Assignment 3 - Some Solutions

These solutions are written with varying levels of detail. They are meant to give you the key ideas needed to solve some of the problems. If you'd like more details, let me know and we can talk about the problems.

- Exercise 2.2.1: First note that $Var(X_m)/m \to 0$ implies that for any $\epsilon > 0$, there is an $A < \infty$ so that $Var(X_m) \leq A + \epsilon m$. Using this estimate and the fact that $\sum_{m=1}^{n} m \leq \sum_{m=1}^{n} (2m-1) = n^2$,

  $$E[(S_n/n - \nu_n)^2] = \frac{1}{n^2}\sum_{m=1}^{n} Var(X_m) \leq A/n + \epsilon.$$

  Since ϵ is arbitrary, this shows the L^2 convergence of $S_n/n - \nu_n$ to 0, and since L^2 convergence implies convergence in probability, we have that type of convergence as well. [Note: Alternatively, you can apply Toeplitz's lemma to estimate $\frac{1}{n^2}\sum_{m=1}^{n} Var(X_m)$.]

- Exercise 2.2.2: Let $\epsilon > 0$ and pick K so that if $k \geq K$, then $r(k) \leq \epsilon$. Noting that Cauchy-Schwarz implies $E[X_i X_j] \leq (E[X_i^2]E[X_j^2])^{1/2} = E[X_k^2] = r(0)$ and breaking the sum into $|i-j| \leq K$ and $|i-j| > K$, we have

  $$E[S_n^2] = \sum_{1 \leq i,j \leq n} E[X_i X_j] \leq n(2K+1)r(0) + n^2\epsilon.$$

  Dividing by n we see lim sup $E[S_n^2/n^2] \leq \epsilon$. Since ϵ is arbitrary we have $S_n/n \to 0$ in L^2 and therefore in probability.

- Exercise 2.2.6: Clearly, $X = \sum_{n=1}^{X} 1 = \sum_{n=1}^{\infty} \mathbb{1}\{X \geq n\}$, so taking expected values proves (i). For (ii), we consider the squares $[0,k]^2$ to get $X^2 = \sum_{n=1}^{\infty} (2n-1)\mathbb{1}\{X \geq n\}$ and then take expected values to get $E[X^2] = \sum_{n=1}^{\infty} (2n-1)P(X \geq n)$.

- Exercise 2.3.9: Pick N_k so that if $m, n \geq N_k$, then $d(X_m, X_n) \leq 2^{-k}$. Given a subsequence $X_{n(m)}$, pick m_k increasing so that $n(m_k) \geq N_k$. Using Chebyshevs inequality with $\phi(z) = z/(1+z)$, we have

  $$P(|X_{n(m_k)} - X_{n(m_{k+1})}| > k^{-2}) \leq (k^2+1)2^{-k}.$$

  The right hand side is summable so the Borel-Cantelli lemma implies that for large k, we have $|X_{n(m_k)} - X_{n(m_{k+1})}| \leq k^{-2}$. Since $\sum_k k^{-2} < \infty$, this and the triangle inequality imply that

$X_{n(m_k)}$ converges a.s. to a limit X . To see that the limit does not depend on the subsequence note that if $X_{n'(m'_k)} \to X'$, then our original assumption implies $d(X_{n(m_k)}, X_{n'(m'_k)}) \to 0$, and the bounded convergence theorem implies $d(X, X') = 0$. The desired result now follows from Theorem 2.3.2 in Durrett.

- Exercise 2.3.16: Note that we can pick $\delta_n \to 0$ so that $P(|X_n - X| > \delta_n) \to 0$. Let $\omega \in \Omega$ with $P(\omega) = p > 0$. For large n we have $P(|X_n - X| > \delta_n) \le p/2$, so $|X_n(\omega) - X(\omega)| \le \delta_n \to 0$. If $\Omega_0 = \{\omega : P(\omega) > 0\}$, then $P(\Omega_0) = 1$ so we have proved the desired result.

  Proof of Lemma 7.4 from the lecture notes:

  We prove this in several steps:

  First let's show that

$$X_n \to X, a.s. \iff P\{\omega : X_n(\omega) \text{ is a Cauchy sequence } \} = 1. \tag{1}$$

Indeed,

$$\sup_{k,l \ge n} |X_k - X_l| \le \sup_{k \ge n} |X_k - X| + \sup_{l \ge n} |X_l - X|,$$

so if $X_n \to X$, a.s., then $\{X_n\}$ is a Cauchy sequence, a.s.

Now suppose that $\{X_n\}$ is a Cauchy sequence, a.s. Let $A = \{\omega : \{X_n(\omega)\} \text{ is not a Cauchy sequence}\}$. Then on $\Omega \setminus A, X_n$ is a Cauchy sequence, which converges since $\mathbb{R}$ is complete. Since $P(\Omega \setminus A) = 1$, X_n converges almost surely. This proves (1)

Second,

$$P\{\omega : X_n(\omega) \text{ is a Cauchy sequence } \} = 1 \iff P\left(\sup_{k,l \ge n} |X_k - X_l| \ge \epsilon\right) \overset{n\to\infty}{\to} 0 \,\forall\, \epsilon > 0. \tag{2}$$

Indeed,

$$\{\omega : X_n(\omega) \text{ is not a Cauchy sequence } \} = \bigcup_{m \ge 1} \bigcap_{n \ge 1} \bigcup_{k,l \ge n} \left\{\omega : |X_k - X_l| \ge \frac{1}{m}\right\}.$$

So

$$\begin{aligned}
P\{X_n \text{ Cauchy}\} = 1 \quad &\iff \quad P\left(\bigcup_{m \ge 1} \bigcap_{n \ge 1} \bigcup_{k,l \ge n} \left\{\omega : |X_k - X_l| \ge \frac{1}{m}\right\}\right) = 0 \\
&\iff \quad P\left(\bigcap_{n \ge 1} \bigcup_{k,l \ge n} \left\{\omega : |X_k - X_l| \ge \frac{1}{m}\right\}\right) = 0 \,\forall\, m \ge 1 \\
&\iff \quad P\left(\bigcup_{k,l \ge n} \left\{\omega : |X_k - X_l| \ge \frac{1}{m}\right\}\right) \overset{n\to\infty}{\to} 0 \,\forall\, m \ge 1 \\
&\iff \quad P\left(\sup_{k,l \ge n} |X_k - X_l| \ge \epsilon\right) \overset{n\to\infty}{\to} 0 \,\forall\, \epsilon > 0
\end{aligned}$$

Finally,

$$P\left(\sup_{k,l \ge n} |X_k - X_l| \ge \epsilon\right) \overset{n\to\infty}{\to} 0 \iff P\left(\sup_{k \ge 0} |X_{n+k} - X_n| \ge \epsilon\right) \overset{n\to\infty}{\to} 0. \tag{3}$$

This just follows from

$$\sup_{k\geq 0}|X_{n+k}-X_n| \leq \sup_{k,l\geq 0}|X_{n+k}-X_{n+l}| \leq 2\sup_{k\geq 0}|X_{n+k}-X_n|.$$

Now one just needs to combine (1), (2), and (3)