Homework 2 Solutions

1. Textbook Problem 2.7: Since \( \{Y_t\} \) is stationary, \( E[Y_t] = \mu_Y \) for all \( t \) and \( \{Y_t\} \) has an autocovariance function \( \gamma_Y \). Therefore,

(a) \[
E[W_t] = E[Y_t - Y_{t-1}] = E[Y_t] - E[Y_{t-1}] = \mu_Y - \mu_Y = 0.
\] (1)

\[
\text{Cov}(W_t, W_{t+h}) = \text{Cov}(Y_t - Y_{t-1}, Y_{t+h} - Y_{t+h-1})
\]
\[
= \text{Cov}(Y_t, Y_{t+h}) - \text{Cov}(Y_t, Y_{t+h-1}) - \text{Cov}(Y_{t-1}, Y_{t+h}) + \text{Cov}(Y_{t-1}, Y_{t+h-1})
\]
\[
= \gamma_Y(h) - \gamma_Y(h-1) - \gamma_Y(h+1) + \gamma_Y(h)
\]
\[
= 2\gamma_Y(h) - \gamma_Y(h-1) - \gamma_Y(h+1),
\]

which depends on \( h \) but not on \( t \), so the autocorrelation function of \( \{W_t\} \) is

\[
\gamma_W(h) = 2\gamma_Y(h) - \gamma_Y(h-1) - \gamma_Y(h+1).
\] (2)

Equations (1) and (2) imply that \( \{W_t\} \) is stationary.

(b) We showed in (a) that if \( \{Y_t\} \) is stationary, so is \( \{\nabla Y_t\} \). Therefore, since by (a), \( \{\nabla Y_t\} \) is stationary, so is \( \{U_t\} = \{\nabla(\nabla Y_t)\} = \{\nabla^2 Y_t\} \).

2. Textbook Problem 2.24: Since \( \{X_t\} \) is stationary, we can write \( E[X_t] = \mu_X \) and \( \text{Cov}(X_t, X_{t+h}) = \gamma_X(h) \). We first note that

\[
E[Y_t] = E[X_t + e_t] = E[X_t] + E[e_t] = E[X_t] + 0 = E[X_t] = \mu_X.
\] (3)

Moreover, since \( \{X_t\} \) and \( \{e_t\} \) are independent processes (which means that \( X_t \) is independent of \( e_s \) for all \( t, s \)), we have

\[
\gamma_Y(h) = \text{Cov}(Y_t, Y_{t+h}) = \text{Cov}(X_t + e_t, X_{t+h} + e_{t+h})
\]
\[
= \text{Cov}(X_t, X_{t+h}) + \text{Cov}(X_t, e_{t+h}) + \text{Cov}(e_t, X_{t+h}) + \text{Cov}(e_t, e_{t+h})
\]
\[
= \gamma_X(h) + 0 + 0 + \gamma_e(h) = \gamma_X(h) + \gamma_e(h) = \gamma_X(h) + \sigma_e^2 \delta(h),
\] (4)

where the last equality follows from the fact that \( \{e_t\} \sim WN(0, \sigma_e^2) \). Note that this last expression in (4) depends only on \( h \), not on \( t \), which, together with (3) implies that \( \{Y_t\} \) is stationary. All the above implies that if we write \( \gamma_X(0) = \sigma_X^2 \), then for \( h \geq 1 \),

\[
\rho_Y(h) = \frac{\gamma_Y(h)}{\gamma_Y(0)} = \frac{\gamma_X(h)}{\gamma_X(0) + \sigma_e^2} = \frac{\gamma_X(h)/\gamma_X(0)}{1 + \sigma_e^2/\gamma_X(0)} = \frac{\rho_X(h)}{1 + \frac{\sigma_e^2}{\sigma_X^2}}.
\]
3. Since $X_t = \phi X_{t-1} + Z_t$, we have

\[
\begin{align*}
X_2 &= \phi X_1 + Z_2 \\
X_3 &= \phi X_2 + Z_3 = \phi^2 X_1 + \phi Z_2 + Z_3 \\
X_4 &= \phi X_3 + Z_4 = \phi^3 X_1 + \phi^2 Z_2 + \phi Z_3 + Z_4,
\end{align*}
\]

so that

\[
\frac{X_1 + X_2 + X_3 + X_4}{4} = \frac{1}{4} \left( (1 + \phi + \phi^2 + \phi^3)X_1 + (1 + \phi + \phi^2)Z_2 + (1 + \phi)Z_3 + Z_4 \right).
\]

Therefore, since $X_1, Z_2, Z_3,$ and $Z_4$ are independent, we get

\[
\begin{align*}
\text{Var} \left( \frac{X_1 + X_2 + X_3 + X_4}{4} \right) &= \frac{1}{16} \left( (1 + \phi + \phi^2 + \phi^3)^2 \text{Var} X_1 \\
&\quad + (1 + \phi + \phi^2)^2 \text{Var} Z_2 + (1 + \phi)^2 \text{Var} Z_3 + \text{Var} Z_4 \right).
\end{align*}
\]

Using the fact (derived in class) that $\text{Var} X_i = \frac{\sigma^2}{1 - \phi^2}$ and that $\text{Var} Z_i = \sigma^2$, we get

\[
\begin{align*}
\text{Var} \left( \frac{X_1 + X_2 + X_3 + X_4}{4} \right) &= \frac{\sigma^2}{16} \left( \frac{(1 + \phi + \phi^2 + \phi^3)^2 - (1 + \phi + \phi^2)^2 - (1 + \phi)^2 + 1}{1 - \phi^2} \right).
\end{align*}
\]

If $\sigma^2 = 1$ and $\phi = 0.9$, we get

\[
\text{Var} \left( \frac{X_1 + X_2 + X_3 + X_4}{4} \right) = 4.6375
\]

and if $\sigma^2 = 1$ and $\phi = -0.9$, we get

\[
\text{Var} \left( \frac{X_1 + X_2 + X_3 + X_4}{4} \right) \approx 0.1256579.
\]

When $\phi = 0.9$, the lag-1 auto-correlation is 0.9, so if $X_t$ is positive, one is more likely to see a positive value of $X_{t-1}$ as well. This means that such AR processes have more variability in the sense that some realizations will spend longer periods of time above the horizontal axis (respectively, below the horizontal axis) than if $\phi = -0.9$ when the process will have a tendency to oscillate between positive and negative values. The sample mean of such an oscillating process (corresponding to negative $\phi$) will typically be close to 0, while the sample mean might be farther from 0 (either positive or negative) when there are runs of positive or negative values, implying more variability in the sample mean in the latter case. This is reflected by the larger value obtained above in the case where $\phi = 0.9$ than when $\phi = -0.9$.

4. If $\{X_t\}$ and $\{Y_t\}$ are stationary, it means that there exist constants $\mu_X$ and $\mu_Y$ such that for any $t \in \mathbb{Z}$, $E[X_t] = \mu_X$ and $E[Y_t] = \mu_Y$. Also, there exist two functions $\gamma_X$ and $\gamma_Y$ such that for all $t, h \in \mathbb{Z}$, $\gamma_X(h) = \text{Cov}(X_{t+h}, X_t)$ and $\gamma_Y(h) = \text{Cov}(Y_{t+h}, Y_t)$. Therefore, for any $t \in \mathbb{Z}$,

\[
E[X_t + Y_t] = E[X_t] + E[Y_t] = \mu_X + \mu_Y,
\]

which is a constant, and since $\{X_t\}$ and $\{Y_t\}$ are uncorrelated, $\text{Cov}(X_{t+h}, Y_t) = \text{Cov}(Y_{t+h}, X_t) = 0$ for any $t, h \in \mathbb{Z}$, so that

\[
\text{Cov}(X_{t+h} + Y_{t+h}, X_t + Y_t) = \text{Cov}(X_{t+h}, X_t) + \text{Cov}(X_{t+h}, Y_t) + \text{Cov}(Y_{t+h}, X_t) + \text{Cov}(Y_{t+h}, Y_t)
\]

\[
= \gamma_X(h) + 0 + 0 + \gamma_Y(h) = \gamma_X(h) + \gamma_Y(h),
\]

which depends only on $h$. This together with (5) implies that $\{X_t + Y_t\}$ is stationary.
5. **Textbook Problem 4.5:** We can draw these ACFs of the four AR processes using the commands below (one might need to fiddle around a bit to find an appropriate maximal lag). Note that the command “abline(h=0)” adds a horizontal line at height 0, which makes the graphs a bit easier to understand. Note also that when using the command “ARMAacf”, R draws the ACF shifted to the right by one unit (starting to index at 1, as opposed to 0). One way around this is to shift the graph of the ACF drawn by R to the left by one unit using the command “ACF1[-1]” as below.

(a) \[ \texttt{ACF1=ARMAacf(ar=0.6,lag.max=12)} \]
    \[ \texttt{plot(x=1:12,y=ACF1[-1],type="h"); abline(h=0)} \]

(b) \[ \texttt{ACF2=ARMAacf(ar=-0.6,lag.max=12)} \]
    \[ \texttt{plot(x=1:12,y=ACF2[-1],type="h"); abline(h=0)} \]

(c) \[ \texttt{ACF3=ARMAacf(ar=0.95,lag.max=20)} \]
    \[ \texttt{plot(x=1:20,y=ACF3[-1],type="h"); abline(h=0)} \]
6. Textbook Problem 4.9:

(a) We are considering the time series defined by

\[ Y_t = 0.6Y_{t-1} + 0.3Y_{t-2} + Z_t \]

with characteristic equation

\[ 1 - 0.6z - 0.3z^2 = 0. \]

We can compute the solutions of this equation using R:

```r
> a=c(1,-0.6,-0.3)
> polyroot(a)
[1] 1.081666-0i -3.081666+0i
> abs(polyroot(a))
[1] 1.081666 3.081666
```

This shows that both roots, 1.081666 and -3.081666, are real and have modulus greater than 1. To generate the ACF of this time series using the recursive equations

\[ \rho(0) = 1, \quad \rho(1) = \frac{\phi_1}{1-\phi_2}, \quad \text{and for } i \geq 2, \rho(i) = \phi_1 \rho(i-1) + \phi_2 \rho(i-2), \]

we use the following code:

```r
> phi1=0.6
> phi2=0.3
> rho[1]=1
> rho[2]=phi1/(1-phi2)
> for (i in 3:30) rho[i]=phi1*rho[i-1]+phi2*rho[i-2]
```

This gives the following data

```plaintext
1.00000000 0.85714286 0.81428571 0.74571429 0.69171429 0.63874286
0.59076000 0.54607886 0.50487531 0.46674885 0.43151190 0.39893179
0.36881265 0.34096713 0.31522407 0.29142458 0.26942197 0.24908056
0.23027492 0.21288912 0.19681595 0.18195631 0.16821857 0.15551803
0.14377639 0.13292124 0.12288566 0.11360777 0.10503036 0.09710055
```

The graph of the ACF is then obtained using the command

```r
> plot(rho[-1],type="h");abline(h=0)
```
(b) We are considering the time series defined by \( Y_t = -0.4Y_{t-1} + 0.5Y_{t-2} + Z_t \) with characteristic equation

\[
1 + 0.4z - 0.5z^2 = 0.
\]

We can compute the solutions of this equation using R:

```r
> b <- c(1, 0.4, -0.5)
> polyroot(b)
[1] -1.069694+0i 1.869694-0i
> abs(polyroot(b))
[1] 1.069694 1.869694
```

This shows that both roots, -1.069694 and 1.869694, are real and have modulus greater than 1.

To generate the ACF of this time series using the recursive equations, we use the same code as in (a) but define

```r
> phi1 <- -0.4
> phi2 <- 0.5
```

and get the data set

\[
\begin{array}{cccccccc}
1.000000 & -0.800000 & 0.820000 & -0.728000 & 0.701200 & -0.644480 \\
0.608392 & -0.565568 & 0.530434 & -0.494972 & 0.463206 & -0.432768 \\
0.404710 & -0.378268 & 0.353662 & -0.330599 & 0.309071 & -0.288928 \\
0.270106 & -0.252506 & 0.236056 & -0.220675 & 0.206298 & -0.192857 \\
0.180292 & -0.168545 & 0.157564 & -0.147298 & 0.137701 & -0.128729 \\
\end{array}
\]

and the graph
(e) We are considering the time series defined by \( Y_t = 0.5Y_{t-1} - 0.9Y_{t-2} + Z_t \) with characteristic equation

\[ 1 - 0.5z + 0.9z^2 = 0. \]

We can compute the solutions of this equation using R:

\begin{verbatim}
> e=c(1,-0.5,0.9)
> polyroot(e)
[1] 0.277778+1.016834i 0.277778-1.016834i
> abs(polyroot(e))
[1] 1.054093 1.054093
\end{verbatim}

This shows that both roots are complex and have modulus greater than 1.

To generate the ACF of this time series using the recursive equations, we use the same code as in (a) but define

\begin{verbatim}
> phi1=-0.4
> phi2=0.5
\end{verbatim}

and get the data set

\begin{verbatim}
 1.00000000 0.26315789 -0.76842105 -0.62105263 0.38105263 0.74947368
 0.03178947 -0.65863158 -0.35792632 0.41380526 0.52903632 -0.10790658
-0.53008597 -0.16792707 0.39311384 0.34769128 -0.17995682 -0.40290056
-0.03948914 0.34286593 0.20697320 -0.20509274 -0.28882225 0.04017234
 0.28002620 0.10385799 -0.20009458 -0.19351948 0.08332538 0.21583022
\end{verbatim}

and the graph

To see a bit more clearly the shape of the damped sine wave, on can use the following command:

\begin{verbatim}
> plot(rho[-1],type="l");abline(h=0)
\end{verbatim}
Since the roots of the characteristic polynomial in this problem are complex, we also look for the damping factor \( R \) and the frequency \( \Theta \): \( R = \sqrt{-\phi_2} = \sqrt{0.9} \approx 0.948633 \). Also, \( \cos \Theta = \frac{\phi_1}{2\sqrt{-\phi_2}} = \frac{0.5}{2\sqrt{0.9}} \Rightarrow \Theta = \arccos \left( \frac{0.5}{2\sqrt{0.9}} \right) \approx 1.304124. \)

Note that one can verify that the expression at the end of Lecture 10 in the notes is correct using the following code and comparing the data with that obtained in the table above:

```r
> R=sqrt(0.9)
> Theta=acos(0.5/(2*R))
> Phi=atan(tan(Theta)*19)
> X=0:30
> Rho=R^X*sin(Theta*X+Phi)/sin(Phi)
```

A third way of obtaining the same values is by simply using the command

```r
> ARMAacf(ar=c(0.5,-0.9),lag.max=30)
```

7. Textbook Problem 4.12: We consider the MA(2) process defined by

\[
Y_t = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-1}.
\]

(6)

(a) We know that if the process is as defined above,

\[
\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad \rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad \text{and} \quad \rho_k = 0 \text{ for } k = 3, 4, 5, \ldots.
\]

Therefore, if \( \theta_1 = \theta_2 = \frac{1}{6} \), we get

\[
\rho_1 = \frac{-1/6 + 1/36}{1 + 1/36 + 1/36} = \frac{-5/36}{38/36} = \frac{-5}{38},
\]

\[
\rho_2 = \frac{-1/6}{1 + 1/36 + 1/36} = \frac{-6/36}{38/36} = \frac{-3}{19}.
\]

Moreover, if \( \theta_1 = -1, \theta_2 = 6 \), we get

\[
\rho_1 = \frac{1 - 6}{1 + 1 + 36} = \frac{-5}{38},
\]

\[
\rho_2 = \frac{-6}{1 + 1 + 36} = \frac{-6}{38} = \frac{-3}{19}.
\]

Since for both processes, \( \rho_k = 0 \) for \( k = 3, 4, \ldots \), we see that they have the same ACF.

(b) The characteristic equation for the time series defined in (6) is \( 1 - \theta_1 z - \theta_2 z^2 = 0 \), so if \( \theta_1 = \theta_2 = \frac{1}{6} \), the equation is

\[
1 - \frac{1}{6} z - \frac{1}{6} z^2 = 0 \iff z^2 + z - 6 = 0 \iff (z + 3)(z - 2) = 0 \iff z = -3 \text{ or } z = 2
\]

and if \( \theta_1 = -1, \theta_2 = 6 \), the equation is

\[
1 + z - 6z^2 = 0 \iff z^2 - z/6 - 1/6 = 0 \iff z = \frac{1}{6} \pm \sqrt{(-1/6)^2 - 4 \cdot 1 \cdot (-1/6)}
\]

\[
\iff z = \frac{1}{12} \pm \frac{\sqrt{1/36 + 24/36}}{2}
\]

\[
\iff z = \frac{1}{12} \pm \frac{5}{12} \Rightarrow z = -\frac{1}{3} \text{ or } z = \frac{1}{2}.
\]
Note that this shows that while the ACF is the same for both sets of parameters, for the second set of parameters, the solutions of the characteristic equation both have modulus less than 1, which means that the second of the two MA processes is not invertible.