Random Fractals

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Outline

1. Fractals

2. Random Fractals
   - Random Walk
   - Percolation
   - Loop-erased Random Walk
   - The Schramm-Loewner Evolution

Smooth curves

Most curves you encounter in your mathematics classes have the same behavior when you zoom in (at least at most points).

http://www.mathopenref.com/graphfunctions.html

If you zoom in repeatedly, they eventually become completely straight. This means you can approximate such curves locally by straight lines and use this to calculate their length.

To calculate the length of a smooth curve, one approximates the curve locally by progressively shorter straight lines. In the limit the approximations converge to a real number.

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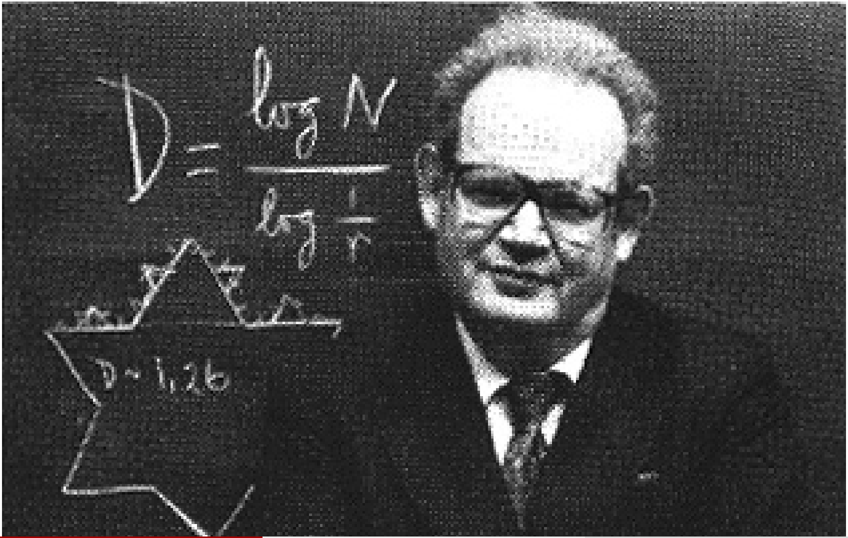
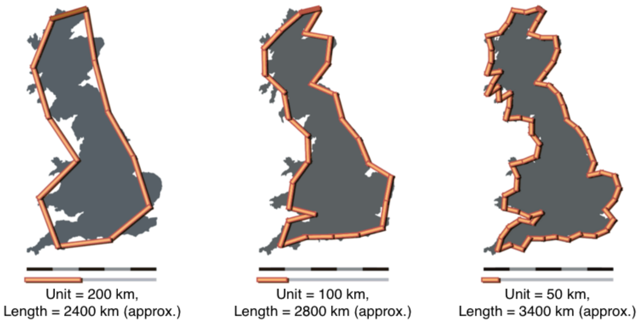
Mandelbrot's Observation

In 1967, Benoît Mandelbrot asked: How long is the coast of Britain?



Mandelbrot's Observation

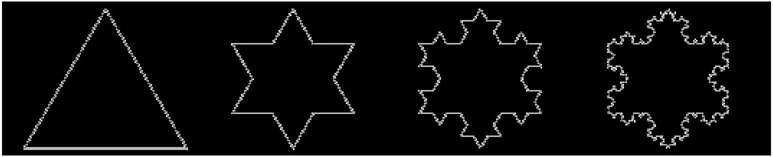
It turns out the answer is not so straightforward:



As one takes shorter and shorter lines to approximate the length, one gets larger and larger values and there appears to be no convergence. This suggests that the length of the coast of Britain is infinite.

The Koch Snowflake

Let's look at the following simple construction:



1. Start with an equilateral triangle of side length 1.
2. Place on the middle third of each existing side of the figure an equilateral triangle, one side of which is the middle third segment. Remove that middle third segment.
3. Repeat Step 2, applying it to the new figure obtained.

How Long Is the Koch Snowflake?

snowflake approx.	#(segments)	length/segment	length(ℓ_n)
S_0	$3 = 3 \cdot 4^0$	1	3
S_1	$12 = 3 \cdot 4^1$	$1/3$	4
S_2	$48 = 3 \cdot 4^2$	$1/3^2$	$48/9$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_n	$3 \cdot 4^n$	$1/3^n$	$3(4/3)^n$

The Koch snowflake is defined to be the limit of the sequence of approximations S_n. We can call it S_∞. The length of S_∞ is therefore

$$\lim_{n\to\infty} \ell_n = \infty.$$

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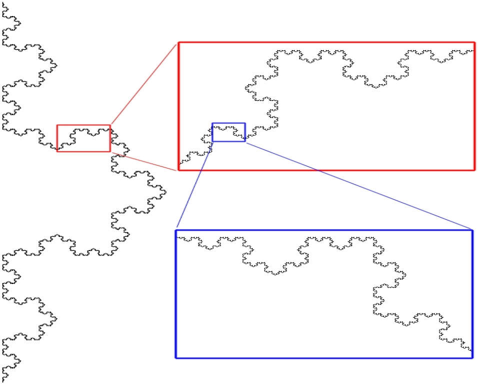
The Koch Snowflake

The Koch snowflake satisfies the *self-similarity* property: It is composed of several smaller versions of itself.



http://upload.wikimedia.org/wikipedia/commons/6/65/Kochsim.gif

Fractals

Mandelbrot introduced the term *fractal* to characterize geometric objects such as those we just saw. There is no absolute consensus as to what defines a fractal, but here are some properties it should satisfy:

- self-similarity (in some possibly vaguer sense than for the Koch Snowflake)
- ruggedness at all scales

Some Examples of Fractals

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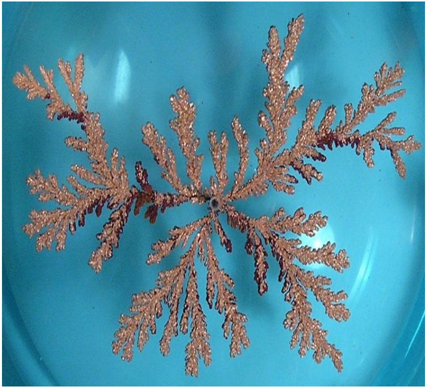
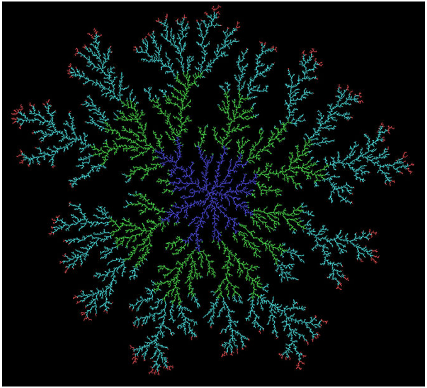
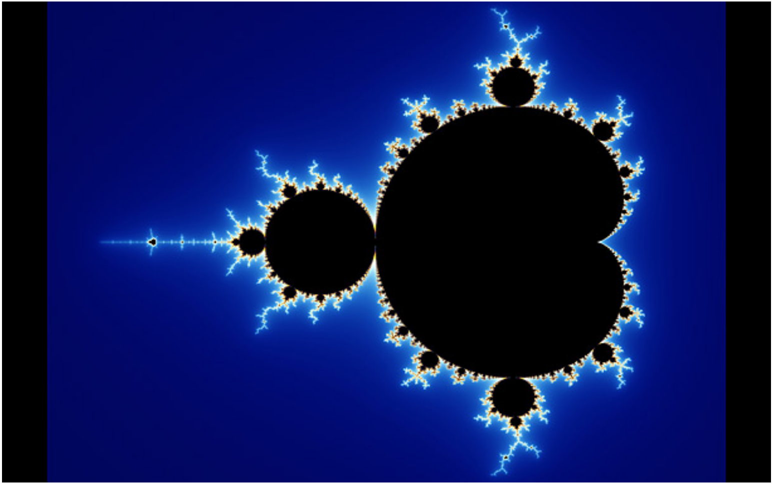


Real World



Computer Simulation

The Mandelbrot Set



http://vimeo.com/12185093

Fractal Dimension (box counting)

Lines are one-dimensional objects, planar domains are two-dimensional, and solids are three-dimensional.
The fact that the Koch curve takes considerably more space than a line of finite length suggests that it should perhaps be considered to be an object of dimension > 1.
So how can we define the dimension of set? We have a very good understanding for what dimensions 0, 1, 2, and 3 mean. Here is one systematic way of determining the dimension of a set S:
If S is a set in $\mathbb{R}^n$, we can count the number of balls of radius $1/k$ (or cubes of side length $1/k$) needed to cover S, for $k = 1, 2, 3, \ldots$. Let's do this for a few very easy sets:

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Fractal Dimension (box counting)

- $[0,1] \subset \mathbb{R}$: A 1-dimensional cube is a segment, so how many segments of length $1/k$ do we need to cover $[0,1]$? The answer is $k = k^{\boxed{1}}$
- $[0,10] \subset \mathbb{R}$: The answer here is $10k = 10 \cdot k^{\boxed{1}}$.
- $[0,1] \times [0,1] \subset \mathbb{R}^2$: Here we need $k \cdot k = k^{\boxed{2}}$ squares (i.e., 2-dimensional cubes).

Note that the boxed numbers, i.e., the *exponents* are the dimension of these objects (in the classical sense). So let's do the same thing for the Koch curve:

Fractal Dimension of the Koch Snowflake

We want to know how many disks of radius $1/k$ are needed to cover the Koch snowflake. Let's simplify the problem a bit and look at S_n, the level n approximation to the snowflake.

S_n is composed of $3 \cdot 4^n$ segments of length $1/3^n$, so we can cover S_n with $\frac{3}{2}4^n$ disks (we can fit two segments in each disk) of radius $\frac{1}{3^n} = \frac{1}{k}$ (we're using 3^n to play the role of k in the general setup).

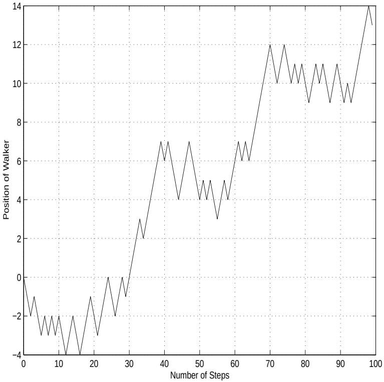
Now note that

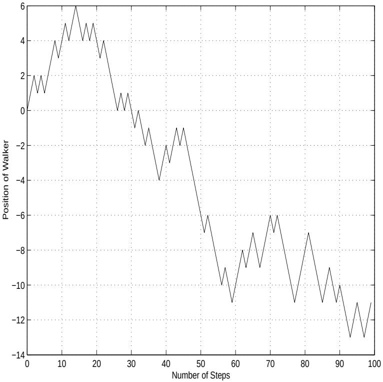
$$\frac{3}{2}4^n = \frac{3}{2}3^{\frac{\log 4}{\log 3}n} = \frac{3}{2}(3^n)^{\frac{\log 4}{\log 3}} = \frac{3}{2}k^{\boxed{\frac{\log 4}{\log 3}}}.$$

So the box counting dimension of the Koch snowflake is $\frac{\log 4}{\log 3}$. This is a measure of how much space the curve takes. It's more than a 1-dim. set, but less than a 2-dim. set.

Random Walk

Let X_i be independent random variables satisfying $P(X_i = 1) = p,\ P(X_i = -1) = 1 - p$. Then $S_n = \sum_{i=1}^{n} X_i$ is called a **simple random walk** on the one-dimensional integer lattice. We will define $S_0 = 0$ (meaning that the random walk starts at the origin).

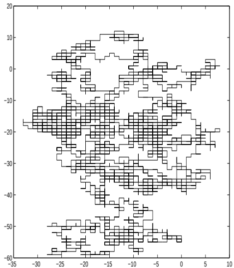


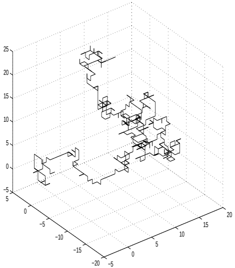


Two realizations of a 100-step one-dimensional random walk.

2 and 3 Dimensions

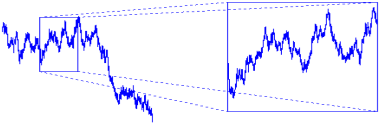
Let X_i be independent random vectors in $\mathbb{Z}^2$ satisfying $P(X_i = \pm\langle 1,0\rangle) = P(X_i = \pm\langle 0,1\rangle) = \frac{1}{4}$. Then $S_n = \sum_{i=1}^{n} X_i$ a **symmetric random walk** in $\mathbb{Z}^2$. Similarly, if X_i are independent random vectors in $\mathbb{Z}^3$ satisfying $P(X_i = \pm\langle 1,0,0\rangle) = P(X_i = \pm\langle 0,1,0\rangle) = P(X_i = \pm\langle 0,0,1\rangle) = \frac{1}{6}$. Then $S_n = \sum_{i=1}^{n} X_i$ a **symmetric random walk** in $\mathbb{Z}^3$.





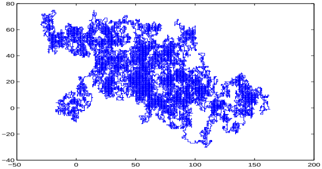
Brownian Motion

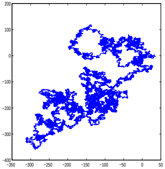
Random walk is not fractal, since it is composed of straight lines. However, if you take a random walk, run it for a very long time and compress it to fit on this screen, in the limit you get a process called *Brownian motion*. It is a *self-similar* process.



Brownian Motion

Brownian motion is a fractal. It is the "continuum limit" of random walk. It isn't restricted to a lattice and is nicer to deal with in the same way integrals are nicer to deal with than sums.





It has been known for a long time that the fractal dimension of d-dimensional Brownian motion is:

- $3/2$ if $d = 1$ (actually this is the dimension of the graph of time versus the Brownian motion)
- 2 if $d \geq 2$.

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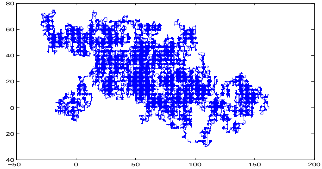

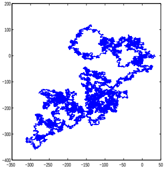

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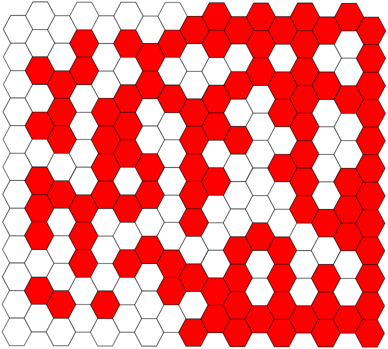
Percolation

Consider a lattice composed of hexagonal faces. Suppose each hexagon is colored red or white with probability 1/2, independently from one hexagon to the next. The configuration you get is called *critical percolation*. If you fix the right-side edges to be red and the left-side edges to be white, you will get a top-bottom interface.



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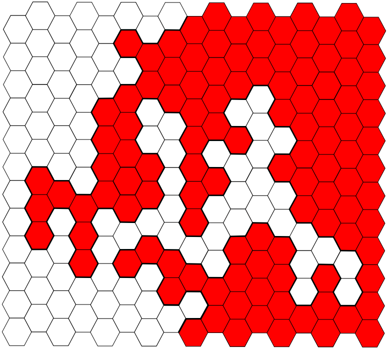


The interface curve you get is obviously not fractal, since it's composed of line segments, but if you take a much larger lattice box and shrink the picture, you will get in the limit a random curve, which it turns out is fractal.

Loop-erased Random Walk

`http://stat.math.uregina.ca/~kozdron/Simulations/LERW/LERW.html`

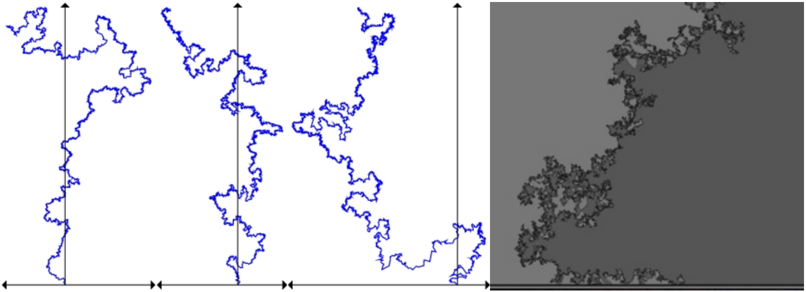
SLE

The scaling limit of the critical percolation exploration path and loop-erased random walk was not well understood until about this last decade. It turns out that both processes (as well as many others: Ising model, FK model, uniform spanning tree Peano curve, Gaussian free field interface, etc.) have scaling limits that belong to a family of curves describe by one same differential equation. This was the brilliant discovery of Oded Schramm, around 1999.

SLE

The chordal Schramm-Loewner Evolution (SLE) is defined to be the the curve (technical details omitted) obtained from the equation:

$$\partial g_t(z) = \frac{2}{g_t(z) - B_{\kappa t}}, \qquad g_0(z) = z.$$



$\kappa = 2$ $\quad \kappa = \frac{8}{3}$ $\quad \kappa = 3$ $\quad \kappa = 6$

SLE

Wendelin Werner received a Fields Medal in 2004 for showing (with Lawler and Schramm), among other things, that the scaling limit of loop-erased random walk is SLE(2). Stas Smirnov received a Fields Medal in 2008 for showing, among other things, that the scaling limit of critical percolation is SLE(6).