r = S_t, \\ &\qquad |\arg_y(S_r) - \arg_y(S_t)| \neq |\arg_z(S_r) - \arg_z(S_t)|\}, \\ T^l_{y,z} &= \inf\{t \geq 0 : \exists 0 \leq s \leq t \text{ with } d(B_s, B_t) \leq 3n^{1/4}\log^2(n), \\ &\qquad |\arg_y(B_s) - \arg_y(B_t)| \neq |\arg_z(B_s) - \arg_z(B_t)|\}, \\ \tilde{T}^l_{y,z} &= \inf\{t \geq 0 : \exists 0 \leq r \leq t \text{ with } d(S_r, S_t) \leq 3n^{1/4}\log^2(n), \\ &\qquad |\arg_y(S_r) - \arg_y(S_t)| \neq |\arg_z(S_r) - \arg_z(S_t)|\}. \end{aligned}$$

Note that y and z lie in different components of $\mathbb{C} \setminus B[0, t]$ if and only if $t \geq T_{y,z}$.

$$\begin{aligned} \mathbb{P}\{y \in & C(z) \setminus \tilde{C}(z); \mathcal{B}(z); \mathcal{B}_z(y); \mathcal{E}(z); \mathcal{P}\} \\ & \leq \mathbb{P}\left\{y \in C(z); \mathcal{B}(z); \mathcal{B}_z(y); \mathcal{P}; T^l_{y,z} < n - n^{2/3}\right\} \\ & \qquad + \mathbb{P}\left\{\tilde{T}_{y,z} \in [n - n^{2/3}, n]\right\}. \end{aligned}$$

Exactly as in Lemma 2.5.3, $\{\mathcal{B}(z); \mathcal{B}_z(y); \mathcal{P}\} \subset \{\tilde{\mathcal{B}}^-(z); \tilde{\mathcal{B}}^-_z(y)\} \subset \{T^l_{y,z} \leq \tilde{T}_{y,z}\}$, and so

$$\mathbb{P}\left\{y \in C(z) \setminus \tilde{C}(z); \mathcal{B}(z); \mathcal{B}_z(y); \mathcal{E}(z)\right\} \leq K n^{\frac{\beta}{2}-\frac{1}{24}} \log n.$$

□

Remark 6. In the last lemma, it is clear that unless $d(y,z) \geq 2n^{\frac{1}{3}+\gamma}$, the probability is 0.

We are now ready to attack the core of the argument. The next lemma contains some of the ideas needed for the proof of the main theorem and we provide it here with its proof.

Lemma 2.5.5. For every $C_1 > 0$, there exists a constant $K > 0$ such that for any $0 < \delta \leq \frac{1}{200}$, any $z \in \mathbb{C}$,

$$\mathbb{P}\left\{\Delta(z) \geq C_1 n^{1-\beta/8}; \mathcal{G}\right\} \leq K n^{-3\beta/8}.$$

Proof. We will show that

$$\mathbb{E}\left[\Delta(z); \mathcal{G}; \mathcal{L}\right] \leq K n^{\frac{23}{24}+\frac{\beta}{2}} \log^3 n. \tag{2.27}$$

Once we have this, the lemma follows from Chebyshev's inequality. We note that

$$\begin{aligned}
\Delta(z) &\leq \#\{y \in \mathbb{Z}^2 : \mathrm{Sq}(y) \cap \partial C(z) \neq \emptyset\} \\
&+ \#\{y \in \mathbb{Z}^2 : \mathrm{Sq}(y) \cap \partial \tilde{C}(z) \neq \emptyset\} \\
&+ \#\{y \in \mathbb{Z}^2 : y \in C(z) \setminus \tilde{C}(z)\} \\
&+ \#\{y \in \mathbb{Z}^2 : y \in \tilde{C}(z) \setminus C(z)\}.
\end{aligned}$$

The first two terms on the right side are related to the Hausdorff dimension of the Brownian frontier. By using Lemma 2.4.3 for those terms, we can show that for every $\delta > 0$,

$$\begin{aligned}
\mathbb{E}\left[\Delta(z); \mathcal{G}\right] &\leq \mathbb{E}\left[\#\{y \in \mathbb{Z}^2 : d(y, \partial C(z)) \leq n^{1/3-\beta}\}; \mathcal{L}_1(z)\right] \\
&+ \mathbb{E}\left[\#\{y \in \mathbb{Z}^2 : d(y, \partial \tilde{C}(z)) \leq n^{1/3-\beta}\}; \mathcal{L}_2(z)\right] \\
&+ \mathbb{E}\left[\sum_{y \in \mathbb{Z}^2} \mathbb{1}\{y \in C(z) \setminus \tilde{C}(z); \mathcal{G}; \mathcal{B}_z(y)\} + \mathbb{1}\{y \in \tilde{C}(z) \setminus C(z); \mathcal{G}; \mathcal{B}_z(y)\}\right]
\end{aligned}$$

$$\le Kn^{1-\beta+\delta} + \sum_{y\in\mathbb{Z}^2} \mathbb{P}\left\{y \in C(z)\setminus \tilde{C}(z); \mathcal{G}; \mathcal{B}_z(y)\right\}$$

$$+ \sum_{y\in\mathbb{Z}^2} \mathbb{P}\left\{y \in \tilde{C}(z)\setminus C(z); \mathcal{G}; \mathcal{B}_z(y)\right\}.$$

We estimate $\sum_{y\in\mathbb{Z}^2} \mathbb{P}\left\{y \in C(z)\setminus \tilde{C}(z); \mathcal{G}; \mathcal{B}_z(y)\right\}$ here. The other term is done in the exact same way.

$$\begin{aligned}\sum_{y\in\mathbb{Z}^2} \mathbb{P}\left\{y \in C(z)\setminus \tilde{C}(z); \mathcal{G}; \mathcal{B}_z(y)\right\} &\le \sum_{\substack{|y|\le\sqrt{n}\log n \\ y\in\mathbb{Z}^2}} \mathbb{P}\left\{y \in C(z)\setminus \tilde{C}(z); \mathcal{G}; \mathcal{B}_z(y)\right\} \\ &\le Kn^{\frac{23}{24}+\frac{\beta}{2}}(\log n)^3,\end{aligned}$$

by Lemma 2.5.4. Thus, for every $\delta \in (0, \frac{1}{200}]$,

$$\mathbb{E}\left[\Delta(z); \mathcal{G}\right] \le Kn^{1-\beta+\delta} + Kn^{\frac{23}{24}+\frac{\beta}{2}}(\log n)^3 \le Kn^{\frac{23}{24}+\frac{\beta}{2}}(\log n)^3,$$

since $\beta - \delta \ge \frac{1}{24} - \frac{\beta}{2}$. Chebyshev's inequality now gives for any $\alpha \le \frac{\beta}{8}$,

$$\begin{aligned}\mathbb{P}\left\{\Delta(z) \ge n^{1-\alpha}; \mathcal{G}\right\} &\le K\frac{n^{\frac{23}{24}+\frac{\beta}{2}}(\log n)^3}{n^{1-\alpha}} \\ &\le Kn^{\alpha-\frac{\beta}{2}} \le Kn^{\beta/8-\beta/2} = Kn^{-3/200}.\end{aligned}$$

□

2.6 Main results

Proposition 2.6.1. If a planar simple random walk S and a planar standard Brownian motion B are coupled as in Section 1.4 and

$$N = \#\{\text{connected components of } \mathbb{C}\setminus B[0,n] \text{ of area larger than } n^{1-\delta}\},$$

$$\tilde{N} = \#\{\text{connected components of } \mathbb{C} \setminus S[0,n] \text{ of area larger than } n^{1-\delta}\},$$

then for every $\epsilon > 0$ and every $0 < \delta \le \frac{1}{200}$,

$$\mathbb{P}\left\{|\tilde{N} - N| > \epsilon N\right\} \to 0 \text{ as } n \to \infty.$$

Remark 7. Recall that in Section 1.4, we decided to re-parametrize the random walk, namely to double its usual velocity, so that B_t is close to S_t, rather than to S_{2t}. The random walk of the Proposition is the re-parametrized version.

Proof. As always in this chapter, we will be using the coupling based on Skorokhod's method (see Chapter 1). First of all, note that it suffices to show this result for ϵ small enough. We set $c = 1 + \epsilon$, we will show that $\mathbb{P}\left\{|\tilde{N} - N| > 13\epsilon N\right\} \to 0$ as $n \to \infty$. The following is an important idea for the proof: Most of the holes of area greater than $n^{1-\delta}$ have an area which is close to $n^{1-\delta}$. For instance, as can be seen from (2.17), there are many more holes of area in the interval $[n^{1-\delta}, n^{1-\frac{\delta}{2}}]$ than in the interval $[n^{1-\frac{\delta}{2}}, \infty)$.

If we let $m = m(n, \epsilon, \delta) = \left[\frac{\delta \log n}{2 \log c}\right]$, then $n^{1-\delta} c^m \le n^{1-\frac{\delta}{2}} \le n^{1-\delta} c^{m+1}$ and so,

$$\begin{aligned}
\mathbb{P}\left\{|\tilde{N} - N| > 13\epsilon N\right\} \;\le\; & \mathbb{P}\left\{|\tilde{N}_{[n^{1-\delta}, n^{1-\delta}c^m)} - N_{[n^{1-\delta}, n^{1-\delta}c^m)}| > 10\epsilon N\right\} \\
+ \; & \mathbb{P}\left\{|\tilde{N}_{[n^{1-\frac{\delta}{2}}, \infty)} - N_{[n^{1-\frac{\delta}{2}}, \infty)}| > \epsilon N\right\} \\
+ \; & \mathbb{P}\left\{\tilde{N}_m > \epsilon N\right\} + \mathbb{P}\left\{N_m > \epsilon N\right\},
\end{aligned}$$

where as before, $\tilde{N}_m = \tilde{N}_{I_m}$ and $N_m = N_{I_m}$. We can show almost immediately that some of the terms on the right side go to 0. For the second term, the idea is that if there are to be many such large holes, either the Brownian motion or the random walk must cover a great distance. By (2.18), we can assume that $N \ge \frac{2n^\delta}{\log^2(n^\delta)}$. By observing that if $N_{[n^{1-\frac{\delta}{2}}, \infty)} > \epsilon \frac{2n^\delta}{\log^2(n^\delta)}$ then the area swept by the Brownian motion

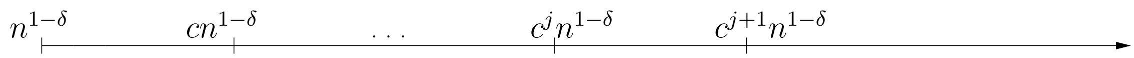

Figure 2.7: Comparing areas over smaller intervals

is greater than $2\epsilon \frac{n^{1+\delta/2}}{\log^2(n^\delta)}$, and that the same holds for random walk, we can conclude that

$$\mathbb{P}\{|\tilde{N}_{[n^{1-\frac{\delta}{2}},\infty)} - N_{[n^{1-\frac{\delta}{2}},\infty)}| > 2\epsilon \frac{n^\delta}{\log^2(n^\delta)}\}$$

$$\leq \mathbb{P}\left\{\sup_{0\leq t\leq n} |B_t| \geq 2\epsilon \frac{n^{(1+\delta)/2}}{\log^2(n^\delta)}\right\}$$

$$+ \mathbb{P}\left\{\sup_{0\leq t\leq n} |S_t| \geq 2\epsilon \frac{n^{(1+\delta)/2}}{\log^2(n^\delta)}\right\} \to 0 \text{ as } n \to \infty.$$

In the same way, we can show that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left\{\tilde{N}_m > \epsilon N\right\} = \lim_{n\to\infty} \mathbb{P}\left\{N_m > \epsilon N\right\} = 0,$$

so that it now suffices to show that

$$\lim_{n\to\infty} \mathbb{P}\left\{|\tilde{N}_{[n^{1-\delta},n^{1-\delta}c^m)} - N_{[n^{1-\delta},n^{1-\delta}c^m)}| > 10\epsilon N\right\} = 0.$$

First, using the notation of Section 2.2,

$$\mathbb{P}\{|\tilde{N}_{[n^{1-\delta},n^{1-\delta}c^m)} - N_{[n^{1-\delta},n^{1-\delta}c^m)}| > 10\epsilon N\}$$

$$\leq \mathbb{P}\left\{\sum_{j=0}^{m-1} |\tilde{N}_j - N_j| > 10\epsilon \sum_{j=0}^{m-1} N_j\right\} \leq \sum_{j=0}^{m-1} \mathbb{P}\left\{|\tilde{N}_j - N_j| > 10\epsilon N_j\right\}.$$

The definition of m shows that it now suffices to prove that for every $\epsilon > 0, 0 < \delta \leq \frac{1}{200}), 0 \leq j \leq m-1$,

$$\mathbb{P}\left\{|\tilde{N}_j - N_j| > 10\epsilon N_j\right\} = o\left((\log n)^{-1}\right), \tag{2.28}$$

where o may depend on ϵ and δ but is uniformly bounded for all $0 \leq j \leq m-1$. Recall that we defined

$$\tilde{Z}_j = \text{ the set of all components of } \mathbb{C} \setminus S[0,n] \text{ with area in } I_j,$$

$$Z_j = \text{ the set of all components of } \mathbb{C} \setminus B[0,n] \text{ with area in } I_j.$$

Unfortunately, we cannot use arguments related to area differences as developed in Lemma 2.5.5 in a direct way to estimate (2.28). The problem is that the fact that $|\tilde{N}_j - N_j| > 10\epsilon N_j$ does not imply anything useful about $\Big||Z_j| - |\tilde{Z}_j|\Big|$ and so estimates about the latter cannot be used to show (2.28). It will become clear in (2.30) why this is the case. To make things work, we need the interval in which lie the areas of random walk holes to be strictly included in the interval in which lie the areas of Brownian motion holes (or vice-versa). The key idea for creating such a situation is to observe that

$$\begin{aligned}
&\mathbb{P}\{|\tilde{N}_j - N_j| > 10\epsilon N_j\} \\
&\qquad \leq \mathbb{P}\left\{\tilde{N}_j \leq (1-2\epsilon)N_j^-\right\} + \mathbb{P}\left\{\tilde{N}_j \geq (1+2\epsilon)N_j^+\right\} \\
&\qquad + \mathbb{P}\left\{N_j^L \geq 4\epsilon N_j^-\right\} + \mathbb{P}\left\{N_j^R \geq 4\epsilon N_j^-\right\} \\
&\qquad + \mathbb{P}\left\{N_{j+1}^L \geq 4\epsilon N_j^-\right\} + \mathbb{P}\left\{N_{j-1}^R \geq 4\epsilon N_j^+\right\}. \qquad (2.29)
\end{aligned}$$

In words, if $\tilde{N}_j$ is much greater than N_j, then either it is somewhat greater than N_j^+, or there are a lot of Brownian motion holes with area in $I_j^+ \setminus I_j$. Things work nicely because the Lebesgue measure of $I_j^+ \setminus I_j$ is of much smaller order than the Lebesgue measure of I_j and the same is true of the number of holes of areas in the corresponding sets, by the work done in Section 2.3.

By (2.14), $\mathbb{P}\left\{N_j \leq \frac{1}{2}\gamma_j\right\} + \mathbb{P}\left\{N_j^- \geq \frac{1}{2}\gamma_j^-\right\} = \mathcal{O}\left((\log n)^{-3/2}\right)$, uniformly for $0 \leq j \leq m-1$. Then

$$\begin{aligned}
&\mathbb{P}\{N_j^L \geq 4\epsilon N_j^-\} \\
&\qquad \leq \mathbb{P}\left\{N_j^L \geq 4\epsilon N_j^-; N_j^L \leq \frac{3}{2}\gamma_j^{LR}; N_j^- \geq \frac{1}{2}\gamma_j^-\right\} \\
&\qquad\qquad + \mathbb{P}\left\{N_j^L \geq \frac{3}{2}\gamma_j^{LR}\right\} + \mathbb{P}\left\{N_j^- \leq \frac{1}{2}\gamma_j^-\right\}
\end{aligned}$$

$$\leq \mathbb{1}\left\{\frac{3}{2}\gamma_j^{LR} \geq 2\epsilon\gamma_j^-\right\} + \mathbb{P}\left\{N_j^L \geq \frac{3}{2}\gamma_j^{LR}\right\} + \mathbb{P}\left\{N_j^- \leq \frac{1}{2}\gamma_j^-\right\}.$$

We know from (2.19) that if ϵ is small enough, $\frac{3}{2}\gamma_j^{LR} \leq 2\epsilon\gamma_j^-$, so that we may discard the first term. The second and the third are $\mathcal{O}\left((\log n)^{-3/2}\right)$.

The last three terms of (2.29) can be treated in the same way.

Recall that we gave the following definitions in Section 2.2:

$$\Delta_j^+ = |\tilde{Z}_j| - |Z_j^+|, \qquad \Delta_j^- = |Z_j^-| - |\tilde{Z}_j|.$$

For the first term on the right hand side of (2.29), note that

$$|Z_j^-| \geq N_j^- c^j n^{1-\delta} \quad \text{and} \quad |\tilde{Z}_j| \leq \tilde{N}_j c^{j+1} n^{1-\delta},$$

so that if $\tilde{N}_j \leq (1-2\epsilon)N_j^-$, then

$$\begin{aligned} \Delta_j^- &\geq N_j^- n^{1-\delta} c^j \left[1 + \epsilon^2 - (1-2\epsilon)c\right] \\ &\geq \epsilon N_j^- n^{1-\delta} c^j \geq \epsilon n^{1-\delta}, \end{aligned}$$

since we can assume that $N_j^- > 0$ by (2.14).

For any measurable sets $A, B \subset \mathbb{C}$,the inequality $|A| - |B| \leq |A \setminus B|$ always holds and implies that

$$\mathbb{P}\left\{\tilde{N}_j \leq (1-2\epsilon)N_j^-\right\} \leq \mathbb{P}\left\{|Z_j^- \setminus \tilde{Z}_j| \geq \epsilon n^{1-\delta}\right\}.$$

But

$$Z_j^- \setminus \tilde{Z}_j \subset \bigcup \mathrm{Sq}(z),$$

where the union is over

$$\{z \in \mathbb{Z}^2 : d(z, \partial Z_j^-) \leq n^{1/3-\beta}\} \cup \{z \in \mathbb{Z}^2 : C(z) \in I_j^-; \tilde{C}(z) \notin I_j; d(z, \partial Z_j^-) \geq n^{1/3-\beta}\}.$$

At this point, let us assume that $\mathcal{P}$ and $\mathcal{N}$ hold. We can do this by (2.3) and Lemma A.1.1. We have the following inequality, where the sums are always over elements of $\mathbb{Z}^2$:

$$\begin{aligned}
\mathbb{E}\left[|Z_j^- \setminus \tilde{Z}_j|; \mathcal{P}; \mathcal{N}\right] &\leq \mathbb{E}\left[\sum_{|z| \leq \sqrt{n}\log n} \mathbb{1}\{d(z, \partial Z_j^-) \leq n^{1/3-\beta}\}\right] \\
&+ \sum_{|z| \leq \sqrt{n}\log n} \mathbb{P}\left\{C(z) \in I_j^-; \tilde{C}(z) \notin I_j; \mathcal{P}; \mathcal{B}(z)\right\} \\
&\leq Kn^{1-\beta+\delta} + \sum_{|z| \leq \sqrt{n}\log n} \mathbb{P}\left\{\Delta(z) \geq \epsilon^2 n^{1-\delta}; \mathcal{G}\right\} \qquad (2.30) \\
&+ \sum_{|z| \leq \sqrt{n}\log n} \mathbb{P}\left\{|C(z)| \in I_j^-; \tilde{C}(z) = \infty; \mathcal{P}; \mathcal{B}(z)\right\} \\
&\leq Kn^{1-\beta+\delta} + Kn\log^2 n(n^{-3\beta/8} + n^{\frac{\beta}{2}-\frac{1}{24}}\log n) \\
&\leq Kn^{1-3\beta/8}\log^2 n,
\end{aligned}$$

by lemmas 2.4.3, 2.5.3, and 2.5.5, and where K may depend on ϵ. Therefore,

$$\mathbb{P}\left\{|Z_j^- \setminus \tilde{Z}_j| \geq \epsilon n^{1-\delta}; \mathcal{P}; \mathcal{N}\right\} \leq K\frac{n^{1-3\beta/8}\log^2 n}{\epsilon n^{1-\delta}} \leq \frac{K}{\epsilon} n^{\delta-3\beta/8} \leq \frac{K}{\epsilon} n^{-1/100}.$$

For every $\epsilon > 0$, this goes to 0 as $n \to \infty$. The second term of (2.29) is done in the same way, which concludes the proof of the proposition.

□

Given this proposition, it is now straightforward to show Theorem 2.1.1:

$$\begin{aligned}
&\mathbb{P}\{|\tilde{N} - 2\pi\gamma| > \epsilon\gamma\} \\
&\quad\leq \mathbb{P}\left\{|\tilde{N} - N| > \frac{\epsilon}{2}\gamma\right\} + \mathbb{P}\left\{|N - 2\pi\gamma| > \frac{\epsilon}{2}\gamma\right\} \\
&\quad\leq \mathbb{P}\left\{|\tilde{N} - N| > \frac{\epsilon}{4}N\right\} + \mathbb{P}\left\{\gamma < N/2\right\} + \mathbb{P}\left\{|N - 2\pi\gamma| > \frac{\epsilon}{2}\gamma\right\}.
\end{aligned}$$

By Proposition 2.6.1, (2.18), and (2.16), we know that for every $\epsilon > 0$, every $\delta \in (0, \frac{1}{200}]$, each of the 3 terms goes to 0 as n goes to infinity. This proves the theorem.

Chapter 3

The Schramm-Löwner Evolution

3.1 Introduction

A long list of models arising in statistical mechanics have been studied for many years by probabilists and physicists alike, some of them with more success than others. Among them we cite percolation, self-avoiding random walk, loop-erased random walk, uniform spanning trees, and the Ising model. By analogy with the case of random walk and Brownian motion, disposing of a scaling limit for these objects would be of great help in their study.

In 1999 (see [29]), Oded Schramm made a breakthrough in the field by proving that if the scaling limit of loop-erased random walk (LERW) from 0 to the unit circle $\partial\mathbb{U}$ is conformally invariant, as many people believed is the case, then it must have the same law as the path $g_t(e^{iW_{2t}})$ where $g_t(z)$ is the solution of the following ordinary differential equation in the open unit disk $\mathbb{U}$:

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{e^{iW_{2t}} + g_t(z)}{e^{iW_{2t}} - g_t(z)}, \quad g_0(z) = z,$$

where W_t is a standard one-dimensional Brownian motion and $g_t(z)$ is defined for $z \in \partial\mathbb{U}$ as the continuous extension of g_t in $\mathbb{U}$. For general $\kappa \geq 0$, he called the solution of

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{e^{iW_{\kappa t}} + g_t(z)}{e^{iW_{\kappa t}} - g_t(z)}, \quad g_0(z) = z$$

the **Stochastic Löwner Evolution** with parameter κ (SLE_κ), but the three letter acronym has now been accepted to be short for "**Schramm-Löwner Evolution**" as well.

It turned out that Schramm's idea did far more than just provide a scaling-limit candidate for loop-erased walk and we refer the reader to [32] or [16] for a list of results proved with the help of SLE. In the next sections, we define two variants of SLE and some associated quantities and give a few important results related to this process.

3.2 Half-plane capacity

Consider the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. For a set $A \subset \mathbb{H}$ we will write $\mathbb{H}_A = \mathbb{H} \setminus A$. A **hull** is a bounded subset $A \subset \mathbb{H}$ such that $A = \mathbb{H} \cap \bar{A}$ and $\mathbb{H}_A$ is a simply connected subset of $\mathbb{C}$. The following result can be found in [17]:

Proposition 3.2.1. For any hull A there exists a unique conformal transformation $g_A : \mathbb{H}_A \to \mathbb{H}$ such that

$$\lim_{z \to \infty} (g_A(z) - z) = 0.$$

For a hull A we define the **half-plane capacity** of A, denoted $\text{hcap}(A)$, by the equation

$$\text{hcap}(A) = \lim_{z \to \infty} z(g_A(z) - z).$$

It follows from the proof of the above proposition that for any hull A, $\text{hcap}(A)$ is finite and that we have the expansion

$$g_A(z) = z + \frac{\text{hcap}(A)}{z} + \mathcal{O}\left(\frac{1}{|z|^2}\right)$$

for $z \to \infty$.

If A is a hull, $r \in \mathbb{R}_+^*, x \in \mathbb{R}$, we then have

$$g_{rA}(z) = r g_A(z/r), \qquad g_{A+x}(z) = g_A(z-x) + x,$$

which gives the following scaling and translation properties for hcap:

$$\text{hcap}(rA) = r^2 \text{hcap}(A), \qquad \text{hcap}(A+x) = \text{hcap}(A).$$

Part (a) of the following Proposition, which can be found in [17] as well, gives a probabilistic characterization of the half-plane capacity which will be particularly useful in the next chapter. There, we define another capacity for discrete subsets of $\mathbb{H}$, in a context where the map g_A is not well defined. We state part (b) of the proposition too, as we will need it in the next chapter.

Proposition 3.2.2. If A is a hull, B_t is a Brownian motion, and $Z = Z_A = \inf\{t \geq 0 : B_t \in \partial \mathbb{H}_A\}$ is the first time the Brownian motion reaches $\partial \mathbb{H}_A$, then for all $z \in \mathbb{H}_A$,

$$(a)\ \text{hcap}(A) = \lim_{y \to \infty} y \mathbb{E}^{iy} \left[\text{Im}(B_Z) \right].$$

$$(b)\ \text{Im}(z) = \text{Im}(g_A(z)) + \mathbb{E}^z \left[\text{Im}(B_Z) \right].$$

In fact, the limit on the right-hand side of (a) exists for all bounded subsets of $\mathbb{H}$, not only hulls. We can therefore use (a) to define the half-plane capacity of general bounded sets $A \subset \mathbb{H}$.

If A is a hull, define $h = h(A) = \sup\{\text{Im}(z) : z \in A\}$ to be the height of A and $R = R(A) = \sup\{|z| : z \in A\}$ to be the radius of A. As the next lemma shows, we can find bounds for hcap(A) in terms of h and R:

Lemma 3.2.3. There exist constants C_1 and C_2 such that for every bounded set $A \subset \mathbb{H}$,

$$C_1 h^2 \leq \text{hcap}(A) \leq C_2 h R.$$

Proof. We will use Proposition 3.2.2 (a) and find a lower and upper bound for $\mathbb{E}^{iy}[\text{Im}(B_Z)]$, uniform for all hulls A.

The upper bound is straightforward. Let $R = R(A)$ and $\mathcal{D}(R) = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$ be the half-circle of radius R.Then, trivially, $y\mathbb{E}^{iy}[\text{Im}(B_Z)] \leq hy\mathbb{P}^{iy}\left\{T_{\mathcal{D}(R)} < T_R\right\}$, so that $\text{hcap}(A) \leq C_2 hR$.

For the lower bound, we choose the leftmost point $w \in A$ with $\text{Im}(w) = h$ and define the segment $L = \{x + ih : |x - \text{Re}(w)| \leq \frac{h}{10}\}$. By looking at the hitting density of $\{x + ih : x \in \mathbb{R}\}$, it is easy to see that there is a $K > 0$ such that for all y large enough,

$$y\mathbb{P}^{iy}\{T_L < T_{\mathbb{R}}\} \geq Kh. \tag{3.1}$$

Recall that we defined $\hat{\Xi}_a = \inf\{t \geq 0 : |B_t - B_0| = a\}$. We have

$$\mathbb{P}^w\left\{B[\hat{\Xi}_{h/10}, \hat{\Xi}_{h/2}] \text{ disconnects } D(z, h/10) \text{ from } \infty\right\} \leq \mathbb{P}^z\{\text{Im}(B_Z) \geq h/2\}.$$

The probability on the left is independent of z and one can easily check that it is strictly positive. Therefore, $\sup_{z \in L} \mathbb{P}^z\{\text{Im}(B_Z) \geq h/2\} \geq K_2 > 0$, which, together with (3.1) gives the lower bound.

□

The next Lemma tells us how far off we are from the value of hcap, when we don't take the limit in the characterization of dhcap given in Lemma 3.2.2 (b). This error depends on both R and h.

Lemma 3.2.4. Let A be a hull. Then

$$\text{hcap}(A) = y\mathbb{E}^{iy}[\text{Im}(B_Z)] + \mathcal{O}\left(\frac{hR^2}{y}\right).$$

Proof. Recall that we defined $\mathcal{D}(R) = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$. If $T = \inf\{t \geq 0 : B_t \in \mathbb{R} \cup \mathcal{D}(R)\}$ and $p(iy, \cdot)$ is the density of B_T with $B_0 = iy$, we have $p(iy, Re^{i\theta}) =$

$\frac{2}{\pi}\frac{R}{y}\sin\theta(1+\mathcal{O}\left(\frac{R}{y}\right))$. Thus, using the strong Markov property and Lemma 3.2.3 for the last inequality, we have

$$\begin{aligned}
|\mathrm{hcap}(A) - y\mathbb{E}^{iy}\left[\mathrm{Im}(B_Z)\right]| &= |\lim_{z\to\infty} z\mathbb{E}^{iz}\left[\mathrm{Im}(B_Z)\right] - y\mathbb{E}^{iy}\left[\mathrm{Im}(B_Z)\right]| \\
&\leq \int_0^\pi |yp(iy, Re^{i\theta}) - \lim_{z\to\infty} zp(iz, Re^{i\theta})|\mathbb{E}^{Re^{i\theta}}\left[\mathrm{Im}(B_Z)\right] \\
&\leq \int_0^\pi \frac{2R}{\pi y}(1+\mathcal{O}\left(\frac{R}{y}\right) - 1)\mathbb{E}^{Re^{i\theta}}\left[\mathrm{Im}(B_Z)\right] \\
&\leq \mathrm{hcap}(A)\cdot\mathcal{O}\left(\frac{R}{y}\right) \leq K\frac{hR^2}{y}.
\end{aligned}$$

□

We also state the following proposition which will be needed later. See [17] for a proof.

Proposition 3.2.5. There exists a constant K such that for every hull A and every $|z| \geq 2R(A)$

$$|z - g_A(z) + \mathrm{hcap}(A)z^{-1}| \leq K\frac{R}{|z|^2}\mathrm{hcap}(A).$$

3.3 Chordal *SLE*

Chordal SLE_κ is the random collection of conformal maps g_t obtained by solving the initial value problem

$$\frac{\partial}{\partial t}g_t(z) = \frac{2}{g_t(z) - B_{\kappa t}}, \quad g_0(z) = z \quad (z \in \mathbb{H}), \tag{3.2}$$

where B_t is a standard one-dimensional Brownian motion started at the origin, and $\kappa \geq 0$.

We denote by H_t the domain of g_t and let $K_t = \mathbb{H}\setminus H_t$. Another way of thinking of K_t is the following: For any $z \in \mathbb{H}$, the solution of (3.2) is well-defined up to a time $T_z = \sup\{t \geq 0 : g_t(z) \text{ is well-defined}\}$, which may be infinite. Then it can be verified that $K_t = \{z : T_z \leq t\}$. The following result by Rohde and Schramm (see [28]) is important in understanding the nature of K_t:

Proposition 3.3.1. For every $\kappa \geq 0$, with probability one, there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.

The $\boldsymbol{SLE_\kappa}$ **path** is the random curve γ of the proposition. In particular, we have $g_t(\gamma(t)) = B_{\kappa t}$ in the sense that $\lim_{z \to \gamma(t)} g_t(z) = B_{\kappa t}$.

Even though $B_{\kappa t}$ has essentially the same behavior for all $\kappa > 0$, the same is not true of SLE as shown in the following proposition:

Proposition 3.3.2. Let γ be an SLE_κ path. Then the following holds:

- If $\kappa \leq 4$, then with probability one, γ is a simple curve with $\gamma(0, \infty) \in \mathbb{H}$.
- If $4 < \kappa < 8$, then with probability one,

  $$\bigcup_{t>0} \bar{K}_t = \bar{\mathbb{H}}, \text{ but } \gamma(0, \infty) \neq \mathbb{H}.$$

  Moreover, $d(0, H_t) \to \infty$ as $t \to \infty$.
- If $\kappa \geq 8$, then γ is a space-filling curve, i.e., $\gamma[0, \infty) = \bar{\mathbb{H}}$.

Another important feature of γ which can be easily verified is that $g_t : H_t \to \mathbb{H}$ has the expansion

$$g_t(z) = z + \frac{2t}{z} + \mathcal{O}\left(|z|^{-2}\right).$$

In particular, $\operatorname{hcap}(\gamma[0, t]) = 2t$.

3.4 Radial SLE

Radial $\boldsymbol{SLE_\kappa}$ is the random collection of conformal maps g_t obtained by solving the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \frac{e^{iB_{\kappa t}} + g_t(z)}{e^{iB_{\kappa t}} - g_t(z)}, \quad g_0(z) = z \quad (z \in \mathbb{U}), \tag{3.3}$$

where B_t is a standard one-dimensional Brownian motion started at the origin, and $\kappa \geq 0$.

We will not use radial SLE in this thesis and do not push its description any further here. For a detailed introduction to the process, the reader may consult [17].

However, we do point out the main difference between chordal and radial SLE. These two processes do not only differ by the fact that they evolve in different domains. The essential difference is that chordal SLE is a process going from a boundary point, 0, to another, ∞, whereas radial SLE grows from a boundary point, 1, to an interior point, 0. For domains other than $\mathbb{H}$ and $\mathbb{U}$, chordal and radial SLE is just defined to be the conformal image of SLE in $\mathbb{H}$ and $\mathbb{U}$, respectively.

3.5 Loop-erased random walk and SLE_2

As mentioned earlier, SLE is a relatively new process which has made it possible to solve numerous conjectures which had been around for some time. We do not discuss them here, but the reader can find a complete discussion of the current state of the art in [17].

However, we give here the result from [23] which will be of interest to us, which shows the convergence of loop-erased random walk in a domain to radial SLE.

Consider a domain $D \underset{\neq}{\subset} \mathbb{Z}^2$ and a point $a \in D$. **Loop-erased random walk** (LERW) in D from a to ∂D is defined as follows. Let S be a simple random walk started at a and stopped at its first hitting time T of ∂D. Let $\beta_0 = a$; for $n \geq 0$, if $\beta_n \in \partial D$, then $n = l$. Otherwise, $\beta_{n+1} = S(k)$, where $k = 1 + \max\{m \leq T : S(m) = \beta_n\}$. Then $\beta = (\beta_0, .., \beta_l)$ is the loop-erasure of S.

Let $D \underset{\neq}{\subset} \mathbb{C}$ be a simply connected domain containing the origin. For $\delta > 0$, let μ_δ be the law of the loop-erasure of simple random walk on the grid $\delta\mathbb{Z}^2$, started at 0 and stopped when it hits ∂D. Let ν be the law of the image of the radial SLE_2 path under a conformal map from the unit disk $\mathbb{U}$ to D, fixing 0.

On the space of parametrized paths in $\mathbb{C}$, consider the metric

$$\rho(\beta, \gamma) = \inf \sup_{t \in [0,1]} |\hat{\beta}(t) - \hat{\gamma}(t)|,$$

where the infimum is over all choices of parameterizations $\hat{\beta}$ and $\hat{\gamma}$ of β and γ.

Theorem 3.5.1 (LERW Scaling Limit). The measures μ_δ converge weakly to ν as $\delta \to 0$ with respect to the metric ρ on the space of curves.

Chapter 4

The Discrete Half-Plane Capacity and Simple Random Walk Excursion

4.1 Introduction

The goal of this chapter is to establish the estimates needed to show that the scaling limit of loop-erased random walk (LERW) excursion in the upper half-plane, denoted by $\hat{S}$, is chordal SLE_2. There exist a number of quantities related to a stochastic process and which depend on the realization of the process. An example of such a quantity is the place where the process leaves a given domain. In some cases, one can recover some of the information about the process from these quantities which we will call "observables". In this chapter, we will define a "discrete half-plane capacity", denoted by dhcap, for subsets of the discrete upper half-plane $\mathcal{H} = \mathbb{H} \cap \mathbb{Z}^2$. This object is the natural analogue, in the discrete setting, of hcap, which we defined in Chapter 3. dhcap will play the role of the time parameter for the LERW excursion. Under this parametrization, we will eventually examine the evolution of the image of the "tip" of LERW excursion under the map g_t taking $\mathbb{H} \setminus \tilde{S}_t$ to $\mathbb{H}$ and satisfying the expansion at infinity

$$g_t(z) = z + \frac{2t}{z} + \mathcal{O}\left(|z|^{-2}\right).$$

The afore-mentioned observable should give us information about this evolution. The goal will then be to show that the scaling limit of this evolution is Brownian motion with speed 2. The method is very much the same as in [17]. However, the observable and the estimates needed to compute it are different.

In Section 4.2, we define the discrete half-plane capacity of a discrete set, in a way very similar to that of Chapter 3 and show that the limit which serves as its definition actually exists. We show in Section 4.3 how this quantity is naturally related to random walk excursion.

In Section 4.4, we give a correspondence between discrete sets in $\mathcal{H}$ and continuous sets in $\mathbb{H}$ and show how the discrete half-plane capacity and the half-plane capacity are related for such corresponding sets.

Section 4.5 provides a list of standard results from complex analysis, which we will need for the technical computation we do in Section 4.6.

In Section 4.7, we compute an observable for random walk excursion, which is the expected number of visits to $-n+i$ by the excursion. Given the close relationship between random walk excursion and LERW excursion, we hope to eventually use this observable to find information about the scaling limit of LERW excursion.

4.2 The discrete half-plane capacity

We start by fixing the notation and giving a few definitions. We will use the complex notation $z = x + iy$ $(x, y \in \mathbb{Z})$ for points in $\mathbb{Z}^2$. For simplicity, the set $\{x + i \cdot 0 : x \in \mathbb{Z}\} \subset \mathbb{Z}^2$ will be denoted by $\mathbb{Z}$. $\mathcal{H} = \{x + iy \in \mathbb{Z}^2 : y > 0\}$ will denote the discrete upper half-plane. For a finite set $A \subset \mathcal{H}$, we let $\mathcal{H}_A = \mathcal{H} \setminus A$ and

$\zeta_A = \inf\{n \geq 1 : S(n) \in \partial \mathcal{H}_A\}$. Recall that we defined in Chapter 3 the following quantities for continuous hulls: $h = h(A) = \sup\{\text{Im}(z) : z \in A\}$, the height of A and $R = R_A = \sup\{|z| : z \in A\}$, the radius of A. We simply extend these definitions to discrete hulls and it will be clear from context which type of object we are dealing with. For any set $D \subset \mathcal{H}_A$, the hitting time of D is $\tau_D = \inf\{n \geq 0 : S_n \in D\}$. Also, if $x, y \in E \subset \mathbb{Z}^2$, we define the discrete Green's function in E by

$$G_E(x,y) = \mathbb{E}^x \left[\sum_{k \geq 0} \mathbb{1}\{S_k = y; k < \tau_{\partial E}\} \right]. \tag{4.1}$$

For $r \in \mathbb{N}$, we let

$$l_r = \{z \in \mathbb{Z}^2 : \text{Im}(z) = r\} \text{ and } \sigma_r = \inf\{n \geq 1 : S(n) \in l_r\}.$$

We will say that a set $D \subset \mathbb{Z}^2$ is connected if for any $x, y \in D$, there exists a path $(\gamma_0 = x, \gamma_1, ..., \gamma_k = y)$ with $\gamma_i \in \mathbb{Z}^2, 0 \leq i \leq k$ and $|\gamma_i - \gamma_{i-1}| = 1, 1 \leq i \leq k$.

The **discrete half-plane capacity** of a finite set $A \subset \mathcal{H}$, denoted by dhcap(A) is defined by

$$\text{dhcap}(A) = \lim_{y \to \infty} y \, \mathbb{E}^{iy} \left[\text{Im}(S(\zeta_A))\right].$$

We will show in Lemma 4.2.3 that this limit exists. Although the definition of dhcap holds for any finite subset A of $\mathcal{H}$, we will be mostly interested in a particular type of subsets of $\mathcal{H}$, which is the discrete analogue of the hulls defined in Section 3.2. A **discrete hull** $A \subset \mathcal{H}$ is a finite subset of $\mathcal{H}$ such that $\mathcal{H}_A$ is simply connected in $\mathbb{Z}^2$, i.e. $\mathbb{Z}^2 \setminus \mathcal{H}_A$ is connected. A discrete hull does not need to be connected. However, if it is disconnected, the boundary of each of its components must intersect $\mathbb{Z}$.

We start by showing that the discrete half-plane capacity exists. The proof will rely on the hitting distribution of $\mathbb{Z}$ by random walk started in the upper half-plane. In the continuous case, this distribution is well known to be the Cauchy distribution.

More precisely, the hitting density of the real line by Brownian motion started at iy is

$$f(x) = \frac{1}{\pi}\frac{y}{y^2 + x^2}. \tag{4.2}$$

In Lemma 4.2.1, we show that for random walk, up to an error term, we get a discrete version of the Cauchy distribution as well.

To prove the existence of dhcap($\cdot$), we also need an estimate on the probability of hitting A before leaving $\mathcal{H}$, when starting at various points in $\mathcal{H}$. We do this in Lemma 4.2.2.

Lemma 4.2.1. For $k \in \mathbb{Z}, y \in \mathbb{Z}_+$,

$$\mathbb{P}^{iy}\left\{S(\sigma_0) = k\right\} = \frac{1}{\pi}\,\frac{y}{y^2 + k^2} + \mathcal{O}\left(\frac{1}{y^2 + k^2}\right),$$

i.e., there is a constant $C < \infty$ such that for all such k, y,

$$\left|\mathbb{P}^{iy}\left\{S(\sigma_0) = k\right\} - \frac{1}{\pi}\,\frac{y}{y^2 + k^2}\right| \leq \frac{C}{y^2 + k^2}.$$

Proof. We find in [30] (p.155) that

$$\mathbb{P}^{iy}\left\{S(\sigma_0) = k\right\} = \frac{1}{4}\left[a\left((y+1)i - k\right) - a\left((y-1)i - k\right)\right].$$

But we know (see e.g. [14]) that

$$a(x) = \frac{2}{\pi}\ln|x| + K + \mathcal{O}\left(|x|^{-2}\right)$$

for some (known) constant K. Hence,

$$\mathbb{P}^{iy}\left\{S(\sigma_0) = k\right\} = \frac{1}{4\pi}\ln\frac{(y+1)^2 + k^2}{(y-1)^2 + k^2} + \mathcal{O}\left(\frac{1}{y^2 + k^2}\right).$$

Using Taylor's series, we can conclude by seeing that

$$\ln \frac{(y+1)^2 + k^2}{(y-1)^2 + k^2} = \frac{4y}{y^2 + k^2} + \mathcal{O}\left(\frac{y^2}{(y^2+k^2)^2}\right).$$

□

Recall that $h = h(A) = \sup\{\text{Im}(z) : z \in A\}$ and $R = R_A = \sup\{|z| : z \in A\}$.

Lemma 4.2.2. There is a constant $C > 0$ such that for any finite set $A \in \mathcal{H}$ and any $|k| > R$,

$$\mathbb{P}^{k+ih}\{S(\zeta_A) \in A\} \leq C_1 \frac{R^2}{k^2}. \tag{4.3}$$

However, if we only consider $|k| \geq 4R$, we have the sharper bound

$$\mathbb{P}^{k+ih}\{S(\zeta_A) \in A\} \leq C_1 \frac{hR}{k^2}, \tag{4.4}$$

where again C_1 is uniform for all finite sets $A \in \mathcal{H}$ and all $|k| \geq 4R$.

Proof. We will assume that $k > R$. The other case is done in the exact same way. Let $v_1 = v_1(A) = \{(x, y) \in \mathbb{Z}^2 : x = R, y \in \{1, .., k-1\}\}$. Then, in order to reach A without leaving $\mathcal{H}$, the random walk must reach l_k or v_1 first:

$$\begin{aligned} \mathbb{P}^{k+ih}\{S(\zeta_A) \in A\} &= \mathbb{P}^{k+ih}\{S(\zeta_A) \in A; \sigma_k < \tau_{v_1} \wedge \sigma_0\} \\ &+ \mathbb{P}^{k+ih}\{S(\zeta_A) \in A; \tau_{v_1} < \sigma_0 \wedge \sigma_k\}. \end{aligned} \tag{4.5}$$

The event considered in the first term on the right-hand side of the equality can be decomposed into two parts: first S must get to height k without hitting A or leaving $\mathcal{H}$; from there it must hit A before leaving $\mathcal{H}$. The strong Markov property and the gambler's ruin estimate give

$$
\begin{aligned}
&\mathbb{P}^{k+ih}\{S(\zeta_A) \in A; \sigma_k < \tau_{v_1} \wedge \sigma_0\} \\
&\qquad = \sum_{l \in \mathbb{Z}} \mathbb{P}^{k+ih}\left\{\sigma_k < \tau_{v_1} \wedge \sigma_0; S(\sigma_k) = l + ik\right\} \cdot \ \mathbb{P}^{l+ik}\left\{S(\zeta_A) \in A\right\} \\
&\qquad \le \max_{l \in \mathbb{Z}} \mathbb{P}^{l+ik}\left\{S(\zeta_A) \in A\right\} \mathbb{P}^{k+ih}\left\{\sigma_k < \sigma_0\right\} \\
&\qquad = \frac{h}{k} \max_{l \in \mathbb{Z}} \mathbb{P}^{l+ik}\left\{S(\zeta_A) \in A\right\}.
\end{aligned}
$$

Now we can use Lemma 4.2.1 to find an upper bound for $\max_{l \in \mathbb{Z}} \mathbb{P}^{l+ik}\left\{S(\zeta_A) \in A\right\}$.

$$
\begin{aligned}
&\mathbb{P}^{l+ik}\{S(\sigma_0) \in [-R, R]\} \\
&\qquad \ge \sum_{w \in A} [\mathbb{P}^{l+ik}\left\{S(\sigma_0) \in [-R, R] | \zeta_A < \sigma_0; S(\zeta_A) = w\right\} \\
&\qquad\qquad\qquad\qquad \cdot \mathbb{P}^{l+ik}\left\{\zeta_A < \sigma_0; S(\zeta_A) = w\right\}] \\
&\qquad = \sum_{w \in A} \mathbb{P}^{w}\left\{S(\sigma_0) \in [-R, R]\right\} \mathbb{P}^{l+ik}\left\{\zeta_A < \sigma_0; S(\zeta_A) = w\right\} \\
&\qquad \ge C \sum_{w \in A} \mathbb{P}^{l+ik}\left\{\zeta_A < \sigma_0; S(\zeta_A) = w\right\} \\
&\qquad = C \mathbb{P}^{l+ik}\left\{S(\zeta_A) \in A\right\},
\end{aligned}
$$

where $C = \inf_{A} \min_{w \in A} \mathbb{P}^{w}\left\{S(\sigma_0) \in [-R, R]\right\} > 0$. The fact that $C > 0$ can be seen from Lemma 4.2.1 and the first equality follows from the strong Markov property. It follows that

$$
\begin{aligned}
\mathbb{P}^{l+ik}\left\{S(\zeta_A) \in A\right\} &\le \frac{1}{C} \mathbb{P}^{l+ik}\left\{S(\sigma_0) \in [-R, R]\right\} = \frac{1}{C} \sum_{m=-R-l}^{R-l} \mathbb{P}^{ik}\left\{S(\sigma_0) = m\right\} \\
&= \frac{1}{\pi C} \sum_{m=-R-l}^{R-l} \frac{k}{k^2 + m^2} + \mathcal{O}\left(\frac{1}{k^2 + m^2}\right) \le C' \frac{2R+1}{k},
\end{aligned}
$$

uniformly in l. Thus the first probability on the right-hand side of the equality in (4.5) is bounded above by $C \frac{hR}{k^2}$, where C is independent of A and k.

We now assume that $k \geq 4R$ and define

$$v_2 = \left\{(x,y) \in \mathbb{Z}^2 : x = \left[\frac{k}{2}\right], y \in \{1,..,k-1\}\right\},$$

as well as the sets

$$\mathcal{D} = \{(x,y) \in \mathbb{Z}^2 : R < x, 0 < y < k\},$$
$$\mathcal{D}_1 = \{(x,y) \in \mathbb{Z}^2 : R < x < [\tfrac{k}{2}], 0 < y < k\},$$
$$\mathcal{D}_2 = \{(x,y) \in \mathbb{Z}^2 : [\tfrac{k}{2}] < x, 0 < y < k\}.$$

Note that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup v_2$. The following decomposition is trivial:

$$\mathbb{P}^{k+ih}\left\{S(\zeta_A) \in A; \tau_{v_1} < \sigma_0 \wedge \sigma_k\right\}$$
$$= \sum_{l=1}^{k} \mathbb{P}^{k+ih}\left\{S(\tau_{v_1} \wedge \sigma_0 \wedge \sigma_k) = R + il\right\} \mathbb{P}^{R+il}\left\{S(\zeta_A) \in A\right\}.$$

We already know from the computation above that $\mathbb{P}^{R+il}\left\{S(\zeta_A) \in A\right\} = \mathcal{O}\left(\frac{R}{l}\right)$.

If we let ρ to be the last time before $\tau_{v_1} \wedge \sigma_0 \wedge \sigma_k$ at which S is in v_2, then

$$\mathbb{P}^{k+ih}\{S(\tau_{v_1} \wedge \sigma_0 \wedge \sigma_k) = R + il\}$$
$$= \sum_{\substack{m=1\\n=1}}^{k} \mathbb{P}^{k+ih}\{\tau_{v_1} < \sigma_0 \wedge \sigma_k; S(\tau_{v_2}) = [\frac{k}{2}] + im; S(\rho) = [\frac{k}{2}] + in; S(\tau_{v_1}) = R + il\}$$
$$= \sum_{\substack{m=1\\n=1}}^{k} \sum_{\omega_1 \in A(m)} \sum_{\omega_2 \in B(m,n)} \sum_{\omega_3 \in C(n)} \mathbb{P}\left\{\omega_1\right\} \mathbb{P}\left\{\omega_2\right\} \mathbb{P}\left\{\omega_3\right\},$$

where

$$A(m) = \{\text{paths from } k + ih \text{ to } [k/2] + im, \text{ inside } \mathcal{D}_2 \text{ except for the endpoint}\},$$

$$B(m,n) = \{\text{paths starting at } [k/2] + im, \text{ in } \mathcal{D}, \text{ ending at } [k/2] + in\},$$

$$C(n) = \{\text{paths starting at } [\frac{k}{2}] + in, \text{ ending at } R + il,$$

$$\text{in } \mathcal{D}_1, \text{ except for the starting and endpoint}\}.$$

Since $\sum_{\omega_2 \in B(m,n)} \mathbb{P}\{\omega_2\} \leq 1$, the last expression is bounded above by

$$\sum_{m=1}^{k} \sum_{\omega_1 \in A(m)} \mathbb{P}\{\omega_1\} \sum_{n=1}^{k} \sum_{\omega_3 \in C(n)} \mathbb{P}\{\omega_3\} = \mathbb{P}^{k+ih}\{\tau_{v_2} < \sigma_0 \wedge \sigma_k\} \sum_{\omega \in D} \mathbb{P}^{R+il}\{\omega\},$$

where $D = \{\omega = [\omega_0, .., \omega_n] : \{\omega_1, .., \omega_{n-1}\} \subset \mathcal{D}_1, \omega_n \in v_2\}$.

The equality $\sum_{n=1}^{k} \sum_{\omega_3 \in C(n)} \mathbb{P}\{\omega_3\} = \sum_{\omega \in D} \mathbb{P}^{R+il}\{\omega\}$ can be seen by reversing paths. We can use Lemma B.2.4 to see that

$$\sum_{\omega \in D} \mathbb{P}^{R+il}\{\omega\} \leq C_1 \frac{l}{([k/2] - R)^2} \leq C_1 \frac{l}{k^2},$$

since $k \geq 4R$. Also, Lemma B.2.5 implies that

$$\mathbb{P}^{k+ih}\{\tau_{v_2} < \sigma_0 \wedge \sigma_k\} \leq C_{\frac{h}{k}},$$

so that we have, for $k \geq 4R$,

$$\mathbb{P}^{k+ih}\{S(\zeta_A) \in A; \tau_{v_1} < \sigma_0 \wedge \sigma_k\} \leq C_1 \sum_{l=1}^{k} \frac{R}{l} \frac{h}{k} \frac{l}{k^2} = C_1 \frac{hR}{k^2}.$$

This gives 4.4. However, it is clear that there exists a constant $C_2 > 0$ such that

$$\sup_{A, |k| > R} \mathbb{P}^{k+ih}\{S(\zeta_A) \in A\} \geq C_2,$$

so that the best uniform bound we can get for $k > R$ is $C_1 \frac{R^2}{k^2}$ with some positive constant C_1. □

We are now ready to show that the discrete half-plane capacity exists.

Lemma 4.2.3. The limit

$$\mathrm{dhcap}(A) := \lim_{y\to\infty} y\,\mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right]$$

exists and satisfies

$$\mathrm{dhcap}(A) = y\,\mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right] + \mathcal{O}\left(\frac{hR^2}{y}\right). \tag{4.6}$$

Moreover, for any $r \geq h$ the following equality holds:

$$\mathrm{dhcap}(A) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \mathbb{E}^{k+ir}\left[\mathrm{Im}(S(\zeta_A))\right]. \tag{4.7}$$

Proof. We will show the stronger fact that for every $w \in A$, the limit

$$q_w = q_w(A) := \lim_{y\to\infty} y\,\mathbb{P}^{iy}\left\{S(\zeta_A) = w\right\}$$

exists. This immediately implies that

$$\lim_{y\to\infty} y\,\mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right] = \sum_{w\in A} q_w \mathrm{Im}(w).$$

By Lemma 4.2.1, for $y > h$,

$$y\mathbb{P}^{iy}\left\{S(\sigma_h) = k + ih\right\} = \frac{1}{\pi}\frac{y(y-h)}{(y-h)^2+k^2}\left(1 + \mathcal{O}\left(\frac{1}{y-h}\right)\right). \tag{4.8}$$

Let $C_{k,w} = C_{k,w}(A) := \mathbb{P}^{k+ih}\left\{S(\zeta_A) = w\right\}$. If $y > h$, the strong Markov property gives

$$y\,\mathbb{P}^{iy}\left\{S(\zeta_A) = w\right\} = y\mathbb{P}^{iy}\left\{S(\sigma_h) = k + ih\right\} C_{k,w}.$$

Hence, by (4.8),

$$y\mathbb{P}^{iy}\left\{S(\zeta_A) = w\right\} = \sum_{k\in\mathbb{Z}} \frac{C_{k,w}}{\pi}\frac{y(y-h)}{(y-h)^2+k^2}\left(1 + \mathcal{O}\left(\frac{1}{y-h}\right)\right).$$

Therefore,

$$\lim_{y\to\infty} y\,\mathbb{P}^{iy}\left\{S(\zeta_A)=w\right\} = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} C_{k,w}.$$

By Lemma 4.2.2, for all A and $|k|>R$, $C_{k,w}\le C\frac{R^2}{k^2}$, and so $\sum_{k\in\mathbb{Z}} C_{k,w}$ converges. This proves the existence of dhcap.

Since by Lemma 4.2.1, for all $k\in\mathbb{Z}$ and all $r\in\mathbb{N}$, $y\,\mathbb{P}^{iy}\left\{S(\sigma_r)=k+ir\right\}\sim 1/\pi$ as $y\to\infty$, we also get for all $r\ge h$:

$$\begin{aligned}
\mathrm{dhcap}(A) &= \lim_{y\to\infty} y\mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right] = \lim_{y\to\infty} y\sum_{w\in A}\mathbb{P}^{iy}\left\{S(\zeta_A)=w\right\}\mathrm{Im}(w)\\
&= \lim_{y\to\infty} y\sum_{w\in A}\sum_{k\in\mathbb{Z}}\mathbb{P}^{iy}\left\{S(\sigma_r)=k+ir\right\}\mathbb{P}^{k+ir}\left\{S(\zeta_A)=w\right\}\mathrm{Im}(w)\\
&= \lim_{y\to\infty} y\sum_{k\in\mathbb{Z}}\mathbb{P}^{iy}\left\{S(\sigma_r)=k+ir\right\}\mathbb{E}^{k+ir}\left[\mathrm{Im}(S(\zeta_A))\right]\\
&= \frac{1}{\pi}\sum_{k\in\mathbb{Z}}\mathbb{E}^{k+ir}\left[\mathrm{Im}(S(\zeta_A))\right].
\end{aligned}$$

This gives (4.7). To get (4.6), we note that for $y\ge h$,

$$\begin{aligned}
&\mathrm{dhcap}(A) - y\mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right]\\
&= \lim_{z\to\infty} z\mathbb{E}^{iz}\left[\mathrm{Im}(S(\zeta_A))\right] - y\mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right]\\
&= \sum_{k,w}\left(\lim_{z\to\infty} z\mathbb{P}^{iz}\{S(\sigma_h)=k+ih\} - y\mathbb{P}^{iy}\{S(\sigma_h)=k+ih\}\right)\mathbb{P}^{k+ih}\{S(\zeta_A)=w\}\mathrm{Im}(w)\\
&= \sum_{k,w}\frac{1}{\pi}\left(1-\frac{y(y-h)}{(y-h)^2+k^2}\left(1+\mathcal{O}\left(\frac{1}{y-h}\right)\right)\right)C_{k,w}\mathrm{Im}(w)\\
&= \frac{1}{\pi}\sum_{k,w}\left(\frac{h^2+k^2-yh}{(y-h)^2+k^2}+\mathcal{O}\left(\frac{y}{(y-h)^2+k^2}\right)\right)C_{k,w}\mathrm{Im}(w).
\end{aligned}$$

The trivial bound $\mathrm{Im}(w)\le h$ for $w\in A$ show that the last term is bounded above by

$$\frac{h}{\pi}\sum_{k}\left(\frac{h^2+k^2-yh}{(y-h)^2+k^2}+\mathcal{O}\left(\frac{y}{(y-h)^2+k^2}\right)\right)\mathbb{P}^{k+ih}\left\{S(\zeta_A)\in A\right\},$$

which by Lemma 4.2.2 is smaller than

$$KhR^2\left(\sum_{k\in\mathbb{Z}^*}\frac{1}{k^2}\left(\frac{h^2+k^2-yh}{(y-h)^2+k^2}+\mathcal{O}\left(\frac{y}{(y-h)^2+k^2}\right)\right)+\mathcal{O}\left(\frac{1}{y}\right)\right)$$

$$\leq K\frac{hR^2}{y}.$$

□

4.3 dhcap and random walk excursion

Now that the existence of dhcap has been established, we introduce the notion of random walk excursion in $\mathcal{H}$ and derive a few estimates for excursions. These will allow us to show that when adding a point v to a bounded set $A \subset \mathcal{H}$, the increase in dhcap can be expressed in terms of the imaginary part of v and quantities related to the random walk excursion started at v, relatively to A.

$\tilde{S}$ will denote a random walk excursion in $\mathcal{H}$, defined informally as random walk conditioned not to leave $\mathcal{H}$. Formally, if $z, z' \in \mathcal{H}$, the transition probability from z to z' is

$$\mathbb{P}\left\{\tilde{S}(n)=z' \mid \tilde{S}(n-1)=z\right\}=\lim_{r\to\infty}\mathbb{P}\left\{S(n)=z' \mid S(n-1)=z;\sigma_r<\sigma_0\right\}.$$

It is straightforward to show that if $y \geq 1$, the transition probabilities for an excursion are as follows:

$$\tilde{p}(x+iy,x+1+iy)=\tilde{p}(x+iy,x-1+iy)=1/4,$$

$$\tilde{p}(x+iy,x+i(y+1))=\frac{y+1}{4y},\qquad \tilde{p}(x+iy,x+(y-1)i)=\frac{y-1}{4y}.$$

For $x \in \mathbb{Z}, \tilde{p}(x, x+i) = 1$. In other words, an excursion may start in $\mathbb{Z}$ but enters $\mathcal{H}$ immediately and from then on never leaves $\mathcal{H}$.

More generally, if $z_0, z_1, \ldots, z_n$ is any nearest-neighbor path with $z_0, z_1, \ldots, z_n \in \mathcal{H}$, then

$$\mathbb{P}^{z_0}\left\{\tilde{S}(1) = z_1, \ldots, \tilde{S}(n) = z_n\right\} = (\frac{1}{4})^n \frac{\text{Im}(z_n)}{\text{Im}(z_0)}. \tag{4.9}$$

Note that this is equivalent to

$$\mathbb{P}^{z_0}\left\{\tilde{S}(1) = z_1, \ldots, \tilde{S}(n) = z_n\right\} = \frac{\text{Im}(z_n)}{\text{Im}(z_0)}\mathbb{P}^{z_0}\left\{S(1) = z_1, \ldots, S(n) = z_n\right\}. \tag{4.10}$$

This simple equality will be very useful since it will allow us to reduce all computations for $\tilde{S}$ to computations for S, a more familiar object.

Like for simple random walk, we define the hitting times $\tilde{\sigma}_r$, $\tilde{\zeta}_A$ and $\tilde{\tau}_A$ for an excursion by

$$\tilde{\tau}_A = \inf\{n \geq 0 : \tilde{S}(n) \in A\}, \qquad \tilde{\zeta}_A = \inf\{n \geq 1 : \tilde{S}(n) \in A\},$$

$$\tilde{\sigma}_r = \inf\{n \geq 1 : \tilde{S}(n) \in l_r\},$$

where l_r is as defined in Section 4.2.

The gambler's ruin formula for excursions is as follows:

If $0 < a < b < c$, and $x \in \mathbb{Z}$,

$$\mathbb{P}^{x+ib}\left\{\tilde{\sigma}_a < \tilde{\sigma}_c\right\} = \frac{a}{b} \cdot \frac{c-b}{c-a}.$$

The second fraction on the right is the simple random walk probability and the extra factor a/b comes from (4.9).

For any $z \in \mathcal{H}$ and $A \subset \mathcal{H}$, we define the hitting probability of A by $\tilde{S}$ and S:

$$\tilde{\phi}_A(z, w) = \mathbb{P}^z\left\{\tilde{S}(\tilde{\zeta}_A) = w; \tilde{\zeta}_A < \infty\right\}, \qquad \phi_A(z, w) = \mathbb{P}^z\left\{S(\zeta_A) = w\right\},$$

and

$$\tilde{\phi}_A(z) = \mathbb{P}^z\left\{\tilde{\zeta}_A < \infty\right\} = \sum_{w \in A} \tilde{\phi}_A(z, w).$$

Note that (4.10) gives

$$\tilde{\phi}_A(z) = \sum_{w \in A} \frac{\mathrm{Im}(w)}{\mathrm{Im}(z)}\, \phi_A(z, w). \tag{4.11}$$

In particular,

$$\lim_{y \to \infty} y^2 \tilde{\phi}_A(iy) = \lim_{y \to \infty} y \mathbb{E}^{iy}\left[\mathrm{Im}(S(\zeta_A))\right] = \mathrm{dhcap}(A).$$

Also, using (4.7) and (4.11), we can show easily that if $r \geq h$, then

$$r \sum_{k \in \mathbb{Z}} \tilde{\phi}_A(k + ir) = \pi\, \mathrm{dhcap}(A). \tag{4.12}$$

When working with the probabilities defined above, it will be useful to consider separately the portion of their path from the last time they are at height r before getting to A to the time they hit A (in other words doing what is known as a last-exit decomposition).

For $z \in \mathcal{H}_A, w \in \partial \mathcal{H}_A$, let

$$\chi_A(z, w) = \mathbb{P}^z\left\{S(\zeta_A) = w; \mathrm{Im}(S_j) < \mathrm{Im}(z), 1 \leq j \leq \zeta_A\right\},$$

$$\tilde{\chi}_A(z, w) = \mathbb{P}^z\left\{\tilde{S}(\tilde{\tau}_A) = w; \mathrm{Im}(\tilde{S}_j) < \mathrm{Im}(z), 1 \leq j \leq \tilde{\tau}_A\right\}$$

and

$$\tilde{\chi}_A(z) = \sum_{w \in A} \tilde{\chi}_A(z, w).$$

Note that by (4.10),

$$\tilde{\chi}_A(z, w) = \frac{\mathrm{Im}(w)}{\mathrm{Im}(z)}\, \chi_A(z, w). \tag{4.13}$$

Lemma 4.3.1.

$$\lim_{r\to\infty} r^2 \sum_{k\in\mathbb{Z}} \tilde{\chi}_A(k+ir) = \frac{\pi}{4}\text{dhcap}(A).$$

Proof. From the definition of $\tilde{\chi}_A(z)$ and (4.13), we see that it suffices to show that

$$\lim_{r\to\infty} r \sum_{k\in\mathbb{Z}} \sum_{w\in A} \chi_A(k+ir,w)\,\text{Im}(w) = \frac{\pi}{4}\,\text{dhcap}(A).$$

Using the last-exit decomposition mentioned above, we see that for each $w \in \partial\mathcal{H}_A \cap A$ and every $r > h$,

$$\mathbb{P}^{k+ir}\left\{S(\zeta_A) = w\right\} = \sum_{j\in\mathbb{Z}} G_{\mathcal{H}_A}(k+ir, j+ir)\,\chi_A(j+ir,w).$$

Summing over k, we get

$$\sum_{k\in\mathbb{Z}} \mathbb{P}^{k+ir}\left\{S(\zeta_A) = w\right\} = \sum_{j\in\mathbb{Z}} \chi_A(j+ir,w) \sum_{k\in\mathbb{Z}} G_{\mathcal{H}_A}(k+ir, j+ir).$$

But by comparison with the cases $A = \emptyset$ and $A = \{z : 1 \leq \text{Im}(z) \leq h\}$, we see that

$$4(r-h) \leq \sum_{k\in\mathbb{Z}} G_{\mathcal{H}_A}(k+ir, j+ir) \leq 4r.$$

Indeed, we have $G_{\mathcal{H}_A}(k+ir, j+ir) = \mathbb{P}^{k+ir}\left\{\zeta_A < \tau_{j+ir}\right\}^{-1}$. For $A = \emptyset$,

$$\begin{aligned}\sum_{k} G_{\mathcal{H}_A}(k+ir, j+ir) &= \mathbb{E}^{ir}\left[\#\text{ of visits to } l_r \text{ before } \zeta_A\right] \\ &= \mathbb{P}^{ir}\left\{\zeta_A < \sigma_r\right\}^{-1} = 4r,\end{aligned}$$

by the gambler's ruin estimate, and we get the other bound in the same way. Hence, as $r \to \infty$,

$$\sum_{k\in\mathbb{Z}} \sum_{w\in A} \mathbb{P}^{k+ir}\left\{S(\zeta_A) = w\right\}\text{Im}(w) \sim 4r \sum_{j\in\mathbb{Z}} \sum_{w\in A} \chi_A(j+ir,w)\text{Im}(w).$$

But (4.7) shows that the left hand side equals $\pi\,\text{dhcap}(A)$.

□

If $w \in \mathcal{H}$, we define the **escape probability** by $\tilde{S}$ from A at w by

$$\mathrm{Es}_A(w) = 1 - \tilde{\phi}_A(w) = \mathbb{P}^w \left\{ \tilde{S}[1, \infty) \cap A = \emptyset \right\}.$$

Then,

$$\begin{aligned} \mathrm{Es}_A(w) &= \lim_{r \to \infty} \mathbb{P}^w \left\{ \tilde{S}[1, \tilde{\sigma}_r] \cap A = \emptyset \right\} \\ &= \lim_{r \to \infty} [r/\mathrm{Im}(w)] \mathbb{P}^w \left\{ S[1, \sigma_r] \cap (A \cup \mathbb{Z}) = \emptyset \right\} \\ &= \lim_{r \to \infty} [r/\mathrm{Im}(w)]^2 \sum_{k \in \mathbb{Z}} \tilde{\chi}_A(k + ir, w), \end{aligned} \tag{4.14}$$

where the last equality can be seen by reversing paths. Note that with this definition, Lemma 4.3.1 and (4.14) give a new characterization of dhcap:

$$\mathrm{dhcap}(A) = \frac{4}{\pi} \lim_{r \to \infty} r^2 \sum_{w \in A} \sum_{k \in \mathbb{Z}} \tilde{\chi}_A(k + ir, w) = \frac{4}{\pi} \sum_{w \in A} \mathrm{Es}_A(w) \, [\mathrm{Im}(w)]^2.$$

If $v \in \mathcal{H}_A$, then again by a last-exit decomposition argument,

$$\mathrm{Es}_{A \cup \{v\}}(v) = \frac{\mathrm{Es}_A(v)}{G_{\mathcal{H}_A}(v, v)}. \tag{4.15}$$

Note that (4.10) implies that $G_{\mathcal{H}_A}(v, v)$ can be interpreted either as the expected number of visits to v of a simple random walk starting at v before leaving $\mathcal{H}_A$ or as the expected number of visits to v of an excursion started at v before visiting A.

The computations done in the last few pages now yield the following expression of $\mathrm{dhcap}(A \cup \{v\}) - \mathrm{dhcap}(A)$ in terms of quantities for $\tilde{S}$ depending on A and v only.

Lemma 4.3.2. For a given set A and a point $v \in \mathcal{H}_A$,

$$\frac{\pi}{4} \left(\mathrm{dhcap}(A \cup \{v\}) - \mathrm{dhcap}(A) \right) = \frac{\mathrm{Im}(v)^2 (\mathrm{Es}_A(v))^2}{G_{\mathcal{H}_A}(v, v)}.$$

Proof. For $v \in \mathcal{H}_A$ and $z \in \mathcal{H}_{A\cup\{v\}}$ with $\mathrm{Im}(z) > \mathrm{Im}(v)$, we have

$$\tilde{\chi}_{A\cup\{v\}}(z) - \tilde{\chi}_A(z) = \tilde{\chi}_{A\cup\{v\}}(z,v)\,\mathrm{Es}_A(v).$$

Hence,

$$\begin{aligned}\frac{\pi}{4}\left[\mathrm{dhcap}(A\cup\{v\}) - \mathrm{dhcap}(A)\right] &= \lim_{r\to\infty} r^2 \sum_{k\in\mathbb{Z}} \tilde{\chi}_{A\cup\{v\}}(k+ir,v)\,\mathrm{Es}_A(v) \\ &= [\mathrm{Im}(v)]^2\,\mathrm{Es}_{A\cup\{v\}}(v)\,\mathrm{Es}_A(v) \\ &= \frac{\mathrm{Im}(v)^2(\mathrm{Es}_A(v))^2}{G_{\mathcal{H}_A}(v,v)}.\end{aligned}$$

where we used Lemma 4.3.1, (4.14), and (4.15) for the first, second, and third equality, respectively.

□

4.4 How close are hcap and dhcap?

In this section, we will define a way to associate continuous sets to discrete sets in such a way that these sets are "as similar as possible". The way in which we create this association will depend slightly on the underlying space, which will be either $\mathbb{Z}^2$ or $\mathcal{H}$.

The **continuous version** $\tilde{E}$ of a set $E \subset \mathbb{Z}^2$ is defined as follows: For each $z \in E$, let $\mathcal{S}_z$ be the closed set in $\mathbb{C}$ bounded by the square centered at z with sides parallel to the axes. Then $\tilde{E} = \bigcup_{z\in E} \mathcal{S}_z$.

We would like the continuous version of a discrete hull $A \in \mathcal{H}$ to be a continuous hull in $\mathbb{H}$. Therefore, we must slightly modify our approach.

For any discrete hull A as defined in Section 4.2, we let $\mathcal{S}_z$ be the closed set in $\mathbb{C}$ bounded by the square centered at z with sides parallel to the axes if $\mathrm{Im}(z) > 1$. However, if $\mathrm{Im}(z) = 1, \mathcal{S}_z$ is the closed rectangle with corners $\mathrm{Re}(z)-1/2$, $\mathrm{Re}(z)+1/2$,

$\mathrm{Re}(z) - 1/2 + 3i/2$, $\mathrm{Re}(z) + 1/2 + 3i/2$. Then we call $\tilde{A} = \bigcup_{z \in A} \mathcal{S}_z$ the continuous version of A and it is easy to check that $\tilde{A}$ is a hull.

Since from here on and for the remainder of this chapter, the sets A and $\tilde{A}$ will go hand in hand, we will simplify the notation and write $\mathbb{H}_A$ for $\mathbb{H} \setminus \tilde{A}$, unless stated otherwise.

Recall the definitions related to Brownian motion and continuous sets, which are the analogues to some definitions of Section 3.2 for discrete objects. If D is any subset of $\mathbb{H}$ and A is a hull in $\mathbb{H}$,

$$T_D = \inf\{t \geq 0 : B_t \in D\}, \qquad Z_A = T_{\partial \mathbb{H}_A} = \inf\{t \geq 0 : B_t \in \partial \mathbb{H}_A\}.$$

For m a positive real, let $\mathcal{A}_m$ be the set of all simply connected subsets A of $\mathcal{H}$ with $R \leq m$.

Recall that in Chapter 1, we introduced the KMT approximation, a coupling in which Brownian motion and random walk are close to each other up to a given time n, with high probability. We will show below that if we define appropriate hitting times for our random walk and Brownian motion, the bound on the distance between the two processes given by the KMT approximation still holds up to the latter of the two hitting times.

Let $A \in \mathcal{A}_m$, $\zeta = \zeta_A$ be the first time S is in $\partial \mathcal{H}_A$, and $Z = Z_A$ the first time B is in $\partial \mathbb{H}_A$.

Lemma 4.4.1. There exist a probability space containing a simple random walk S and a standard Brownian motion B with $B_0 = S_0$ and a constant K such that for every $\lambda > 0$ there exists a constant $C_1 = C_1(\lambda) < \infty$ such that

$$\mathbb{P}^{im} \left\{ \sup_{0 \leq t \leq \zeta \vee Z} |B_t - S_t| \geq C \log m \right\} \leq K m^{-\lambda}.$$

Proof. We know from Theorem 1.4.2 that for some $K_2 > 0$ and for all $\lambda > 0$ there is a $C_2 = C_2(\lambda)$ such that for all $m \geq 1$,

$$\mathbb{P}^{im}\left\{\sup_{0\leq t\leq m}|B_t - S_t| \geq C_2 \log m\right\} \leq K_2 m^{-\lambda}.$$

We let $C = C(\lambda) = C_2 \cdot (4\lambda + 3)$ and, for $d \in \mathbb{R}_+^*$, let ξ_d and Ξ_d be as defined in Chapter 0. The probability in the lemma is bounded above by

$$\mathbb{P}^{im}\left\{Z > m^{4\lambda+3}\right\} + \mathbb{P}^{im}\left\{\zeta > m^{4\lambda+3}\right\}$$
$$+ \mathbb{P}^{im}\left\{\sup_{0\leq t\leq \zeta\vee Z}|B_t - S_t| \geq C_2(4\lambda+3)\log m; \zeta \vee Z \leq m^{4\lambda+3}\right\}$$

$$\leq \mathbb{P}^{im}\left\{Z > m^{4\lambda+3}; \sup_{0\leq t\leq Z}|B_t - B_0| \leq m^{2\lambda+1}\right\} + \mathbb{P}^{im}\left\{Z > \Xi_{m^{2\lambda+1}}\right\}$$
$$+ \mathbb{P}^{im}\left\{\zeta > m^{4\lambda+3}; \sup_{0\leq t\leq \zeta}|S_t - S_0| \leq m^{2\lambda+1}\right\} + \mathbb{P}^{im}\left\{\zeta > \xi_{m^{2\lambda+1}}\right\}$$
$$+ \mathbb{P}^{im}\left\{\sup_{0\leq t\leq m^{4\lambda+3}}|B_t - S_t| \geq C_2(4\lambda+3)\log m\right\}$$
$$\leq \mathbb{P}^{im}\left\{\Xi_{m^{2\lambda+1}} > m^{4\lambda+3}\right\} + \mathbb{P}^{im}\left\{Z > \Xi_{m^{2\lambda+1}}\right\}$$
$$+ \mathbb{P}^{im}\left\{\xi_{m^{2\lambda+1}} > m^{4\lambda+3}\right\} + \mathbb{P}^{im}\left\{\zeta > \xi_{m^{2\lambda+1}}\right\}$$
$$+ \mathbb{P}^{im}\left\{\sup_{0\leq t\leq m^{4\lambda+3}}|B_t - S_t| \geq C_2 \log m^{4\lambda+3}\right\}.$$

By Lemmas A.1.2 and A.1.6, the first and third terms are bounded above by $\exp\{-Km\}$ for some positive constant K. The second and fourth terms are bounded above by $K_1 m^{-\lambda}$ by Beurling's continuous and discrete estimates (see Section A.3). Finally, Theorem 1.4.2 tells us that the last term is bounded above by $K_2(m^{4\lambda+3})^{-\lambda}$.

Since K, K_1, K_2 are independent of r or λ, we can choose a K, independent of r and λ, such that the Lemma holds.

□

Lemma 4.4.2. There is a constant K such that if $A \in \mathcal{A}_m$, then

$$|\text{dhcap}(A) - \text{hcap}(\tilde{A})| \leq K m^{3/2} \log m.$$

Remark 8. Recall that we showed in Chapter 3 that there is a constant $C_1 > 0$ such that $\text{hcap}(\tilde{A}) \geq C_1 \, h(A)^2$. Hence, for sets A such that $h(A) \geq R(A)$ (i.e., sets which grow as much in the vertical direction as in the horizontal direction), $\text{hcap}(\tilde{A})$ is of order R^2. For such sets the result of the lemma is nontrivial.

Proof. In what follows, we assume that m is an integer. The proof for real m only requires minor modifications.

Lemma 4.4.1 guarantees that we can define B and S on the same probability space and choose a constant C_1 so that

$$\mathbb{P}^z \left\{ \sup_{0 \leq t \leq \zeta \vee Z} |B_t - S_t| \geq C_1 \, \log m \right\} \leq C_1 \, m^{-10}. \tag{4.16}$$

We assume for the rest of the proof that B and S are coupled in this way.

Recall that $h = h(A) = \sup\{\text{Im}(z) : z \in A\}, \zeta_A = \inf\{n \geq 1 : S_n \in \partial \mathcal{H}_A\}, Z_A = \inf\{t \geq 0 : B_t \in \partial \mathbb{H}_A\}$, and let $z = im^2$. We know from Lemmas 3.2.4 and 4.2.3 that

$$\text{hcap}(\tilde{A}) = m^2 \, \mathbb{E}^z \left[\text{Im}(B_Z)\right] + \mathcal{O}\left(hR/m^2\right),$$

$$\text{dhcap}(A) = m^2 \mathbb{E}^z \left[\text{Im}(S_\zeta)\right] + \mathcal{O}\left(hR^2/m^2\right).$$

Hence it suffices to show that

$$\mathbb{E}^z \left[|\text{Im}(B_Z) - \text{Im}(S_\zeta)|\right] \leq K \, m^{-1/2} \, \log m.$$

Note that since $A \in \mathcal{A}_m$, we obviously have $|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)| \leq m$, so that trivially,

$$\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|\right] = \mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)| \wedge m\right]. \tag{4.17}$$

This observation may seem irrelevant for now, but will be essential later.

We define the set

$$V = \{z \in \mathcal{H} : |\mathrm{Re}(z)| \geq \frac{3m}{2}; 0 < \mathrm{Im}(z) \leq 2C_1 \log m\},$$

where C_1 is the constant of the coupling, and the stopping times

$$\beta = \inf\{j \geq 0 : d(S_j, A \cup \mathbb{Z}) \leq 2C_1 \log m; |S_j| \leq 3m\}, \qquad \gamma = \inf\{j \geq 0 : S(j) \in V\}.$$

We also define the following events which will appear naturally below in the sense that we will evaluate $\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|\right]$ over these events and their complements separately.

$$E_1 = \left\{\sup_{0 \leq t \leq \zeta \vee Z} |B_t - S_t| \geq C_1 \log m\right\}, \qquad E_2 = \{Z < \zeta\},$$

$$E_3 = \{|\mathrm{Re}(B_Z)| \geq 2m\}, \qquad E_4 = \{\beta < \zeta\}.$$

Note that E_1 is the event considered in (4.16), so that by the observation of (4.17), we have

$$\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|\,;\, E_1\right] \leq m\,\mathbb{P}(E_1) \leq C_1\, m^{-9}. \tag{4.18}$$

We will show that there exists a constant K, independent of m, such that

$$\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|\,;\, E_2 \cap E_1^c\right] \leq K\, m^{-1/2} \log m, \tag{4.19}$$

where E_1^c denotes the complement of E_1. A similar argument (reversing the roles of random walk and Brownian motion), which we omit, shows that

$$\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|\,; E_2^c \cap E_1^c\right] \leq K\, m^{-1/2} \log m. \tag{4.20}$$

This then implies that $\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|\,; E_1^c\right] \leq K\, m^{-1/2}\, \log m$, which, together with (4.18) gives the lemma.

To show (4.19), we look at the expectation separately over the events E_3 and E_3^c. For the first case, we observe that $E_3 \subset \{\mathrm{Im}(B_Z) = 0\}$ and $E_2 \cap E_3 \cap E_1^c \subset \{\gamma < Z < \zeta\}$, and so

$$\begin{aligned}\mathbb{E}^z\left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)|; E_2 \cap E_3 \cap E_1^c\right] &\leq \mathbb{E}^z\left[\mathrm{Im}(S_\zeta); E_2 \cap E_3 \cap E_1^c\right] \\ &\leq \mathbb{E}^z\left[\mathrm{Im}(S_\zeta); \gamma < \zeta\right].\end{aligned}$$

Note that this is a quantity for random walk only. Fix $w \in A$ and recall that for $a \in \mathbb{N}, \sigma_a = \inf\{n \geq 1 : \mathrm{Im}(S_n) = a\}$. Since $\gamma \geq \sigma_{[2C_1 \log m]}$, the strong Markov property gives

$$\begin{aligned}&\mathbb{P}^z\{S_\zeta = w; \gamma < \zeta\} \\ &\quad\leq \mathbb{P}^z\left\{S_\zeta = w; \gamma = \sigma_{[2C_1 \log m]} < \zeta\right\} + \mathbb{P}^z\left\{S_\zeta = w; \sigma_{[2C_1 \log m]} < \gamma < \zeta\right\} \\ &\quad\leq \sum_{|k| \geq 3m/2} \mathbb{P}^z\left\{S(\sigma_{[2C_1 \log m]}) = k + i[2C_1 \log m]\right\} \mathbb{P}^{k+i[2C_1 \log m]}\left\{S(\zeta) = w\right\} \\ &\qquad\qquad + \mathbb{P}^z\left\{|\mathrm{Re}(S(\sigma_{[2C_1 \log m]}))| \leq 3m/2\right\} \cdot \sup_{x \in V} \mathbb{P}^x\left\{S(\zeta) = w\right\} \\ &\quad\leq \sum_{|k| \geq 3m/2} \mathbb{P}^z\left\{S(\sigma_{[2C_1 \log m]}) = k + [2C_1 \log m]i\right\} \mathbb{P}^{[2C_1 \log m]i}\left\{\hat{\xi}_{|k| - m/2} < \sigma_0\right\} \\ &\qquad\qquad + \mathbb{P}^z\left\{|\mathrm{Re}(S(\sigma_{[2C_1 \log m]}))| \leq 3m/2\right\} \cdot \sup_{x \in V} \mathbb{P}^x\left\{S(\zeta) = w\right\},\end{aligned}$$

where for $a \in \mathbb{R}_+^*, \hat{\xi}_a = \inf\{n \geq 0 : |S_n - S_0| \geq a\}$. We can find some constant K and $|k| \geq \frac{3m}{2}$ such that

$$\mathbb{P}^{[2C_1 \log m]i}\left\{\hat{\xi}_{|k| - m/2} < \sigma_0\right\} \leq K \frac{\log m}{|k| - m/2} \leq K \frac{\log m}{|k|}. \tag{4.21}$$

This can be shown by inscribing a rectangle in the half-circle and proceeding as in the proof of Lemma B.2.4. Similarly,

$$\sup_{x \in V} \mathbb{P}^x \{S(\zeta) = w\} \leq K \frac{\log m}{m}.$$

This, together with

$$\mathbb{P}^z \left\{ \mathrm{Re}(S(\sigma_{[2c_1 \log m]})) \leq 3m/2 \right\} \leq \frac{K}{m},$$

which follows from Lemma 4.2.1, allows us to see that

$$\begin{aligned} &\mathbb{P}^z\{S_\zeta = w; \gamma < \zeta\} \\ &\quad \leq \sum_{|k| \geq 3m/2} \left(\frac{1}{\pi} \frac{m^2 - [2C_1 \log m]}{(m^2 - [2C_1 \log m])^2 + k^2} + \mathcal{O}\left(\frac{1}{m^4 + k^2} \right) \right) \frac{K \log m}{|k|} + K \frac{\log m}{m^2} \\ &\quad \leq K \log m \sum_{|k| \geq 3m/2} \left(\frac{m^2}{m^4 + k^2} + \mathcal{O}\left(\frac{1}{m^4 + k^2} \right) \right) \frac{1}{|k|} + K \frac{\log m}{m^2} \end{aligned}$$

For the first inequality, we used Lemma 4.2.1. The sum can be estimated by computing

$$\int_{3m/2}^{\infty} \frac{m^2}{x(m^4 + x^2)}\, dx \leq m^2 \left[\int_{3m/2}^{m^2} \frac{dx}{m^4 x} + \int_{m^2}^{\infty} \frac{dx}{x^3} \right] \leq K \frac{\log m}{m^2}.$$

Therefore,

$$\mathbb{P}^z \{S_\zeta = w; \gamma < \zeta\} \leq C \frac{(\log m)^2}{m^2},$$

and since every $w \in A$ satisfies $\mathrm{Im}(w) \leq m$,

$$\mathbb{E}^z [S_\zeta; \gamma < \zeta] \leq C \frac{(\log m)^2}{m}.$$

We still have to compute

$$\mathbb{E}^z \left[|\mathrm{Im}(B_Z) - \mathrm{Im}(S_\zeta)| \,;\, E_2 \cap E_3^c \cap E_1^c \right].$$

We use the fact that $E_2 \cap E_3^c \cap E_1^c \subset \{\beta \le Z \le \zeta\}$. ($\beta \le Z$ because of $E_3^c \cap E_1^c$ and $Z \le \zeta$ because of E_2.)

$$\mathbb{E}^z\left[|\mathrm{Im}(B_Z - S_\zeta)| \wedge m; E_2 \cap E_3^c \cap E_1^c\right]$$

$$\begin{aligned}
&\le \mathbb{E}^z\left[|\mathrm{Im}(B_Z - S_Z)| + (|\mathrm{Im}(S_Z - S_\zeta)| \wedge m); E_2 \cap E_3^c \cap E_1^c\right] \\
&\le C_1 \log m + \mathbb{E}^z\left[|\mathrm{Im}(S_Z - S_\zeta)| \wedge m; \beta \le Z \le \zeta\right] \\
&\le C_1 \log m + \mathbb{E}^z\left[2 \sup_{\beta \le t \le \zeta} |\mathrm{Im}(S_t - S_\beta)| \wedge m; \beta \le \zeta\right].
\end{aligned}$$

Observe that this is where we need the remark made in (4.17), since, although $|\mathrm{Im}(S_Z) - \mathrm{Im}(S_\zeta)| \le 2r$, $\sup_{\beta \le t \le \zeta} |\mathrm{Im}(S_t - S_\beta)|$ may be arbitrarily large.

It now suffices to show that

$$\mathbb{E}^z\left[\sup_{\beta \le t \le \zeta} |\mathrm{Im}(S_t - S_\beta)| \wedge m\,;\, \beta \le \zeta\right] < c\, m^{-1/2}\, \log m. \tag{4.22}$$

Note again that this is a quantity for random walk only. If we recall the definition of B, we see that it follows from the discrete Beurling estimate (see Section A.3) that there is a positive constant C_2 such that for every $\alpha > 0$,

$$\mathbb{P}^z\left\{\sup_{\beta \le j \le \zeta} |S_j - S_\beta| \ge \alpha\, \log m\right\} \le C_2\, \alpha^{-1/2}.$$

We now use the strong Markov property to see that

$$\mathbb{P}^z\{\beta \le \zeta; \sup_{\beta \le t \le \zeta} |S_t - S_\beta| \ge \alpha \log m\}$$

$$\begin{aligned}
&= \sum_{w \in \partial U^c} \mathbb{P}^z\{\beta \le \zeta; S_\beta = w\}\, \mathbb{P}^w\left\{\sup_{0 \le t \le \zeta} |S_t - S_\beta| \ge \alpha \log m\right\} \\
&\le C_2 \alpha^{-1/2} \mathbb{P}^z\{\beta \le \zeta\} = \alpha^{-1/2} \mathcal{O}\left(\frac{1}{m}\right),
\end{aligned}$$

where

$$U^c = \{w : d(w, A \cup \mathbb{Z}) > 2C_1 \log m \text{ or } |w| > 3m;\}$$

is the complement of U and the $\mathcal{O}\left(\frac{1}{r}\right)$ term can be seen from Lemma 4.2.1. Therefore, a change of variables and straightforward observations give

$$\mathbb{E}^z[\sup_{\beta\le t\le\zeta}|\mathrm{Im}(S_t-S_\beta)|\,;\,\beta\le\zeta]$$

$$\le \int_{\alpha\log m=0}^{\infty}\mathbb{P}^z\left\{\beta\le\zeta;\ \sup_{\beta\le t\le\zeta}|\mathrm{Im}(S_t-S_\beta)|\wedge m\ge\alpha\log m\right\}$$

$$\le \int_{\alpha\log m=0}^{m}\mathbb{P}^z\left\{\beta\le\zeta;\ \sup_{\beta\le t\le\zeta}|\mathrm{Im}(S_t-S_\beta)|\ge\alpha\log m\right\}$$

$$\le \int_{\alpha\log m=0}^{m}\mathbb{P}^z\left\{\beta\le\zeta;\ \sup_{\beta\le t\le\zeta}|S_t-S_\beta|\ge\alpha\log m\right\}$$

$$= \log m\int_{0}^{m/\log m}\mathbb{P}^z\left\{\beta\le\zeta;\ \sup_{\beta\le t\le\zeta}|S_t-S_\beta|\ge\alpha\log m\right\}\,d\alpha$$

$$\le K\frac{\log m}{m}\int_0^{m/\log m}\alpha^{-1/2}\,d\alpha = K\frac{\log m}{m}\left(\frac{m}{\log m}\right)^{1/2} = K\left(\frac{\log m}{m}\right)^{1/2}.$$

This gives (4.22), which implies (4.19). (4.20) can be done in the exact same way. The lemma then follows directly from (4.18), (4.19), and (4.20).

□

4.5 Standard results for conformal maps

At the center of our estimates in the next section will be bounds related to the derivative of conformal maps. We state the results in this section without their proofs, as they are rather standard. They can be found in [6] or [17].

Let $\mathcal{S}$ to be the set of analytic one-to-one maps f defined on $\mathbb{U}$, the open unit disk centered at the origin, with $f(0)=0$ and $f'(0)=1$.

Theorem 4.5.1. [Koebe One-quarter Theorem] If $f\in\mathcal{S}$ and $0<r<1$, then $D(0,r/4)\subset f(r\mathbb{U})$.

Corollary 4.5.2. Suppose $f : \mathbb{U} \to D'$ is a conformal transformation with $f(z) = z'$. Then

$$\frac{d'}{4d} \leq |f'(z)| \leq \frac{4d'}{d},$$

where $d = d(z, \mathbb{U}), d' = d(z', D')$.

Theorem 4.5.3. [Distortion Theorem] If $f \in \mathcal{S}$ and $z \in \mathbb{U}$,

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Theorem 4.5.4. [Growth Theorem] If $f \in \mathcal{S}$ and $z \in \mathbb{U}$,

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)}.$$

Corollary 4.5.5. If $f : D \to D'$ is a conformal transformation, $z, w \in D$, with $f(z) = z'$, then for all $r \in (0,1)$, all w satisfying $|w - z| \leq r d(z, \partial D)$,

$$|f(w) - z'| \leq \frac{4d(z', \partial D')}{1-r^2} |w - z|.$$

4.6 Getting away from the boundary

We now turn to a toilsome computation which will be essential in our main estimate of this chapter, i.e. the computation up to error terms of

$$\mathbb{P}^w \{S(\zeta_A) = -n\} \tag{4.23}$$

where A is a discrete hull, $\mathcal{H}_A = \mathcal{H} \setminus A, w \in \partial \mathcal{H}_A \cap A$, and ζ_A is as always $\inf\{n \geq 1 : S_n \in \partial \mathcal{H}_A\}$.

In this section, A will always denote a discrete hull with $R(A) \leq n^\beta$, where $\beta < 1$. The continuous version of A, denoted by $\tilde{A}$, is as defined in section 4.4, and we write

$\mathbb{H}_A$ for $\mathbb{H} \setminus \tilde{A}$. g_A will be the conformal map taking $\mathbb{H}_A$ to $\mathbb{H}$, satisfying

$$g_A(z) = z + \frac{\text{hcap}(\tilde{A})}{z} + \mathcal{O}\left(|z|^{-2}\right). \tag{4.24}$$

Recall that $Z_A = \inf\{t \geq 0 : B_t \in \partial\mathbb{H}_A\}$. In what follows, we will show that the image under g_A of a random walk in $\mathcal{H}_A$ does not move too much laterally before covering a certain vertical distance. More precisely, the horizontal displacement of $g_A(S_n)$ is of the same order of magnitude as its vertical displacement, with high probability. We should note that showing this fact for random walk itself, rather than its image, is trivial and nothing more than solving the Dirichlet problem. Unfortunately, the question we consider here is not as straightforward.

We start with a general result about domains in $\mathbb{Z}^2$ and their continuous version. Let $D \subset \mathbb{Z}^2$ be a bounded set and $\tilde{D}$ its continuous version, as defined at the beginning of the previous section. Let $T = \inf\{t \geq 0 : B_t \in \partial\tilde{D}\}$ be the first time Brownian motion leaves $\tilde{D}$ and $\tau = \inf\{n \geq 0 : S_n \in \partial D\}$ the first time random walk leaves D. If $V \subset \partial D$, we define $\tilde{V}$ to be the union of all the boundary edges of $\tilde{D}$ which intersect an edge between neighboring points $a \in V$ and $b \in D$. For any such boundary edge $e \in \tilde{V}$, we say that a is the corresponding boundary point. Note that an edge has exactly one corresponding point $a \in V$, but a boundary point $a \in V$ can correspond to more than one edge of $\tilde{V}$. The following result, which we do not prove here can be found in [13]:

Proposition 4.6.1. For every $\epsilon > 0$, there exists a $\delta > 0$ such that if D is a finite connected subset of $\mathbb{Z}^2$, $V \subset \partial A$ and $x \in A$ with $\mathbb{P}^x\left\{B_T \in \tilde{V}\right\} \geq \epsilon$, then $\mathbb{P}^x\{S_\tau \in V\} \geq \delta$.

For points $z \in \mathbb{H}_A$, we introduce the notation

$$i_A(z) = \text{Im}(g_A(z)) \text{ and } r_A(z) = \text{Re}(g_A(z)). \tag{4.25}$$

The purpose of the next lemma is to find bounds on $|g_A(x) - g_A(y)|$ for points $x, y \in \mathbb{H}_A$ which are close to each other, but x is away from $\partial\mathbb{H}_A$. In particular, if we assume that $d(x, \partial\mathbb{H}_A) \geq K > 0$, we can show that the bound depends only on $i_A(s)$.

Lemma 4.6.2. Let A be a discrete hull and $w, w' \in \mathcal{H}_A$ with $|w - w'| = 1$. There exist constants L_1, L_2, L_3 such that

$(a)|g_A(w) - g_A(w')| \leq L_1 i_A(w)$,

(b) for any $z \in \mathrm{Sq}(w), |g_A(z) - g_A(w)| \leq L_2 i_A(w)$,

(c) for any $z, z' \in \mathrm{Sq}(w'), |g_A(z) - g_A(w)| \leq L_3 |g_A(z') - g_A(w)|$.

Remark 9. In particular, the lemma implies that if $w, w' \in \mathcal{H}_A$ with $|w - w'| = 1$ and $z \in \mathrm{Sq}(w)$,

$$i_A(z) \leq (1 + L_1) i_A(w) \text{ and } i_A(w') \leq (1 + L_1) i_A(w) \tag{4.26}$$

Proof. Take a sequence of four open disks $\{D_i\}_{i=1,..,4}$ of radius $1/3$, such that D_1 is centered at w, for $i = 2, 3, 4, D_i$ is centered at a point inside D_{i-1}, and $w' \in D_4$. By applying the Growth and Distortion Theorems from Section 3.2 to each of these disks, we see that there exists a universal constant K such that $|g_A(w) - g_A(w')| \leq K|g'_A(w)|$. Since $d(w, \partial\mathbb{H}_A) \geq \frac{1}{2}$, Corollary 4.5.2 gives (a).

To show (b), we use the same argument, but in the case where $d(z, \partial\mathbb{H}_A)$, we also need Schwarz reflection to extend the result to the boundary of $\mathrm{Sq}(w)$.

Finally, (c) can be shown by letting noting that on one hand, by part (a), the Growth Theorem, the Distortion Theorem, and Corollary 4.5.2,

$$\begin{aligned} |g_A(z) - g_A(w)| &\leq |g_A(w') - g_A(w)| + |g_A(z) - g_A(w')| \\ &\leq L_1 i_A(w) + K_1 |g'_A(w')| \leq L_1 i_A(w) + K_2 |g'_A(w)| \\ &\leq L_1 i_A(w) + K_3 i_A(w) = K_4 i_A(w). \end{aligned}$$

On the other hand, since z' is not in the disk of radius 1/3 about w, the Koebe One-quarter Theorem and Corollary 4.5.2 give

$$|g_A(z') - g_A(w)| \geq K_5 |g_A'(w)| \geq K_6 i_A(w).$$

Choosing $L_3 = \frac{K_4}{K_5}$ concludes the lemma.

$\square$

If A is a hull, $w \in \mathcal{H}_A$, and c_1, c_2 are two positive constants with $c_1 \geq 1$, we let

$$\mathcal{R} = \mathcal{R}_A(w, c_1, c_2) = \{q \in \mathbb{H} : \text{Im}(q) \leq c_1 i_A(w) \text{ and } |\text{Re}(q) - r_A(w)| \leq c_2 i_A(w)\}$$

be the rectangle sitting in the upper half-plane, around $g_A(w)$, of height $c_1 i_A(w)$, width $2c_2 i_A(w)$, and define

$$\eta = \eta_A(w, c_1, c_2) = \min\{j \geq 0 : S_j \notin g_A^{-1}(\mathcal{R}_A(w, c_1, c_2))\}.$$

Lemma 4.6.3. There exist constants $c_1, c_2, \hat{c}, \epsilon > 0$ such that if A is a discrete hull and $w \in \mathcal{H}_A$, then

(a) $\mathbb{P}^w \{S(\eta(w, c_1, c_2)) \notin \mathcal{H}_A\} \geq \epsilon$.

(b) $\mathbb{P}^w \{i_A(S(\eta(w, c_1, c_2))) \geq c_1 i_A(w); |r_A(S(\eta(w, c_1, c_2))) - r_A(w)| \leq c_2 i_A(w)\} \geq \epsilon$.

(c) $\mathbb{P}^w \{|r_A(S(\eta(w, c_1, c_2))) - r_A(w)| \leq \hat{c} i_A(w) \,|\, S(\eta(w, c_1, c_2)) \in \mathcal{H}_A\} = 1$.

Proof. Fix A, $w \in \mathcal{H}_A$, $1 < C_1 < C_2$, and define

$$J = \{z \in \mathcal{H}_A : g_A(z) \in \mathcal{R}(w, C_1, C_2)\}, \quad J^+ = \{z \in \mathcal{H}_A : d(z, J) \leq 1\}, \quad \partial J = J^+ \setminus J.$$

Let $\tilde{J}$ be the continuous version of J^+ in $\mathbb{Z}^2$. Clearly, if $z \in \partial J, |g_A(z) - g_A(w)| \geq (C_1 - 1) i_A(w)$. Lemma 4.6.2 (c) now implies that if $\tilde{z} \in \partial \tilde{J}$, $|g_A(\tilde{z}) - g_A(w)| \geq \frac{(C_1 - 1)}{L_3} i_A(w)$.

It is clear that there is a $\delta_1 > 0$ (independent of w and A) such that

$$\begin{aligned}
\delta_1 &= \mathbb{P}^{g_A(w)}\left\{B_t \text{ leaves } \mathcal{R}(w, \frac{(C_1-1)}{2L_3}, \frac{(C_1-1)}{2L_3}) \text{ at } \mathbb{R}\right\} \\
&\leq \mathbb{P}^{g_A(w)}\left\{B(T_{\partial g_A(\tilde{J})}) \in \mathbb{R}\right\} \\
&= \mathbb{P}^{w}\left\{B(T_{\partial \tilde{J}}) \in \partial \mathbb{H}_A\right\}.
\end{aligned}$$

By Proposition 4.6.1, there is an $\epsilon_1 > 0$ such that $\mathbb{P}^w\{S(\tau_{\partial J^+}) \in \partial \mathcal{H}_A\} \geq \epsilon_1$. Since by Lemma 4.6.2 (a), every $z \in \partial J$ satisfies $|g_A(z) - g_A(w)| \leq (C_2 + L_1) i_A(w)$, this proves that there exist c_1 and c_2 such that

$$\mathbb{P}^w\{S(\eta(w, c_1, c_2)) \notin \mathcal{H}_A\} \geq \epsilon_1.$$

This proves (a). Note that this construction works for any constants $C_2 > C_1 > 1$, but that ϵ_1 may depend on them.

Now take $C_1' > 0$ and $C_2' \geq (L^2 + L + K)C_1'$, where K is defined below, let $\mathcal{R}' = \mathcal{R}(w, C_1', C_2')$, define L, R, and T to be respectively the left, right, and top side of $\mathcal{R}'$, and define $I = \{z \in \mathcal{H}_A : g_A(z) \in \mathcal{R}'\}$. If $z \in \partial_{int} I$ and we let e_z be the set of edges leading from z to a point $z' \in \partial I$ with $d(z, z') = 1$, two things can occur. Either the image of at least one of the edges of e_z intersects $\mathcal{R}'$ at $L \cup R$, or none of them does. We call the set of points $z \in \partial_{int} I$ which satisfy the first condition E_1 and the set of those that satisfy the second condition E_2. We let $L = \max\{L_1, L_2, L_3\}$, where L_1, L_2, L_3 are as in Lemma 4.6.2. If $z \in E_1$, $|r_A(z) - C_2' i_A(w)| \leq L^2 i_A(z) \leq L^2 C_1' i_A(w)$, by Lemma 4.6.2 (a) and (c). Also, by using the same lemma one more time, we see that if $\tilde{E}_1 = \bigcup\{\tilde{z} \in \mathrm{Sq}(z) \cap \partial \tilde{I}\}$, where the union is over all $z \in E_1$, then for $\tilde{z} \in \tilde{E}_1$, $|r_A(\tilde{z}) - C_2' i_A(w)| \leq L^2 C_1' i_A(w) + L C_1' i_A(w) = (L^2 + L) C_1' i_A(w)$. In particular, $|r_A(\tilde{z})| \geq (C_2' - (L^2 + L)C_1') i_A(w)$. Also, with the definition $\tilde{E}_2 = \tilde{I} \setminus \tilde{E}_1$, we can easily see that any $\tilde{z} \in \tilde{E}_2$ satisfies $i_A(\tilde{z}) \leq L C_1' i_A(w)$. Also, there exists a constant

K such that if $C'_2 \geq KC'_1$, $\tilde{E}_2$ is nonempty. Now it is clear that there exists a $\delta_2 > 0$ such that

$$\begin{aligned} \delta_2 &\leq \mathbb{P}^{g_A(w)}\left\{B_t \text{ leaves } g_A(\tilde{I}) \text{ at } g_A(\tilde{E}_2)\right\} \\ &= \mathbb{P}^{w}\left\{B_t \text{ leaves } \tilde{I} \text{ at } \tilde{E}_2\right\}. \end{aligned}$$

If $z \in E_2$, one can show by contradiction that $i_A(z) \geq \frac{C'_1}{L^2} i_A(w)$. Proposition 4.6.1, now implies that (b) holds with $c_1 = \frac{C'_1}{L^2}, c_2 = C'_2 - (L^2 + L)C'_1$, and $\epsilon = \epsilon_2$.

It is easy to see that we can choose c_1, c_2, and ϵ so that both (a) and (b) hold. (c) just follows from Lemma 4.6.2 (a).

□

We define a new rectangle whose main difference with the rectangle $\mathcal{R}$ above is that the side lengths do not need to scale with $i_A(w)$ anymore,

$$\mathcal{Q}_A(w, a, b) = \{q \in \mathbb{H} : \text{Im}(q) \leq a, |\text{Re}(q) - r_A(w)| \leq b\}$$

and the exiting time of its inverse image under g_A^{-1} in $\mathcal{H}_A$,

$$\theta_A(w, a, b) = \min\{j \geq 0 : S_j \notin g_A^{-1}(\mathcal{Q}(w, a, b))\}.$$

In what follows, the constants c_1 and c_2 are those from Lemma 4.6.3. Note that $c_1 \geq 2$.

Corollary 4.6.4. There exist constants $c_3, c_4 > 0$ such that for any $a > 0$, any discrete hull A, and any w with $i_A(w) < \frac{a}{c_1}$,

$$\mathbb{P}^w\left\{i_A(S(\theta_A(w, a, c_3 a))) \geq a; |r_A(S(\theta_A(w, a, c_3 a))) - r_A(w)| \leq c_3 a\right\} \geq \left(\frac{i_A(w)}{a}\right)^{c_4}.$$

Proof. We define a sequence of points. Let $z_0 = g_A(w)$. For $i \geq 1$, let $w_i = S(\eta(z_{i-1}, c_1, c_2))$ and $z_i = g_A(w_i)$. In what follows we work with the assumption that at each point w_i the event described in Lemma 4.6.3 (b) occurs. Then $\mathrm{Im}(z_{k_0}) \geq a$, where $k_0 = \left[\frac{\log(a/i_A(w))}{\log c_1}\right]$. However, it could happen that $\mathrm{Im}(z_k) \geq a$ for some $k \leq k_0$. For a given realization, suppose k_1 is the smallest integer for which $\mathrm{Im}(z_{k_1}) \geq a$. We will now find a bound on $|\mathrm{Re}(z_{k_1}) - r_A(w)|$, independent on the realization.

For $i \geq 1$, let $v_i = \mathrm{Im}(z_i) - \mathrm{Im}(z_{i-1})$ and $h_i = |\mathrm{Re}(z_i) - \mathrm{Re}(z_{i-1})|$. Then, if the event described in Lemma 4.6.3 (b) occurs, $v_i \geq (c_i - 1) i_A(w_{i-1})$ and $h_i \leq c_2 i_A(w_{i-1})$. Therefore, $h_i \leq \frac{c_2}{c_1 - 1} v_i$. This means that the total horizontal displacement between two points z_i and z_j is bounded above by $\frac{c_2}{c_1-1}$ times the total vertical displacement between those points. Since $\mathrm{Im}(z_{k_1}) \leq a$ by definition of k_1, $|\mathrm{Re}(z_{k_1}) - r_A(w)| \leq \frac{c_2}{c_1-1}(a - i_A(w)) \leq \frac{c_2}{c_1-1} a$. Now since $\mathrm{Im}(z_{k_1}) \leq a, |\mathrm{Re}(z_{k_1}) - \mathrm{Re}(z_{k_1 - 1})| \leq \frac{c_2}{c_1-1} a$. Therefore, $|\mathrm{Re}(z_{k_1}) - r_A(w)| \leq 2\frac{c_2}{c_1-1} a$.

The sequence of points in our construction has length at most k_0. Therefore, by Lemma 4.6.3,

$$\mathbb{P}^w \left\{ i_A(S(\theta_A(w, a, 2\frac{c_2}{c_1 - 1} a))) \geq a; |r_A(S(\theta_A(w, a, 2\frac{c_2}{c_1 - 1} a))) - r_A(w)| \leq \frac{2c_2}{c_1 - 1} a \right\}$$

$$\geq \epsilon^{k_0} \geq \left(\frac{a}{i_A(w)}\right)^{\log \epsilon / \log c_1},$$

from which the corollary follows if we choose $c_3 = 2\frac{c_2}{c_1-1}$ and $c_4 = -\frac{\log \epsilon}{\log c_1}$.

□

Proposition 4.6.5. There exists constants $c, \epsilon > 0$ such that for all $a > 0$, for any hull A, and any $w \in \mathcal{H}_A$ with $i_A(w) < \frac{a}{c_1}$,

$$\mathbb{P}^w \left\{ i_A(S(\theta(w, a, ca))) \geq a \,|\, S(\theta(w, a, ca)) \in \mathcal{H}_A \right\} \geq \epsilon.$$

Proof. For any $m \geq 1$, let

$$\mathcal{S}_m = \sup \mathbb{P}^w \left\{ i_A(S(\theta(w,a,c_3a))) \geq a \,|\, S(\theta(w,a,c_3a)) \in \mathcal{H}_A \right\},$$

where the sup is over $\{w : \frac{a}{c_1^m} \leq i_A(w) < \frac{a}{c_1}\}$ and c_3 is as in Corollary 4.6.4. Then, since for any $c > 0$ and $a > 0$,

$$\mathbb{P}^w \left\{ i_A(S(\theta(w,a,ca))) \geq a \,|\, S(\theta(w,a,ca)) \in \mathcal{H}_A \right\} \geq \mathbb{P}^w \left\{ i_A(S(\theta(w,a,ca))) \geq a \right\},$$

Corollary 4.6.4 implies that for any $m \geq 1$ we can find $\epsilon_m > 0$ such that

$$\mathcal{S}_m \geq \epsilon_m.$$

Let c_1 be as in Lemma 4.6.3 and fix $a > 0$. Suppose $c_1^{-(m+1)}a \leq i_A(w) < c_1^{-m}a$ and for such a w consider the rectangle

$$\mathcal{R} = \mathcal{R}(w, c_1, m) = \mathcal{Q}(w, c_1 i_A(w), m i_A(w)) \tag{4.27}$$

and the corresponding exiting time

$$\eta = \eta(w, c_1, m^2 a) = \min\{j \geq 0 : S_j \notin g_A^{-1}(\mathcal{R})\}. \tag{4.28}$$

Then, if m is an integer greater than c_2, Lemma 4.6.3 (b) gives that

$$\mathbb{P}^w \left\{ i_A(S(\eta)) \geq c_1^{-m}a; |r_A(S(\eta)) - r_A(w)| \leq m c_1^{-m} a \right\} \geq \epsilon.$$

In particular, repeated iterations of Lemma 4.6.3 (a) give

$$\mathbb{P}^w \left\{ |r_A(S(\eta)) - r_A(w)| \geq c_1^{-1}a \right\} \leq (1-\epsilon)^{m^2/\hat{c}} = e^{-\beta m},$$

where $\hat{c}$ is as in Lemma 4.6.3 (c), and therefore β is independent of m or a.This implies that

$$\begin{aligned} &\mathbb{P}^w\{|r_A(S(\eta)) - r_A(w)| \geq m c_1^{-m} a \,|\, S(\eta)) \in \mathcal{H}_A\} \\ &\qquad \leq \frac{\mathbb{P}^w \left\{ |r_A(S(\eta)) - r_A(w)| \geq m c_1^{-m} a \right\}}{\mathbb{P}^w \left\{ |r_A(S(\eta)) - r_A(w)| \geq m c_1^{-m} a \right\} + \mathbb{P}^w \left\{ i_A(S(\eta)) \geq c_1 i_A(w) \right\}} \\ &\qquad \leq \epsilon^{-1} e^{-\beta m}, \end{aligned}$$

where, again, ϵ is as in Lemma 4.6.3. In particular, for $m \geq \hat{c}$,

$$\mathbb{P}^w \left\{ |r_A(S(\eta)) - r_A(w)| \leq m i_A(w) \,|\, S(\eta) \in \mathcal{H}_A \right\} = 1,$$

and so we get for all w satisfying $i_A(w) \in (c_1^{-(m+1)}a, c_1^{-m}a]$, with the definition $c' = \sum_{j \geq 1} j c_1^{-j}$, and $\theta = \theta(w, a, (c' + c_3)a)$,

$$\mathbb{P}^w \left\{ i_A(S(\theta)) \geq a \,|\, S(\theta) \in \mathcal{H}_A \right\} \geq \prod_{j=m_0}^{m} (1 - \epsilon^{-1} e^{-\beta m}) \cdot \mathcal{S}_{m_0},$$

where m_0 is the smallest integer greater than $\sup\{c_2, \hat{c}\}$.

Since $\prod_{j=1}^{\infty}(1 - \epsilon^{-1}e^{-\beta m}) > 0$ and $\mathcal{S}_{m_0} > 0$, this gives the proposition.

□

Corollary 4.6.6. There exist constants $d, \beta > 0$ such that for any $a > 0$, any hull A, any $r > 0$, and any $x \in A$,

$$\mathbb{P}^w \left\{ i_A(S(\theta(w, a, ra))) \geq a | S(\theta(w, a, ra)) \in \mathcal{H}_A \right\} \geq 1 - de^{-\beta r}.$$

In particular, there is a constant d' such that

$$\mathbb{P}^w \left\{ r_A(S(\theta(w, a, ra))) \geq ra \right\} \leq d' e^{-\beta r} \mathbb{P}^w \left\{ i_A(S(\theta(w, a, ra))) \geq a \right\}.$$

Proof. We just iterate Proposition 4.6.5 as often as is needed.

□

4.7 A hitting probability

We now have most of the tools needed to compute

$$\mathbb{P}^w \left\{ S(\zeta_A) = -n \right\}.$$

A continues to denote a discrete hull. We fix $\gamma < 1$ and define the continuous and discrete **halo** associated with A: $\tilde{h}_\gamma = \{z \in \mathbb{H}_A : i_A(z) < n^\gamma\}$ and $h_\gamma = \tilde{h}_\gamma \cap \mathcal{H}$. The halo's boundary is $\tilde{b}_\gamma = \partial \tilde{h}_\gamma$ in the continuous case and $b_\gamma = \partial h_\gamma$ in the discrete case. τ_γ will be short for $\tau_{b_\gamma} = \inf\{n \geq 1 : S_n \in b_\gamma\}$.

Recall the definition of the discrete Green's function in a domain $E \subset \mathbb{Z}^2$ for $x, y \in E$:

$$G_E(x,y) = \mathbb{E}^x \left[\sum_{k \geq 0} \mathbb{1}\{S_k = y; k < \tau_{\partial E}\} \right].$$

Since for the remainder of this section we will mostly be working in the domain $\mathcal{H}_A$, we will simplify the notation and write G_A for $G_{\mathcal{H}_A}$.

If $E \subset \mathbb{C}$ is conformally equivalent to the open unit disk $\mathbb{U}$, $z, w \in E, z \neq w$, and $f : E \to \mathbb{U}$ is a conformal transformation with $f(z) = 0$, then we define the Green's function in E to be

$$g_E(z, w) = -\log |f(w)|.$$

Note that since f is unique up to a rotation, this is well-defined.

Like for the discrete case, we will write $g_A(z,w)$ for $g_{\mathbb{H}_A}(z,w)$ until the end of this section. A word of caution is needed here since we use the same symbol -g_A- for the continuous Green's function in $\mathbb{H}_A$ and the map defined in (4.24). However, it will always be clear from context what is meant. If $A = \emptyset$, we will just write $G(z,w)$ and $g(z,w)$ for the discrete and continuous Green's functions in the upper half-plane.

We now prove a few ancillary lemmas. The first three give information about the thickness of a halo, the image of its boundary, and the escape probability from A by random walk excursion started at its boundary.

Recall that for $a \in \mathbb{R}_+$, $D(a)$ is the disk of radius a centered at the origin and $C(a) = \partial D(a)$ is the circle making up its boundary. For the purpose of the following lemma, we define the upper half-disk and upper half-circle

$$D_u(a) = D(a) \cap \mathbb{H}, \ \ C_u(a) = C(a) \cap \mathbb{H}. \tag{4.29}$$

Lemma 4.7.1. For all $\beta, \gamma < 1$, there exist constants $N > 0, K_1 > 0, K_2 > 0$, such that for all $n \geq N$, for every continuous hull $\tilde{A}$ with $R(\tilde{A}) \leq n^\beta$, and for all $z \in b_\gamma$,

$$d(z, \tilde{A} \cup \mathbb{R}) \geq K_1 n^{2\gamma - \max\{\gamma,\beta\}}. \tag{4.30}$$

In particular,

$$\mathrm{Im}(z) \geq K_1 n^{2\gamma - \max\{\gamma,\beta\}}.$$

Moreover, if $\beta < \gamma < 1$,

$$d(z, \tilde{A} \cup \mathbb{R}) \leq K_2 n^\gamma. \tag{4.31}$$

Proof. (4.31) follows immediately from Proposition 4.5.5, so we can concentrate on (4.30).

We start with the case $\beta \geq \gamma$. If n is large enough and $|z| \geq n^{3/2}$, the fact that

$$\mathrm{Im}(z) \geq i_A(z) = n^\gamma, \tag{4.32}$$

which follows from Proposition 3.2.2 (b), gives the result.

For $|z| \leq n^{3/2}$, we will proceed by contradiction. First suppose that $\forall N > 0$, we can find $n \geq N$, a hull $\tilde{A}$ with $R(\tilde{A}) \leq n^\beta$ and a point $z \in g_A^{-1}(b_\gamma)$ such that $d(z, \tilde{A} \cup \mathbb{R}) \leq \frac{1}{64 C_B^2} n^{2\gamma-\beta}$, where C_B is the constant in the continuous Beurling Estimate (Theorem A.3.2). We can assume that $C_B \geq 1$. Then, the fact that $|a^2 - b^2| \leq |a-b| \cdot |a+b|$ implies that

$$d(z^2, \tilde{A}^2 \cup \mathbb{R}_+) \leq \frac{1}{64 C_B^2} n^{2\gamma-\beta} \cdot 2n^\beta = \frac{n^{2\gamma}}{32 C_B^2}.$$

Here, $\tilde{A}^2$ denotes $\{z^2 : z \in \tilde{A}\}$. Thus, conformal invariance of Brownian motion and the continuous Beurling Estimate give

$$\begin{aligned}\mathbb{P}^z \{\Xi_{n^2} < Z_A\} &\leq \mathbb{P}^{z^2} \left\{\hat{\Xi}_{n^4/2} < T_{\tilde{A}^2 \cup \mathbb{R}}\right\} \\ &\leq C_B \left(\frac{\frac{1}{32C_B^2} n^{2\gamma}}{n^4/2}\right)^{1/2} = \frac{1}{4} n^{\gamma-2}\end{aligned}$$

On the other hand, it follows from Proposition 3.2.5 that there is a constant K such that for all $w \in C_u(n^2)$, $|w - g_A(w) - \frac{\text{hcap}(\tilde{A})}{n^2}| \leq K\frac{R(\tilde{A})\text{hcap}(\tilde{A})}{n^4}$. In particular, $\exists N > 0$ such that $\forall\ n \geq N$, for every hull $\tilde{A}$ with $R(\tilde{A}) \leq n^\beta$ and all $w \in C_u(n^2)$, $|g_A(w) - w| \leq 2n^{2(\beta-1)}$. Therefore there exists $N > 0$ such that for all $n \geq N$, for all $\tilde{A}$ with $R(\tilde{A}) \leq n^\beta$,

$$D_u(n^2/2) \subset g_A(D_u(n^2)) \subset D_u(2n^2).$$

This implies that

$$\begin{aligned}\mathbb{P}^{g_A(z)} \left\{T_{g_A(C(n^2))} < \sigma_0\right\} &\geq \mathbb{P}^{g_A(z)} \{\xi_{2n^2} < \sigma_0\} \\ &\geq \mathbb{P}^{g_A(z)} \{\sigma_{2n^2} < \sigma_0\} = \frac{1}{2} n^{\gamma-2},\end{aligned}$$

by Gambler's ruin and since $i_A(z) = n^\gamma$.

Since $\mathbb{P}^z \{\xi_{n^2} < T_{A\cup\mathbb{R}}\} = \mathbb{P}^{g_A(z)} \left\{T_{g_A(C(n^2))} < \sigma_0\right\}$, this gives a contradiction.

If $\beta < \gamma$, the lemma is straightforward, since (4.32) implies that we can find an N so that for all $n \geq N$, $d(z, A \cup \mathbb{R}) \geq \frac{1}{2} n^\gamma$.

□

Lemma 4.7.2. For every $0 < \beta < \gamma$, there exists a constant K such that for all $z \in b_\gamma$ and every discrete hull A with radius $R(A) \leq n^\beta$,

$$0 \leq i_A(z) - n^\gamma \leq K n^{2\beta-\gamma}.$$

Proof. Note that by lemma 4.7.1, there exist constants $K_1, K_2 > 0$ such that $\text{Im}(z) \in [K_1 n^\gamma, K_2 n^\gamma]$.

The first inequality is obvious since if $z \in b_\gamma$, then $z \in h_\gamma^c$, so that $i_A(z) \geq n^\gamma$.

We know from Proposition 3.2.2 (*b*) that $\text{Im}(z) = i_A(z) + \mathbb{E}^z[\text{Im}(B(Z_A))]$. Clearly, if we recall the definition of D_u in (4.29), use the conformal invariance of Brownian motion, and since $R(A) \leq n^\beta$, then,

$$\begin{aligned}
\mathbb{E}^z[\text{Im}(B(Z_A))] &\leq n^\beta \mathbb{P}^z\{B(Z_A) \in A\} \\
&\leq n^\beta \mathbb{P}^z\left\{B(Z_{D_u(n^\beta)}) \in D_u(n^\beta)\right\} \\
&= n^\beta \mathbb{P}^{g_{D_u(n^\beta)}(z)}\left\{B(T_\mathbb{R}) \in [-2n^\beta, 2n^\beta]\right\} \\
&\leq K' n^{2\beta-\gamma}.
\end{aligned}$$

Lemma 4.7.1 implies that $\text{Im}(z) \geq K_1 n^\gamma$ and one can check easily by considering the explicit map $g_{D_u(n^\beta)}$ that this implies $\text{Im}(g_{D_u(n^\beta)}(z)) \geq K_2 n^\gamma$ for some constant $K_2 > 0$. From this and the hitting distribution of $\mathbb{R}$ for Brownian motion (see (4.2)), we get the last inequality. K' is independent of A, n, and z. Therefore,

$$0 \leq \text{Im}(z) - i_A(z) \leq K' n^{2\beta-\gamma}.$$

Since $z \in b_\gamma$, there exists $z' \in h_\gamma$ with $|z - z'| = 1$ and clearly, since $\text{Im}(z) \geq K_1 n^\gamma, \text{Im}(z') \geq K_1 n^\gamma - 1 \geq K_3 n^\gamma$. In the same way as above, we then get

$$0 \leq \text{Im}(z') - i_A(z') \leq K'' n^{2\beta-\gamma}.$$

Therefore,

$$\begin{aligned}
|i_A(z) - i_A(z')| &\leq |i_A(z) - \text{Im}(z)| + |\text{Im}(z) - \text{Im}(z')| + |\text{Im}(z') - i_A(z')| \\
&\leq K' n^{2\beta-\gamma} + 1 + K'' n^{2\beta-\gamma} \leq K n^{2\beta-\gamma}
\end{aligned}$$

Since $i_A(z') \leq n^\gamma$, this implies that $i_A(z) \leq n^\gamma + K n^{2\beta-\gamma}$.

□

Recall that we defined for $z \in \mathcal{H}$ and $A \subset \mathcal{H}$ a discrete hull,

$$\mathrm{Es}_A(z) = \mathbb{P}^z \left\{ \tilde{S}[1,\infty) \cap A = \emptyset \right\}.$$

Recall also that we defined for $r \in \mathbb{N}$, $l_r = \{z \in \mathbb{Z}^2 : \mathrm{Im}(z) = r\}$ and $\sigma_r = \inf\{n \geq 1 : S_n \in l_r\}$. For the next lemma, we will simplify the notation and write, for $\alpha \geq 0$,

$$l_\alpha = \{z \in \mathbb{Z}^2 : \mathrm{Im}(z) = [n^\alpha]\} \text{ and } \sigma_\alpha = \inf\{n \geq 1 : S_n \in l_\alpha\}.$$

The corresponding quantities for Brownian motion are

$$L_\alpha = \{z \in \mathbb{C} : \mathrm{Im}(z) = [n^\alpha]\} \text{ and } \Sigma_\alpha = \inf\{t \geq 0 : \mathrm{Im}(B_t) \in L_\alpha\}.$$

If random walk excursion started at $w \in A$ escapes from A, it does so in two steps. First it gets to b_γ without hitting A. Then it goes to infinity without hitting A. We give a precise estimate on the probability of the second step here.

Lemma 4.7.3. For $z \in h_\gamma^c$,

$$\mathrm{Es}_A(z) = \frac{i_A(z)}{\mathrm{Im}(z)}(1 + \mathcal{O}\left(n^{2\beta-2}\right)).$$

Proof. We start by computing $\mathbb{P}^z\left\{\Sigma_1 < Z_A\right\}$ and will then show that this is very close to $\mathbb{P}^z\left\{\sigma_1 < \zeta_A\right\}$.

By conformal invariance of Brownian motion, Lemma 4.7.2 with the value $\gamma = 1$ and the gambler's Ruin estimate (see Section A.2),

$$\mathbb{P}^z\left\{\Sigma_1 < Z_A\right\} = \mathbb{P}^{g_A(z)}\left\{T_{g_A(L_1)} < T_{\mathbb{R}}\right\} = \frac{i_A(z)}{n}(1 + \mathcal{O}\left(n^{2\beta-2}\right)).$$

We use the KMT approximation to couple a random walk with a Brownian motion so that before they reach height n or 0, they are within $c \log n$ of each other, up to a probability of less than n^{-10}.

We introduce the stopping times

$$M = \inf\{t \geq 0 : d(B_t, A \cup \mathbb{R}) \leq c \log n\}, \ N = \inf\{t \geq 0 : \mathrm{Im}(B_t) = n - c \log n\},$$

where the constant c is that of the coupling, and the corresponding stopping times μ, ν for random walk.

$$\begin{aligned} &\mathbb{P}^z \left\{ (\sigma_1 < \zeta_A; \Sigma_1 > Z_A) \cup (\sigma_1 > \zeta_A; \Sigma_1 < Z_A) \right\} \\ &\leq \mathbb{P}^z \left\{ N < Z_A < \Sigma_1 \right\} + \mathbb{P}^z \left\{ M < \Sigma_1 < Z_A \right\} \\ &+ \mathbb{P}^z \left\{ \nu < \zeta_A < \sigma_1 \right\} + \mathbb{P}^z \left\{ \mu < \sigma_1 < \zeta_A \right\} + K n^{-10}. \end{aligned}$$

By the strong Markov property and the gambler's ruin estimate, the first and third probabilities are bounded above by

$$\frac{n - (n - c \log n)}{n - K n^{\gamma}} \leq K \frac{\log n}{n}.$$

If $d(z, A \cup \mathbb{R}) \leq c \log n$ and if we let $\lambda = \frac{1+\beta}{2}$, the gambler's ruin and Beurling estimates give

$$\begin{aligned} \mathbb{P}^z \left\{ \Sigma_1 < Z_A \right\} &\leq \mathbb{P}^z \left\{ \Sigma_\lambda < Z_A \right\} \mathbb{P}^{S(\Sigma_\lambda)} \left\{ \Sigma_1 < \Sigma_\beta \right\} \\ &\leq K \left(\frac{\log n}{n} \right)^{1/2} \frac{n^\lambda - n^\beta}{n - n^\beta} \leq K n^{-\frac{1}{2}}. \end{aligned}$$

The exact same argument works for random walk too. This shows that

$$\mathbb{P}^z \left\{ \sigma_1 < \zeta_A \right\} = \mathbb{P}^z \left\{ \Sigma_1 < Z_A \right\} + \mathcal{O}\left(n^{-1/2} \right)$$

and so

$$\mathbb{P}^z \left\{ \sigma_1 < \zeta_A \right\} = \frac{i_A(z)}{n} (1 + \mathcal{O}\left(n^{2\beta - 2} \right)).$$

From this we easily find, for $R \geq n$,

$$\mathbb{P}^z \left\{ \sigma_R < \zeta_A \right\} = \frac{i_A(z)}{R} (1 + \mathcal{O}\left(n^{2\beta - 2} \right)).$$

Using (4.10) we can translate this equation for simple random walk into an equation for simple random walk excursion:

$$\mathbb{P}^z\left\{\tilde{\sigma}_R < \tilde{\zeta}_A\right\} = \frac{i_A(z)}{R}\frac{R}{\mathrm{Im}(z)}(1+\mathcal{O}\left(n^{2\beta-2}\right)) = \frac{i_A(z)}{\mathrm{Im}(z)}(1+\mathcal{O}\left(n^{2\beta-2}\right)).$$

□

Our last preparatory lemma is just a continuous Green's function computation. Recall that for $z, w \in \mathbb{H}$, we chose the notation $g(z,w)$ for the Green's function in $\mathbb{H}$.

Lemma 4.7.4. Suppose $x, y \in \mathbb{R}$ satisfy $|x-y| \geq \frac{n}{2}$. Then,

$$g(x+in^\gamma, y+in^\gamma) = 2\frac{n^{2\gamma}}{(y-x)^2}\left(1+\mathcal{O}\left(n^{2\gamma-2}\right)\right).$$

Proof. One can easily verify that

$$g(x+in^\gamma, y+in^\gamma) = g(in^\gamma, y-x+in^\gamma).$$

Since we know that $f(z) = \frac{z-i}{z+i}$ maps the upper half-plane $\mathbb{H}$ onto the unit disk $\mathbb{U}$, we can reduce the question to one about the Green's function in the disk, $g_{\mathbb{U}}$, for which the solution is well known (see for instance [3]).

$$g(in^\gamma, y-x+in^\gamma) = g_{\mathbb{U}}\left(\frac{n^\gamma-1}{n^\gamma+1}, \frac{(y-x)+i(n^\gamma-1)}{(y-x)+i(n^\gamma+1)}\right) = \log\left|\frac{uv-1}{u-v}\right|,$$

where $u = \frac{n^\gamma-1}{n^\gamma+1} = 1-2\frac{i}{n^\gamma+1}$, $v = \frac{(y-x)+i(n^\gamma-1)}{(y-x)+i(n^\gamma+1)} = 1-2\frac{i}{(y-x)+i(n^\gamma+1)}$.To evaluate this, we need precise estimates for $|uv-1|$ and $|u-v|$.

$$\begin{aligned}uv-1 &= -\frac{2}{n^\gamma+1} - 2\frac{n^\gamma+1}{(y-x)^2+(n^{\gamma+1})^2} + \frac{8}{(y-x)^2} \\ &\qquad + (-2\frac{y-x}{(y-x)^2+(n^\gamma+1)^2} + \frac{8}{(n^\gamma+1)(y-x)})i \\ &= -\frac{2}{n^\gamma+1}\left(1+\frac{(n^\gamma+1)^2}{(y-x)^2+(n^\gamma+1)^2} - \frac{4(n^\gamma+1)}{(y-x)^2}\right) \\ &\qquad - 2\frac{y-x}{(y-x)^2+(n^\gamma+1)^2}\left(1-4\frac{(y-x)^2+(n^\gamma+1)^2}{(y-x)^2(n^\gamma+1)}\right)i.\end{aligned}$$

Also,

$$u - v = \frac{2}{n^\gamma + 1}\left(1 - \frac{(n^\gamma+1)^2}{(y-x)^2 + (n^\gamma+1)^2}\right) - 2\frac{y-x}{(y-x)^2+(n^\gamma+1)^2}.$$

This gives

$$|uv-1|^2 = \tfrac{4}{(n^\gamma+1)^2}\left(1 + \tfrac{2(n^\gamma+1)^2}{(y-x)^2+(n^\gamma+1)^2} + \tfrac{(y-x)^2(n^\gamma+1)^2}{((y-x)^2+(n^\gamma+1)^2)^2} + \mathcal{O}\left(n^{\gamma-2}\right)\right)$$

and

$$|u-v|^2 = \tfrac{4}{(n^\gamma+1)^2}\left(1 - \tfrac{2(n^\gamma+1)^2}{(y-x)^2+(n^\gamma+1)^2} + \tfrac{(y-x)^2(n^\gamma+1)^2}{((y-x)^2+(n^\gamma+1)^2)} + \mathcal{O}\left(n^{4\gamma-4}\right)\right),$$

which, with the help of Taylor's expansion for $\log(1+x)$, gives

$$g(x+in^\gamma, y+in^\gamma) = 2\frac{n^{2\gamma}}{(y-x)^2}\left(1+\mathcal{O}\left(n^{2\gamma-2}\right)\right).$$

□

Since the continuous Green's function has the nice property of conformal invariance which the discrete Green's function does not possess, we will find good use in a result from [13] which says that the discrete Green's function is close to the continuous Green's function when both points are away from the boundary. This will allow us to reduce a discrete Green's function in $\mathcal{H}_A$ to a continuous Green's function in $\mathbb{H}$, which we just handled in Lemma 4.7.4. The result from [13] is the following:

Proposition 4.7.5. Suppose a set $E \subset \mathbb{Z}^2$ satisfies $\min_{z\in\partial E}|z| \in [n, 2n]$. Then for $x \in E$,

$$G_E(0,x) = \frac{2}{\pi} g_E(0,x) + \mathcal{O}\left(|x|^{-\frac{3}{2}}\right) + \mathcal{O}\left(n^{-\frac{1}{3}}\log(n)\right).$$

Note that this result does not include the sets $\mathcal{H}_A$, since their continuous version is constructed in a slightly different way. However, the result still holds in that case.

If we assume for $u, v \in \mathcal{H}_A$ that $|u-v| \geq \frac{n}{2}$ and $u, v \in b_\gamma$, we get from this, Lemma 4.7.1, and conformal invariance of the Green's function that

$$G_A(u,v) = \frac{2}{\pi} g_A(u,v) + \mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right) = \frac{2}{\pi} g(g_A(u), g_A(v)) + \mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right), \quad (4.33)$$

Proposition 4.7.6. If $0 < \beta < \gamma < 1$, A is a hull with radius $R(A) \leq n^\beta$, $w \in \partial A \cap \mathcal{H}$, and $X = r_A(w)$, then

$$\mathbb{P}^w \{S(\zeta_A) = -n\} = \frac{1}{\pi} \mathrm{Es}_A(w) \mathrm{Im}(w) \frac{1}{|X+n|^2} (1 + \mathcal{O}(\phi(n))),$$

where $\phi(n) = n^{2\beta - 2\gamma} + n^{\gamma - 1} \log^2 n + n^{\frac{6-7\gamma}{3}}$.

Proof. A last-exit decomposition gives

$$\begin{aligned} \mathbb{P}^w \{S(\zeta_A) = -n\} &= \sum_{u,v \in b_\gamma} \mathbb{P}^w \left\{S_{\tau_\gamma} = u; \tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{S_{\tau_\gamma} = v; \tau_\gamma < \zeta_A\right\} G_A(u,v) \\ &+ \mathbb{P}^w \{S_{\zeta_A} = -n; \zeta_A < \tau_\gamma\}. \end{aligned}$$

Also, if we let $\xi = \inf\{n \geq 1 : |S_n| \geq n^{\frac{\gamma+\beta}{2}}\}$, we see that

$$\begin{aligned} &\frac{\mathbb{P}^w \{S(\zeta_A) = -n; \zeta_A < \tau_\gamma\}}{\mathbb{P}^w \{S(\zeta_A) = -n; \tau_\gamma < \zeta_A\}} \\ &\leq \frac{\mathbb{P}^w \{\xi < \zeta_A\} \sup_z \mathbb{P}^z \{S(\zeta_A) = -n; \zeta_A < \tau_\gamma\}}{\mathbb{P}^w \{\xi < \zeta_A\} \inf_{z'} \mathbb{P}^{z'} \{S(\zeta_A) = -n; \tau_\gamma < \zeta_A\}} \\ &= \frac{\sup_z \mathbb{P}^z \{S(\zeta_A) = -n; \zeta_A < \tau_\gamma\}}{\inf_{z'} \mathbb{P}^{z'} \{S(\zeta_A) = -n; \tau_\gamma < \zeta_A\}}, \end{aligned}$$

where the sup and inf are both over $\{z \in \mathcal{H} : n^{\frac{\gamma+\beta}{2}} \leq |z| < n^{\frac{\gamma+\beta}{2}} + 1\}$. One can check easily that the supremum is bounded above by $K_1 \exp\{-K_2 n^{1-\gamma}\}$ for some positive constants K_1 and K_2 and that the infimum is greater than $K_3 n^{-2}$ for some positive constant K_3. Therefore,

$$\begin{aligned} \mathbb{P}^w \{S(\zeta_A) = -n\} &= \sum_{u,v \in b_\gamma} \mathbb{P}^w \left\{S_{\tau_\gamma} = u; \tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{S_{\tau_\gamma} = v; \tau_\gamma < \zeta_A\right\} G_A(u,v) \\ &\cdot (1 + \mathcal{O}(\exp\{-K n^{1-\gamma}\})). \end{aligned} \quad (4.34)$$

This, together with (4.33) and Corollary 4.6.6, gives

$$\mathbb{P}^w\{S(\zeta_A) = -n\}$$

$$= \sum \mathbb{P}^w \left\{S_{\tau_\gamma} = u; \tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{S_{\tau_\gamma} = v; \tau_\gamma < \zeta_A\right\}(1 + \mathcal{O}\left(e^{-\beta r}\right))$$

$$\cdot \left[\frac{2}{\pi} g(g_A(u), g_A(v)) + \mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right)\right] (1 + \mathcal{O}\left(\exp\{-Kn^{1-\gamma}\}\right)),$$

where the sum is over all $u, v \in b_\gamma$ such that $|r_A(u) - r_A(w)| \leq rn^\gamma$ and $|r_A(v) - r_A(-n)| \leq rn^\gamma$. Letting $r = \log^2 n$, we get from Corollary 4.6.6 (for the first equality) and Lemmas 4.7.2 and 4.7.4 (for the second),

$$\mathbb{P}^w\{S(\zeta_A) = -n\}$$

$$= \mathbb{P}^w \left\{\tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{\tau_\gamma < \zeta_A\right\} (1 + \mathcal{O}\left(\exp\{-kn^{1-\gamma}\}\right))(1 + \mathcal{O}\left(n^{-\beta \log n}\right))$$

$$\cdot \left[\frac{2}{\pi} g(r_A(w) + \mathcal{O}\left(n^\gamma \log^2 n\right) + i_A(u)i, r_A(-n) + i_A(v)i) + \mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right)\right]$$

$$= \mathbb{P}^w \left\{\tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{\tau_\gamma < \zeta_A\right\} (1 + \mathcal{O}\left(\exp\{-kn^{1-\gamma}\}\right))(1 + \mathcal{O}\left(n^{-\beta \log n}\right))$$

$$\cdot \left[\frac{2}{\pi} g\left(r_A(w) - r_A(-n) + \mathcal{O}\left(n^\gamma \log^2 n\right) + i(n^\gamma + \mathcal{O}\left(n^{2\beta-\gamma}\right)), i(n^\gamma + \mathcal{O}\left(n^{2\beta-\gamma}\right))\right)\right.$$

$$\left. + \mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right)\right].$$

Proposition 3.2.5 shows that $r_A(-n) = -n(1 + \mathcal{O}\left(n^{2\beta-2}\right))$. If we let $X = r_A(w)$ and note that $\exp\{-kn^{1-\gamma}\} = o(n^{-\beta \log n})$, Lemma 4.7.4 gives

$$\mathbb{P}^w\{S(\zeta_A) = -n\}$$

$$= \mathbb{P}^w \left\{\tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{\tau_\gamma < \zeta_A\right\} (1 + \mathcal{O}\left(n^{-\beta \log n}\right))$$

$$\cdot \left[\mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right) + \frac{4}{\pi} \frac{n^{2\gamma}(1 + \mathcal{O}\left(n^{2\beta-2\gamma}\right))}{|X + n + \mathcal{O}\left(n^\gamma \log^2 n\right)|^2} \cdot \frac{1 + \mathcal{O}\left(n^{2\gamma-2}\right)}{1 + \mathcal{O}\left(n^{2\beta-2}\right)}\right]$$

$$= \mathbb{P}^w \left\{\tau_\gamma < \zeta_A\right\} \mathbb{P}^{-n} \left\{\tau_\gamma < \zeta_A\right\} (1 + \mathcal{O}\left(n^{-\beta \log n}\right))$$

$$\cdot \left[\mathcal{O}\left(n^{-\frac{\gamma}{3}} \log n\right) + \frac{4}{\pi} \frac{n^{2\gamma}}{|X + n|^2}(1 + \mathcal{O}\left(n^{2\beta-2\gamma}\right))(1 + \mathcal{O}\left(n^{\gamma-1} \log^2 n\right))\right]$$

$$= \mathbb{P}^w\left\{\tau_\gamma < \zeta_A\right\}\mathbb{P}^{-n}\left\{\tau_\gamma < \zeta_A\right\}(1 + \mathcal{O}\left(n^{-\beta\log n}\right))$$

$$\cdot \left[\frac{4}{\pi}\frac{n^{2\gamma}}{|X+n|^2}(1+\mathcal{O}\left(n^{\frac{6-7\gamma}{3}}\right))(1+\mathcal{O}\left(n^{\gamma-1}\log^2 n\right))\right]$$

$$= \mathbb{P}^w\left\{\tau_\gamma < \zeta_A\right\}\mathbb{P}^{-n}\left\{\tau_\gamma < \zeta_A\right\}\frac{4}{\pi}\frac{n^{2\gamma}}{|X+n|^2}(1+\mathcal{O}\left(n^{\frac{6-7\gamma}{3}}\right))(1+\mathcal{O}\left(n^{\gamma-1}\log^2 n\right)).$$

We also have

$$\begin{aligned}
\mathrm{Es}_A(w) &= \mathbb{P}^w\left\{\tilde{S}[1,\infty)\cap A = \emptyset\right\} \\
&= \sum_{x\in b_\gamma}\mathbb{P}^w\left\{S[1,\tau_\gamma]\cap(A\cup\mathbb{Z}) = \emptyset; S(\tau_\gamma) = x\right\}\frac{\mathrm{Im}(x)}{\mathrm{Im}(w)}\mathbb{P}^x\left\{\tilde{S}[1,\infty)\cap A = \emptyset\right\} \\
&= \sum_{x\in b_\gamma}\mathbb{P}^w\left\{S[1,\tau_\gamma]\cap(A\cup\mathbb{Z}) = \emptyset; S(\tau_\gamma) = x\right\}\frac{\mathrm{Im}(x)}{\mathrm{Im}(w)}\frac{i_A(x)}{\mathrm{Im}(x)}(1+\mathcal{O}\left(n^{2\beta-2}\right)) \\
&= \mathbb{P}^w\left\{\tau_\gamma < \zeta_A\right\}\frac{1}{\mathrm{Im}(w)}n^\gamma(1+\mathcal{O}\left(n^{2\beta-2\gamma}\right))(1+\mathcal{O}\left(n^{2\beta-2}\right)).
\end{aligned}$$

Here, we used (4.10) for the second equality, Lemma 4.7.3 for the third, and Lemma 4.7.2 for the last. This implies that

$$\mathbb{P}^w\left\{\tau_\gamma < \zeta_A\right\} = \frac{\mathrm{Es}_A(w)\mathrm{Im}(w)}{n^\gamma}(1+\mathcal{O}\left(n^{2\beta-2\gamma}\right)).$$

Also, by Lemma 4.7.2 and the gambler's ruin estimate (see Section A.2),

$$\mathbb{P}^{-n}\left\{\tau_\gamma < \zeta_A\right\} = \frac{1}{4}\frac{1}{n^\gamma}(1+\mathcal{O}\left(n^{2\beta-2\gamma}\right)).$$

Therefore,

$$\mathbb{P}^w\left\{S(\zeta_A) = -n\right\} = \frac{1}{\pi}\mathrm{Es}_A(w)\mathrm{Im}(w)\frac{1}{|X+n|^2}(1+\mathcal{O}\left(\phi(n)\right)),$$

where $\phi(n) = n^{2\beta-2\gamma} + n^{\gamma-1}\log^2 n + n^{\frac{6-7\gamma}{3}}$.

$\square$

Appendix A

Standard Estimates for Brownian Motion and Random Walk

This appendix will be devoted to state and prove some elementary results on random walk and Brownian motion which are needed in this thesis. All of them are rather standard. They include large deviations estimates for random walk and Brownian motion, the gambler's ruin formula, Beurling estimates, and the Harnack principle. Although the Harnack principle is formulated in terms of harmonic functions, the relationship between harmonic functions and hitting distributions makes the Harnack principle a statement about the latter as well.

A.1 Large deviations

It is easy to check that in time n, Brownian motion and random walk are expected to reach a distance of roughly $\sqrt{n}$. In this section we show how likely it is that they travel far (i.e. reach a distance which is much greater than $\sqrt{n}$) or are very constrained (i.e. remain in a disk which has a radius much smaller than $\sqrt{n}$). We only give upper bounds for these probabilities, except in the case of Brownian motion travelling too far, where the lower bound is straightforward enough for us to write it here as well at no cost.

Lemma A.1.1. If B is a planar Brownian motion and $r \geq 1$,

$$\mathbb{P}\left\{\sup_{0\leq t\leq n}|B_t| \geq rn^{1/2}\right\} \asymp r^{-1}\exp\{-\frac{r^2}{2}\}.$$

Proof. If we write $B_t = (B_t^1, B_t^2)$, simple geometric considerations, Brownian scaling, and the reflection principle give

$$\begin{aligned}
&\mathbb{P}\left\{B_1^1 \geq r\right\} \\
&= \mathbb{P}\{B_n^1 \geq rn^{1/2}\} \\
&\leq \mathbb{P}\left\{\sup_{0\leq t\leq n}|B_t| \geq rn^{1/2}\right\} \\
&\leq 2\mathbb{P}\left\{\sup_{0\leq t\leq n}|B_t^1| \geq rn^{1/2}\right\} \\
&\leq 4\mathbb{P}\left\{\sup_{0\leq t\leq n}B_t^1 \geq rn^{1/2}\right\} \\
&= 2\mathbb{P}\left\{B_n^1 \geq rn^{1/2}\right\} \\
&= 2\mathbb{P}\left\{B_1^1 \geq r\right\}.
\end{aligned}$$

If $\phi(x)$ is the normal density and $r > 0$,

$$\frac{1}{r+r^{-1}}\phi(r) \leq \mathbb{P}\left\{B_1^1 \geq r\right\} \leq \frac{1}{r}\phi(r),$$

from which the result follows immediately

□

Remark 10. The following is an equivalent formulation of the lemma:

$$\mathbb{P}\left\{\xi_{rn^{1/2}} \leq n\right\} \asymp r^{-1}\exp\{-\frac{r^2}{2}\}.$$

We have evaluated the probability that Brownian motion goes farther than expected. We now look at the probability that it moves less than expected.

Lemma A.1.2. There exists a constant $K > 0$ such that for all $n \geq 1, r \geq 2$,

$$\mathbb{P}\left\{\sup_{0\leq t\leq n}|B_t| \leq r^{-1}n^{1/2}\right\} \leq \exp\{-Kr^2\}.$$

Proof. For $l \geq 1$, we define

$$I_l = I_l(r, n) = [(l-1)\frac{n}{r^2}, l \cdot \frac{n}{r^2}].$$

Then, if $k = [r^2]$,

$$\cup_{l=1}^{k} I_l \subset [0, n].$$

A simple geometric argument, the Markov property, and Brownian scaling give:

$$\begin{aligned}
\mathbb{P}\left\{\sup_{0\leq t\leq n} |B_t| \leq r^{-1}n^{1/2}\right\} &\leq \prod_{l=1}^{k} \mathbb{P}\left\{\sup_{t\in I_l} |B_t - B_{(l-1)\frac{n}{r^2}}| \leq 2r^{-1}n^{1/2}\right\} \\
&\leq \prod_{l=1}^{k} \mathbb{P}^0\left\{\sup_{0\leq t\leq 1} |B_t| \leq 2\right\} \\
&\leq \rho^k \leq \exp\{-Kr^2\},
\end{aligned}$$

where $\rho < 1$ is independent of r and n, and $K = -\frac{\ln \rho}{2} > 0$.

□

We need the same results for random walk. The nice scaling properties of Brownian motion cannot be used here and we must do a little bit more work. We first prove the result for random walk under slightly milder assumptions than needed for simple random walk. These are still quite restrictive, but will be sufficient for our needs.

Lemma A.1.3. Suppose $(X_i)_{i\geq 1}$ are independent random variables with mean 0 such that for some $a > 0$, the moment generating function

$$M(t) = \mathbb{E}\left[e^{tX_1}\right] < \infty \text{ for } |t| \leq a. \tag{A.1}$$

Then there exists a constant K_1 depending on a and the distribution of X_1 such that if $S_n = \sum_{i=1}^{n} X_i$, for all $n \geq 1, r > 0$,

$$\mathbb{P}\left\{|S_n| \geq r\sqrt{n}\right\} \leq K_1 e^{-ar}.$$

Proof. Choose a distribution X_1 and an a for which (A.1) holds. It suffices to show that there is a $K_1 > 0$ such that $\mathbb{P}\left\{S_n \geq r\sqrt{n}\right\} \leq K_1 e^{-ar}$. By Chebyshev's inequality,

$$\mathbb{P}\left\{\frac{S_n}{\sqrt{n}} \geq r\right\} \leq \frac{\mathbb{E}\left[\exp\{aS_n/\sqrt{n}\}\right]}{e^{ar}}.$$

But by expanding $\mathbb{E}\left[e^{tX_1}\right]$ about 0, we get for $|t| \leq a$,

$$M(t) = 1 + \frac{\mathbb{E}\left[X_1^2\right]}{2}t^2 + \mathcal{O}\left(t^3\right),$$

so that we can find a constant K such that for all $n \geq 1$,

$$M(\frac{a}{\sqrt{n}}) \leq 1 + \frac{K}{n}.$$

Thus,

$$\mathbb{E}\left[\exp\{aS_n/\sqrt{n}\}\right] = M(\frac{a}{\sqrt{n}})^n \leq (1 + \frac{K}{n})^n \leq K_1,$$

where $K_1 = e^K$. This implies that

$$\mathbb{P}\left\{\frac{S_n}{\sqrt{n}} \geq r\right\} \leq K_1 e^{-ar}.$$

□

The next Corollary is an immediate consequence. We point out that it is not optimal in the case of simple random walk, but as it will be sufficient for the purpose of this thesis, we content ourselves with it.

Corollary A.1.4. Under the hypotheses of Lemma A.1.3, there exists a constant $C_1 > 0$ such that for all $n \geq 1, r > 0$,

$$\mathbb{P}\left\{\max_{1\leq k\leq n} |S_k| \geq r\sqrt{n}\right\} \leq C_1 n e^{-ar}.$$

The following particular case is of special interest:

Corollary A.1.5. For one-dimensional standard Brownian motion started at 0, define $T_1 = \inf\{t \geq 0 : |B_t| = 1$ and for $j \geq 2, T_j = \inf\{t \geq T_{j-1} : |B_t - B_{T_{j-1}}| = 1\}$. Then there exist constants $C_1, a > 0$ such that for all $n \geq 1, r > 0$,

$$\mathbb{P}\left\{\max_{1\leq k\leq n} |T_k - k| \geq r\sqrt{n}\right\} \leq C_1 n e^{-ar}.$$

Proof. The fact that $\mathbb{E}[T_1] = 1$ is well known and since for all $j \geq 1, T_j - T_{j-1} \stackrel{\mathcal{D}}{=} T_1$, it suffices to show that there is an $a > 0$ such that $\mathbb{E}\left[e^{a(T_1-1)}\right] < \infty$. It follows from

$$\mathbb{P}\{T_1 \geq k+1 | T_1 \geq k\} \leq \mathbb{P}\{|B_{k+1} - B_k| \leq 2\} = \rho < 1$$

that $\mathbb{P}\{T_1 \geq k\} \leq \rho^k$, and it suffices to choose $a < \ln(\rho^{-1})$ to ensure that $\mathbb{E}\left[e^{aT_1}\right] < \infty$. □

We now complete the picture by giving the analogue to Lemma A.1.2 for planar simple random walk.

Lemma A.1.6. There exists a constant $K > 0$ such that for all $n \geq 1, r \geq 2$,

$$\mathbb{P}\left\{\max_{0\leq k\leq n} |S_k| \leq r^{-1} n^{1/2}\right\} \leq \exp\{-Kr^2\}.$$

A.2 Gambler's ruin

It is often useful to know what the chance is that Brownian motion or random walk move much more in one direction than in the opposite. The following facts are well known and their proofs can be found in any basic book on stochastic processes:

Let r_1, r_2 be positive reals and n_1, n_2 be positive integers. Define

$$\sigma = \inf\{k \geq 0 : \mathrm{Im}(S_k) = n_1 \text{ or } -n_2\}$$

and

$$\Sigma = \inf\{t \geq 0 : \mathrm{Im}(B_t) = r_1 \text{ or } -r_2\}.$$

Lemma A.2.1. If $z \in \mathbb{Z}^2$ satisfies $\mathrm{Im}(z) = 0$. Then

$$(a)\mathbb{P}^z\left\{\mathrm{Im}(S(\sigma)) = n_1\right\} = \frac{n_2}{n_1 + n_2} \text{ and } \mathbb{P}^z\left\{\mathrm{Im}(S(\sigma)) = -n_2\right\} = \frac{n_1}{n_1 + n_2}.$$

$$(b)\mathbb{P}^z\left\{\mathrm{Im}(B(\Sigma)) = r_1\right\} = \frac{r_2}{r_1 + r_2} \text{ and } \mathbb{P}^z\left\{\mathrm{Im}(B(\Sigma)) = -r_2\right\} = \frac{r_1}{r_1 + r_2}.$$

A.3 Beurling estimates

It is often useful to know how likely it is for Brownian motion to get to distance R without hitting a set A with $d(B_0, A) = r$ and $\mathrm{rad}(A) \geq 2R$, when $\frac{r}{R}$ gets small. The probability of this event can be bounded above by a power function of the ratio, uniformly for all sets A. The same question for random walk is of course of equal interest. Given the Beurling Projection Theorem which we state below, it is easy to find the best possible exponent of this power function so that we do it in this section. The discrete case is more difficult and we just refer the reader to [10], where the proof is given.

The first result of this section is the Beurling Projection Theorem. It says that among all connected sets of a given radius, that which Brownian motion will most likely avoid is a straight line. Consider a set $E \subset R\mathbb{D} = \{z \in \mathbb{C} : |z| \leq R\}$. The circular projection of E is $\gamma(E) = \{|z| : z \in E\}$.

Theorem A.3.1 (Beurling Projection Theorem).

$$\mathbb{P}^{-1}\left\{\Xi_R < T_E\right\} \leq \mathbb{P}^{-1}\left\{\Xi_R < T_{\gamma(E)}\right\}.$$

For a proof, see [3] or [1].

We will be interested in the case where E satisfies $\gamma(E) = [0, R]$. Now that we have Theorem A.3.1, we know that finding an upper bound for $\mathbb{P}^{-1}\left\{\Xi_R < T_{[0,R]}\right\}$ also

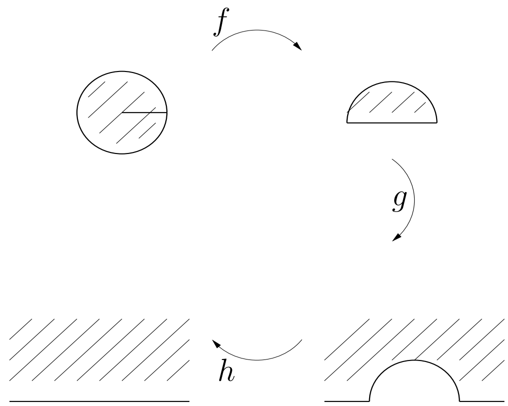

Figure A.1: The sequence of conformal transformations leading to the Beurling estimate.

provides an upper bound for $\mathbb{P}^{-1}\{\Xi_R < T_E\}$ for all sets $E \subset R\mathbb{D}$ with $\gamma(E) = [0, R]$. We can compute such a bound via a sequence of conformal maps, the fact that the exit distribution of the upper half-plane is a Cauchy distribution, and the fact that harmonic measure is conformally invariant (see [3] for a proof of this). It turns out that the bound we find is optimal up to a multiplicative constant.

Consider the following domains, where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$: the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, the slit unit disk $\mathbb{U}_s = \mathbb{U} \setminus \{z \in \mathbb{C} : 0 \leq \text{Re}(z) < 1; \text{Im}(z) = 0\}$, the upper half-disk $\mathbb{U}_u = \mathbb{U} \cap \mathbb{H}$, and the complement of the closed upper half-disk $\mathbb{H}_{\mathbb{U}} = \mathbb{H} \cap \{z \in \mathbb{C} : |z| > 1\}$.

We have the following conformal transformations (surjective conformal maps):

$$\mathbb{U}_s \xrightarrow{f} \mathbb{U}_u \xrightarrow{g} \mathbb{H}_{\mathbb{U}} \xrightarrow{h} \mathbb{H},$$

where $f(z) = \sqrt{z}, g(z) = -\frac{1}{z}, h(z) = z + \frac{1}{z}$. Then $h \circ g \circ f(-\epsilon) = (\frac{1}{\sqrt{\epsilon}} - \sqrt{\epsilon})i$ and $h \circ g \circ f(\mathbb{U}) = [-2, 2]$.

Conformal invariance of Brownian motion implies that

$$\mathbb{P}^{\epsilon}\{B(T_{\partial \mathbb{U}_s}) \in \mathbb{U}\} = \mathbb{P}^{(\frac{1}{\sqrt{\epsilon}} - \sqrt{\epsilon})i}\{B(T_{\mathbb{R}}) \in [-2, 2]\}.$$

Using the fact that the exit distribution of the upper half-plane is a Cauchy distribution and the Beurling Projection Theorem, Brownian scaling gives the following:

Theorem A.3.2 (Continuous Beurling Estimate). There exists a constant $K > 0$ such that for any $R \geq 1$, any x with $|x| \leq R$, any set A with $[0, R] \subset \gamma(E)$,

$$\mathbb{P}^x \{\xi_R \leq T_A\} \leq K \left(\frac{|x|}{R}\right)^{1/2}.$$

As we pointed out earlier, showing that this result is true for random walk as well is not as easy, mainly because none of the conformal invariance techniques are available. In [10], Kesten first showed that the Beurling estimate holds in the discrete case as well. We state the the theorem here without a proof.

Theorem A.3.3 (Discrete Beurling Estimate). Let $R \in \mathbb{R}$ be positive and define $\mathcal{A}_R$ to be the set of subsets of $\mathbb{Z}^2$ for which $\sup\{|x| : x \in A\} = R$. Let $\tau_A = \inf\{n \geq 1 : S_n \in A\}$ and $\xi_R = \inf\{k \geq 0 : |S_k| \geq R\}$. Then there exists a constant $K > 0$ such that if $A \in \mathcal{A}_R$, $|x| < R$,

$$\mathbb{P}^x \{\tau_A > \xi_R\} \leq K \left(\frac{|x|}{R}\right)^{1/2}.$$

Note that although the exponents are the same in the continuous and discrete case, it is not clear that in a given discrete disk, a straight line is the easiest set to avoid for random walk.

A.4 Harnack inequalities

Harmonic functions play a fundamental role in analysis and have a strong connection to exiting probabilities for random walk and Brownian motion. Indeed, finding a harmonic function with specific boundary conditions (in other words, solving a Dirichlet problem) is equivalent to computing exiting probabilities. This is true in the continuous setting as well as in the discrete.

A.4.1 The continuous case

Recall that a function $h : \mathbb{R}^2 \to \mathbb{R}$ is C^2 if its second order partial derivatives exist. The **Laplacian** of a C^2 function $h : \mathbb{R}^2 \to \mathbb{R}$ is the operator Δ defined by

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}.$$

If D is a domain and $h : D \to \mathbb{R}$ is C^2, we say that h is harmonic on D if $\Delta h \equiv 0$ in D.

The following is an important but well-know result about nonnegative harmonic functions. Recall that for $a \in \mathbb{R}_+$, $D(a)$ is the disk of radius a, centered at the origin.

Theorem A.4.1. Suppose $r < R$ are positive reals. There exists a constant c such that if u is nonnegative and harmonic in $D(R)$ and $x, y \in D(r)$, then

$$u(x) \leq cu(y).$$

For a proof, see [3].

A.4.2 The discrete case

If $f : \mathbb{Z}^2 \to \mathbb{R}$, the discrete Laplacian of f at $x \in \mathbb{Z}^2$ is defined by

$$\Delta f(x) = \frac{1}{4} \sum (f(x') - f(x)),$$

where the sum is over $\{x' \in \mathbb{Z}^2 : |x' - x| = 1\}$. If $A \subset \mathbb{Z}^2$, we say that f is **discrete harmonic** on A if for each $x \in A, \Delta f(x) = 0$.

Let $C_n = \{x \in \mathbb{Z}^2 : |x| < n\}$ and recall that $\bar{C}_n = \{z \in \mathbb{Z}^2 : d(z, C_n) \leq 1\}$. Then the following theorem, which can be found with its proof in [14], is the discrete analogue of Theorem A.4.1:

Theorem A.4.2. For every $r < 1$, there exists a constant c_r such that if $f : \bar{C}_n \to [0, \infty)$ is harmonic on C_n and x_1, x_2 satisfy $|x_1| \leq rn, |x_2| \leq rn$, then

$$f(x_1) \leq c_r f(x_2).$$

Appendix B

Less Standard Estimates for Random Walk

B.1 Introduction

The Laplacian Δ of a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}(x, y).$$

If in a domain $D, \Delta f \equiv 0$, we say that f is harmonic in D. Similarly, recall that we defined in Appendix A for any function $f : \mathbb{Z}^2 \to \mathbb{R}$ the discrete Laplacian

$$\Delta f(x, y) = \frac{1}{4} \sum (f(x', y') - f(x, y)),$$

where the sum is over $\{(x', y') : |(x', y') - (x, y)| = 1\}$. By analogy with the continuous case, we say that $f : \bar{D} \to \mathbb{R}$ is discrete harmonic in a set D if $\Delta f \equiv 0$ in D.

To solve the Dirichlet problem in a domain D with boundary condition ϕ, where $\phi : \partial D \to \mathbb{R}$ is to find a function $u : \bar{D} \to \mathbb{R}$ such that u is harmonic in D and $u \equiv \phi$ on ∂D. The discrete Dirichlet problem is defined in the natural analogous way.

The Dirichlet problem is intimately related to Brownian motion and so is the discrete Dirichlet problem to random walk. Solving the Dirichlet problem with appropriate boundary conditions is equivalent to computing the probability that Brownian motion (or random walk in the discrete case) leaves a domain at a given subset of the boundary. More precisely, if A and B are disjoint subsets of the boundary of D and $A \cup B = \partial D$, solving the Dirichlet problem with boundary value 1 on A and 0 on B is equivalent to finding the probability that Brownian motion (in the continuous case) or random walk (in the discrete case) leaves D at A. (See [8] for a discussion of this in the continuous case and [14] for the discrete case.) We will need upper bounds for various such exiting probabilities in this thesis and solve the corresponding Dirichlet problems here. We also prove a "difference estimate" which gives a bound on the difference between the exiting distribution of a domain when starting at neighboring points. While working on the problems of this thesis, we initially believed that we would need this estimate, which eventually was replaced by another. As the result is of interest by itself and is not, to our knowledge, anywhere in the litterature, we present it here as well.

B.2 Discrete Dirichlet problem in the finite and infinite rectangles

We will solve the Dirichlet problem with boundary conditions a general funtion ϕ on one side and 0 on the others. We will then use this to find bounds for the problem with specific ϕ and starting points.

We start with a trivial lemma which will be needed in our study of the discrete Dirichlet problem in some specific domains.

Lemma B.2.1. If for $1 \leq j \leq n-1$, $a_j = a_j(n)$ is defined by the equation

$$\cosh(a_j) = 2 - \cos(\frac{\pi j}{n}),$$

then

$$a_j = \frac{\pi j}{n}(1 + \mathcal{O}\left(\frac{j}{n}\right)).$$

Moreover, for any $1 \leq j \leq n-1$,

$$\frac{j}{2n} \leq a_j \leq \frac{\pi j}{n}.$$

Proof. The equality

$$\sum_{k\geq 1} \frac{a_j^{2k}}{(2k)!} = \sum_{k\geq 1} (-1)^{k+1} \frac{(\pi j/n)^{2k}}{(2k)!},$$

obtained by Taylor-expanding the equation for a_j, allows us to see that $\forall\, n > 0, \forall\, j \leq n/\pi$,

$$\frac{\pi j}{2n} \leq a_j \leq \frac{\pi j}{n}.$$

Indeed, suppose that $a_j > \frac{\pi j}{n}$. Then

$$\sum_{k\geq 1} \frac{(\pi j/n)^{2k}}{(2k)!} < \sum_{k\geq 1} (-1)^{k+1} \frac{(\pi j/n)^{2k}}{(2k)!}.$$

This is clearly impossible. Also, if we suppose that $a_j \leq \frac{\pi j}{2n}$, we get the inequality

$$0 \leq \sum_{k\geq 1} \frac{(\pi j/n)^{2k} \left((1/2)^{2k} - (-1)^{k+1}\right)}{(2k)!}.$$

The first term in this sum is $\frac{-3}{8}(\frac{\pi j}{n})^2$. The sum of the positive terms is

$$\leq \frac{17}{16}(\frac{\pi j}{n})^4 \sum_{k\geq 1} \frac{1}{(4k)!} \leq \frac{1}{10}(\frac{\pi j}{n})^4,$$

and we get a contradiction. It is also easy to see directly from the cos and cosh functions that for $n/\pi \leq j \leq n-1$, $1/2 \leq a_j \leq \pi$. This is not optimal but sufficient for our needs. The lemma now follows easily.

$\square$

We let

$$R(L,N) = \{(x,y) \in \mathbb{Z}^2 : 1 \leq x \leq L-1, 1 \leq y \leq N-1\},$$

be the discrete rectangle of "side lengths" $L-1$ and $N-1$, with boundary $\partial R(L,N)$ and closure $\bar{R}(L,N) = R(L,N) \cup \partial R(L,N)$.

Lemma B.2.2. Let $\phi : \{1,..,N-1\} \to \mathbb{R}$ be a given function. Then the unique function $f : \bar{R}(L,N) \to \mathbb{R}$ satisfying

$$\Delta f(x,y) = 0 \text{ in } R(L,N),$$

$$f(x,y) = \begin{cases} \phi(y) & \text{on } \{(L,y) : 1 \leq y \leq N-1\} \\ 0 & \text{on } \partial R(L,N) \setminus \{(L,y) : 1 \leq y \leq N-1\} \end{cases},$$

is given by

$$f(x,y) = \sum_{j=1}^{N-1} b_j(\phi) \frac{\sinh(a_j x)}{\sinh(a_j L)} \sin(\frac{\pi y j}{N}), \tag{B.1}$$

where a_j is the positive solution of

$$\cosh(a_j) = 2 - \cos(\frac{\pi j}{N}), \tag{B.2}$$

and

$$b_j(\phi) = \frac{2}{N-1} \sum_{y=1}^{N-1} \phi(y) \sin(\frac{\pi y j}{N}). \tag{B.3}$$

Proof. It suffices to check that the given function is harmonic and that it has the right values on $\partial R(L,N)$. Uniqueness follows from [14, Theorem 1.4.5].

We first check the boundary conditions. It is clear that

$$f(0,y) = f(x,0) = f(x,N) = 0 \ \ \forall\, x, y \in \mathbb{Z}.$$

Also,

$$
\begin{aligned}
f(L,y) &= \sum_{j=1}^{N-1} b_j(\phi)\sin(\frac{\pi y j}{N}) \\
&= \frac{2\phi(y)}{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\sin(\frac{\pi k j}{N})\sin(\frac{\pi y j}{N}).
\end{aligned}
$$

It is easy to see that if $1 \le y \le N-1$, then

$$
\sum_{j=1}^{N-1}\sin(\frac{\pi k j}{N})\sin(\frac{\pi y j}{N}) = \frac{N-1}{2}\delta_{y,k},
$$

from which it follows that

$$
f(L,y) = \phi(y) \ \ \forall\, y \in \{1,..,N-1\}.
$$

To see that f is harmonic in $R(L,N)$, we do a straightforward computation: Fix $(x,y) \in R(L,N)$. Then, using the fact that $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$, $\Delta f(x,y)$

$$
\begin{aligned}
&= f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \\
&= \sum_{j=1}^{N-1} b_j \left[2\sin(\frac{\pi y j}{N})(\cos(\frac{\pi j}{N}-1))\frac{\sinh(a_j x)}{\sinh(a_j L)} + 2\sin(\frac{\pi y j}{N})\sinh(a_j x)(\cosh(b_j)-1)\right] \\
&= 2\sum_{j=1}^{N-1} b_j \left[\sin(\frac{\pi y j}{N})\frac{\sinh(a_j x)}{\sinh(a_j L)}\left((\cos(\frac{\pi j}{N})-1) + (\cosh(a_j)-1)\right)\right].
\end{aligned}
$$

We see that this is 0 if $\cos(\frac{\pi j}{N}) + \cosh(a_j) = 2$.

□

We now turn to the infinite rectangle

$$
\mathcal{R}(n) = \{(x,y) \in \mathbb{Z}^2 : x \ge 1, 1 \le y \le n-1\}
$$

with boundary $\partial\mathcal{R}(n)$ and closure $\bar{\mathcal{R}}(n) = \mathcal{R}(n) \cup \partial\mathcal{R}(n)$.

Lemma B.2.3. Let $\phi : \{1, .., n-1\} \to \mathbb{R}$ be a given function. Then the unique bounded function $f(x, y) : \bar{\mathcal{R}}(n) \to \mathbb{R}$ satisfying

$$\Delta f(x, y) = 0 \text{ in } \mathcal{R}(n),$$

$$f(x, y) = \begin{cases} \phi(y) & \text{on } \{(0, y) : 1 \leq y \leq n-1\} \\ 0 & \text{on } \partial\mathcal{R}(n) \setminus \{(0, y) : 1 \leq y \leq n-1\} \end{cases}$$

is given by

$$f(x, y) = \sum_{j=1}^{n-1} b_j(\phi) \exp(-a_j x) \sin(\frac{\pi y j}{n}), \tag{B.4}$$

where a_j is the positive solution of

$$\cosh(a_j) = 2 - \cos(\frac{\pi j}{n}),$$

and

$$b_j(\phi) = \frac{2}{n-1} \sum_{y=1}^{n-1} \phi(y) \sin(\frac{\pi y j}{n}).$$

Proof. We invoke [14, Theorem 1.4.8] to show uniqueness. We can check that f has the right boundary conditions exactly as in Lemma B.2.2. Harmonicity follows from

$$\Delta f(x, y) = \sum_{j=1}^{n-1} b_j(\phi) \exp(-a_j x) \sin(\frac{\pi y j}{n}) \left(2\cos(\frac{\pi j}{n}) + 2\cosh(a_j) - 4 \right).$$

□

We now find upper bounds for solutions of the Dirichlet problem in a finite and infinite rectangle at particular points.

Lemma B.2.4. If $f(x, y)$ is the solution of the Dirichlet problem in $R([an], n)$ with $\phi(y) \equiv 1$, then there exists a positive constant K, depending on a, such that for all y and all n,

$$f(1, y) \leq C \frac{y}{n^2}.$$

Proof. This is a particular case of Lemma B.2.2. First note that a_j and b_j depend on n.

$$b_j := b_j(\phi) = \frac{2}{n-1}\sum_{k=1}^{n-1} \sin\left(\frac{\pi k j}{n}\right) \overset{n\to\infty}{\longrightarrow} 2\int_0^1 \sin(\pi j x)\, dx = \begin{cases} \frac{4}{\pi j}, & j \text{ odd} \\ 0, & j \text{ even}, \end{cases} ,$$

so that $\exists\, C > 0,$ s.t. $\forall\, n > 0,\ \forall\, j \in \{1, .., n-1\}$,

$$b_j \leq \frac{C}{j}. \tag{B.5}$$

Finding a bound for the term $\frac{\sinh(a_j x)}{\sinh(a_j [an])}$ is more delicate. We first note that for all $x \geq 0$, $\sinh(x) \geq x$, and for $x \leq 1,\ \sinh(x) \leq 2x$. We also recall from Lemma B.2.1 that $a_j \geq \frac{j}{2n}$.

- If $1 \leq j < \frac{4}{a\pi}$, then $a_j[an] \geq C$ for some $C > 0$, and a_j is small, so that

$$\frac{\sinh(a_j)}{\sinh(a_j[an])} \leq \frac{C_1 a_j}{C_2} \leq C\frac{j}{n}.$$

- If $\frac{4}{a\pi} \leq j < \frac{n}{\pi}$ and n is large enough,

$$a_j[an] \geq [an]\frac{\pi j}{2n} \geq cj \geq 1, \text{ for some } c > 0, \text{ and } a_j < 1,$$

  so that

$$\frac{\sinh(a_j)}{\sinh(a_j[an])} \leq C\frac{j/n}{\exp(cj)}.$$

- Finally, if $j \geq n/\pi$, then

$$\frac{\sinh(a_j)}{\sinh(a_j[an])} \leq C\exp(-n/2).$$

We also note that

$$\sin(x) \leq x, \ \forall x \geq 0. \tag{B.6}$$

Plugging these bounds into $\frac{\sinh(a_j)}{\sinh(a_j[an])}$, as well as (B.5) and (B.6) into (B.1), we get

$$\begin{aligned} f(x,y) &= \sum_{j=1}^{n-1} b_j(\phi)\frac{\sinh(a_j x)}{\sinh(a_j L)}\sin(\frac{\pi y j}{n}) \\ &\leq C\left(\sum_{j=1}^{[4/a\pi]}\frac{\pi y j}{\pi j n^2} + \sum_{j=[4/a\pi]}^{[n/\pi]}\frac{\pi y j^2}{\pi j n^2\exp(cj)} + \sum_{j=[n/\pi]}^{n-1}\frac{\pi y j\exp(-n/2)}{\pi j n}\right) \\ &\leq C\left(\frac{y}{n^2} + \frac{y}{n^2}\sum_{j=[4/a\pi]}^{[n/\pi]}(\frac{j}{\exp(cj)}) + 4y\exp(-n/2)\right) \leq C\frac{y}{n^2} \end{aligned}$$

□

Lemma B.2.5. If $f(x,y)$ is the solution of the Dirichlet problem in $\mathcal{R}(n)$ with $\phi(y) \equiv 1$, then there exists a constant $K > 0$ such that for all n and all $y \in \{1,..,n-1\}$,

$$f(n,a) \leq K\frac{a}{n}.$$

Proof. This is a particular case of Lemma B.2.3. From the proof of Lemma B.2.4, we know that

$$b_j \leq \frac{C}{j} \text{ and } \sin(\frac{\pi a j}{n}) \leq \frac{\pi a j}{n}.$$

We also know from Lemma B.2.1 that for $j \leq [\frac{n}{\pi}], a_j \geq \frac{\pi j}{2n}$, so that

$$\exp(-a_j n) \leq \exp(-\pi j/2) \leq \exp(-j/2).$$

For $[n/\pi] \leq j \leq n-1, a_j \geq 1/2$, so that we also have

$$\exp(-a_j n) \leq \exp(-n/2) \leq \exp(-j/2).$$

Plugging all this into (B.4) gives

$$f(n,a) \leq K \sum_{j=1}^{n-1} \frac{1}{j} \exp(-j/2) \frac{aj}{n} \leq K \frac{a}{n}.$$

$\square$

B.3 Difference estimates

Let $\alpha < 1, D(r) = \{z \in \mathbb{Z}^2 : |z| \leq r\}$ be the discrete disk of radius r, centered at 0, and $\mathcal{D}_\alpha = D([n^\alpha]) \cap \mathcal{H}$. $\mathcal{D}$ will be short for $\mathcal{D}_1$.

The purpose of this section is to show that the probability that a random walk starting at $\mathcal{D}_\alpha$ leaves $\mathcal{D}$ at $w_0 \in \mathcal{H}$, given that it leaves $\mathcal{D}$ at some point of $\mathcal{H}$, depends mostly on the imaginary part of the starting point.

Since we have an exact expression for the solution of the Dirichlet problem in a square, we will prove the result in the case where we replace the half-disks by squares. Once this is done, extending the result to half-disks is straightforward.

Recall from (B.1) that the solution of the discrete Dirichlet problem in $R(n,n)$ with $\phi(y) = \delta_{k^*}$ is

$$\tilde{H}(j+ik, n+ik^*) = \frac{2}{n-1} \sum_{m=1}^{n-1} \sin(\pi m \frac{k^*}{n}) \frac{\sinh(a_m j)}{\sinh(a_m n)} \sin(\pi m \frac{k}{n}), \tag{B.7}$$

where a_m is as in (B.2). Here, j, k, and k^* are integers. This series is unfortunately hard to estimate precisely. Instead of analyzing it directly, we will show that it is very close to another series which corresponds to a continuous Dirichlet problem for which we can find a very precise solution.

Let Σ be the open domain bounded by the square with vertices $(0,0), (0,1), (1,1)$, and $(1,0)$. The solution of the continuous Dirichlet problem in Σ, with boundary condition the delta function at $1+iy^*$ can be found by separation of variables to be

$$H_\Sigma(x+iy, 1+iy^*) = 2\sum_{m\geq 1} \sin(\pi m y^*)\frac{\sinh(\pi m x)}{\sinh(\pi m)}\sin(\pi m y). \tag{B.8}$$

Note that this is just the Poisson kernel. Here, x, y, and y^* are real. The following Lemma shows that the solution of the discrete Dirichlet problem in $R(n,n)$ with a certain boundary value can be expressed precisely in terms of the solution of the continuous Dirichlet problem in Σ with the "corresponding" boundary value.

Lemma B.3.1. There exist constants $C, N > 0$ such that $\forall\, n \geq N$, $j \leq \frac{n}{(\log n)^3}$, and $k \in [\frac{n}{2} - \frac{n}{\log n}, \frac{n}{2} + \frac{n}{\log n}]$,

$$|n\tilde{H}(j+ik, n+ik^*) - H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + \frac{k^*}{n})| \leq \frac{C}{n} H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + \frac{k^*}{n})|.$$

Proof. We may suppose without loss of generality that $k^* \leq \frac{n}{2}$. We first show that in the sums for H and $\tilde{H}$, most of the terms bring no contribution. For H we have

$$\begin{aligned}
&\Big|\sum_{m\geq(\log n)^2} \sin(\pi m\frac{k^*}{n})\frac{\sinh(\pi m\frac{j}{n})}{\sinh(\pi m)}\sin(\pi m\frac{k}{n})\Big| \\
&\qquad \leq \sum_{m\geq(\log n)^2}\frac{\sinh(\pi m\frac{j}{n})}{\sinh(\pi m)} \leq \sum_{m\geq(\log n)^2}\exp(\pi m(\frac{j}{n}-1))(1+\mathcal{O}\left(e^{-\pi m}\right)) \\
&\qquad \leq C\sum_{m\geq(\log n)^2}\exp(-\frac{\pi m}{2}) \leq C\int_{(\log n)^2-1}^{\infty}\exp(-\frac{\pi x}{2})\,dx \leq C\epsilon_n,
\end{aligned} \tag{B.9}$$

where ϵ_n decays faster than any power of n. In the case of $\tilde{H}$, we use Lemma B.2.1 to see that

$$\Big|\sum_{m=(\log n)^2}^{n-1}\sin(\pi m\frac{k^*}{n})\frac{\sinh(a_m j)}{\sinh(a_m n)}\sin(\pi m\frac{k}{n})\Big|$$

$$\leq \sum_{m=(\log n)^2}^{n-1}\frac{\sinh(a_m j)}{\sinh(a_m n)} \leq C\sum_{m=(\log n)^2}^{n-1}\exp(-\frac{a_m}{2}n)$$

$$\leq C\sum_{m\geq(\log n)^2}\exp(-\frac{m}{4}) \leq C\int_{(\log n)^2-1}^{\infty}\exp(-\frac{x}{4})\,dx \leq C\epsilon_n, \tag{B.10}$$

where again ϵ_n decays faster than any power of n.

We know from Lemma B.2.1 and the Taylor expansion of sinh that for any m and any $j \leq n$,

$$\sinh(a_m j) = \frac{\pi m j}{n}(1+\mathcal{O}\left(\frac{m}{n}\right))(1+\mathcal{O}\left(\left(\frac{jm}{n}\right)^2\right)) = \sinh(\frac{\pi m j}{n})(1+\mathcal{O}\left(\frac{m}{n}\right)),$$

so that

$$\left|\sum_{m=1}^{(\log n)^2}\sin(\pi m\frac{k^*}{n})\frac{\sinh(\pi m\frac{j}{n})}{\sinh(\pi m)}\sin(\pi m\frac{k}{n}) - \sin(\pi m\frac{k^*}{n})\frac{\sinh(a_m j)}{\sinh(a_m n)}\sin(\pi m\frac{k}{n})\right|$$

$$\leq \left|\sum_{m=1}^{(\log n)^2}\sin(\pi m\frac{k^*}{n})\left(\frac{\sinh(\pi m\frac{j}{n})}{\sinh(\pi m)} - \frac{\sinh(a_m j)}{\sinh(a_m n)}\right)\sin(\pi m\frac{k}{n})\right|$$

$$\leq \tfrac{C}{n}\left|\sum_{m=1}^{(\log n)^2} m\sin(\pi m\frac{k^*}{n})\frac{\sinh(\pi m\frac{j}{n})}{\sinh(\pi m)}\sin(\pi m\frac{k}{n})\right|. \tag{B.11}$$

We now note that

$$\sum_{m=1}^{(\log n)^2} m \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n})$$

$$= \sin(\pi \frac{k^*}{n}) \frac{\sinh(\pi \frac{j}{n})}{\sinh(\pi)} \sin(\pi \frac{k}{n})$$

$$\cdot \left[1 + \sum_{m=2}^{(\log n)^2} m \frac{\sin(\pi m \frac{k^*}{n})}{\sin(\pi \frac{k^*}{n})} \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi \frac{j}{n})} \frac{\sinh(\pi)}{\sinh(\pi m)} \frac{\sin(\pi m \frac{k}{n})}{\sin(\pi \frac{k}{n})} \right].$$

Using the fact that $\sin(\pi m \frac{k^*}{n}) \leq \pi m \frac{k^*}{n}, \sin(\pi \frac{k^*}{n}) \geq \frac{k^*}{2n}, \sin(\pi \frac{k}{n}) \geq C$, and recalling that for small $x, \sinh(x) \asymp x$, we see that this last sum is smaller, in absolute value, than $C \sum_{m=2}^{(\log n)^2} m^3 e^{-\pi m}$, which itself is bounded by a constant. Therefore,

$$\sum_{m=1}^{(\log n)^2} m \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n}) = \sin(\pi \frac{k^*}{n}) \frac{\sinh(\pi \frac{j}{n})}{\sinh(\pi)} \sin(\pi \frac{k}{n})(1 + \mathcal{O}(1)),$$

and similarly,

$$\sum_{m=1}^{(\log n)^2} \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n}) = \sin(\pi \frac{k^*}{n}) \frac{\sinh(\pi \frac{j}{n})}{\sinh(\pi)} \sin(\pi \frac{k}{n})(1 + \mathcal{O}(1)),$$

which implies that

$$\left| \sum_{m=1}^{(\log n)^2} m \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n}) \right| \leq \left| \sum_{m=1}^{(\log n)^2} \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n}) \right|.$$

This together with (B.11) gives

$$\left|\sum_{m=1}^{(\log n)^2} \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n}) - \sin(\pi m \frac{k^*}{n}) \frac{\sinh(a_m j)}{\sinh(a_m n)} \sin(\pi m \frac{k}{n})\right|$$

$$\le \tfrac{C}{n} \left|\sum_{m=1}^{(\log n)^2} \sin(\pi m \frac{k^*}{n}) \frac{\sinh(\pi m \frac{j}{n})}{\sinh(\pi m)} \sin(\pi m \frac{k}{n})\right|.$$

$$= \tfrac{C}{n} \left[\left|\tfrac{1}{2} H_\Sigma(\tfrac{j}{n} + i\tfrac{k}{n}, 1 + \tfrac{k^*}{n})\right| + \mathcal{O}\left(\epsilon_n\right)\right] \le \tfrac{C}{n} H_\Sigma(\tfrac{j}{n} + i\tfrac{k}{n}, 1 + \tfrac{k^*}{n}), \tag{B.12}$$

since $H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + \frac{k^*}{n})$ is positive and decays like a power of n. (B.9), (B.10), and (B.12) show that

$$|n\tilde{H}(j + ik, n + ik^*) - H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + \frac{k^*}{n})|$$
$$\le \frac{C}{n} H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + \frac{k^*}{n}) + 2\mathcal{O}\left(\epsilon_n\right) \le \frac{C}{n} H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + \frac{k^*}{n}),$$

which proves the Lemma.

□

Recall that $R(n,n) = \{(x,y) \in \mathbb{Z}^2 : 1 \le x \le n-1; 1 \le y \le n-1\}$. Call the four sides of $\partial R(n,n)$

$$E_1 = \{(x,y) \in \partial R(n,n) : x = 0\} \qquad E_2 = \{(x,y) \in \partial R(n,n) : y = 0\}$$
$$E_3 = \{(x,y) \in \partial R(n,n) : x = n\} \qquad E_4 = \{(x,y) \in \partial R(n,n) : y = n\}$$

Let $\lambda = \inf\{n \ge 1 : S_n \in \partial R(n,n)\}$. For $l \in \{1,..,4\}$, $1 \le j,k \le n-1$, and $w \in E_l$, define

$$\phi_l(j + ik, w) = \mathbb{P}^{j+ik}\left\{S(\lambda) = w \,|\, S(\lambda) \in E_l\right\}.$$

Lemma B.3.2. If $\alpha < 1$, $j, j' \le n^\alpha$, $\frac{n}{2} - n^\alpha \le k, k' \le \frac{n}{2} + n^\alpha$, then for any $w \in E_3$,

$$\phi_3(j + ik, w) = \phi(j' + ik', w)(1 + \mathcal{O}\left(n^{\alpha-1}\right)).$$

Proof. A first step in the proof is to estimate $\tilde{H}(j+ik, n+ik^*)$, where $\tilde{H}$ is as defined in (B.7). Lemma B.3.1 allows us to estimate H, given by (B.8), instead, since up to an error term the relationship between H and $\tilde{H}$ is known. The series for H is not easier to compute than the one for $\tilde{H}$. However, knowing that it is the Poisson kernel in the square will allow us to derive it from the Poisson kernel in the upper half plane, which, as noted in (4.2), is

$$H(x+iy, x') = \frac{1}{\pi}\frac{y}{(x-x')^2+y^2},$$

where $x, x' \in \mathbb{R}, y \in \mathbb{R}_+$. To do this, we use the following fact which can be found for instance in [3]:

If $D, D' \subset \mathbb{C}$ are domains and $f : D \to D'$ a conformal transformation, then

$$H_{D'}(f(z), f(w)) = |f'(w)|^{-1} H_D(z, w). \tag{B.13}$$

By mapping $\mathbb{H}$ to Σ we can derive (up to error terms) the Poisson kernel in Σ from the Poisson kernel in $\mathbb{H}$, at least for a specific selection of starting points. A map from the half-plane to a square is given by the Schwarz-Christoffel formula (see [2]):

$$F(w) = \int_0^w \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

where $k = (\sqrt{2}-1)^2$. F maps the points 1 and -1 to $K/2$ and $-K/2$, respectively, and $1/k$ and $-1/k$ to $K/2 + iK$ and $-K/2 + iK$, respectively, where

$$K = \int_{-1}^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \in [3.1, 3.2].$$

Let $G(z) = \frac{1}{K}(z + K/2)$. Then $h = G \circ F$ sends $\mathbb{H}$ to Σ. Numerical computations give $M = h^{-1}(i/2) \in [-2.5, -2.4]$. This will be precise enough for our purpose. Let

$$Q = \{z \in \mathbb{C} : -3 \leq \mathrm{Re}(z) \leq -2\ ,\ 0 \leq \mathrm{Im}(z) \leq 1\}$$

and

$$Q_n = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq n^{\alpha-1}, 1/2(1-n^{\alpha-1}) \leq \mathrm{Im}(z) \leq 1/2(1+n^{\alpha-1})\}.$$

We also let R_n be the region bounded by the closed rectangle defined by the points

$$P_1 = (M + 100Kn^{\alpha-1}, 0), \qquad P_2 = (M - 100Kn^{\alpha-1}, 0),$$

$$P_3 = (M - 100Kn^{\alpha-1}, 100Kn^{\alpha-1}), \qquad P_4 = (M + 100Kn^{\alpha-1}, 100Kn^{\alpha-1}).$$

Q and R_n are thought of as subsets of $\bar{\mathbb{H}}$ and Q_n as a subset of Σ. Note that

$$F_x(z) = \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{-i}{\sqrt{|1-z^2||1-k^2z^2|}}$$

and

$$F_y(z) = \frac{1}{\sqrt{|1-z^2||1-k^2z^2|}}.$$

Clearly, if $z \in R_n$, then

$\sqrt{|1-z^2|} = |1-M^2|^{1/2}(1+\mathcal{O}\left(n^{\alpha-1}\right))$ and $\sqrt{|1-k^2z^2|} = |1-k^2M^2|^{1/2}(1+\mathcal{O}\left(n^{\alpha-1}\right))$.

Therefore the partial derivatives of h in R_n are

$$h_x = \frac{-i}{K|1-k^2M^2|^{1/2}|1-M^2|^{1/2}}(1+\mathcal{O}\left(n^{\alpha-1}\right))$$

and

$$h_y = \frac{1}{K|1-k^2M^2|^{1/2}|1-M^2|^{1/2}}(1+\mathcal{O}\left(n^{\alpha-1}\right)).$$

It is easy to see that

$$\frac{1}{K|1-k^2M^2|^{1/2}|1-M^2|^{1/2}} \geq \frac{1}{80}.$$

By integrating h_x and h_y along paths on ∂R_n, going from M to each of the points P_i, we get we get, for n large enough,

$$\mathrm{Im}(h(P_1)) \leq \frac{1}{2} - \frac{10}{9}n^{\alpha-1}, \qquad \mathrm{Im}(h(P_2)) \geq \frac{1}{2} + \frac{10}{9}n^{\alpha-1},$$

$$\mathrm{Re}(h(P_3)) \geq \frac{10}{9}n^{\alpha-1}, \qquad \mathrm{Re}(h(P_4)) \geq \frac{10}{9}n^{\alpha-1},$$

and $h^{-1}(Q_n) \subset R_n$. Let $z \in R_n$ and $h(z) = x + iy$. Then $z = (M + ixK|1 - k^2M^2|^{1/2}|1 - M^2|^{1/2})(1 + \mathcal{O}(n^{\alpha-1}))$ and if for $0 \leq y^* \leq 1$, we write $v = 1 + iy^*$, (B.13) gives

$$\begin{aligned} &H_\Sigma(x+iy, v) \\ &\quad = |h'(h^{-1}(v))|^{-1} H((M + ixK|1 - k^2M^2|^{1/2}|1 - M^2|^{1/2})(1 + \mathcal{O}\left(n^{\alpha-1}\right)), h^{-1}(v)) \\ &\qquad = C_v x(1 + \mathcal{O}\left(n^{\alpha-1}\right)), \end{aligned} \tag{B.14}$$

where C_v is a constant depending on v. In particular, for any $x + iy \in Q_n, v = 1 + iy^*, 0 \leq y^* \leq 1$,

$$H_\Sigma(x+iy, v) = \frac{x}{x'} H_\Sigma(x' + iy', v)(1 + \mathcal{O}\left(n^{\alpha-1}\right)).$$

Lemma B.3.1 gives the following relationship between the continuous and discrete Dirichlet problem in a square:

$$\tilde{H}(j + ik, n + ik^*) = \frac{1}{n} H_\Sigma(\frac{j}{n} + i\frac{k}{n}, 1 + i\frac{k^*}{n})(1 + \mathcal{O}\left(n^{-1}\right)).$$

Together with (B.14), this implies that

$$\tilde{H}(j + ik, n + ik^*) = \frac{j}{j'} \tilde{H}(j' + ik', n + ik^*)(1 + \mathcal{O}\left(n^{\alpha-1}\right)). \tag{B.15}$$

We now go through the same procedure for the discrete and continuous Dirichlet problem with boundary value 1 on the right side of the square, 0 everywhere else.This gives the probability that either random walk or Brownian motion leave a square on its right side. The solutions are given by

$$\tilde{f}(j + ik) = \sum_{\substack{m=1 \\ m \text{ odd}}}^{n-1} \frac{4}{\pi m} \frac{\sinh(a_m j)}{\sinh(a_m n)} \sin(\pi m \frac{k}{n})$$

for the discrete problem in $R(n,n)$ with $\phi \equiv 1$, and

$$f_\Sigma(x+iy) = \sum_{\substack{m\geq 1 \\ m \text{ odd}}} \frac{4}{\pi m} \frac{\sinh(\pi m x)}{\sinh(\pi m)} \sin(\pi m y)$$

for the continuous problem in Σ with boundary value 1 on $\{1+iy : 0 \leq y \leq 1\}$ and 0 everywhere else.

The same analysis as above, which we omit here, gives

$$\tilde{f}(j+ik) = f_\Sigma(\frac{j}{n} + i\frac{k}{n})(1 + \mathcal{O}\left(n^{\alpha-1}\right)),$$

$$f_\Sigma(\frac{j}{n} + i\frac{k}{n}) = \frac{j}{j'} f_\Sigma(\frac{j'}{n} + i\frac{k'}{n})(1 + \mathcal{O}\left(n^{-1}\right)),$$

and thus

$$\tilde{f}(j+ik) = \frac{j}{j'}\tilde{f}(j'+ik')(1 + \mathcal{O}\left(n^{\alpha-1}\right)).$$

The fact that $\phi_3(j+ik,w) = \frac{\tilde{H}(j+ik,w)}{\tilde{f}(j+ik)}$ now gives the lemma.

□

Remark 11. Using the same ideas, one can show the same result for ϕ_2 and ϕ_4.

If we let $\mathcal{D}_\alpha = \{z \in \mathcal{H} : |z| \leq n^\alpha\}, \mathcal{D} = \{z \in \mathcal{H} : |z| \leq 2n\}$, we can now show that if we start a random walk inside $\mathcal{D}_\alpha$, the probability that it leaves $\mathcal{D}$ at a point of $\mathcal{H}$, depends mostly on the imaginary part of the starting point.

Corollary B.3.3. If $z, z' \in \mathcal{D}_\alpha, v \in \partial\mathcal{D}$, and $\tilde{K}(z,v) = \mathbb{P}^z\left\{S(\tau_{\partial\mathcal{D}}) = v\right\}$, then

$$\frac{\tilde{K}(z,v)}{\operatorname{Im}(z)} = \frac{\tilde{K}(z',v)}{\operatorname{Im}(z')}(1 + \mathcal{O}\left(n^{\alpha-1}\right)).$$

Proof. Let $R = \{z \in \mathcal{H} : -[n/2] + 1 \leq \text{Re}(z) \leq [(n+1)/2] - 1; 1 \leq \text{Im}(z) \leq n - 1\}$. This is a discrete square composed of $(n-1) \times (n-1)$ points. Then the strong Markov property yields

$$\frac{\tilde{K}(z,v)}{\tilde{K}(z',v)} = \frac{\sum_{x \in \partial R} \tilde{H}(z,x)\tilde{K}(x,v)}{\sum_{x \in \partial R} \tilde{H}(z',x)\tilde{K}(x,v)}$$

By (B.15) and the corresponding results for E_2 and E_4, since $\mathcal{D}_\alpha$ is included in a square of $[n^\alpha] + 1 \times [n^\alpha] + 1$ points, we have for all $z, z' \in \mathcal{D}_\alpha$, all $x \in \partial R \cap \mathcal{H}$,

$$\tilde{H}(z,x) = \frac{\text{Im}(z)}{\text{Im}(z')}\tilde{H}(z',x)(1 + \mathcal{O}\left(n^{\alpha-1}\right)).$$

This gives the result.

□

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Biography

Christian Gabriel Beneš was born on March 5, 1975 in Geneva, Switzerland. During his young years, he spent much time running after a soccer ball and making music, first with his voice and later with a piano. He graduated from the Centre d'Enseignement Secondaire Supérieur de l'Ouest Vaudois in 1993 with a high-school degree in classical languages. In 1998, he received his Bachelor degree in Mathematics from the University of Geneva and decided the Alps were obstructing his view. Stranded in North Carolina, he discovered the joys of Graduate School at Duke University, which included learning mathematics, but also teaching it, an activity he enjoys very much. He received with great pleasure the L.P. Smith Award for excellence in teaching during the academic years 2000-2001 and 2002-2003. His adviser's growing fame gave him the opportunity to discover new horizons, such as Stockholm where he spent a semester in the fall of 2001, enjoying the romantic atmosphere of the Institut Mittag-Leffler, and Cornell University in Ithaca, NY, a wonderful town on the rare warm sunny days, where he spent the following three semesters. He returned to the South for the last year of his doctoral studies during which he was offered and accepted an Assistant Professor position at Tufts University. He was recently offered the 2004 Laha Award from the Institute of Mathematical Statistics.