Assignment 2 - Some Solutions

- Exercise 1.6.14*: $\frac{y}{X}\mathbb{1}_{(X>y)} \le 1$ and converges to 0 a.s. as $y \to \infty$, so the first result follows from the bounded convergence theorem. To prove the second result, we use our first observation to see that if $0 < y < \epsilon$,

  $$E(y/X; X > y) \le P(0 < X < \epsilon) + E(y/X; X \ge \epsilon).$$

  On $\{X \ge \epsilon\}, y/X \le y/\epsilon \le 1$ and $y/X \to 0$ so the bounded convergence theorem implies

  $$\limsup_{y\to 0} E(y/X; X > y) \le P(0 < X < \epsilon).$$

  Since ϵ is arbitrary,

  $$\lim_{y\to 0} E(y/X; X > y) = 0.$$

- Exercise 2.1.7*: $E[X_n] = \int_0^1 \sin(2\pi nx)\,dx = -(2\pi n)^{-1}\cos(2\pi nx)|_0^1 = 0$. Integrating by parts twice,

  $$\begin{aligned} E[X_m X_n] &= \int_0^1 \sin(2\pi mx)\sin(2\pi nx)\,dx \\ &= \frac{m}{n}\int_0^1 \cos(2\pi mx)\cos(2\pi nx)dx = \frac{m^2}{n^2}\int_0^1 \sin(2\pi mx)\sin(2\pi nx)\,dx, \end{aligned}$$

  so if $m \ne n, E[X_m X_n] = 0 = E[X_m]E[X_n]$.

  To see that X_m and X_n are not independent, note that $X_m(x) = 0$ when $x = k/2m, 0 \le k < 2m$, and on this set, $X_n(x)$ takes on the values $V_n = \{y_0, y_1, ...y_{2m-1}\}$. Let $[a,b] \subset [-1,1] \setminus V_n$ with $a < b$. Continuity of sin implies that if $\epsilon > 0$ is suffciently small, we have

  $$P(X_m \in [0,\epsilon], X_n \in [a,b]) = 0 < P(X_m \in [0,\epsilon])P(X_n \in [a,b]).$$

- Additional Exercise 1*

  1. We need the following

     Lemma: If $X_n \xrightarrow{P} X$, as $n \to \infty$, then there exists a subsequence X_{n_k} such that $X_{n_k} \to X$, almost surely, as $k \to \infty$.

     Proof. (Idea) We just need to construct a sequence $\{n_k\}$ such that $P(|X_{n_k} - X| > 2^{-k}) < 2^{-k}$ (it's easy to show that this is possible. Then, by Borel-Cantelli,

     $$P(|X_{n_k} - X| > 2^{-k}, \text{ i.o.}) = 0,$$

     which implies that $X_{n_k} \xrightarrow{a.s.} X$. □

Since X_n is increasing, for every $\omega \in \Omega$, the sequence $\{X_n(\omega)\}_{n\geq 1}$ has a limit $X(\omega)$. By the lemma, there exist a sequence $\{n_k\}_{k\geq 1}$ such that the set $C = \{\omega : X_{n_k}(\omega) \text{ converges}\}$ satisfies $P(C) = 1$. By uniqueness of limits, for all $\omega \in C, X_n(\omega) \to X(\omega)$.

2. For $\epsilon > 0$, define $A_n^\epsilon = \{\omega : |X_n - X| \geq \epsilon\}, A^\epsilon = \limsup_{n\to\infty} A_n^\epsilon$. Then

$$\{\omega : X_n \not\to X\} = \bigcup_{m\geq 1} A^{1/m}.$$

So

$$\begin{aligned} X_n \to X \text{ a.s.} &\iff P(X_n \not\to X) = 0 \iff P\left(\cup_{m\geq 1} A^{1/m}\right) = 0 \\ &\iff P(A^{1/m}) = 0 \,\forall\, m \geq 1 \iff P(A^\epsilon) = 0 \,\forall\, \epsilon > 0 \\ &\iff \lim_{n\to\infty} P\left(\bigcup_{k\geq n} A_k^\epsilon\right) = 0 \iff \lim_{n\to\infty} P\left(\sup_{k\geq n} |X_k - X| \geq \epsilon\right) = 0 \end{aligned}$$

3. (a) Define $Y_n = X_1^2 \mathbb{1}\{X_1^2 < \epsilon n^2\}$. Then $Y_n \to X_1^2$, as $n \to \infty$ and $|Y_n| \leq X_1^2$. Since $E[X_1^2] < \infty$ (since $Var(X_1) < \infty$), we can use the dominated convergence theorem to conclude that

$$\int Y_n \, dP \to \int X_1^2 \, dP. \tag{1}$$

Now if we define $A_n = \{\omega : |X_1(\omega)| \geq \epsilon\sqrt{n}\}$, we get

$$E[X_1^2] = \int_{A_n} X_1^2 \, dP + \int_{A_n^c} X_1^2 \, dP \geq \epsilon_n^2 P(A_n) + \int Y_n \, dP.$$

Therefore, $\epsilon^2 n P(A_n) \leq \int X_1^2 \, dP - \int Y_n \, dP$, which with the help of (1) implies that $nP(|X_1| \geq \epsilon\sqrt{n}) \to 0$, as $n \to \infty$.

(b) This follows almost immediately from part (a):

$$\begin{aligned} P(\max_{1\leq i\leq n} X_i > \epsilon\sqrt{n}) &= P(\bigcup_{1\leq i\leq n} \{X_i > \epsilon\sqrt{n}\}) \leq \sum_{i=1}^n P(X_i > \epsilon\sqrt{n}) \\ &= nP(X_1 > \epsilon\sqrt{n}) \leq nP(|X_1| > \epsilon\sqrt{n}) \overset{n\to\infty}{\to} 0. \end{aligned}$$

4. Let $\phi(z) = |z|/(1 + |z|)$.

(a) Since $\phi(z) > 0$ for $z \neq 0, E[\phi(|X - Y|) = 0$ implies $\phi(|X - Y|) = 0$ a.s. and hence $X = Y$ a.s.

(b) It is obvious that $d(X, Y) = d(Y, X)$.

(c) The triangle inequality follows by noting that Exercise 4.10 in Durrett implies that $\phi(|X - Y|) + \phi(|Y - Z|) \geq \phi(|X - Z|)$ and then taking expected values.

Note that if $X_n \to X$ in probability, then since $\phi \leq 1$, the dominated convergence theorem implies $d(X_n, X) = E[\phi(|X_n - X|) \to 0$. To prove the converse let $\epsilon > 0$ and note that Chebyshev's inequality implies $P(|X_n - X| > \epsilon) \leq d(X_n, X)/\phi(\epsilon) \to 0$.