Assignment 2

(Due on Tuesday, 10-15)

Below is the list of problems I expect you to do. Please hand in only those marked with an asterisk. All numbers of non-additional exercises refer to the fourth edition of "Probability: Theory and Examples".

- Exercises 1.6.3, 1.6.6, 1.6.14*, 1.6.15, 2.1.2, 2.1.7*
- Additional Exercise 1* (Convergence: almost sure and in probability):
  1. Suppose $\{X_n\}_{n\geq 1}$ is an increasing sequence of r.v.'s satisfying $X_n \overset{P}{\to} X$. Show that $X_n \overset{a.s.}{\to} X$.
  2. Suppose $\{X_n\}_{n\geq 1}$ is a sequence of r.v.'s. Show that
  $$X_n \overset{a.s.}{\to} X \iff \sup_{k\geq n} |X_k - X| \overset{P}{\to} 0, \text{ as } n \to \infty.$$
  3. Suppose $\{X_n\}_{n\geq 1}$ is a sequence of identically distributed r.v.'s with finite variance. Show that
     (a) for every $\epsilon > 0, nP(|X_1| \geq \epsilon\sqrt{n}) \to 0$, as $n \to \infty$
     (b) $\max_{1\leq i\leq n} |X_i|/\sqrt{n} \overset{P}{\to} 0$, as $n \to \infty$.
  4. Given two random variables X and Y, define
  $$d(X,Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right].$$
  Show that d is a metric and that it metrizes convergence in probability:
  $$X_n \overset{P}{\to} X \iff d(X_n, X) \to 0.$$
- Additional Exercise 2* (L^p spaces and convergence):
  1. Let $\{X_n\}_{n\geq 1}$ be a sequence of nonnegative random variables such that $X_n \overset{P}{\to} X$ and $E[X_n] \to E[X]$. Show that $X_n \overset{L^1}{\to} X$.
  2. Give an example of a sequence $\{X_n\}_{n\geq 1}$ such that X_n converges in L^1, but does not converge in L^2.
  3. Show that if $q < \infty, \|X\|_q \leq \|X\|_\infty$.
  4. Show that Hölder's and Minkowski's inequalities hold also if $q = \infty$: If X, Y are r.v.'s,
  $$E[|XY|] \leq \|X\|_1 \|Y\|_\infty \text{ and } \|X+Y\|_\infty \leq \|X\|_\infty + \|Y\|_\infty.$$