A Random Stroll through Random Lattice Models

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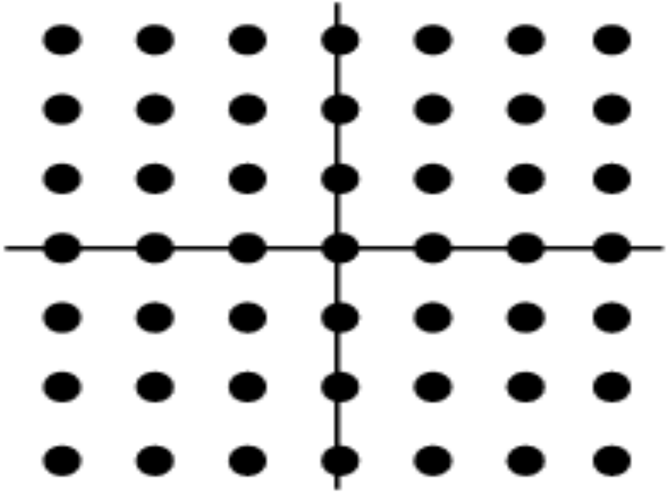
05-10-2011

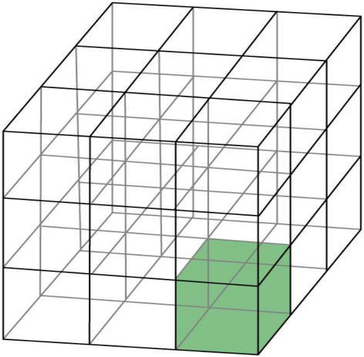
Outline

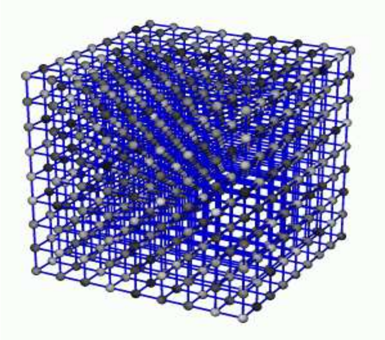
1. Lattices
2. Random Walk
3. Voter Model
4. Percolation

Lattices: $\mathbb{Z}, \mathbb{Z}^2$

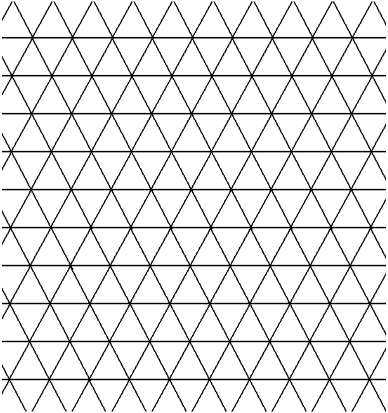


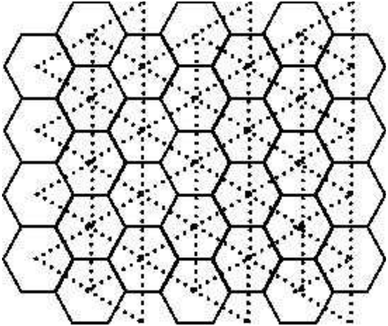
Lattices: $\mathbb{Z}, \mathbb{Z}^2$ and $\mathbb{Z}^3$



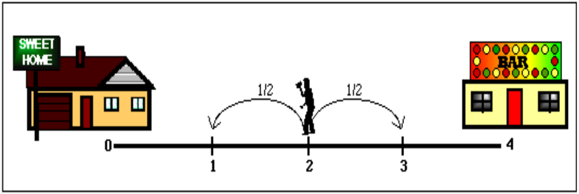


More lattices: triangular and honeycomb

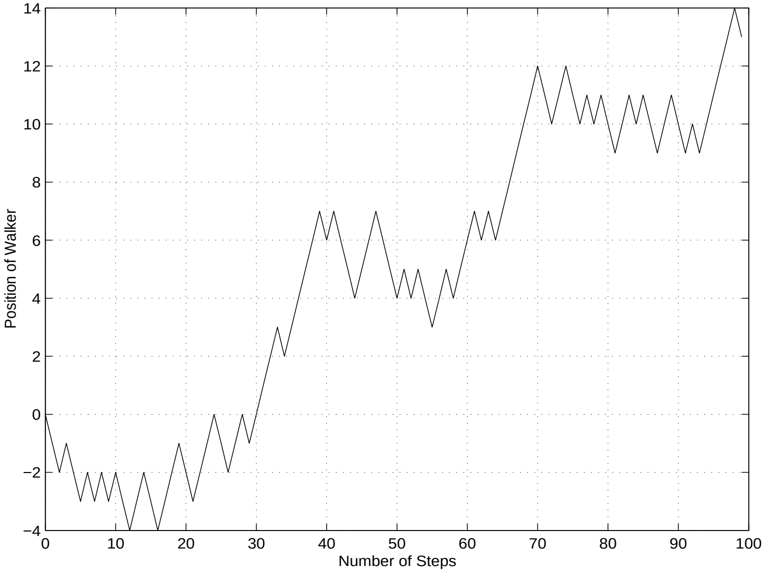




Random Walk



Let X_i be independent random variables satisfying $\mathbb{P}\{X_i = 1\} = p$, $\mathbb{P}\{X_i = -1\} = 1 - p$. Then $S_n = \sum_{i=1}^{n} X_i$ is called a **simple random walk** on the one-dimensional integer lattice. We will define $S_0 = 0$ (meaning that the random walk starts at the origin).


14
12
10
8
6
4
2
0
−2
−4
Position of Walker
0
10
20
30
40
50
60
70
80
90
100
Number of Steps

A random walk is called **recurrent** if

$$\mathbb{P}\{S_n = 0 \text{ for some } n > 0\} = 1.$$

Let

$$r = \mathbb{P}\{\text{returning to the origin}\} = \mathbb{P}\{S_n = 0 \text{ for some } n > 0\}$$

and

$$m = \text{the expected number of returns to the origin.}$$

The quantities r and m are related:

$$r = \mathbb{P}\{\text{returning to the origin}\} = \mathbb{P}\{S_n = 0 \text{ for some } n > 0\},$$

$$m = \text{the expected number of returns to the origin.}$$

$\mathbb{P}\{\text{returning to the origin exactly } k \text{ times}\} = r^k(1-r).$

$$\begin{aligned}
m &= \mathbb{E}[\text{number of returns to the origin }] \\
&= 1 \cdot \mathbb{P}\{\text{returning to the origin exactly 1 time}\} \\
&+ 2 \cdot \mathbb{P}\{\text{returning to the origin exactly 2 times}\} \\
&+ 3 \cdot \mathbb{P}\{\text{returning to the origin exactly 3 times}\} \\
&+ 4 \cdot \mathbb{P}\{\text{returning to the origin exactly 4 times}\} \\
&+ \ldots \\
&= r(1-r) + 2r^2(1-r) + 3r^3(1-r) + 4r^4(1-r) + \ldots \\
&= (1-r)\left(r + 2r^2 + 3r^3 + 4r^4 + \ldots\right) = \frac{r}{1-r}
\end{aligned}$$

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Since

$$m = \frac{r}{1-r},$$

we see that $r = 1$ if and only if m is infinite. So **the walker is certain to return to the origin if and only if m is infinite.**

We define $u_n = \mathbb{P}\{S_n = 0\}$,

$$A_n = \begin{cases} 1, & \text{if } S_n = 0 \\ 0, & \text{otherwise} \end{cases}$$

and $A = \sum_{i \geq 0} A_i$.

Note that $\mathbb{E}[A_n] = 1 \cdot u_n + 0 \cdot (1 - u_n) = u_n$ and that A is the total number of times the random walk is at the origin.

So

$$m = \mathbb{E}[A] = \sum_{i \geq 0} \mathbb{E}[A_i] = \sum_{i \geq 0} u_i.$$

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The random walk can only be at the origin at even times, so

$$m = u_0 + u_2 + u_4 + u_6 + \ldots = \sum_{n \geq 0} u_{2n}.$$

$$u_{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!} p^n (1-p)^n.$$

Stirling's formula ($n! \sim \sqrt{2\pi n} e^{-n} n^n$) implies

$$u_{2n} \sim \frac{\sqrt{2\pi 2n} e^{-2n} (2n)^{2n} p^n (1-p)^n}{(\sqrt{2\pi n} e^{-n} n^n)^2} = \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

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$$m = \sum_{n \geq 0} u_{2n} \sim \sum_{n \geq 0} \frac{(4p(1-p))^n}{\sqrt{\pi n}}.$$

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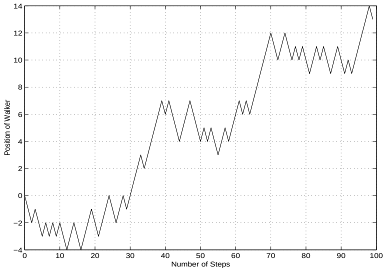
This means that $r = 1$ if and only if $p = \frac{1}{2}$, and so the random walk is certain to come back to the origin if and only if $p = \frac{1}{2}$. This gives the following result due to Pólya:

Theorem

If $p = \frac{1}{2}$, the random walk returns to the origin infinitely often. If $p \neq \frac{1}{2}$, the random walk returns to the origin only finitely many times.



Simple Random Walk of 100 steps

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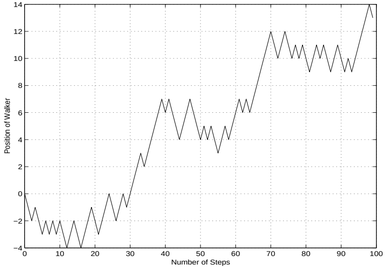

Simple Random Walk of 100 steps

Implications for (American) Roulette

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Expected gains if you bet one dollar on red:

$$\frac{18}{38} \cdot 1 + \frac{20}{38}(-1) = -\frac{1}{19}\$.$$

Thus on average, you will lose $\frac{1}{19}$ dollars per game! You will of course have the privilege of losing more (on average) if you bet more.

Polya's theorem implies that eventually, your gains (a random walk with $p = \frac{18}{38}$) will never be 0 again.

Gambler's Ruin

Suppose you go to Las Vegas with \$200 and play roulette by betting on red at every spin. You do this until you win \$200 or lose all your money. What is the chance that you'll win \$200?

Theorem

If you start a random walk (with probability p of going up, $q = 1 - p$ of going down by 1) at an integer $x > 0$ and decide to let the walk run until it reaches a pre-determined value of $y > x$ or 0, the probability of reaching y before 0 is

- $\frac{x}{y}$ if $p = \frac{1}{2}$.
- $\frac{\left(\frac{q}{p}\right)^x - 1}{\left(\frac{q}{p}\right)^y - 1}$ if $p \neq q$.

So all we have to do to answer the question is use $x = 200, y = 400, p = \frac{9}{19}, q = \frac{10}{19}$. This gives a probability of

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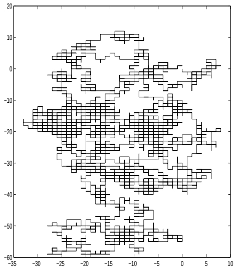
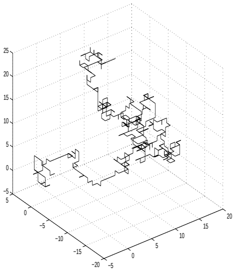
Gambler's Ruin

$$\boxed{\approx 7 \cdot 10^{-10}}$$

2 and 3 Dimensions

Let X_i be independent random vectors in $\mathbb{Z}^2$ satisfying $\mathbb{P}\{X_i = \pm\langle 1,0\rangle\} = \mathbb{P}\{X_i = \pm\langle 0,1\rangle\} = \frac{1}{4}$. Then $S_n = \sum_{i=1}^{n} X_i$ a **symmetric random walk** in $\mathbb{Z}^2$. Similarly, if X_i are independent random vectors in $\mathbb{Z}^3$ satisfying $\mathbb{P}\{X_i = \pm\langle 1,0,0\rangle\} = \mathbb{P}\{X_i = \pm\langle 0,1,0\rangle\} = \mathbb{P}\{X_i = \pm\langle 0,0,1\rangle\} = \frac{1}{6}$. Then $S_n = \sum_{i=1}^{n} X_i$ a **symmetric random walk** in $\mathbb{Z}^3$.





2 and 3 Dimensions

http://stat.math.uregina.ca/~kozdron/Simulations/LERW/LERW.html

2 and 3 Dimensions

We can use the same analysis as for the one-dimensional case:
In the 2-d case,

$$
\begin{aligned}
u_{2n} &= \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!}(\frac{1}{4})^{2n} \\
&= (\frac{1}{4})^{2n}\frac{(2n)!}{n!n!}\sum_{k=0}^{n}\frac{n!n!}{k!k!(n-k)!(n-k)!} \\
&= (\frac{1}{4})^{2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2 = \left((\frac{1}{2})^{2n}\binom{2n}{n}\right)^2 .
\end{aligned}
$$

This is the square of the 1-d case, and so

$$u_{2n} \sim \frac{1}{\pi n},$$

the sum of which diverges.

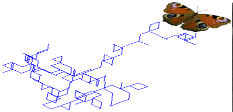
2 and 3 Dimensions

For the 3d case, we get (for some constant K)

$$u_{2n} \leq \frac{K}{n^{3/2}},$$

Theorem (Pólya)

In two dimensions, the symmetric random walk returns to the origin infinitely often. In three dimensions, the symmetric random walk returns to the origin only finitely many times.



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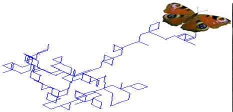
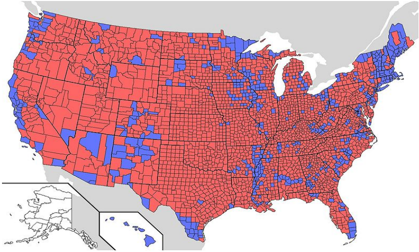
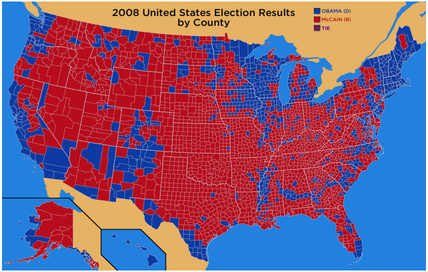


Voter Model

The voter model on a lattice is as follows: Color each lattice point in red or blue (or let each individual on a vertex vote red or blue). Now at each new time, pick an individual randomly and let her change opinions with a probability proportional to the number of neighbors with that opinion.



2004 presidential election map



2008 presidential election map

Voter Model Simulations

http://www.math.umaine.edu/~hiebeler/java/CA/Majority/Majority.html

http://www.math.utah.edu/~rbutler/votermodel/index.html

Voter Model

Theorem

- In 1 and 2 dimensions, the voter model converges to absolute consensus.
- In 3 dimensions and higher, differences of opinion can persist.

Percolation

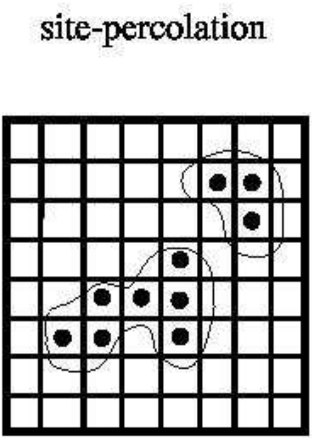
Consider one of the lattices seen at the beginning of this talk. It is composed of vertices and edges.

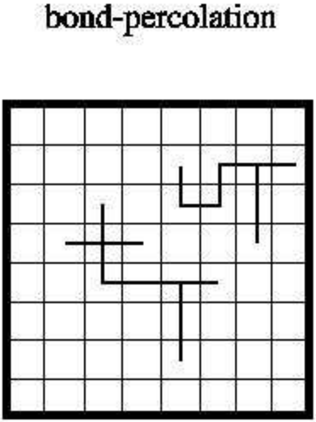
Imagine that you paint each vertex in one of two colors, randomly, and independently for each vertex. This is called *site percolation*.

The same thing can be done with edges rather than vertices. This is called *bond percolation*.

Often, one of the colors is the “invisible” color, that is, we either keep or remove the vertices (or edges).

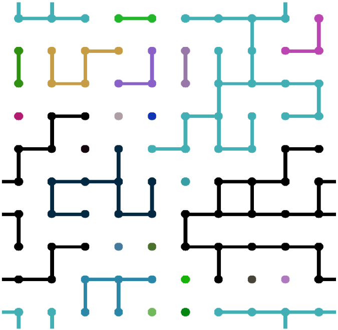
Percolation

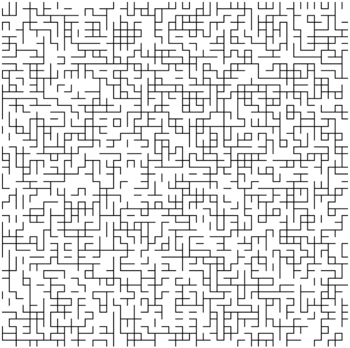




Percolation

When is there a path from left to right?





Questions

A related (but hard) question is the following:
When is there an infinite open cluster? The answer depends on the probability p of keeping an edge.
There is a value of p called p_c (p-critical) such that

- If $p < p_c$, there is no infinite cluster.
- If $p > p_c$, there is an infinite cluster.

The big questions of percolation theory are:

1. For a given lattice, what is p_c?
2. Is there an infinite cluster at p_c?

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Simulations

http://www.physics.buffalo.edu/gonsalves/Java/Percolation.html

http://www.svengato.com/forestfire.html

http://stat.math.uregina.ca/~kozdron/Simulations/Perc.html

http://www.ibiblio.org/e-notes/Perc/perc.htm

Answers

1.

lattice	p_c (site)	p_c (bond)
square	0.592746	$\frac{1}{2}(Kesten, 80's)$
triangular	$\frac{1}{2}$	$2\sin(\frac{\pi}{18})$
honeycomb	0.6962	$1 - 2\sin(\frac{\pi}{18})$

2. There are no infinite clusters at p_c in dimensions 2 and ≥ 19. Physicists believe that it is true in all dimensions, but no rigorous proof exists.

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