Assignment 1

Selected Solutions

These solutions are meant to give you the key ideas needed to solve some of the problems. If you'd like more details, let me know and we can talk about the problems.

- Exercise 1.1.6*: $\mathcal{A}$ is not an algebra and therefore not a sigma-algebra. Indeed, define $A = \{2k\}_{k\in\mathbb{N}}$. Then if $n = 2k$,
$$\frac{|A \cap \{1, 2, \dots, n\}|}{n} = \frac{1}{2}$$
and if $n = 2k + 1$,
$$\frac{|A \cap \{1, 2, \dots, n\}|}{n} = \frac{k}{2k+1}.$$
Therefore,
$$\lim_{n\to\infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n} = \frac{1}{2}.$$
Now for $n \geq 0$, let B_n be the set of even integers in $[2^{2n}, 2^{2n+1}]$ and odd integers in $(2^{2n+1}, 2^{2n+2})$ and define $B = \bigcup_{n\geq 0} B_n$. Then if $n = 2^{2k+1}$,
$$|B \cap \{1, \dots, n\}| = 1 + \sum_{i=1}^{k} 2^{2i-1} + 1 + \sum_{i=0}^{k-1} 2^{2i} = 2^{2k} + k,$$
implying that
$$\frac{|B \cap \{1, \dots, n\}|}{n} \to \frac{1}{2}, \text{ as } k \to \infty.$$
Similarly, if $n = 2^{2k+2}$,
$$|B \cap \{1, \dots, n\}| = 1 + \sum_{i=1}^{k} 2^{2i-1} + 1 + \sum_{i=0}^{k} 2^{2i} = 2^{2k+1} + k,$$
implying that
$$\frac{|B \cap \{1, \dots, n\}|}{n} \to \frac{1}{2}, \text{ as } k \to \infty.$$
Monotonicity now implies that
$$\frac{|B \cap \{1, \dots, n\}|}{n} \to \frac{1}{2}, \text{ as } n \to \infty.$$
Now define $C = A \cap B$. Then $C = \bigcup_{n\geq 0} C_n$, where C_n is the set of even integers in $[2^{2n}, 2^{2n+1}]$. Then you can check (and should do it as carefully as in the previous steps) as above that if $n = 2^{2k+1}$,
$$\frac{|C \cap \{1, \dots, n\}|}{n} \to \frac{1}{3}, \text{ as } k \to \infty,$$

and if $n = 2^{2k+2}$,

$$\frac{|C \cap \{1, \ldots, n\}|}{n} \to \frac{1}{6}, \text{ as } k \to \infty,$$

Since $\limsup \frac{|C \cap \{1,\ldots,n\}|}{n} \neq \liminf \frac{|C \cap \{1,\ldots,n\}|}{n}$, the limit doesn't exist, so $C \notin \mathcal{A}$.

- Exercise 1.2.4*: The key here is to define appropriately F^{-1} (which should not completely be thought of as the inverse of F, as F may not be strictly increasing). Let

$$F^{-1}(y) = \sup\{x : F(x) = y\}.$$

Then continuity of F implies that $F(F^{-1}(y)) = y$ (though $F^{-1}(F(y)) \neq y$ in general). The definition of F^{-1} implies that $F(X) \leq y \iff X \leq F^{-1}(y)$. Therefore, if $0 \leq y \leq 1$,

$$P(Y \leq y) = P(F(X) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y.$$

- Exercise 1.2.7: (i) Let $F(x) = P(X \leq x)$. Then if $y \geq 0$, $P(X^2 \leq y) = F(\sqrt{y}) - F(-\sqrt{y})$. Differentiating gives the density function g for X^2:

$$g(y) = \frac{f(\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}}.$$

(ii) If f is the normal density function, the previous expression gives the density function for the χ^2 distribution:

$$g(x) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}.$$

- Exercise 1.3.3: Define $\Omega_0 = \{\omega : X_n(\omega) \to X(\omega)\}$. If $\omega \in \Omega_0$, the definition of continuity implies that $f(X_n) \to f(X)$. Since $P(\Omega_0) = 1$, $f(X_n) \stackrel{a.s.}{\to} f(X)$.

- Additional Exercise 2:

Suppose that $B \in \mathcal{B}$ so that

$$X^{-1}(B) = \begin{cases} \Omega, & \text{if } 1 \in B, 0 \in B, \\ \{a, b\}, & \text{if } 1 \in B, 0 \notin B, \\ \{c, d, e\}, & \text{if } 1 \notin B, 0 \in B, \\ \emptyset, & \text{if } 1 \notin B, 0 \notin B. \end{cases}$$

Thus, the σ-algebra generated by X is $\sigma(X) = \{\emptyset, \Omega, \{a, b\}, \{c, d, e\}\}$. Similarly,

$$Y^{-1}(B) = \begin{cases} \Omega, & \text{if } 2 \in B, 0 \in B, \\ \{a, c\}, & \text{if } 2 \in B, 0 \notin B, \\ \{b, d, e\}, & \text{if } 2 \notin B, 0 \in B, \\ \emptyset, & \text{if } 2 \notin B, 0 \notin B. \end{cases}$$

so that $\sigma(Y) = \{\emptyset, \Omega, \{a, c\}, \{b, d, e\}\}$.