Assignment 4

(Due on Monday, 12-16 (if you can't send it to me electronically, please let me know))

Below is the list of problems I recommend you do. Please hand in only those marked with an asterisk.

1. * Prove that if $X_n \Rightarrow X$ and $X = C$ a.s., where C is a constant, then $X_n \xrightarrow{P} X$.

2. A distribution F is called *infinitely divisible* if, whenever a random variable X has distribution F, for every $n \in \mathbb{N}$, there exists a family of n i.i.d random variables $\{X_i\}_{i=1}^n$ such that X has the same distribution as $\sum_{i=1}^{n} X_i$.

   (a) * Use characteristic functions to show that if $X \sim Po(\lambda)$, then X is infinitely divisible. [Note: You may use the characteristic function of the Poisson distribution, without deriving it.]

   (b) * Let Y_n be a Poisson random variable with parameter n. Show that

   $$\frac{Y_n}{n} \to 1, \text{ a.s.}$$

   (c) * Prove that the Cauchy distribution is infinitely divisible. [Note: You may use the characteristic function of the Cauchy distribution, without deriving it.]

   (d) A random variable X has the gamma distribution with parameters $\alpha, \beta > 0$ if it has density

   $$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0.$$

   Use contour integration on the boundary of the bounded region determined by the curves $y = 0, x = c, y = -\frac{tx}{a}, x = d$, where $0 < c < d$, to show that

   $$\phi_X(t) = \frac{\beta^\alpha}{(\beta - it)^\alpha}.$$

   Use this to prove that X is infinitely divisible. [Note: Two particular cases of the gamma distribution are the exponential distribution (when $\alpha = 1$) and the chi-square distribution with k degrees of freedom (when $\alpha = k/2, k \in \mathbb{N}$ and $\beta = 1/2$).]

3. A distribution F is called *stable* if for every $n \in \mathbb{N}$, there exist constants a_n, b_n such that if $\{X_i\}_{i=1}^n$ is a family of n i.i.d random variables with distribution F, then

$$b_n + a_n^{-1}\sum_{i=1}^{n} X_i$$

also has distribution F.

   (a) Show that a stable law is infinitely divisible.

   (b) Show that an infinitely divisible law is not necessarily stable.

   (c) Show that the Cauchy distribution is stable.

   (d) Show that the normal distribution is stable.

4. Suppose S and T are stopping times with respect to a same filtration $\mathcal{F}_t$. Prove that $S \vee T$ is a stopping time with respect to $\mathcal{F}_t$.

5. Suppose that $\{X_i\}_{i\geq 1}$ are independent standard normal random variables. Show that the process M defined by

$$M_n = \exp\left\{(\sum_{i=1}^{n} X_i) - n/2\right\}$$

is a martingale with respect to the filtration generated by $\{X_i\}_{i\geq 1}$.

6. * Suppose that $\{X_i\}_{i\geq 1}$ are independent and identically distributed random variables with $P\{X_1 = 1\} = 1 - P\{X_1 = -1\} = p$ for some $0 < p < 1/2$. Let $S_n = \sum_{i=1}^{n} X_i$ and define $T = \inf\{n \geq 0 : S_n = 1\}$.

   (a) Give a heuristic argument for whether $E[T] < \infty$ or $E[T] = \infty$.

   (b) Give a rigorous argument.

7. * Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables with $P(X_0 = 0) = 1$ and for all $n \geq 1, \mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and

$$P(X_{n+1} = X_n + \frac{1-|X_n|}{2}|\mathcal{F}_n) = P(X_{n+1} = X_n - \frac{1-|X_n|}{2}|\mathcal{F}_n) = \frac{1}{2}.$$

   (a) Prove that X is a martingale.

   (b) Prove that X converges almost surely.

   (c) What is the distribution of $X_\infty := \lim_{n\to\infty} X_n$?