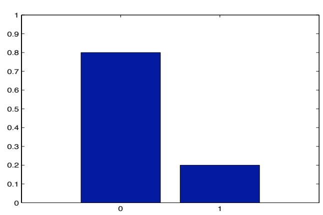
All Your Favorite Random Variables

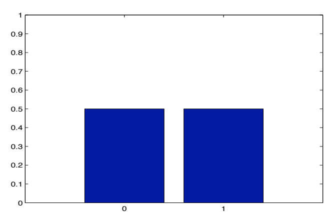
This is a list of some of the named random variables encountered this semester, with their most important properties, their connections to other random variables, and some pictures. Note that it is not always possible to determine from a picture if a discrete r.v. takes an infinite or only a finite number of values. However, the equation of the p.m.f. always answers that question.

The Bernoulli random variable $Be(p), 0 \leq p \leq 1$.

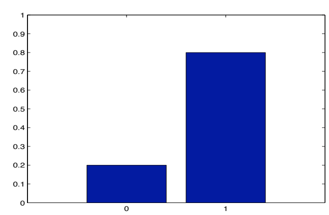
- Used to model an experiment which can only result in one of two outcomes (associated with the numerical values 0 and 1).
- Has p.m.f. defined by $P(X = 1) = p, P(X = 0) = 1 - p$.
- Is connected to the binomial random variable as follows: If $X_i, i = 1, 2, ...$ are i.i.d. Bernoulli $Be(p)$, then $\sum_{i=1}^{n} X_i \sim Bin(n, p)$.
- Has mean p and variance $p(1 - p)$.



$Be(.2)$



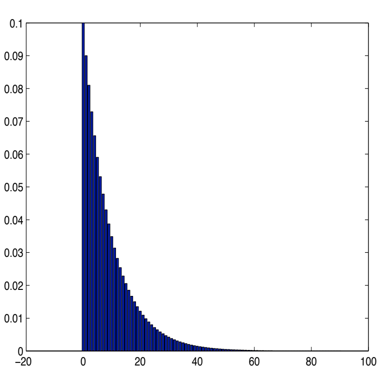
$Be(.5)$



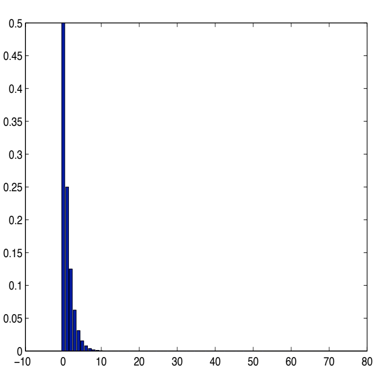
$Be(.8)$

The Geometric random variable $Geo(p)$

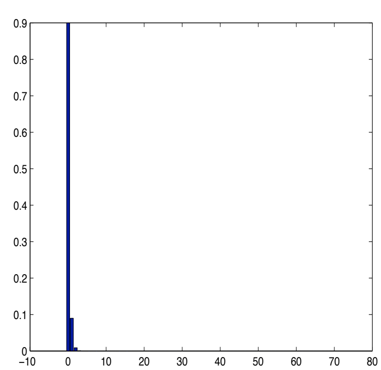
- Used to model the number of trials needed until a success occurs in independent Bernoulli experiments.
- Has p.m.f. defined by $P(X = i) = (1 - p)^{i-1}p, i = 1, 2, ...$
- Is connected to the negative binomial r.v. See below.
- Decays exponentially.
- Has mean $\frac{1}{p}$, variance $\frac{1-p}{p^2}$.
- Has the lack of memory property: $P(X > n + m | X > n) = P(X > m)$.



$Geom(.1)$



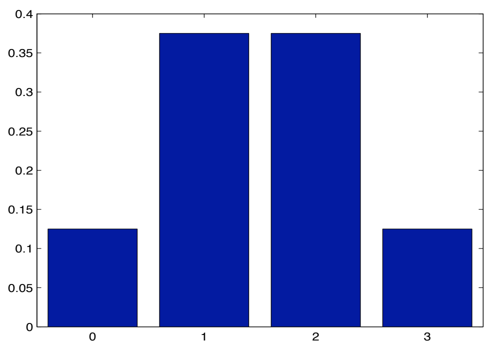
$Geom(.5)$



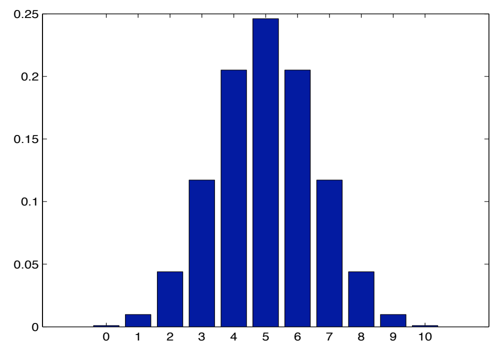
$Geom(.9)$

The Binomial random variable $Bin(n,p), n \in \mathbb{N}, 0 \leq p \leq 1$

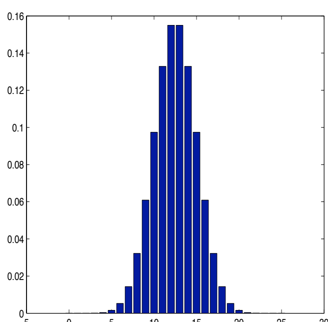
- Used to model the number of successes in n repeated independent identical Bernoulli experiments.
- Has p.m.f. defined by $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, .., n.$
- In particular, if $X \sim Bin(1,p)$, then $X \sim Be(p)$.
- Is connected to the Bernoulli random variable. See above.
- Is connected to the negative binomial random variable as follows: If $Y \sim Bin(n,p)$ and $X \sim NegBin(r,p)$, then $P(X > n) = P(Y < r)$.
- Has mean np and variance $np(1-p)$ (which can be shown in one line using the fact that a binomial is a sum of independent Bernoullis).



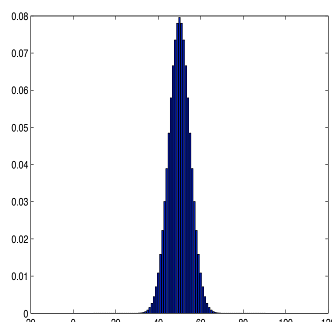
$Bin(3, .5)$



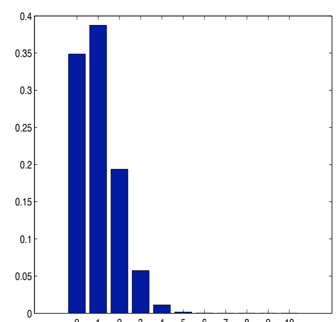
$Bin(10, .5)$



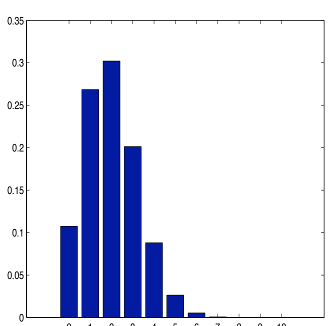
$Bin(25, .5)$



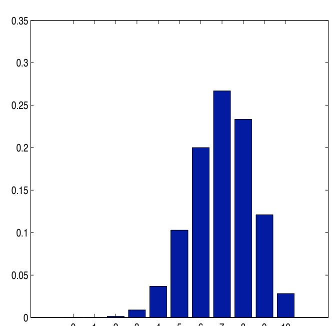
$Bin(100, .5)$



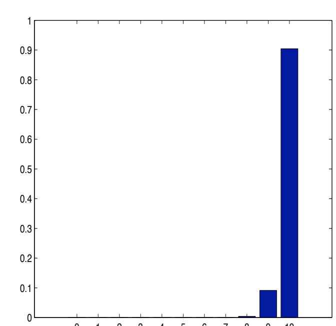
$Bin(10, .1)$



$Bin(10, .2)$



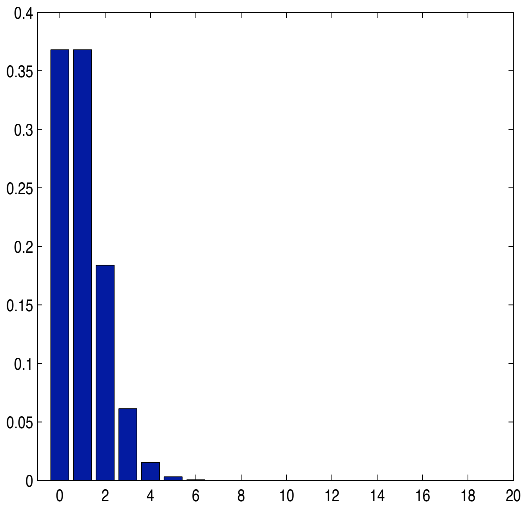
$Bin(10, .7)$



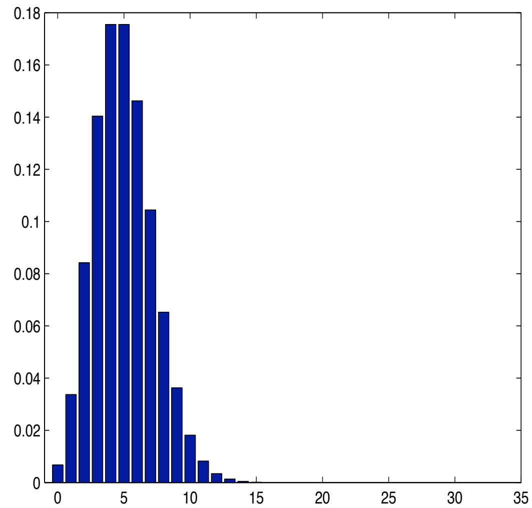
$Bin(10, .99)$

The Poisson random variable $Po(\lambda), \lambda > 0$

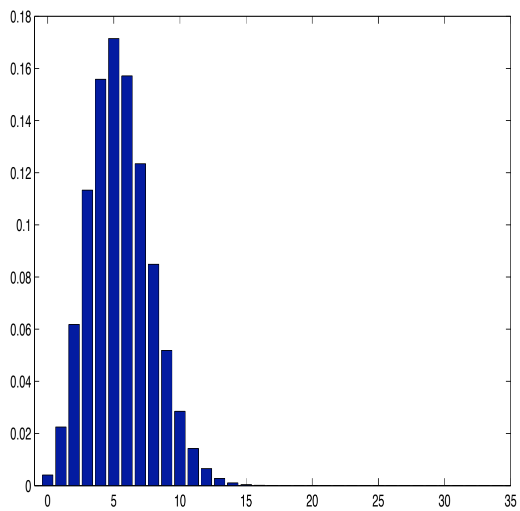
- Is used in different settings, usually when modeling the number of occurrences of some kind in a given interval of space or time.
- Has p.m.f. defined by $P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, i = 0, 1, 2, ...$
- Has mean λ and variance λ.



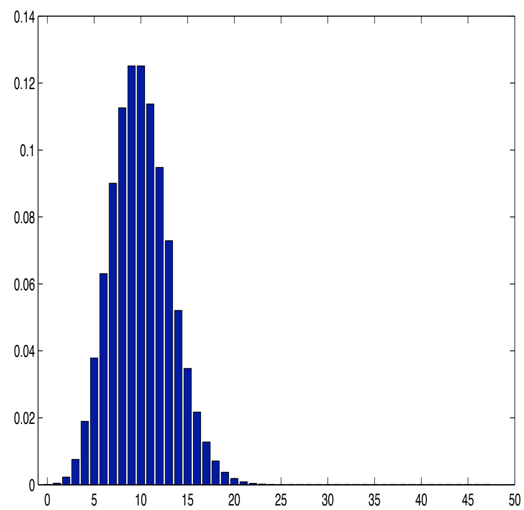
$Po(1)$



$Po(5)$



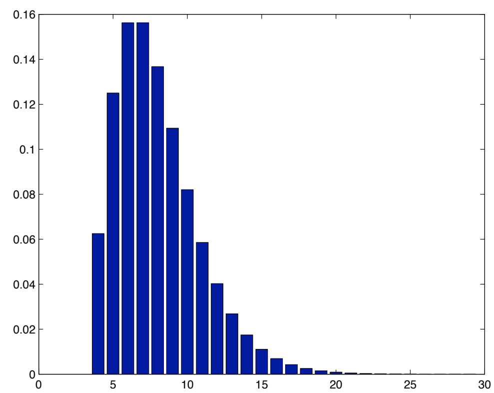
$Po(5.5)$



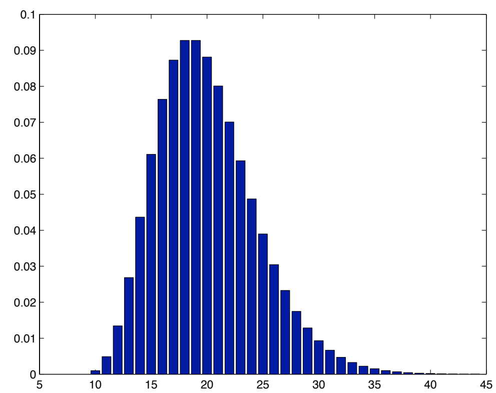
$Po(10)$

The Negative Binomial random variable $NegBin(r, p), r \in \mathbb{N}, 0 \leq p \leq 1$

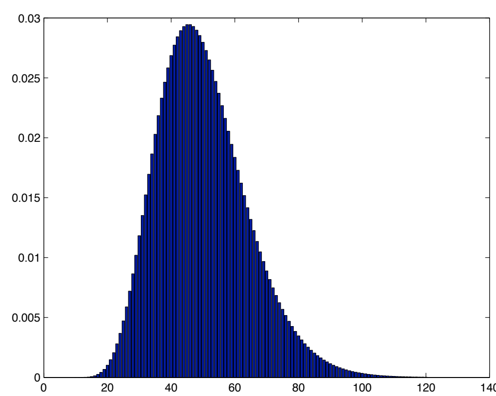
- Used to model the amount of tries needed until r successes occur in independent identical Bernoulli experiments.
- Has p.m.f.
$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, n = r, r+1, ...$$
(Note that unlike the binomial, the negative binomial can take infinitely many values.)
- Is connected to the binomial random variable. See above.
- Is connected to the geometric random variable as follows. If $X_i \sim Geo(p), i = 1, 2, ...$ are i.i.d. geometric, $\sum_{i=1}^{r} X_i \sim NegBin(r, p)$. In particular, if $X \sim NegBin(1, p)$, then $X \sim Geo(p)$.
- Has mean $\frac{r}{p}$, variance $\frac{r(1-p)}{p^2}$ (which can be shown in one line using the fact that a neg. binomial is a sum of independent geometrics).



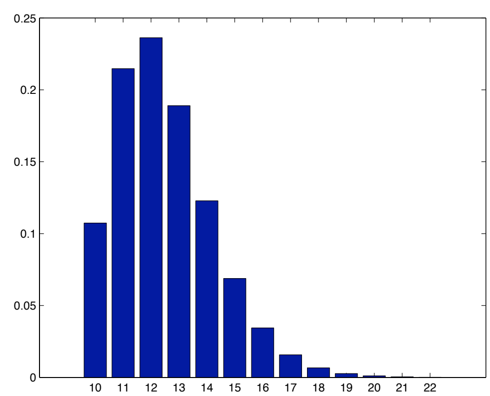
$NegBin(4, .5)$



$NegBin(10, .5)$



$NegBin(10, .2)$



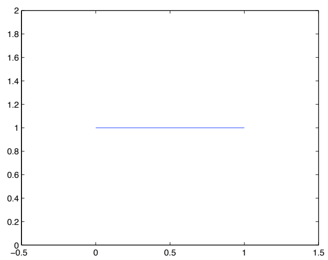
$NegBin(10, .8)$

The Uniform random variable $\mathcal{U}(a,b), a,b \in \mathbb{R}, a<b$

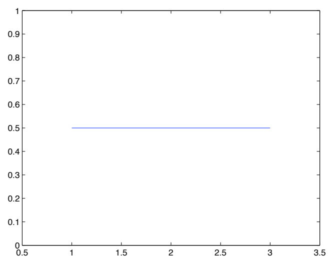
- Used to model experiments whose outcomes are real and where the probability of drawing a number from each interval contained in (a,b) is proportional to the length of that interval.
- Has density

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & otherwise \end{cases}$$

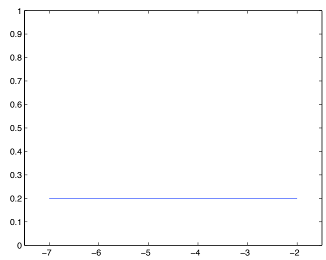
- Has mean $\frac{a+b}{2}$, variance $\frac{(b-a)^2}{12}$.



$\mathcal{U}(0,1)$



$\mathcal{U}(1,3)$



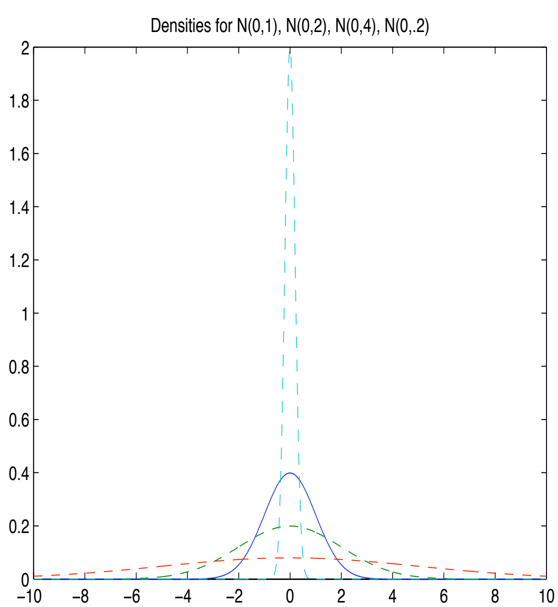
$\mathcal{U}(-7,-2)$

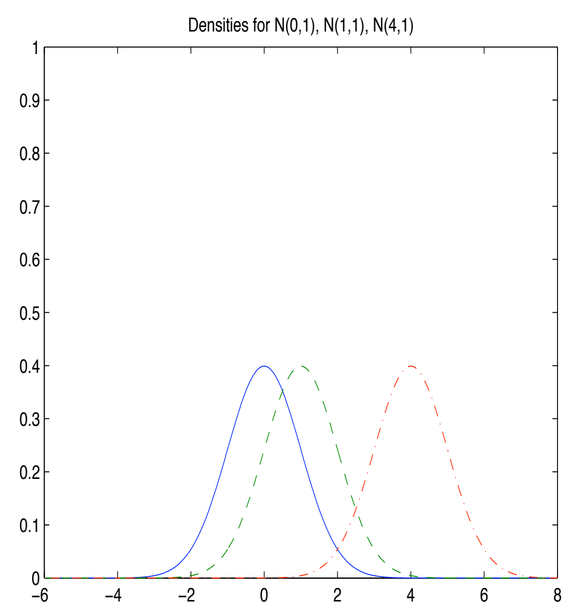
The Normal random variable $\mathcal{N}(\mu,\sigma^2), \mu,\sigma \in \mathbb{R}$

- Is used to model experiments which consist of sums of independent random experiments.
- Is therefore related to pretty much *every* random variable, through the Central Limit Theorem.
- Has density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}.$$

- The normal density is symmetric about the axis $x=\mu$.
- Has mean μ, variance σ^2.



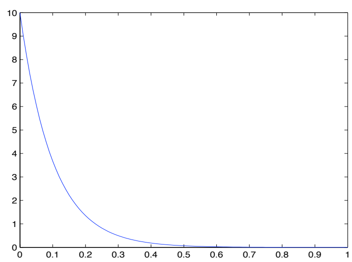


The Exponential random variable $Exp(\lambda), \lambda > 0$

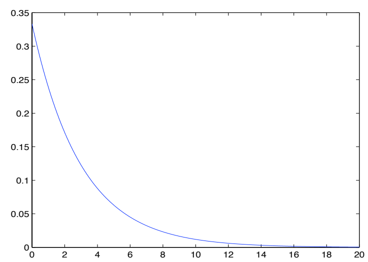
- Used to model the amount of time until an event occurs.
- Has density

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

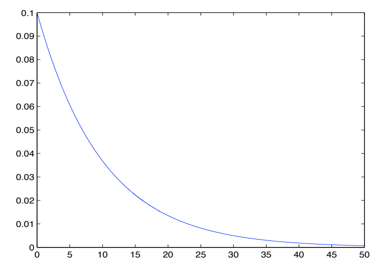
- Has the lack of memory property.
- Is related to the Gamma random variable as follows. $Exp(\lambda)$ has the same distribution as $\Gamma(1, \lambda)$. Moreover, if $X_i \sim Exp(\lambda), i = 1, 2, ...$ are i.i.d. , then $\sum_{i=1}^{n} X_i \sim \Gamma(n, \lambda)$
- Has mean $\frac{1}{\lambda}$, variance $\frac{1}{\lambda^2}$.



$Exp(.1)$



$Exp(3)$



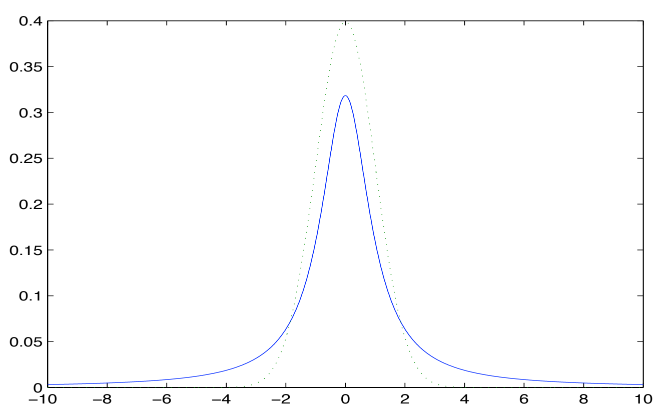
$Exp(10)$

The Standard Cauchy random variable

- Has density

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}$$

- Is symmetric about the y-axis.
- Is connected to the normal: If X and Y are independent standard normals, then $\frac{X}{Y}$ is Cauchy. (See TE 35, Chapter 6.)
- If X is a Cauchy r.v., $\frac{1}{X}$ is Cauchy as well. (See self-test problem 5.16.)



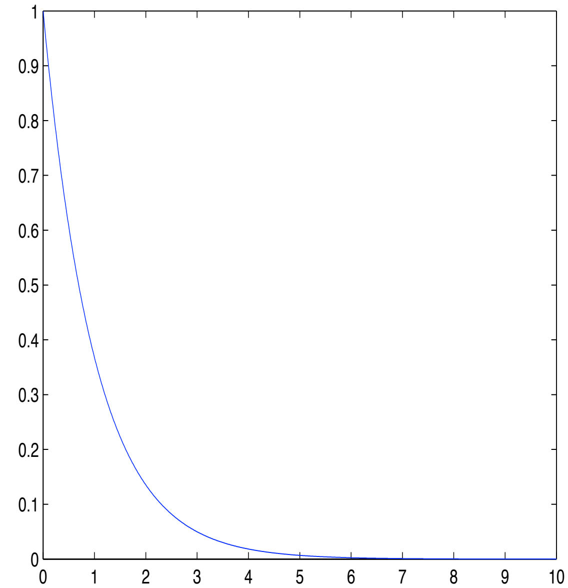
The Cauchy density drawn together with the standard normal. (Which is which?)

The Gamma random variable $\Gamma(\alpha, \lambda)$ $(\alpha > 0, \lambda > 0)$

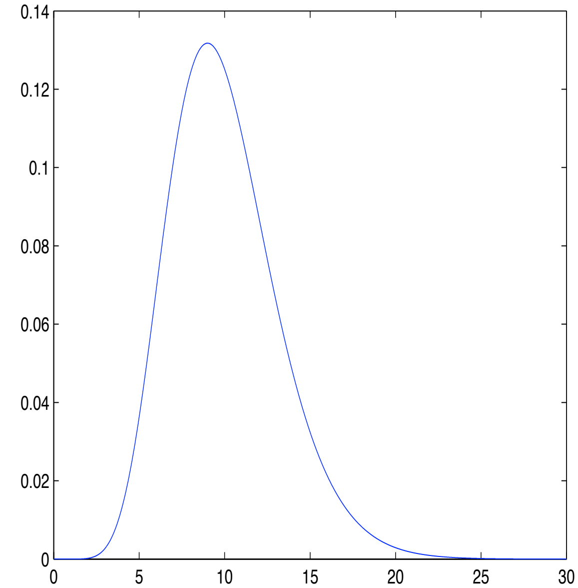
- When α is an integer, the Gamma r.v. is a sum of Exponential r.v.s, in which case it can be used to model the amount of time until α events occur.
- Has density

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

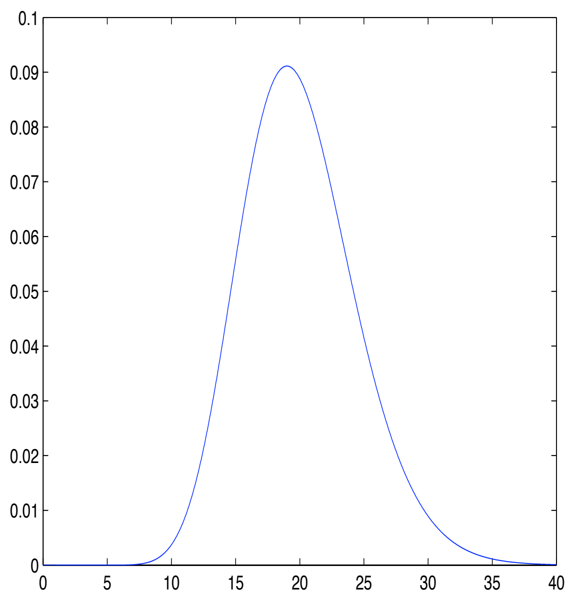
- Has mean $\frac{\alpha}{\lambda}$, variance $\frac{\alpha}{\lambda^2}$.



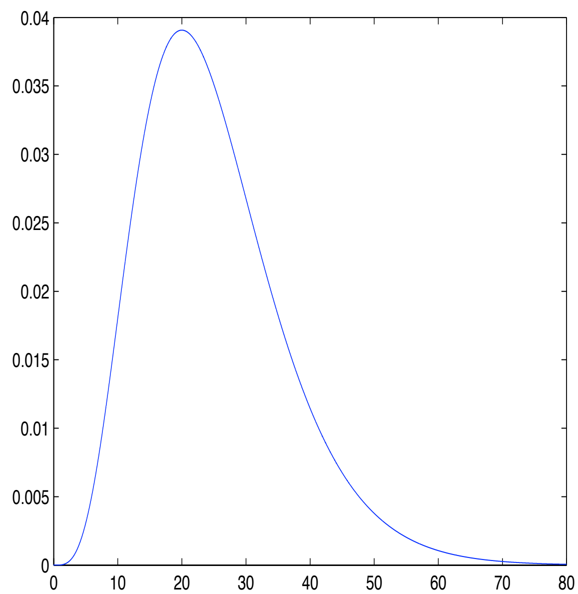
$\Gamma(1,1)$ $(= Exp(1))$



$\Gamma(10,1)$



$\Gamma(20,1)$



$\Gamma(5,5)$

The Beta random variable $B(a,b), a,b > 0$

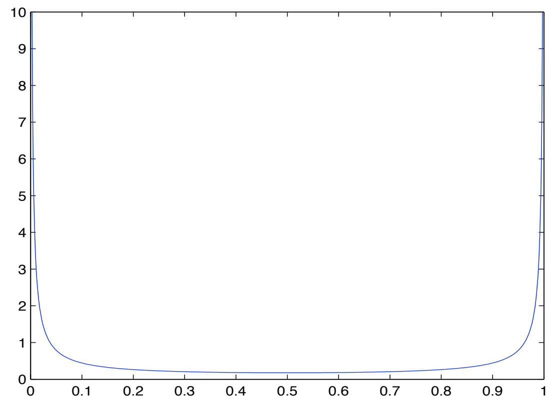
- Has density

$$f(x) = \begin{cases} \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, & 0 < x < 1 \\ 0, & otherwise \end{cases},$$

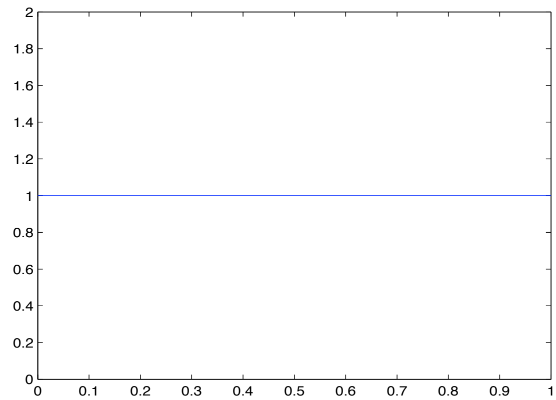
where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

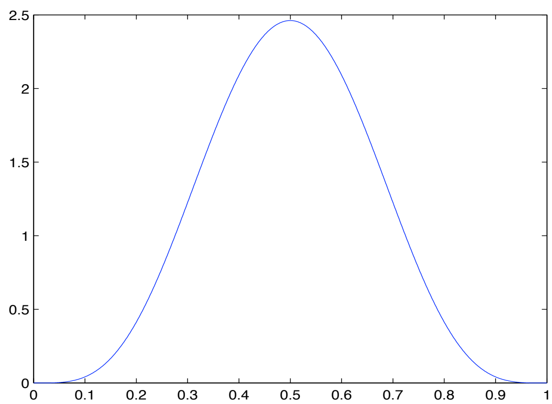
- Has mean $\frac{a}{a+b}$, variance $\frac{ab}{(a+b)^2(a+b+1)}$.



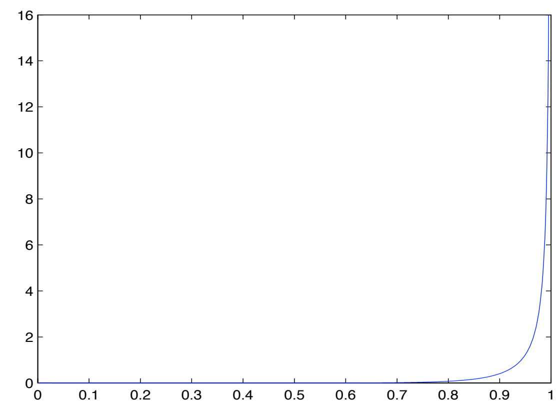
$Beta(.1,.1)$



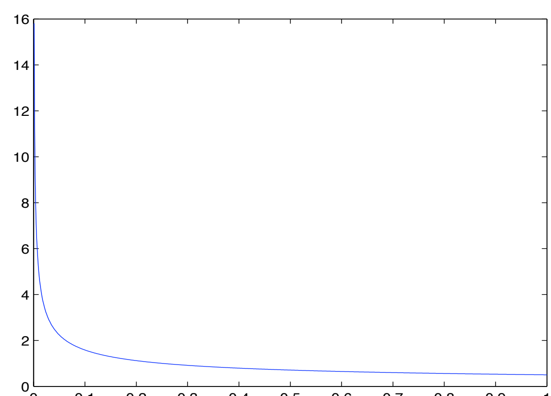
$Beta(1,1)\;\;(=\mathcal{U}(0,1))$



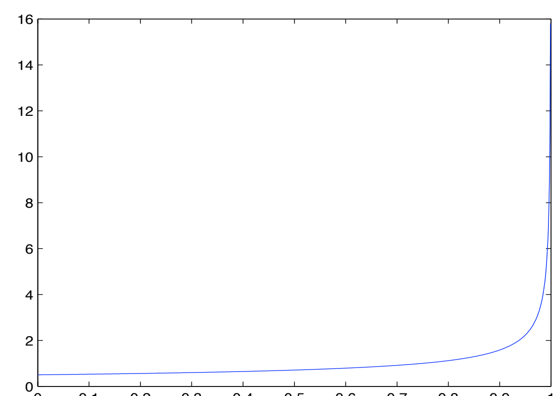
$Beta(5,5)$



$Beta(10,.1)$



$Beta(.5,1)$



$Beta(1,.5)$